Sheet 34: Linear maps

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The simplest example of a finite dimensional vectorspace over a field K is K itself. More generally, one can take K^n, the set of n-tuples over K. The aim of this sheet is to show that in a certain sense, thats all one can really do.

Definition 1 *Let V, W be vector spaces over the field K. The function $f : V \to W$ is a* linear map *(or homomorphism) if for all $v, v' \in V$ and $\alpha \in K$ we have*

$$f(v + v') = f(v) + f(v') \text{ and } f(\alpha v) = \alpha f(v).$$

We denote the set of homomophisms from V to W by $Hom_K(V, W)$.

Let $f : V \to W$ be a linear map. Then let

$$Imf = \{f(v) \mid v \in V\}$$

be the *image* and and let

$$\ker f = \{v \in V \mid f(v) = 0\}$$

the *kernel* of the map.

Exercise 2 *We have $Imf \leq W$ and $\ker f \leq V$ (that is they are subspaces).*

Exercise 3 *We have*

$$f(\langle X \rangle) = \langle f(X) \rangle$$

for any subset X of V. In particular, the image of a generating set generates the image subspace.

Exercise 4 *Let B be a basis in V and let $f : B \to W$ be an arbitrary map. Then f uniquely extends to a homomorphism from V to W.*

What is funny is that the set of homomorphisms themselves naturally form a vector space over K: if $f, g \in Hom_K(V, W)$ then let $(f + g)(v) = f(g) + g(v)$ and let $(\alpha f)(v) = \alpha f(v)$.

Exercise 5 *If V and W are finite dimensional, then*

$$\dim_K Hom_K(V, W) = \dim_K V \dim_K W.$$

Definition 6 *$f \in Hom_K(V, W)$ is an* isomorphism*, if it is bijective.*

If two vector spaces are isomorphic, then they are the same (you can not dictinguish them on the language of linear algebra).

Theorem 7 *The following are equivalent for $f \in Hom_K(V, W)$:*
1) f is injective;
2) $\ker f = \{0\}$;
3) the image of every independent subset is independent;
4) there exists a basis B in V such that f is injective on B and $f(B)$ is independent.

Theorem 8 *V and W are isomorphic if and only if they have the same dimension.*

That is, it makes sense to talk about 'the' k-dimensional vector space over K.