Sheet 30: Uniform limits

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The aim of this sheet is to provide preliminaries for Taylor series.

We have defined on a previous sheet when a sequence of functions converges uniformly to another function and proved that the uniform limit of continuous functions is continuous. To make the sheet complete, we quote them here.

Definition 1 *Let (f_n) be a sequence of functions defined on A and let f be defined on A. Then f is the uniform limit of (f_n) (or $\lim_{n\to\infty} f_n = f$) if for all $\epsilon > 0$ there exists N such that for all $n > N$ and for all $x \in [a,b]$ we have $|f(x) - f_n(x)| < \epsilon$.*

Theorem 2 *Let (f_n) be a sequence of continuous functions on $[a,b]$ that uniformly converges to f on $[a,b]$. Then f is continuous on $[a,b]$.*

We have a slightly weaker theorem for integrals.

Theorem 3 *Let (f_n) be a sequence of functions which are integrable on $[a,b]$ and that (f_n) uniformly converges to f on $[a,b]$ which is integrable on $[a,b]$. Then*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

Of course, this makes one ask if the uniform limit of integrable functions is always integrable.

Exercise 4 *Let (f_n) be a sequence of functions which are integrable on $[a,b]$ and that (f_n) uniformly converges to f on $[a,b]$. Is f integrable on $[a,b]$?*

Sadly, derivatives are not so nice in this sense.

Exercise 5 *Find a sequence of differentiable functions that uniformly converge to $f(x) = |x|$ on $[-1,1]$.*

Even if the uniform limit is differentiable, we may be in trouble.

Exercise 6 *Let*

$$f_n(x) = \frac{1}{n}\sin(n^2 x)$$

Then f_n uniformly converges to $f = 0$ but $\lim_{n\to\infty} f_n'$ does not exist.

We can still save something ugly which will be enough for power series. Make sure you completely understand the assumptions and why they are needed.

Theorem 7 *Let* (f_n) *be a sequence of functions which are differentiable on* $[a,b]$*, with integrable derivatives* f_n' *and that* (f_n) *pointwise converges to* f *on* $[a,b]$*. Suppose that* f_n' *uniformly converges on* $[a,b]$ *to some continuous function* g*. Then* f *is differentiable and for all* $x \in [a,b]$ *we have*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Hint: the proof is shorter than the statement.

Now we can define convergence of a series of functions and prove the corresponding results.

Definition 8 *The series* $\sum_{n=1}^{\infty} f_n$ *converges uniformly to* f *on* A *if the sequence of partial sums* $s_n = \sum_{i=1}^{n} f_i$ *converges to* f *uniformly.*

Theorem 9 *Let* $\sum_{n=1}^{\infty} f_n$ *converge uniformly to* f *on* $[a,b]$*.*
1) If each f_n *is continuous on* $[a,b]$ *then* f *is continuous on* $[a,b]$*.*
2) If f *and each* f_n *is integrable on* $[a,b]$ *then*

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$$

3) If each f_n *has integrable derivative and* $\sum_{n=1}^{\infty} f_n'$ *converges unformly on* $[a,b]$ *to some continuous function then for all* $x \in [a,b]$ *we have*

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

To be able to use this we will need a strong condition that ensures uniform convergence.

Theorem 10 (Weierstrass M-test) *Let* f_n *be a sequence of functions defined on* A *and suppose* $|f_n|$ *is bounded by* M_n *on* A*. Suppose that* $\sum_{n=1}^{\infty} M_n$ *converges. Then for all* $x \in A$ *the series* $\sum_{n=1}^{\infty} f_n(x)$ *absolutely converges and* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on* A *to the function*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Among other things, this theorem allows one to construct a function which is continuous everywhere and differentiable nowhere.