Sheet 32: Complex power series

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The best thing about complex numbers is that $d(z, w) = |z - w|$ turns it into a metric space, which is isometric to $\mathbb{R}^2$ endowed with d_2. So, we can apply all the topology, like compactness of bounded closed sets, all that we learned about sequences and series, uniform and pointwise convergence of functions and power series. In particular, all the theorems from the last sheet hold. Check which definitions make sense for complex numbers from this year. Prove the corresponding theorems.

For instance, a complex function (a function from $\mathbb{C}$ to $\mathbb{C}$) is differentiable at z, if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \text{ exists}$$

Of course, h is now also a complex number approaching 0. This is a much more restrictive notion than in the real setting; it turns out that even some of the nicest functions are not differentiable.

Proposition 1 *Let $f(z) = \overline{z}$. Then f is not differentiable at 0.*

Just as for real power series, it turns out that the set of convergence of a complex power series is always a ball (containing the interior and contained in the closure). In particular, show that the following power series converge everywhere.

Definition 2 *For a complex z let*

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \\ e^z &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \end{aligned}$$

Here goes one of the best jokes about complex numbers.

Theorem 3 *For all complex z we have*

$$e^{iz} = \cos z + i \sin z$$

What does this mean geometrically for real z?

In particular, the following hold:

Corollary 4 *We have*

$$e^{i\pi} = -1$$

Corollary 5 *For all complex z we have*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

This allows you to prove ALL the trigonomeric equalities with ease. Let us try.

Exercise 6 *Show that for arbitrary real numbers a, b, c the following holds:*

$$4\cos a\cos b\cos c = \cos(a+b+c) + \cos(a+b-c) + \cos(a-b+c) + \cos(-a+b+c)$$

Now it makes sense why the Taylor series

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots ?$$

has radius of convergence 1. Does it?

Complex analysis is one of the most beautiful subjects you will encounter in your regular studies. Let us mention just one theorem and a striking consequence.

Theorem 7 (Liouville) *Let f be complex function that is differentiable everywhere. If f is bounded then f is constant.*

Theorem 8 *Let $f(z) = \sum_{i=0}^{n} a_n z^n$ be a polynomial of degree $n \geq 0$. Then for each $\varepsilon > 0$ there exists $R > 0$ such that for all $|z| > R$ we have*

$$\left| \frac{1}{f(z)} \right| < \varepsilon.$$

Theorem 9 (Fundamental theorem of algebra) *Let $f(z) = \sum_{i=0}^{n} a_n z^n$ be a polynomial of degree $n \geq 1$. Then there exists a complex w such that $f(w) = 0$.*