Sheet 24: Planar graphs

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On this sheet, all graphs are finite and simple. We start with a (vague) definition.

Definition 1 *A* planar graph *is a graph that can be drawn on the plane so that no edges intersect.*

Note that the edges do not have to be straight lines; they are allowed to be curves.

Exercise 2 *Find a graph that is not planar.*

Draw some connected planar graphs. Form conjectures. Define the faces of such a graph, following the intuition coming from polyhedra. Note that there is an infinite face.

Exercise 3 *A planar graph is a tree if and only if it has one face.*

Let V denote the set of vertices, E the set of edges and F the set of faces. Let $v = |V|$, $e = |E|$ and $f = |F|$.

Just like for polyhedra, in a connected planar graph, every face has at least three edges and every edge is connected to at most two faces.

Theorem 4 *In a connected planar graph on at least* 3 *vertices, we have* $3f \leq 2e$.

The following theorem is extremaly important.

Theorem 5 (Euler's formula) *In a connected planar graph we have*

$$v - e + f = 2$$

Corollary 6 *In a connected planar graph on at least* 3 *vertices, we have* $e \leq 3v - 6$.

Counting again.

Theorem 7 *In a connected planar graph there is always a vertex of degree at most* 5.

Definition 8 *We say that a graph* can be colored by c colors *if one can color the vertices with* c *colors in a way such that no neighbours have the same color.*

Theorem 9 *Let* (V, E) *be a graph and let* D *denote the maximal degree of a vertex. Then* G *can be colored by* $D + 1$ *colors.*

Definition 10 *A graph is* bipartite *if it can be colored by* 2 *colors.*

Exercise 11 *What is the maximal number of edges of a bipartite graph on* n *vertices?*

Exercise 12 *Show that a graph is bipartite if and only if it has no cycles of odd length.*

The most famous theorem about colorings is the following.

Theorem 13 (Four-color theorem) *Every planar graph can be colored by* 4 *colors.*

We will not prove this, however, using one of the previous theorems, one can show the following:

Theorem 14 (Six-color theorem) *Every planar graph can be colored by* 6 *colors.*