Sheet 7: Compactness

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The aim of this sheet is to continue discussing C, the *continuum* by introducing compactness, one of the most important concepts in mathematics. Just as before,

Only axioms and already proved theorems may be used in proving things about C.

Definition 1 (Open cover) *Let $X \subseteq C$ be a set and let $\mathcal{A}$ be a set of subsets of C. We say that $\mathcal{A}$ is an* open cover for X *if for all $A \in \mathcal{A}$ the set A is open and*

$$X \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Important: the subsets in the cover need not be disjoint!

Exercise 2 *Let $p \in C$ be a point and let*

$$\mathcal{A} = \{\text{ext}(a;b) \mid p \in (a;b)\}$$

Show that $\mathcal{A}$ is an open cover for $C \backslash p$.

Of course, a cover need not be 'efficient'; it may happen that a proper subset of the cover still covers X.

Definition 3 (Subcover) *Let $\mathcal{A}$ be an open cover for X. A subset $\mathcal{B} \subseteq \mathcal{A}$ is a* subcover *if*

$$X \subseteq \bigcup_{B \in \mathcal{B}} B.$$

Let us understand this phenomenon via an example. Take $(\mathbb{Q}, <)$. For a moment do not worry about whether it is a model of C or not. Let

$$A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$$

Exercise 4 *Show that A is closed.*

Exercise 5 *Prove that every open cover of A has a finite subcover.*

Definition 6 (Compact set) *A set X is* compact *if every open cover of X has a finite subcover.*

As we just proved, the set X defined above is compact. Let us see some examples for non-compact sets.

Exercise 7 *Let $\mathcal{A}$ be the set of all regions. Show that no finite subset of $\mathcal{A}$ covers C.*

In particular, C is not compact. Note that some other open covers of C may have finite subcovers (find one), but one 'bad' open cover already ruins compactness.

Another example of a non-compact set is $C \backslash p$ for any point p.

Exercise 8 *Let $p \in C$ be a point and let*

$$\mathcal{A} = \{\text{ext}(a;b) \mid p \in (a;b)\}$$

Show that no finite subset of $\mathcal{A}$ covers $C \backslash p$.

We are ready to point out two main obstructions to being compact.

Theorem 9 (Compact sets are bounded) *If $X \subseteq C$ is not bounded then X is not compact.*

Theorem 10 (Compact sets are closed) *If $X \subseteq C$ is not closed then X is not compact.*

As it will turn out, these are the only obstructions, namely, closed and bounded subsets of C are compact!

To achieve this, first we have to understand covers consisting of regions. Moreover, we will first focus our attention to a special type of sets called closed intervals.

Definition 11 *For $a < b$ let the closed interval $[a;b]$ be defined as*

$$[a;b] = (a;b) \cup \{a\} \cup \{b\}$$

Exercise 12 *Closed intervals are closed.*

Definition 13 (Chain or regions) *Let $a < b$. A* chain of regions going from a to b *is defined as a finite sequence $R_1, R_2, \ldots, R_n$ of regions such that $a \in R_1$, $b \in R_n$ and for $1 \leq i \leq n-1$ we have $R_i \cap R_{i+1} \neq \emptyset$.*

Note that a chain is finite by definition.

Exercise 14 *A chain of regions from a to b covers the closed interval $[a;b]$.*

Theorem 15 *Let $a < b$ and let $\mathcal{A}$ be a set of regions that covers $[a; b]$. Let*

$$X = \{x \in [a; b] \mid \textit{there is a chain of regions } R_1, R_2, \ldots, R_n \in \mathcal{A} \textit{ going from } a \textit{ to } x\}$$

Then $\sup X = b$. Moreover, $b \in X$.

Theorem 16 (Closed intervals are compact with respect to regions) *Let $a < b$. Then any set of regions that covers $[a; b]$ has a finite subcover.*

Now all we have to do is to extend the result to arbitrary open covers (not just covers by regions) and then to covers of bounded closed sets (not just of closed intervals). Both are way easier than they sound. First we need a trick that allows us to substitute our open sets from the cover with suitable subregions, still a cover.

Let $a < b$ and let $\mathcal{A}$ be a set of open sets that covers $[a; b]$. Let

$$S = \{(c; d) \mid c < d, \text{ there exists } A \in \mathcal{A} \text{ with } (c; d) \subseteq A\}.$$

It turns out that these 'good' regions already cover $[a; b]$.

Theorem 17 *We have*

$$[a; b] \subseteq \bigcup_{(c;d) \in S} (c; d)$$

For $(c; d) \in S$ let

$$A_{(c;d)} \in \mathcal{A} \text{ such that } (c; d) \subseteq A_{(c;d)}.$$

Note that there may be many elements of $\mathcal{A}$ containing $(c; d)$; just choose one randomly.

Corollary 18 *We have*

$$[a; b] \subseteq \bigcup_{(c;d) \in S} A_{(c;d)}$$

And there we go.

Theorem 19 (Closed intervals are compact) *For $a < b$ the closed interval $[a; b]$ is compact.*

What is left is to show that bounded closed sets are compact as well. This is easier than it sounds. We need the following two trivialities.

Theorem 20 *Let $X \subseteq C$ be a closed set and let $\mathcal{A}$ be an open cover of X. Then $\mathcal{A} \cup \{C \backslash X\}$ is an open cover of C.*

Theorem 21 *Let $X \subseteq C$ be a set and let $\mathcal{B}$ be an open cover of X such that $C \backslash X \in \mathcal{B}$. Then $\mathcal{B} \backslash \{C \backslash X\}$ is an open cover of X.*

Theorem 22 (Bounded closed sets are compact) *Let $X \subseteq C$ be a bounded closed set. Then X is compact.*

Hint: of course C is not compact and at some point we have to use that X is bounded. (If you find this hint confusing, disregard it.)