Sheet 25: Complex numbers

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We already defined complex numbers as $\mathbb{R}[x]$ modulo x^2+1 but now we do it again.

Definition 1 *A complex number is an ordered pair of real numbers. The set of complex numbers is denoted by* $\mathbb{C}$.

So $\mathbb{C}$ as a set is simply $\mathbb{R}\times\mathbb{R}$. It is multiplication that distinguishes the two rings.

Definition 2 *For* $z_1=(a_1,b_1)$ *and* $z_2=(a_2,b_2)\in\mathbb{C}$ *let*

$$z_1+z_2=(a_1+a_2,b_1+b_2)$$

and let

$$z_1\cdot z_2=(a_1a_2-b_1b_2,a_1b_2+a_2b_1)$$

Now we have to check that this definition makes sense, that is, we get a ring.

Theorem 3 $\mathbb{C}$ *endowed with* $+$ *and* $\cdot$ *is a ring.*

Actually, this is the step that we could omit using the more abstract definition.

Definition 4 *The imaginary number*

$$\mathrm{i}=(0,1)$$

Now we identify real numbers with a subset of the complexes.

Let $\varphi:\mathbb{R}\to\mathbb{C}$ be defined by

$$\varphi(x)=(x,0)$$

Theorem 5 *The function* φ *is injective and for all* $x,y\in\mathbb{R}$ *we have* $\varphi(x+y)=\varphi(x)+\varphi(y)$ *and* $\varphi(xy)=\varphi(x)\cdot\varphi(y)$.

So from now on we treat real numbers as a special kind of complex numbers. The point is to get rid of the parentheses.

Lemma 6 *We have*

$$\mathrm{i}\cdot\mathrm{i}=-1$$

Definition 7 *Let $z = (a, b)$ be a complex number. Then the real part*

$$\operatorname{Re} z = a$$

and the imaginary part

$$\operatorname{Im} z = b$$

Lemma 8 *Let z be a complex number. Then we have*

$$z = \operatorname{Re} z + \mathrm{i} \cdot \operatorname{Im} z$$

As you will see, the form $a + b \cdot \mathrm{i}$ is much more convenient than the use of pairs. From now on we will omit $\cdot$ as well.

Definition 9 *Let $z = a + b\mathrm{i}$ be a complex number. Then the* complex conjugate *of z is*

$$\overline{z} = a - b\mathrm{i}$$

This implies that

$$z\overline{z} = a^2 + b^2$$

is a non-negative real number which is zero only if $z = 0$. This allows us to conveniently compute the multiplicative inverse of a nonzero complex number.

Lemma 10 *For $0 \neq z \in \mathbb{C}$ we have*

$$z \frac{\overline{z}}{z\overline{z}} = 1$$

That is,

$$z^{-1} = \frac{\overline{z}}{z\overline{z}}.$$

This turns complex numbers into a field.

Exercise 11 *Compute $(1 + \mathrm{i})/(2 + 3\mathrm{i})$.*

Definition 12 *For $z \in \mathbb{C}$ let the* absolute value *of z be*

$$|z| = \sqrt{z\overline{z}}$$

Figure out the geometric meaning of the conjugate and the absolute value.

Theorem 13 *Let z, w be complex numbers. Then the following hold:*
1) $\overline{\overline{z}} = z$;
2) $\overline{z} = z$ if and only if z is real;
3) $\overline{z + w} = \overline{z} + \overline{w}$;
4) $-\overline{z} = \overline{-z}$;
5) $\overline{zw} = \overline{z} \cdot \overline{w}$;
6) $\overline{z^{-1}} = \overline{z}^{-1}$ if $z \neq 0$;
7) $|z| = 0$ if and only if $z = 0$;
8) $|z + w| \leq |z| + |w|$;
9) $|zw| = |z|\,|w|$.

Let us go into the geometric meaning of multiplication now. For $0 \neq z \in \mathbb{C}$ the absolute value

$$\left|\frac{z}{|z|}\right| = 1$$

which means that $\frac{z}{|z|} = a + b\mathrm{i}$ where $a^2 + b^2 = 1$. This implies that there exists a real number α such that

$$\cos\alpha = \mathrm{Re}\,\frac{z}{|z|} \text{ and } \sin\alpha = \mathrm{Im}\,\frac{z}{|z|}$$

In other terms

$$z = |z|\,(\cos\alpha + \mathrm{i}\sin\alpha)$$

This is called the *trigonometric form* of a complex number.

Definition 14 *Any real number α satisfying the above equality is called an* argument *of z.*

The argument is of course not unique, but

Theorem 15 *If α and β are arguments of z then $\alpha - \beta = 2k\pi$ for some integer k.*

Note that 0 does not have an argument.

With this notation multiplication becomes beautiful.

Theorem 16 *Let $z = |z|\,(\cos\alpha + \mathrm{i}\sin\alpha)$ and $w = |w|\,(\cos\beta + \mathrm{i}\sin\beta)$. Then*

$$zw = |z|\,|w|\,(\cos(\alpha+\beta) + \mathrm{i}\sin(\alpha+\beta))$$

So when we multiply two complex numbers, we get the result by simply multiplying their absolute values and adding up their arguments.

Corollary 17 *Let $z = |z|\,(\cos\alpha + \mathrm{i}\sin\alpha)$. Then*

$$z^n = |z|^n\,(\cos n\alpha + \mathrm{i}\sin n\alpha)$$

Now we discuss taking n-th roots.

Definition 18 *A complex number z is an n-th* root of unity *if it satisfies*

$$z^n = 1$$

Theorem 19 *Let n be a natural number. Then there are exactly n n-th roots of unity, namely*

$$\varepsilon_{n,k} = \cos(k\frac{2\pi}{n}) + \mathrm{i}\sin(k\frac{2\pi}{n}) \quad (0 \leq k \leq n-1)$$

Draw these roots on the plane.

Just as for 1, every nonzero complex number has exactly n n-th root.

Theorem 20 *Let* $0 \neq z \in \mathbb{C}$ *and let* n *be an integer. Then there are exactly* n *complex numbers satisfying the equiality*

$$w^n = z$$

The following more general theorem also holds and simplifies the proof.

Theorem 21 *Let* $p(x) \in \mathbb{C}[x]$ *be a complex polynomial of degree* n*. Then* $p(x)$ *has at most* n *roots.*

In fact it will turn out that (counting multiplicities) $p(x)$ always has exactly n roots!

Exercise 22 *Where is the mistake in the following?*

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1$$

Exercise 23 *Let* u, w *be complex numbers. Find the complex numbers* z *such that* u, w, z *form a perfect (equilateral) triangle. Express the centers of these triangles.*

Exercise 24 *Take an arbitrary triangle. Draw a perfect triangle on all sides looking outside. Prove that the centers of these triangles form a perfect triangle.*

Exercise 25 *Compute* $(1 + \mathrm{i})^{2006}$.

Exercise 26 *What is the sum of the* n*-th roots of unity?*

Try to find as many proofs for this as you can.

Exercise 27 *What is the product of the* n*-th roots of unity?*

Exercise 28 *What is the sum of the squares of the* n*-th roots of unity?*