Sheet 31: Taylor series

Miklós Abért

On this sheet we introduce power series and the Taylor series of a function.

Definition 1 *A function of the form*

$$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$$

is called a power series centered at a.

The Weierstrass M-test is ideal to analyze functions of this form.

Theorem 2 *Suppose that the series*

$$\sum_{n=0}^{\infty}a_n x_0^n$$

converges and let $0<a<|x_0|$. *Then on* $B(0,a)$ *the series*

$$f(x)=\sum_{n=0}^{\infty}a_n x^n$$

and

$$g(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$$

uniformly and absolutely converge. Also f *is differentiable and*

$$f'(x)=g(x)$$

for all $x\in B(0,a)$.

Hint: look at $(|x|/a)^n$.

Why do we use the balls $B(0,a)$ in this theorem instead of just saying $[-a,a]$? We shall see later.

The theorem tells us that the set of points where a power series converges is basically a ball.

Theorem 3 *For a power series* $\sum_{n=0}^{\infty} a_n x^n$ *let*

$$A = \left\{ x \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}$$

be the set of convergence *of the power series. Then either* A *is everything or there exists* a *such that*

$$B(0,a) \subseteq C \subseteq \overline{B(0,a)}$$

This a *is called the* radius of convergence *of the power series.*

So, inside the ball the series absolutely converges while outside it diverges. What happens at the boundary? Almost anything.

Exercise 4 *Find real power series centered at* 0 *with sets of convergence being:* $0, \mathbb{R}, (-1,1), [-1,1), [-1,1]$.

The above theorem allows us to play with power series with no difficulty. Say, the sum of two power series will be the formal sum of the two series. For the product, a similar result holds, but for that we need to extend our knowledge on absolutely convergent series.

Theorem 5 *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge absolutely and* $\{c_n\}$ *is a sequence containing the products* $a_i b_j$ *for each pair* (i,j) *then*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right)$$

For power series this gives the following.

Theorem 6 (Cauchy product) *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *be power series with radius of convergence at least* a*. Let*

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

Then the power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

has radius of convergence at least a *and for* $x \in B(0,a)$ *we have*

$$h(x) = f(x)g(x)$$

Now we can introduce Taylor series and have some fast results.

Definition 7 *Let f be a real function such that $f^{(n)}(a)$ exists for all n. Then the* Taylor series of f at a *is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Of course, the Taylor series may or may not be equal to $f(x)$. Equation holds if and only if the remainder terms in the Taylor polynomials tend to 0.

Theorem 8 *For all real x we have*

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\end{aligned}$$

It is strange that e^x looks so similar to a trigonometric series.. this is not a coincidence.

Theorem 9 *For $x \in [-1,1)$ we have*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

It also turns out that the Taylor series of a power series is the power series itself.

Theorem 10 *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ be convergent in $B(a,r)$ for some $r > 0$. Then the Taylor series of $f(x)$ at a equals $\sum_{n=0}^{\infty} a_n(x-a)^n$.*

There is another strangeness here: the radius of convergence for $\sum_{n=0}^{\infty} x^n$ is 1 which makes sense since $\frac{1}{1-x}$ has a discontinuity at 1. But how about

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots ?$$

The Taylor series has radius of convergence 1 again, while $f(x)$ itself is a nice, smooth real function everywhere on the real line. So what causes its Taylor series to 'stop' at 1? Go figure!