Sheet 17: Moore about metric spaces

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The aim of this sheet is to continue discussing topological and metric spaces, in particular, $\mathbb{R}^n$.

Warmup.

Theorem 1 *Let* (X, d) *be a metric space and let* (a_n) *be a sequence in* X*. Then* $\lim_{n\to\infty} a_n = a$ *if and only if* $\lim_{n\to\infty} d(a_n, a) = 0$*.*

Definition 2 *Let* $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$ *denote the set of real* n*-tuples.*

One can define many different metrics on $\mathbb{R}^n$.

Definition 3 *For* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ *let*

$$d_0(\mathbf{a}, \mathbf{b}) = \max_{1\le i\le n} |a_i - b_i|$$

let

$$d_1(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} |a_i - b_i|$$

and let

$$d_2(\mathbf{a}, \mathbf{b}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

Theorem 4 *The functions* d_0*,* d_1 *and* d_2 *are all metrics on* $\mathbb{R}^n$*.*

Actually, d_2 is what we call the *standard metric* on $\mathbb{R}^n$. So, from now on, when we talk about $\mathbb{R}^n$ in itself, we mean $\mathbb{R}^n$ endowed with the metric d_2.

The following theorem shows that although balls are not the same in these metrics, they fit nicely into one another. Draw a picture for the $n = 2$ for all the balls of radius 1 centered at the origin!

Theorem 5 *For all* $0 \le i, j \le 2$*, for all* $x \in \mathbb{R}^n$ *and* $r > 0$ *there exists* $r' > 0$ *such that*

$$B_{d_i}(x, r') \subseteq B_{d_j}(x, r)$$

Corollary 6 *The metrics d_0, d_1 and d_2 generate the same topology on $\mathbb{R}^n$, namely, a subset $A \subseteq \mathbb{R}^n$ is open in $(\mathbb{R}^n, d_i)$ if it is open in $(\mathbb{R}^n, d_j)$ ($0 \leq i, j \leq 2$).*

Just as in $\mathbb{R}$, a subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed. One can derive this directly, playing with coverings and showing that the product of compact spaces is compact, or using the following general notion of 'having no holes' in a metric space.

Definition 7 *Let (X, d) be a metric space. A sequence (a_n) on X has the* Cauchy property *(or, it is a Cauchy-sequence) if for all $\epsilon > 0$ there exists N such that for all $n, m > N$ we have $d(a_n, a_m) < \epsilon$.*

Definition 8 *A metric space (X, d) is* complete *if every Cauchy-sequence on X is convergent.*

As we already proved, $\mathbb{R}$ is complete. Also, $[0, 1]$ with the absolute value metric is complete, while $(0, 1)$ is not.

Theorem 9 *$\mathbb{R}^n$ is complete.*

Hint: Take a Cauchy-sequence of n-tuples. What will be the i-th coordinate of the limit?

One can also derive this as a corollary of a much more general theorem, once you know that bounded closed subsets of $\mathbb{R}^n$ are compact.

Theorem 10 *Every compact metric space is complete.*

Hint: this is an old story – you have seen it.

OK, so let us try and turn it around, just for $\mathbb{R}^n$.

Theorem 11 *Let (a_n) be a bounded sequence in $\mathbb{R}^d$. Show that (a_n) has a convergent subsequence.*

Hint: you have already proved this, I guess.

Now we make our lifes easier (and more miserable at the same time). As you may have observed, this happens very often in math.

A ball in $\mathbb{R}^d$ is *rational*, if all the coordinates of its center are rational and its ratio is rational.

Theorem 12 *Show that a set $A \subseteq \mathbb{R}^d$ is open if and only if for all $x \in A$ there is a rational ball O such that $x \in O$ and $O \subseteq A$.*

Theorem 13 *Let C be a closed, bounded subset of $\mathbb{R}^d$ and let $\mathcal{A}$ be an open cover for C. Then $\mathcal{A}$ has a countable subcover.*

And there you go.

Theorem 14 *Closed, bounded subsets of* $\mathbb{R}^d$ *are compact.*

Hint: assume not. Then you can find a nice subcover.. then a nice sequence.. then a nice contradiction.

Remark. If you have any metric space, you can make it complete using the so-called Cauchy completion. The deal is the following: let (X, d) be a metric space. Let Y be the set of Cauchy sequences from X. Define a relation $\backsim$ on Y as follows: $(a_n) \backsim (b_n)$ if $\lim_{n\to\infty} d(a_n, b_n) = 0$. Then $\backsim$ is an equivalence relation. Let X be the set of equivalence classes. Now we can define a metric d' on Z as follows:

$$d'(\overline{(a_n)}, \overline{(b_n)}) = \lim_{n\to\infty} d(a_n, b_n)$$

(this exists since $\mathbb{R}$ is complete). Also, X has a natural injective map into Z using constant sequences and d and d' coincide on this. Finally, X is dense in Z and Z is complete. When X has some operators, like addition or multiplication, it will naturally extend to Z. Work out the details for yourself.

This method also gives you an alternate way to define real numbers.