Sheet 8: How to prove connectedness

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The aim of this sheet is to finish discussing C, the *continuum*.

The first, extremely important result is a corollary of compactness of closed intervals.

Theorem 1 *Let A be a bounded infinite set. Then A has a limit point.*

Hint: A is a subset of $[a;b]$ for some $a < b$. Assume indirectly that no point in $[a;b]$ is a limit point of A.

We have shown last quarter that Axiom 4 (together with the first three axioms) implies the following:

1) every bounded non-empty point set has a least upper bound;

2) For all points a, b with $a < b$ there exists c such that $a < c < b$.

In fact, it also works the other way round, that is, 1) and 2) together imply Axiom 4. This is very useful because these conditions are usually easier to check.

In the following we only assume that the first three axioms and 1) and 2) hold for C.

Theorem 2 *Let O be a nonempty bounded open set. Then $\sup O$ is a limit point of both O and the complement of O.*

Now assume that there is an open set $A \subseteq C$ which is open and closed, but $A \neq C$ and $A \neq \emptyset$. Let B be the complement of A. Let $a \in A$ and let $b \in B$. By changing the role of A and B, if necessary, we can assume that $a < b$.

Theorem 3 *Let*

$$s = \sup(A \cap (a;b)).$$

Then s is a limit point of both A and B.

And there we go.

Theorem 4 *If every bounded non-empty point set has a least upper bound and regions are non-empty, then the only sets that are open and closed at the same time are $\emptyset$ and C.*