Sheet 14: Sequences again

Miklós Abért

The aim of this sheet is to continue discussing sequences.

If you are given a sequence and you can only look at its elements, how do you decide whether it is convergent? Here is a cool way.

Definition 1 (Cauchy sequence) *We say that a sequence (a_n) is a* Cauchy sequence*, if the following holds. For each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $|a_n - a_m| < \varepsilon$.*

Lemma 2 *Every convergent sequence has the Cauchy property.*

Lemma 3 *Let (a_n) be a Cauchy sequence, and let $(b_k = a_{n_k})$ be a subsequence. Prove that if (b_k) converges then so does (a_n) itself.*

Lemma 4 *Every Cauchy sequence is bounded.*

Theorem 5 (Cauchy convergence theorem) *A sequence is convergent if and only if it is Cauchy.*

Now we will define the liminf and limsup of a sequence and state some basic results about them.

Definition 6 *Let (a_n) be a bounded sequence, and A the set of its accumulation points. We define its* limes inferior*, $\liminf_{n\to\infty} a_n$, to be the first point of A, and the* limes superior*, $\limsup_{n\to\infty} a_n$, to be the last point of A.*

Corollary 7 *Let (a_n) be a bounded sequence. Then $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ and equality holds if and only if the sequence is convergent.*

Like often in math, the name is not a coincidence.

Theorem 8 *Let (a_n) be a bounded sequence. Then*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf\{a_k \mid k > n\}\right)$$

and

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup\{a_k \mid k > n\}\right)$$

Theorem 9 *Let (a_n) be a bounded sequence. Then*

$$\liminf_{n\to\infty} a_n = \sup\{x \mid \textit{there are finitely many } n \textit{ with } a_n \in (-\infty, x)\}$$

and

$$\limsup_{n\to\infty} a_n = \inf\{x \mid \textit{there are finitely many } n \textit{ with } a_n \in (x, \infty)\}$$