Sheet 2: The continuum

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The aim of this sheet is to start discussing C, the *continuum* and a relation $<$ on C. We call elements of C *points*. We will only describe C through some fundamental properties called *axioms*, which will be introduced one by one.

Only these axioms may be used in proving things about C.

Axiom 1 *There is at least one point in C.*

Axiom 2 *The relation $<$ is an ordering on C.*

Definition 1 (First and last point) *If $A \subseteq C$ is a subset, then a point $a \in A$ is called a* first point *of A if for all $x \in A$ either $x = a$ or $a < x$. "Last" may be defined analogously.*

Lemma 2 *Every non-empty finite set of points of C has a first point and a last point.*

Theorem 3 *Let A be a nonempty finite set of points of C. If A contains n points, then we can label them as $a_1, a_2 \ldots, a_n$ in such a way that $a_1 < a_2$, $a_2 < a_3$, $\ldots, a_{n-1} < a_n$. (In other words, $a_i < a_{i+1}$ for $i = 1, 2, \ldots, n-1$.)*

Definition 4 (Betweenness) *Let x, y, z be points of C. We say that y lies* between *x and z if $x < y$ and $y < z$.*

Corollary 5 *Of three distinct points, one always lies between the other two.*

Axiom 3 *The continuum C has no first point and no last point.*

Corollary 6 *C is infinite.*

Definition 7 (Region) *Let a, b be points in C such that $a < b$. The set of all points that lie between a and b is called the* region $(a; b)$.

Theorem 8 *For every point $p \in C$, there exists a region containing p.*

Definition 9 (Limit point) *Let $A \subseteq C$ be a subset. A point p is called a* limit point of *A if for every region R that contains p, R has at least one point in common with A other than p. In other words, for every region R such that $p \in R$, we have*

$$R \cap (A \setminus p) \neq \emptyset.$$

Note that a limit point p of A may or may not be an element of A; it makes no difference.

Theorem 10 *Let $A \subseteq B \subseteq C$ be subsets. If some point p is a limit point of A, then it is also a limit point of B.*

Definition 11 (Exterior) *Let $(a;b)$ be a region. Then $C \setminus (a;b) \setminus a \setminus b$ is called the* exterior *of $(a;b)$; the symbol is $\mathrm{ext}(a;b)$.*

Lemma 12 *For any region $(a;b)$, we have*

$$C = \{x \mid x < a\} \cup \{a\} \cup (a;b) \cup \{b\} \cup \{x|x > b\}.$$

Lemma 13 *For any region $(a;b)$, we have*

$$C = \mathrm{ext}(a;b) \cup (a;b) \cup \{a\} \cup \{b\}.$$

Lemma 14 *No point of the exterior of a region is a limit point of the region; no point of a region is a limit point of the exterior of the region.*

Theorem 15 *If two regions A and B have a point x in common, then $A \cap B$ is also a region containing x.*

Corollary 16 *If n regions $R_1, \ldots, R_n$ have a point x in common, then their intersection $R_1 \cap R_2 \cap \cdots \cap R_n$ is also a region containing x.*

Theorem 17 *Let A, B be arbitrary sets of points. If p is a limit point of the union $A \cup B$, then p is a limit point of at least one of the sets A, B. The converse implication is also true: if p is a limit point of A or of B, then it is a limit point of $A \cup B$.*

Corollary 18 *Let $A_1, \ldots, A_n$ be arbitrary subsets of C. If p is a limit point of the union $A_1 \cup A_2 \cup \cdots \cup A_n$, then it is also a limit point of at least one of the sets A_k. The converse implication is also true.*

Problem 19 *Find realizations of $(C, <)$, that is, concrete sets endowed with a relation $<$ that satisfies all the axioms so far. Are they the same? What does this question mean?*