Sheet 35: Linear transformations

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Let V be a vector space over the field K. Linear maps from V to V are called *linear transformations*. Here is a typical way to produce one. Fix a basis $b_1, b_2, \ldots, b_n$ in V.

Let $A = (\alpha_{i,j})$ be an n by n matrix such that $\alpha_{i,j} \in K$. Let us define $f : B \to V$ by

$$f(b_i) = \sum_{j=1}^{n} a_{i,j} b_j$$

This uniquely extends to a linear transformation of V; let us call it f_A.

As it turns out, this method is very typical.

Let $f : V \to V$ be a linear transformation. Then for all i, $f(b_i)$ is a linear combination of some b_j-s, that is, there exist $\alpha_{i,j}$ such that

$$f(b_i) = \sum_{j=1}^{n} a_{i,j} b_j$$

We can put this data into an n by n matrix: let us denote $M_f = (\alpha_{i,j})$.

So, once we fixed a basis, linear transformations and matrices are just the same. Of course, one can write up the same transformation in different bases and get very different matrices.

Let us analyze the case when $K = \mathbb{R}$ and $n = 2$. It is easy to see that the Euclidean transformations fixing 0 are linear. Let us fix the basis $b_1 = (0, 1)$ and $b_2 = (1, 0)$.

Exercise 1 *Write up the matrices for the following:*
1) rotation by degree α*;*
2) reflection to the line spanned by (a, b)*;*
3) reflection to 0*.*

Of course, the linear transformation belonging to the zero matrix is not Euclidean; in general, Euclidean transformations are bijections of V. But this is not the only problem.

Exercise 2 *Find a* 2 *by* 2 *matrix* A *such that* f_A *is bijective but is not a Euclidean transformation.*

So, as of now, we can not really describe linear transformations for dimension 2. How about 1?

Exercise 3 *Let W be a 1 dimensional vector space over K. Let $f : W \to W$ a linear transformation. Then there exists $\alpha \in K$ such that $f(v) = \alpha v$ for all $v \in W$.*

Definition 4 *A subspace $W \leq V$ is* invariant under f *if $f(W) \subseteq W$.*

Definition 5 *A vector $0 \neq v \in V$ is an eigenvector with eigenvalue α if $f(v) = \alpha v$.*

Equivalently, if $\langle v \rangle$ is an invariant subspace under f with eigenvalue α.
Let us again analyze 2 by 2 matrices.

Exercise 6 *Find a real matrix that has no eigenvalues.*

However, this can not happen over the complex numbers.

Exercise 7 *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary complex matrix.
1) When is 0 an eigenvalue of A? (that is, when is A not a bijection)?
2) Find the eigenvalues of A;

Let us go back to just linear maps $f : V \to W$. There again, one can associate a matrix to f by fixing a basis *both* in V and W. Define it for yourself.

Let U, V, W be vector spaces of finite dimension over K and let $f : U \to V$, $g : V \to W$ be linear maps. Fix bases B_U, B_V, B_W in the corresponding spaces. Then $g \circ f : U \to W$ is also a linear map. What is the matrix of $g \circ f$ with respect to the fixed bases B_U, B_W?

Definition 8 *Let $A = (\alpha_{i,j})$ be an n by m matrix and let $B = (\beta_{i,j})$ be a m by l matrix. Then $AB = (\gamma_{i,j})$, the* product *of A and B is an n by l matrix such that*

$$\gamma_{i,j} = \sum_{k=1}^{m} \alpha_{i,k}\beta_{k,j}$$

Draw a picture that allows you to do matrix multiplication on a piece of paper!

Exercise 9 *We have*

$$M_{g \circ f} = M_f M_g$$

When f is bijective, it has an inverse f^{-1} that is also a linear transformation (check this).

Exercise 10 *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that 0 is not an eigenvalue of A. Write up the matrix of the inverse of A.

How to compute the matrix of the inverse in general? How to find the eigenvalues/eigenvectors of a transformation? To aswer these questions, one needs the notion of determinant.

Finally, here is a good joke.

Exercise 11 *Let*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and let $v = (a, b)$*. What is* vA*? What is* $(1, 1)A^n$*?*