RESEARCH NOTES

VOLUME I

Research Notes

Volume I

Important Note:

These notes, in this and later volumes are not a complete record, are not corrected and are not entirely chronological. They do start in 1971.

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$A_5 \simeq L_2(5)$

$$
\begin{array}{r|ccccc}
 & 1 & 2 & 3 & 5 & 5 \\
\hline
 & 1 & 1 & 1 & 1 & 1 \\
 & 3_1 & -1 & 0 & 1+\lambda+\lambda^4 & 1+\lambda^2+\lambda^3 \\
\text{ord} & 3_2 & -1 & 0 & 1+\lambda^2+\lambda^3 & 1+\lambda+\lambda^4 \\
 & 5 & 1 & -1 & 0 & 0 \\
 & 4 & 0 & 1 & -1 & -1 \\
\hline
 & 1 & \cdot & 1 & 1 & 1 \\
\text{mod} & 2_1 & \cdot & -1 & \lambda+\lambda^4 & \lambda^2+\lambda^3 \\
 & 2_2 & \cdot & -1 & \lambda^2+\lambda^3 & \lambda+\lambda^4 \\
 & 4 & \cdot & 1 & -1 & -1
\end{array}
\qquad
D \quad \begin{array}{|cccc}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\qquad
C = \begin{pmatrix}
4 & 2 & 2 & 0 \\
2 & 2 & 1 & 0 \\
2 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$2_1 \otimes 2_2 = 4 \qquad 2_1 \otimes 2_1 \doteq 1 \oplus 1 \oplus 2_2$

$\mathrm{Hom}(2_1 \otimes 2_1, 1) \simeq \mathrm{Hom}(2_1, 2_1^* \otimes 1) \cong \mathrm{Hom}(2_1, 2_1 \otimes 1) \simeq K$

Also $\mathrm{Hom}(1, 2_1 \otimes 2_1) \simeq K$.

$\mathrm{Hom}(2_1 \otimes 2_1, 2_2) \simeq \mathrm{Hom}(2_1, 2_1 \otimes 2_2) = 0$.

Thus get $2_1 \otimes 2_1 = \boxed{\begin{smallmatrix} 1 \\ 2_2 \\ 1 \end{smallmatrix}}$ uniserial, using self-dualness.

Thus get existence of $\begin{smallmatrix} & 1 & \\ 2_1 & \oplus & 2_2 \\ 1 & & 1 \end{smallmatrix}$. By dimension ker of P_1 to this is non-trivial so has bottom 1 in it. Also by duality, $P_1/1$ has 2_1 and 2_2 in submodule. P_1/Ker does not so these are in kernel. Hence P_1 is ext of $\frac{2_1 \;\; 2_2}{1}$ by $\begin{smallmatrix} & 1 & \\ 2_1 & \oplus & 2_2 \\ 1 & & 1 \end{smallmatrix}$

What about P_{2_1}. Get, by similar args $\begin{smallmatrix} 2_1 \\ 1 \\ 2_2 \\ 1 \\ 2_1 \end{smallmatrix}$ or $\begin{smallmatrix} 2_1 \\ 1+1+2_2 \\ 2_1 \end{smallmatrix}$

But in latter case get $\frac{1}{2_1 \; 2_1}$ and $\frac{1}{2_2 \; 2_2}$ i.e. ext of P_1 given above must split. i.e. $\frac{1 \;\; 1}{1}$ at bottom, contradicting $A_5 = O^2(A_5)$. Hence get P_{2_1}, P_{2_2} now get P_1 easily using bottom part of P_{2_1}, P_{2_2}.

$A_6 \simeq L_2(9)$

ord

	1	2	3_1	3_2	4	5	5
	1	1	1	1	1	1	1
	5	1	2	−1	−1	0	0
	5	1	−1	2	−1	0	0
	9	1	0	0	1	−1	−1
	10	−2	1	1	0	0	0
	8	0	−1	−1	0	$1+\lambda+\lambda^4$	$1+\lambda^2+\lambda^3$
	8	0	−1	−1	0	$1+\lambda^2+\lambda^3$	$1+\lambda+\lambda^4$

mod (GF(2))

	1	2	3_1	3_2	4	5	5
	1	·	1	1	·	1	1
	4	·	1	−2	·	−1	−1
	4	·	−2	1	·	−1	−1
	8	·	−1	−1	·	$1+\lambda+\lambda^4$	$1+\lambda^2+\lambda^3$
	8	·	−1	−1	·	$1+\lambda^2+\lambda^3$	$1+\lambda+\lambda^4$

} perm reps $\subseteq L_4(2)$? ∴ abs irred & irred.

Dec #'s $D_0(2)$:

	1	4	4
1	1		
5	1	1	
5	1		1
9	1	1	1
10	2	1	1

$$C = \begin{array}{c} \\ 1 \\ 4 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 4 & 4 \end{array} \\ \begin{pmatrix} 8 & 4 & 4 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{pmatrix} \end{array}$$

Guess: P_4: $\begin{array}{c} 4 \\ 1 \\ 4' \\ 1 \\ 4 \\ 1 \\ 4' \\ 1 \\ 4 \end{array}$ $P_{4'}$: $\begin{array}{c} 4' \\ 1 \\ 4 \\ 1 \\ 4' \\ 1 \\ 4 \\ 1 \\ 4' \end{array}$ then ⇒ P_1: $\begin{array}{c} 1 \\ \hline \begin{array}{c|c} 4 & 4' \\ 1 & 1 \\ 4' & 4 \\ 1 \;\oplus & 1 \\ 4 & 4' \\ 1 & 1 \\ 4' & 4 \end{array} \\ \hline 1 \end{array}$

say $q \equiv 1\ (4)$

$N(S_q) = N$, $\lambda^2 = 1$, char of N

$\lambda^{\uparrow G} = \chi_1 + \chi_2$ irred of deg $\frac{q+1}{2}$

But mod 2, $\lambda^G = 1^G = 1 + \chi$

χ_1, χ_2 conj under aut $\Rightarrow$ χ_i mod 2 $= 1 + \ldots$

But min degree is $\frac{q-1}{2}$ by N. $\therefore$ $\chi_i = 1 +$ irred
deg $\frac{q-1}{2}$ & not'l

Now have $4 \otimes 4' = 8_1 \oplus 8_2$ by check of Brauer characters.

$\therefore$ $\mathrm{Ext}(4', 4) \simeq \mathrm{Ext}(1, \underbrace{4'^{*}}_{4'} \otimes 4) \simeq \mathrm{Ext}(1, 8 \oplus 8') = 0$. (*)

$\therefore$ no extension $\frac{4'}{4}$, $\frac{4}{4'}$, etc.

Also by direct calc, $4 \otimes 4$ has comp factors $1, 1, 1, 1, 4, 4', 4'$.

$\mathrm{Hom}(4 \otimes 4, 1) \simeq \mathrm{Hom}(4, 4^*) \simeq \mathrm{Hom}(4, 4) \simeq GF(2)$

$\mathrm{Hom}(4 \otimes 4, 4') \simeq \mathrm{Hom}(4, 4^* \otimes 4') \simeq \mathrm{Hom}(4, 4 \otimes 4') = 0$.

$\mathrm{Hom}(4 \otimes 4, 4) \simeq \mathrm{Hom}(4, 4 \otimes 4)$. Hence as 4 appears once, either both are zero or 4 is direct summand of $4 \otimes 4$.

(*) Hence Z_4 is rep by J_4 in 4 & $4'$
(Or else is rep by $J_3 + J_1$ in both — as conj under aut gp —
and $(J_3 \oplus J_1) \otimes (J_3 \oplus J_1) = J_3 \otimes J_3 \oplus J_3 \oplus J_3 \oplus J_1$
$\neq 4 J_4$)

Suppose 4 is direct summand of $4 \otimes 4$. Then

have $4 \otimes 4 = 4 \oplus \dfrac{1}{\overline{\ \ }\ 1}$ } comp factors $4', 4', 1, 1$. } call this M.

now $\mathrm{Hom}(M,1) = \mathrm{Hom}(1,M) = 0$. M is self dual so get $M = \dfrac{4'}{\dfrac{1 \oplus 1}{4'}}$

we noted both maps

Now Z_4 is sup of J_4 on 4 / so is $4J_4$ on $4 \otimes 4$. $\therefore$ is $3J_4$

on $\begin{array}{c} 1 \\ \hline 4' \\ \hline 1 \oplus 1 \\ \hline 4' \\ \hline 1 \end{array}$ Hence as $4'$ splits off for J_4, at most 2 f.p.

in $\begin{array}{c} 1 \oplus 1 \\ \hline 4' \\ \hline 1 \end{array}$. But then for Z_4 is $4' \oplus \dfrac{1 \oplus 1}{1}$ so get ≥ 3 f.p.

a contrad. Hence, $\mathrm{Hom}(4 \otimes 4, 4) = \mathrm{Hom}(4, 4 \otimes 4) = 0$.

But there is a better approach.

__Lemma 1__ In no comp series of $4 \otimes 4$ are there two successive 1's.

Pf If so, regarding as Z_4 module, pull out the 4 dim subs so $4 \otimes 4 = A \oplus B$, A is free, B has $1 \oplus 1$ section (as 1 followed by 1 in $4 \otimes 4$ must be $1 \oplus 1$), $\therefore$ B is not J_4 so has 2 dim of f.p. $\therefore$ ≥ 5 dim of f.p. in $4 \otimes 4$ contradicting $4 \otimes 4$ is free as Z_4 module.

__Lemma 2__ In no comp series of $4 \otimes 4$ are there two successive 4-dim factors.

Pf Say $4 \otimes 4 \supseteq A \supset B \supseteq 0$, $A \supset X \supset B$, A/X, X/B 4 dim irred. $\therefore$ $4 \otimes 4/A$ and B share one four between them. $\therefore$ the one with no four is at most one 1. $\therefore$ other has a four and at least three 1's. $\therefore$ this violates Lemma 1.

Lemma 3 $4 \otimes 4$ is uniserial.

Pf. Start a comp series and go as far as possible as long as there is no choice. If get started then get to situation $\dfrac{X}{Y \oplus Z}$ which violates the previous lemmas. If not have $X \oplus Y$ at top so get $\dfrac{X \oplus Y}{2}$ also a contradiction.

Now can finish same Lemma 1 at top and bottom so get

$$4 \otimes 4 = \begin{array}{c} 1 \\ \hline 4' \\ | \\ 4 \\ | \\ 4' \\ \hline 1 \end{array} \quad \text{as } (4 \otimes 4)^* = 4 \otimes 4.$$

Now want structure of projectives. From $4 \otimes 4$ & $4' \otimes 4'$ get

$$\begin{array}{cc} \multicolumn{2}{c}{1} \\ \hline 4 & 4' \\ | & | \\ 4' & 4 \\ | & | \\ 4 & 4' \\ 1 & 1 \end{array}$$

and as for A_5 get P_1 ext of $\dfrac{4\ \ 4'}{1}$ by that.

$\dfrac{4\,4'}{1}$ is at bottom or else a 4 drops by and get $\dfrac{1}{1}$ at bottom of P_1

and get that $Ext(1,4) \simeq Ext(1,4') \simeq \mathbb{Z}_2$.

Let's look at P_4:

$$\begin{array}{c} 4 \\ \hline | \\ 4' \\ | \\ 4 \\ | \\ \\ \\ \hline 4 \end{array} \left.\vphantom{\begin{array}{c} | \\ 4' \\ | \\ 4 \\ | \\ \\ \end{array}}\right\} M_4$$

where $\begin{array}{c} | \\ 4' \\ | \\ 4 \\ | \end{array} = M_4/K$. Now there is $1/4$ submodule. If that intersects K ~~trivially~~ (i.e. in 4) then M_4 has 1 and - so $K/4$ has comp factors $4', 1$ — 1 or 4! But $Ext(1,4) \simeq \mathbb{Z}_2$, $ext(4',4) = 0$. $\therefore$ have $1 \subseteq K$.

(see page 27 for another argument).

Better: as no 4' submodule of M_4 by structure so far obtained for P_4, get M_4 has unique 1 at bottom. ∴ above that is 4' and that is either unique or there is 1 over 1 also impossible. ∴ get P_4 as desired. also $P_{4'}$. Now P_1 also follows.

Perhaps also works when q is Fermat. Try $q = 17$.

Guess 1, 8, 8'.

$$C = \begin{pmatrix} 16 & 8 & 8 \\ 8 & 5 & 4 \\ 8 & 4 & 5 \end{pmatrix}$$

Z_4 on top of Z_{17} gets Z_2 acting freely so Z_8 does too as 1 class of invol.

again get $8 \otimes 8$ uniserial. But what is order of factors. Have four 8''s and three 8's.

Nevertheless from $8 \otimes 8 + 8' \otimes 8'$ get P_1 is ~~of~~ $\frac{8\ 8'}{1}$ at top with no "1" slipping by so $Ext(1,8) = Ext(1,8') \simeq Z_2$.

<u>Lemma</u> Let U be uniserial with alternate 1 & 8 dim factors with 1 at the bottom. Then $Ext(1,U) \simeq 0$ or Z_2 as 1 or 8 dim module is at top.

<u>Pf.</u> By induction on comp length of U as $U = 1$ is ok as G is simple. Let V be max submodule of U so

$$Hom(1, U/V) \to Ext(1,V) \to Ext(1,U) \to Ext(1, U/V)$$

is exact. Say U/V is 8 dim so V has 1 at top by hypothesis so $Ext(1,V) = 0$. also $Ext(1, U/V) = Z_2$ by above so

$$0 \to Ext(1,U) \to Z_2$$

Say U/V is 1 dim. Now have then $Hom(1,U) \to Hom(1,U/V)$ and this sends $Hom(1,V) \to 0$ as 1 in U is at bottom in V.

to get $\quad 0 \to Z_2 \to Ext(1,V) \to Ext(1,U) \to 0$.

By ind, $Ext(1,V)$ is 0 or Z_2 so (get is Z_2) $Ext(1,U) = 0$.

<u>Cor</u> $Ext(1, 8 \otimes 8) = Ext(1, 8' \otimes 8') = 0$

$Ext(8,8) = Ext(8',8') = 0$. How to finish? see p28

Let's look at $q \equiv 3\ (8)$ so $B_0(2)$ consists of characters of degrees $1, \frac{q-1}{2}, \frac{q-1}{2}, q$ with 2 complex conjugate. ∴ mod degrees are $1, \frac{q-1}{2}, \frac{q-1}{2}$ and by calc (should do it out)

$$D: \begin{array}{c|ccc} & 1 & \frac{q-1}{2} & \frac{q-1}{2} \\ \hline 1 & 1 & & \\ \frac{q-1}{2} & & 1 & \\ \frac{q-1}{2} & & & 1 \\ q & 1 & 1 & 1 \end{array} \qquad C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Now $\frac{q-1}{2}, \frac{q-1}{2}$ are duals, so standard arguments give

$$P_1 : \frac{\;1\;}{\dfrac{V \oplus V^*}{1}} \qquad P_V : \frac{V}{\dfrac{1 + V^*}{V}} \qquad P_{V^*} : \frac{V^*}{\dfrac{1 + V}{V^*}}$$

Proj mods:

Proj: $\begin{matrix} V^* \\ 1\ V \\ V^* \end{matrix} \quad \begin{matrix} 1 \\ V\ V^* \\ 1 \end{matrix} \quad \begin{matrix} V \\ V^*\ 1 \\ V \end{matrix} \quad \Bigg| \quad \begin{matrix} V \\ V^*\ 1 \\ V \end{matrix} \quad \begin{matrix} V^* \\ 1\ V \\ V^* \end{matrix} \quad \Bigg| \quad \begin{matrix} 1 \\ V\ V^* \\ 1 \end{matrix} \quad \Bigg| \quad 1$

Kernels: etc. $\leftarrow$ $\quad \begin{matrix} 1\ V \\ V^* \end{matrix} = \begin{matrix} V\ V^* \\ 1 \end{matrix} = \begin{matrix} V^*\ 1 \\ V \end{matrix} \quad \Bigg| \quad \dfrac{V^*\ 1}{V} = \dfrac{1\ V}{V^*} \quad \Bigg| \quad \dfrac{V\ V^*}{1}$

Remark: $V \otimes V$, $V \otimes V^*$ should be easy from Brauer char, just need in B_0 + other blocks (undoubtedly projectives)

A_7

A_6	#	1	45	40	40	90	72	72
		1	2	3_1	3_2	4	5	5
red/cho		1	1	1	1	1	1	1
		5	1	2	-1	-1	0	0
		5	1	-1	2	-1	0	0
		9	1	0	0	1	-1	-1
		10	-2	1	1	0	0	0
		8	0	-1	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
		8	0	-1	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
Proj/cho		40	0	4	4	0	0	0
		24	0	3	0	0	-1	-1
		24	0	0	3	0	-1	-1

$L_3(2)$	1	21	56	42	24	24
	1	2	3	4	7	7
	1	1	1	1	1	1
ord ind	3	-1	0	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$
	3	-1	0	1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
	6	2	0	0	-1	-1
	7	-1	1	-1	0	0
Proj/cho	8	0	-1	0	1	1
	8	0	2	0	1	1
	16	0	1	0	$\frac{-3+\sqrt{-7}}{2}$	$\frac{-3-\sqrt{-7}}{2}$
	16	0	1	0	$\frac{-3-\sqrt{-7}}{2}$	$\frac{-3+\sqrt{-7}}{2}$

A_7		1	105	70	280	630	504	210	360	360		
		1	2	3_1	3_2	4	5	6	7	7	$B_0(2)$	$B_1(2)$
Ord irrd	χ_1	1	1	1	1	1	1	1	1	1	✓	
	χ_2	6	2	3	0	0	1	−1	−1	−1		✓
	χ_3	14	2	2	−1	0	−1	2	0	0	✓	
	χ_4	15	−1	3	0	−1	0	−1	1	1	✓	
	χ_5	14	2	−1	2	0	−1	−1	0	0		✓
	χ_6	35	−1	−1	−1	1	0	−1	0	0	✓	
	χ_7	21	1	−3	0	−1	1	1	0	0	✓	
	χ_8	10	−2	1	1	0	0	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$		✓
	$\bar{\chi}_8=\chi_9$	10	−2	1	1	0	0	1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$		✓
Induced proj from A_6		280	0	16	4	0	0	0	0	0		
		168	0	12	0	0	−2	0	0	0		
		168	0	0	3	0	−2	0	0	0		
		56	0	−4	−1	0	1	0	0	0		
		56	0	−4	−1	0	1	0	0	0		

Projectives (remainder): $\chi_6+\chi_7$, $\chi_2+2\chi_3+2\chi_4+\chi_5+2\chi_6+\chi_8+\chi_9$, $\chi_3+\chi_4+2\chi_5+2\chi_6+\chi_7+\chi_8+\chi_9$, $\chi_1+2\chi_2+2\chi_3+3\chi_4+2\chi_5+3\chi_6+\chi_7+2\chi_8+2\chi_9$

Induced proj from $L_3(2)$	120	0	0	−3	0	0	0	1	1
	120	0	0	6	0	0	0	1	1
	240	0	0	3	0	0	0	$\frac{-3+\sqrt{-7}}{2}$	$\frac{-3-\sqrt{-7}}{2}$
	240	0	0	3	0	0	0	$\frac{-3-\sqrt{-7}}{2}$	$\frac{-3+\sqrt{-7}}{2}$

Now perhaps we can guess the indecomposable projectives. Say:

$$\chi_6 + \chi_7, \quad \chi_3 + \chi_4 + \chi_6, \quad \chi_1 + \chi_4 + \chi_6 + \chi_7$$

Would give

D: ← deduce last two modular degrees in $B_0(2)$, using entries.

	1	14	20
χ_1	1	0	0
χ_3	0	1	0
χ_4	1	1	0
χ_6	1	1	1
χ_7	1	0	1

Then

$$C: \quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

Can deal with $B_1(2)$ in any case.

	1	2	3_1	3_2	4	5	6	7	7
	1		1	1		1		1	1
V	4		-2	1		-1		$\frac{1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$
V^*	4		-2	1		-1		$\frac{1-\sqrt{-7}}{2}$	$\frac{1+\sqrt{-7}}{2}$
$V\wedge V$	6		3	0		1		-1	-1

Now, V, V^* writable / $GF(2)$ so $\therefore$ abs irreducible as must be irred / $GF(2)$. Claim: $V, V^* \in B_1(2)$. Otherwise $1, V, V^*$ are the mod irred of $B_0(2)$. Can't then calculate D. Now $\overline{\chi_2} = V\wedge V$ and $\chi_2 \in B_1(2)$ so $V\wedge V$ is also. Just need it is absolutely irred. First, is given in $GF(2)$ and by using 7-elt it is irred or $3 \oplus 3$. But $A_7 \notin GL(3,2)$. $\therefore$ is irred / $GF(2)$. Now can't be reducible over larger field or get 2 irred in $B_1(2)$ giving too many.

Now we calc.:

$$D: \quad \begin{array}{c|c|c|c|} & V & V^* & V\wedge V \\ \chi_2 & 0 & 0 & 1 \\ \chi_5 & 1 & 1 & 1 \\ \chi_8 & 1 & 0 & 1 \\ \chi_9 & 0 & 1 & 1 \end{array} \qquad C: \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Next, claim $\bar{\chi}_3$ is irreducible. First, note that $V \otimes V^*$ has a middle — throw away 1 at top and 1 at bottom — with this Brauer character. Now $V \otimes V^*$ writable in $GF(2)$ as V, V^* are $\therefore$ have $GF(2)$ reps with this Brauer character:

1	3_1	3_2	5	7	7
14	2	-1	-1	0	0

Restricting to $L_3(2)$

1	3_2	7	7
1	1	1	1
3	0	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
3	0	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
8	-1	1	1
14	-1	0	0

as $14/L_3(2) \doteq 3_1 \oplus 3_2 \oplus 8$. Now A_7 not in $L_3(2)$ so $\therefore$ 14 is irred as stated over $GF(2)$ and $\doteq$ abs irred, again using $14 = 3+3+8$, or the 14 is a 6 and an 8. Both must be in B_0 as χ_3 is in B_0. Hence, have so far

1	3_1	3_2	5	7	7
1	1	1	1	1	1
8		-1		1	1
6		0		-1	-1

Now all these must be abs irred as no more Brauer char in B_0.

If we try and express χ_7 in terms of these, linear equations are $21 = a + 8b + 6c$, $0 = a - b$, $0 = a + b - c$ so $a = 1$, $b = 1$, $c = 2$.
$\therefore$ if 8 + 6 take values β, γ on 3_1 get $\beta + 2\gamma = -3$. But the six is writable over $GF(2)$ so by rep of Z_3 over $GF(2)$ get $\gamma = -3, 0, 3, 6$.
Hence $\gamma = 0$, $\beta = -3$. But also, $\beta = -4, -1, 2, 5$ or 8.
a contradiction.

Have so far

$$D: \begin{array}{c|ccc} & 1 & 14 & ? \\ \hline \chi_1 & 1 & 0 & 0 \\ \chi_3 & 0 & 1 & 0 \\ \chi_4 & 1 & 1 & 0 \\ \chi_6 & & & 1 \\ \chi_7 & & & 1 \end{array}$$

But $2\chi_3 + 2\chi_4 + 2\chi_6$ is proj. so must have $P_{14} = \chi_3 + \chi_4 + \chi_6$.
Two cases only now using projectives constructed:

(deduce this from D)

Case 1

$$D \quad \begin{array}{c|ccc} & 1 & 14 & 20 \\ \hline \chi_1 & 1 & 0 & 0 \\ \chi_3 & 0 & 1 & 0 \\ \chi_4 & 1 & 1 & 0 \\ \chi_6 & 1 & 1 & 1 \\ \chi_7 & 1 & 0 & 1 \end{array}$$

$$C = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

Case 2

	1	14	21
χ_1	1	0	0
χ_3	0	1	0
χ_4	1	1	0
χ_6	0	1	1
χ_7	0	0	1

$$C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Let's eliminate this case. On A_6 we have

	1	40	40	72	72
	1	3_1	3_2	5	5
$4_2^{A_6}$	4	-2	1	-1	-1

Hence, on A_7

	1	70	280	504		
	1	3_1	3_2	5	7	7
$4_2^{A_7}$	28	-8	1	-2	0	0
modular characters deduced	1	1	1	1	1	1
	14	2	-1	-1	0	0
	21	-3	0	1	0	0
	4	-2	1	-1	$\frac{1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$
	4	-2	1	-1	$\frac{1-\sqrt{-7}}{2}$	$\frac{1+\sqrt{-7}}{2}$
	6	3	0	1	-1	-1

$(= -1 + 21 + 4 + 4$ by inspection, R.A.A.$)$

Must solve:

$$28 = a + 14b + 21c + 8d + 6e$$
$$-8 = a + 2b - 3c - 4d + 3e$$
$$1 = a - b + 2d$$
$$-2 = a - b + c - 2d + e$$
$$0 = a + d - e$$

$$\left.\begin{aligned} 28 &= 7a + 14b + 21c + 14d \\ -8 &= 4a + 2b - 3c - d \\ 1 &= a - b + 2d \\ -2 &= 2a - b + c - d \end{aligned}\right\} \Rightarrow \begin{aligned} -12 &= 9a + 6b - 3c \\ -15 &= 9a + 3b - 6c \\ -3 &= 5a - 3b + 2c \end{aligned}$$

Hence $1 = b + c$. Since want $b, c \geq 0$, 2 cases. If $b=1$, $c=0$ get $a=0$ from last equation, $a=-2$ from second. Hence, $b=0$, $c=1$ so $a=-1$ contradiction.

Hence have

$$D \quad \begin{array}{ccc} 1 & 14 & 20 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array} \qquad C = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

		1	20	280	504	360	360	
		1	3_1	3_2	5	7	7	
	φ_1	1	1	1	1	1	1	
modular characters	φ_2	14	2	-1	-1	0	0	$B_0(2)$
	φ_3	20	-4	-1	0	-1	-1	
	φ_4	4	-2	1	-1	$\frac{1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$	
	φ_5	4	-2	1	-1	$\frac{1-\sqrt{-7}}{2}$	$\frac{1+\sqrt{-7}}{2}$	$B_1(2)$
	φ_6	6	3	0	1	-1	-1	

<u>Prop</u> $P_1 : \begin{array}{c} 1 \\ \hline 20 \quad 14 \\ 1 \quad\;\; 1 \\ 14 \quad 20 \\ \hline 1 \end{array}$; $P_{14} : \begin{array}{c} 14 \\ \hline 1 \\ 20 \quad 14 \\ 1 \\ \hline 14 \end{array}$; $P_{20} \begin{array}{c} 20 \\ \hline 1 \\ 14 \\ 1 \\ \hline 20 \end{array}$

<u>Pf</u> First suppose P_{20} is uniserial. Since all multiplicities are self-dual get P_{20} as given above. Next, let's get structure of P_{14}. Existence of $\begin{array}{c}14\\1\\20\end{array}$ and fact 20 only appears once in P_{14} means middle of P_{14} has no 20 at top. ∴ has 1 and perhaps a 14. But if 14 at top then at bottom so is direct summand so only appears once. Then other summand has top 1's & a 20; a quotient of it is $\begin{array}{c}1\\20\end{array}$ and it is self dual. ∴ is $\begin{array}{c}1\\20\\1\end{array}$ and P_{14} is as desired. ∴ to get P_{14} have to eliminate the possibility that only 1 appears at the top of P_{14}. Self-duality now forces the middle to be $\begin{array}{c}1\\ \hline 14 \quad 20 \\ \hline 1\end{array}$. <u>Let's come back to this</u>.

Now we consider other possible structures for P_{20}. Self-duality now implies P_{20} : $20/1+1+14/20$. Now we consider P_{14}. We have $14/20$ and only 1 "20" in P_{14} so self-duality implies 20 is a direct summand of the "middle" of P_{14}. Two cases:

<u>Case a</u> $P_{14} : \begin{array}{c} 14 \\ \hline 20+1+1+14 \\ \hline 14 \end{array}$ Now using P_{20} & P_{14} get $P_1 : \begin{array}{c} 1 \\ \hline 14+14+20+20 \\ \vdots \end{array}$

all direct summands of the middle; ∴ contradict no $\frac{1}{1}$ by simplicity.

<u>Case b</u> $P_{14} : \begin{array}{c} 14 \\ \hline 20 + \begin{array}{c}1\\14\\1\end{array} \\ \hline 14 \end{array}$ Now $P_1 : \begin{array}{c}1\\14\\1\\14\end{array} + \begin{array}{c}1\\ 20\;20\end{array}$ gives: $\begin{array}{c} 1 \\ \boxed{\begin{array}{c}14\\1\\14\end{array}} + 20 + 20 \\ \hline 1 \end{array}$

where the square is ext of 1 by $\begin{array}{c}14\\14\end{array}$, as one "1" is left over. Contradicts no $\frac{1}{1}$.

Hence, P_{20} is as stated in first paragraph of the proof.

Now back to case in first paragraph. Note that if P_{14} is right the P_1 follows easily from existence of $\begin{array}{c}1\\14\\1\\20\end{array}$ $\begin{array}{c}1\\20\\1\\14\end{array}$.

∴ left: <u>Eliminate</u> $P_{14} : \begin{array}{c} 14 \\ \hline 1 \\ 14+20 \\ 1 \\ \hline 14 \end{array}$.

Let's determine relevant restrictions to A_6.

First we look at $\varphi_2 | A_6$. First note that

$\varphi_4 \varphi_5 = 2\varphi_1 + \varphi_2$. Since $\varphi_5 = \overline{\varphi_4}$, if V_4, V_5 are corresponding modules, then $V_4 \otimes V_4^*$ has 1 at "top" and "bottom" - as being isomorphic to $\mathrm{Hom}(V_4, V_4)$. Hence, $\varphi_2 | A_6$ is in the heart of $V_4 | A_6 \otimes (V_4 | A_6)^*$ - at least is the "middle" of it.

But $\varphi_4 | A_6 = \varphi_5 | A_6 =$ "4_2" and $4_2 \otimes 4_2$ is uniserial with factors $1, 4_1, 1, 4_2, 1, 4_1, 1$. Hence

$$V_2 | A_6 = \boxed{\begin{matrix} 4_1 \\ | \\ 4_2 \\ | \\ 4_1 \end{matrix}}$$

(Note: Another way to get some info. is as follows. Calc comp factors of $V_2 | A_6$ readily. Then want, as a start $\mathrm{Hom}_{KA_6}(1, V_2 | A_6) = 0$. That is, $\mathrm{Hom}_{KA_7}(1^{A_7}, V_2) = 0$, ie. $\mathrm{Hom}_{KA_7}(1 + 6, V_2) = 0$ etc.)

Next consider φ_3. Have easily $\varphi_3 | A_6 = 4_2 + 8_1 + 8_2$ so as have three different blocks, get

$$V_3 | A_6 = 4_2 \oplus 8_1 \oplus 8_2.$$

Now, let calculate projectives of A_6

	1	3_1	3_2	5	5
P_1	40	4	4	0	0
P_{4_1}	24	3	0	-1	-1
P_{4_2}	24	0	3	-1	-1
P_{8_1}	8	-1	-1	$\frac{1+\sqrt5}{2}$	$\frac{1-\sqrt5}{2}$
P_{8_2}	8	-1	-1	$\frac{1-\sqrt5}{2}$	$\frac{1+\sqrt5}{2}$
$P_{14}(A_7) \| A_6$	64	4	-2	-1	-1

Hence $P_{14}(A_7)\,|\,A_6 = P_{4_1} \oplus P_{4_1} \oplus P_{8_1} \oplus P_{8_2}$.

$$= \begin{matrix} 4_1 \\ 1 \\ 4_2 \\ 1 \\ 4_1 \\ 1 \\ 4_2 \\ 1 \\ 4_1 \end{matrix} \oplus \begin{matrix} 4_1 \\ 1 \\ 4_2 \\ 1 \\ 4_1 \\ 1 \\ 4_2 \\ 1 \\ 4_1 \end{matrix} \oplus 8_1 \oplus 8_2.$$

This doesn't seem to help at least as far as distinguishing the possibilities.

More on A_7 - see pages 29, 30.

See page 31 for a proof of the prop of p16 on structure of projections in $B_0(2)$.

A_8

#	1	112	210	1344	2520	1120	1680	2880	2880	1260	3360	1344	1344	105
Order	1	3_s	2_s	5	4_s	3_e	6_s	7	7	4_e	6_e	15	15	2_e
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	4	3	2	1	1	0	0	0	-1	-1	-1	-1	-1
χ_3	20	5	4	0	0	-1	1	-1	-1	0	1	0	0	4
χ_4	21	6	1	1	-1	0	-2	0	0	1	0	1	1	-3
χ_5	28	1	4	-2	0	1	1	0	0	0	-1	1	1	-4
χ_6	64	4	0	-1	0	-2	0	1	1	0	0	-1	-1	0
χ_7	35	5	-5	0	-1	2	1	0	0	-1	0	0	0	3
χ_8	14	-1	2	-1	0	2	-1	0	0	2	0	-1	-1	6
χ_9	70	-5	2	0	0	1	-1	0	0	-2	1	0	0	-2
χ_{10}	56	-4	0	1	0	-1	0	0	0	0	-1	1	1	8
χ_{11}	45	0	-3	0	1	0	0	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$	1	0	0	0	-3
χ_{12}	45	0	-3	0	1	0	0	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$	1	0	0	0	-3
χ_{13}	21	-3	1	1	-1	0	1	0	0	1	0	$-\frac{1}{2}+\frac{\sqrt{-15}}{2}$	$-\frac{1}{2}-\frac{\sqrt{-15}}{2}$	-3
χ_{14}	21	-3	1	1	-1	0	1	0	0	1	0	$-\frac{1}{2}-\frac{\sqrt{-15}}{2}$	$-\frac{1}{2}+\frac{\sqrt{-15}}{2}$	-3
φ_1	1	1		1		1		1	1			1	1	
φ_2	4	-2		-1		1		$\frac{1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$			$\frac{1+\sqrt{-15}}{2}$	$\frac{1-\sqrt{-15}}{2}$	
$\overline{\varphi_2} = \varphi_3$	4	-2		-1		1		$\frac{1-\sqrt{-7}}{2}$	$\frac{1+\sqrt{-7}}{2}$			$\frac{1-\sqrt{-15}}{2}$	$\frac{1+\sqrt{-15}}{2}$	
$\varphi_2 \wedge \varphi_3 \wedge \varphi_3 = \varphi_4$	6	3		1		0		-1	-1			-2	-2	
see page 20 φ_5	20	-4		0		-1		-1	-1			$-\frac{3}{2}+\frac{\sqrt{-15}}{2}$	$-\frac{3}{2}-\frac{\sqrt{-15}}{2}$	
φ_6	20	-4		0		-1		-1	-1			$-\frac{3}{2}-\frac{\sqrt{-15}}{2}$	$-\frac{3}{2}+\frac{\sqrt{-15}}{2}$	
(from $\varphi_2 \otimes \varphi_3$) φ_7	14	2		-1		-1		0	0			2	2	
(induced by A_7) φ_8	64	4		-1		-2		1	1			-1	-1	

D:

	1	4	4	6	??	?	14	64	Notes: all modular reps.
1	1	0	0	0	0	0	0	0	
7	1	0	0	1	0	0	0	0	
20	0	0	0	1	0	0	1	0	
21	1	0	0	1	0	0	1	0	
28	0	1	1	1	0	0	1	0	
64	0	0	0	0	0	0	0	1	
35	1	1	1	2	0	0	1	0	
14	0	1	1	1	0	0	0	0	
70	2	1	1	1	1	1	1	0	$\overline{70} = \overline{56} + \overline{14}$
56	2	0	0	0	1	1	1	0	$\overline{56} = 14 + \overline{21}_1 + \overline{21}_2$
45	1	0	1	1	1	0	1	0	$\overline{45}_1 = 4_2 + 6 + 14 + \overline{21}_1$
45	1	1	0	1	0	1	1	0	$\overline{45}_2 = 4_1 + 6 + 14 + \overline{21}_2$
21	1	0	0	0	1	0	0	0	$\overline{21}\|_{A_7} = 1+6+14$ so no 4 dim bits.
21	1	0	0	0	0	1	0	0	

see page 21 see next page see page 18 see page 20

C:

	1	4	4	6	20	20	14	64
1	16	4	4	8	6	6	8	0
4	4	5	4	6	1	2	4	0
4	4	4	5	6	2	1	4	0
6	8	6	6	12	2	2	8	0
20	6	1	2	2	4	2	3	0
20	6	2	1	2	2	4	3	0
14	8	4	4	8	3	3	8	0
64	0	0	0	0	0	0	0	1

In $\underline{\underline{A_7}}$ let $P = \chi_2 + \chi_5 + \chi_8 + \chi_9$, a projective indecomposable.

	1	70	504	280	360	360
	1	3_1	5	3_2	7	7
P	40	4	0	4	-2	-2

	1	3_1	3_2	7	7
$\therefore P^{A_8}$	320	20	8	-2	-2

To calculate decomposition

	1	3_1	3_2	7+7	mult
	320	2240	8960	-11520	
χ_1	1	1	1	1	0
χ_2	7	4	1	0	1
χ_3	20	5	-1	-1	1
χ_4	21	6	0	0	1
χ_5	28	1	1	0	1
χ_6	64	4	-2	1	0
χ_7	35	5	2	0	2
χ_8	14	-1	2	0	1
χ_9	70	-5	1	0	1
χ_{10}	56	-4	-1	0	0
χ_{11}	45	0	0	$-1/2$	1
χ_{12}	45	0	0	$-1/2$	1
χ_{13}	21	-3	0	0	0
χ_{14}	21	-3	0	0	0

Easy inspection of D now yields this is proj. indec for 6 dim mod.

In A_7 let $Q = \chi_6 + \chi_7$, a projective indecomposable.

	1	70	504	280	360	360
	1	3_1	5	3_2	7	7
Q	56	-4	1	-1	0	0

	1	3_1	5	3_2
$Q^{A_8} = P$	448	-20	3	-2

To calculate inner products:

	1	3_1	5	3_2	mult
	448	-2240	4032	-2240	
χ_1	1	1	1	1	0
χ_2	7	4	2	1	0
χ_3	20	5	0	-1	0
χ_4	21	6	1	0	0
χ_5	28	1	-2	1	0
χ_6	64	4	-1	-2	1
χ_7	35	5	0	2	0
χ_8	14	-1	-1	2	0
χ_9	70	-5	0	1	2
χ_{10}	56	-4	1	-1	2
χ_{11}	45	0	0	0	1
χ_{12}	45	0	0	0	1
χ_{13}	21	-3	1	0	1
χ_{14}	21	-3	1	0	1

Now easily get columns in D for unknown degrees.
But could get directly, as $\overline{\chi_{13}}$, $\overline{\chi_{14}}$ each have one new character, a complex conjugate pair by values of $\chi_{11} - \chi_{14}$.

In A_7 let $R = \chi_1 + \chi_4 + \chi_6 + \chi_7$ so R is projective

$$\begin{array}{c|cccccc} & 1 & 70 & 504 & 280 & 360 & 360 \\ & 1 & 3_1 & 5 & 3_2 & 7 & 7 \\ \hline R & 72 & 0 & 2 & 0 & 2 & 2 \end{array}$$

$$\begin{array}{c|ccccc} & 1 & 3_1 & 5 & 7 & 7 \\ \hline R^{A_8} & 576 & 0 & 6 & 2 & 2 \end{array}$$

To calc inner products - in 7+7 here put sum not average

	1	5	7+7	mult
	576	8064	5760	
χ_1	1	1	2	1
χ_2	7	2	0	1
χ_3	20	0	-2	0
χ_4	21	1	0	1
χ_5	28	-2	0	0
χ_6	64	-1	2	2
χ_7	35	0	0	1
χ_8	-14	-1	0	0
χ_9	-70	0	0	2
χ_{10}	56	1	0	2
χ_{11}	45	0	-1	1
χ_{12}	45	0	-1	1
χ_{13}	21	1	0	1
χ_{14}	21	1	0	1

So P_7 or $P_7 + P_? + P_?$. But not clear yet.
(modulo 2 χ_6)

In A_7 let $S = \chi_3 + \chi_4 + \chi_6$, a projective

	1	70	504	280	360	360
	1	3_1	5	3_2	7	7
S	64	4	-1	-2	1	1

	1	3_1	5	3_2	7	7
S^{A_8}	512	20	-3	-4	1	1

To calculate inner products – in 7+7 use sum

	1	3_1	5	3_2	7+7	mult
	512	2240	4032	-4480	2880	
χ_1	1	1	1	1	2	0
χ_2	7	4	2	1	0	0
χ_3	20	5	0	-1	-2	1
χ_4	21	6	1	0	0	1
χ_5	28	1	-2	1	0	1
χ_6	64	4	-1	-2	2	3
χ_7	35	5	0	2	0	1
χ_8	14	-1	-1	2	0	0
χ_9	70	-5	0	1	0	1
χ_{10}	56	-4	1	-1	0	1
χ_{11}	45	0	0	0	-1	1
χ_{12}	45	0	0	0	-1	1
χ_{13}	21	-3	1	0	0	0
χ_{14}	21	-3	1	0	0	0

Must be P_{14} (less $3 \cdot \chi_6$)

Only ambiguity left is P_1. Say $\bar{\chi}_{13}, \bar{\chi}_{14}$ irreducible. Then $P_1 = \chi_1 + \chi_2 + \chi_4 + \chi_7$ (see page 21). Let's calculate non-zero values:

	1	3_1	5	3_2	7	7	15	15
P_1	64	16	4	4	1	1	1	1

Thus

	1	3_1	3_2	5	7	7
$P_1 \mid A_7$	64	16	4	4	1	1

also $P_1 \mid A_7$ is projective. Now indecomposable projectives for A_7 have degrees as follows: 72, 64, 56, 24, 24, 40. Hence $P_1 \mid A_7$ is the "64" or the sum of a 24 and the 40. But only the 24's are irrational, Hence $P_1 \mid A_7$ is the 64. But the values of the 64 are:

1	3_1	3_2	5	7	7
64	4	-2	-1	1	1

Thus have P_1 and rest follows.

another approach: Say two unknown degrees are less than 20. (Know they are non-real Brauer characters so degrees are equal) Since there is mod irred of A_8 restricted to A_7 having 20-dim component, get $\varphi_8 | A_7$ contains 20-dim mod irred of A_7 and no other φ_i does.

now on A_7

	1	3_1	3_2	5	7	7	mult
$\varphi_1 \mid A_7$	64	4	-2	-1	1	1	
	1	1	1	1	1	1	2
	4	-2	1	-1	$\frac{1+\sqrt{-7}}{2}$	$\frac{1-\sqrt{-7}}{2}$	0
	4	-2	1	-1	$\frac{1-\sqrt{-7}}{2}$	$\frac{1+\sqrt{-7}}{2}$	0
	6	3	0	1	-1	-1	0
	14	2	-1	-1	0	0	3
	20	-4	-1	0	-1	-1	1

∴ reg rep of A_8 restricted to A_7 has 20-dim irred exactly 64 times. ∴ reg rep of A_7 has 20 dim exactly 8 times. But $8 \neq 20$, a contradiction.

<u>On A_6</u>:

We know $4 \otimes 4$ is $\begin{smallmatrix}4'\\|\\4\\|\\4'\end{smallmatrix}$ uniserial. Hence we have existence of a uniserial $\begin{smallmatrix}1\\4\\1\end{smallmatrix}$. Now consider

$\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4$. It has $1 \otimes 1$ as submodule extended by $\begin{smallmatrix}1\\4\end{smallmatrix} \otimes 4$ which is $4 \otimes 4$ extended by $1 \otimes 4$. Claim: It is uniserial. The picture

$$\begin{matrix}\underline{\underline{4}}\\|\\4'\\|\\4\\|\\4'\\|\\ \hline 4\end{matrix}$$

Enough to show $\mathrm{Hom}(1, \begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4) = \mathrm{Hom}(\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4, 1) = 0$.

But $\mathrm{Hom}(1, \begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4) \simeq \mathrm{Hom}(4^*, \begin{smallmatrix}1\\4\\1\end{smallmatrix}) = \mathrm{Hom}(4', \begin{smallmatrix}1\\4\\1\end{smallmatrix}) = 0$.

also $\mathrm{Hom}(\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4, 1) \simeq \mathrm{Hom}(\begin{smallmatrix}1\\4\\1\end{smallmatrix}, 4^*) = \mathrm{Hom}(\begin{smallmatrix}1\\4\\1\end{smallmatrix}, 4') = 0$.

Done now as this uniserial is image of P_4 and has same dimension.

On $L_2(q)$, q a Fermat number.

We do $q = 17$ though the argument is general. We know $8 \otimes 8$ is uniserial. As on page 27 we have $\begin{array}{c}1\\8\\1\end{array} \otimes 8$ uniserial so it is therefore P_8. Similarly get $P_{8'}$. Thus, suppose $\begin{array}{c}8\\|\\8\end{array}$ exists. Then no $\begin{array}{c}8\\|\\8'\end{array}$ so P_8: $\begin{array}{c}8\\|\\8\\|\\8\end{array}\Big\} \longrightarrow$ as can't be $\begin{array}{c}8\\|\\8'\end{array}$.

Now everything falls into place easily.

On A_7 again

		1	3_1	3_2	5	7	7
1	P_1	72	0	0	2	2	2
	P_{14}	64	4	-2	-1	1	1
	P_{20}	56	-4	-1	1	0	0
	P_4	24	0	3	-1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
	$\bar{P}_4$	24	0	3	-1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
	P_6	40	4	4	0	-2	-2

projectives

Restriction to A_6

	Q_1	Q_4	Q_4'	Q_8	$\bar{Q}_8$
P_1	1	0	0	2	2
P_{14}	0	2	0	1	1
P_{20}	0	0	1	2	2
P_4	0	0	1	0	0
$\bar{P}_4$	0	0	1	0	0
P_6	1	0	0	0	0

Now, dot product of subscripts of P's & given col. under Q_i should be $7 \cdot i$, that is 7 times Q subscript. Checks!

Now get structure of P_V, P_{V^*}, $P_{V\wedge V}$ in other block.

$V|A_6 = V^*|A_6 = 4'$, $\quad V\wedge V\,|\,A_6 \doteq 1+1+4$.

Now

$$P_V|A_6 = Q_{4'} = \begin{array}{c} 4' \\ | \\ 4 \\ | \\ 4' \\ | \\ 4 \end{array}$$

Hence, $P_V = \boxed{\begin{array}{c} V \\ \hline V\wedge V \\ \hline V^* \\ \hline V\wedge V \\ \hline V \end{array}}$ $\quad\therefore\quad$ $P_{V^*} = \boxed{\begin{array}{c} V^* \\ \hline V\wedge V \\ \hline V \\ \hline V\wedge V \\ \hline V^* \end{array}}$

Thus,

$$P_{V\wedge V} = \begin{array}{c} \overline{\hspace{4em} V\wedge V \hspace{4em}} \\ \left|\begin{array}{c} V \\ V\wedge V \\ V^* \end{array}\right| \oplus \left|\begin{array}{c} V^* \\ V\wedge V \\ V \end{array}\right| \\ \underline{\hspace{4em} V\wedge V \hspace{4em}} \end{array},$$

Get a periodic resolution:

$$V \longrightarrow \begin{array}{c} V \\ V\wedge V \\ V^* \\ V\wedge V \\ V \end{array} \longrightarrow \begin{array}{cc} \multicolumn{2}{c}{V\wedge V} \\ V & V^* \\ V\wedge V & V\wedge V \\ V^* & V \\ \multicolumn{2}{c}{V\wedge V} \end{array} \longrightarrow \begin{array}{c} V \\ V\wedge V \\ V^* \\ V\wedge V \\ V \end{array} \longrightarrow V$$

Say P_{14} has structure $\begin{matrix}14\\ \hline 1\\ \hline 14+20\\ \hline 1\\ \hline 14\end{matrix}$ by which we mean P_{14} has exactly two composition series - or, what is the same here, the middle of P_{14} has a unique maximal submodule and a unique minimal submodule. Hence $\begin{matrix}1\\ \hline 14+20\\ \hline 1\\ \hline 14\end{matrix}$ is a quotient of P_1. We know Cartan matrix so the kernel of this quotient of P_1 is $\frac{20}{1}$. We now claim we can "push" the 20 in $\frac{20}{1}$ to the top of the middle of P_1. This will then be a contradiction, as then the middle of P_1 will have $20 \oplus 20$ as a direct factor the $\frac{20}{1}$ at the top of the middle.

Now we have the middle M_1 of P_1 as an extension of 20 by $\begin{matrix}14+20\\ \hline 1\\ \hline 14\end{matrix}$. Since M_1 has 14 as a submodule, by self-duality, get that M_1 has a quotient which is an extension of 20 by $\begin{matrix}14+20\\ \hline 1\end{matrix}$. The "1" must split over the 20 or else get a $\begin{matrix}20\\ \hline 1\\ \hline 20\end{matrix}$, which is out by structure of P_{20}. $\therefore$ M_1 has a quotient which is an extension of 20 by $14+20$. By structure of P_{20} and what we know of P_{14} get this splits. Hence done.

Suppose now $q \equiv 5$ (modulo 8) and we get projectives in $B_0(2)$ for $L_2(q)$. Irreducibles in $B_0(2)$ / ch 0 are of degrees $1, \frac{q+1}{2}, \frac{q+1}{2}, q$. Since $1+q$ and $\frac{q+1}{2}+\frac{q+1}{2}$ have same Brauer characters easily get the picture:

$$D: \begin{array}{c|ccc} & 1 & \frac{q-1}{2} & \frac{q-1}{2} \\ \hline 1 & 1 & 0 & 0 \\ \frac{q+1}{2} & 1 & 1 & 0 \\ \frac{q+1}{2} & 1 & 0 & 1 \\ q & 1 & 1 & 1 \end{array} \qquad C: \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Have the following:

$$\begin{array}{c} \frac{q-1}{2} \\ \hline 1 \\ \hline \left(\frac{q-1}{2}\right)' \\ \hline 1 \\ \hline \frac{q-1}{2} \end{array}, \qquad \begin{array}{c} \left(\frac{q-1}{2}\right)' \\ \hline 1 \\ \hline \left(\frac{q-1}{2}\right) \\ \hline 1 \\ \hline \left(\frac{q-1}{2}\right)' \end{array}, \qquad \begin{array}{c} 1 \\ \hline \begin{array}{ccc} \frac{q-1}{2} & & \left(\frac{q-1}{2}\right)' \\ \hline 1 & \oplus & 1 \\ \hline \left(\frac{q-1}{2}\right)' & & \left(\frac{q-1}{2}\right) \end{array} \\ \hline 1 \end{array}$$

Indeed, if not, using the outer aut and duality get

$$P_{\frac{q-1}{2}}: \quad \begin{array}{c} \frac{q-1}{2} \\ \hline 1 \oplus 1 \oplus \left(\frac{q-1}{2}\right)' \\ \hline \frac{q-1}{2} \end{array}$$

and so

$$P_{\left(\frac{q-1}{2}\right)'}: \quad \begin{array}{c} \left(\frac{q-1}{2}\right)' \\ \hline 1 \oplus 1 \oplus \frac{q-1}{2} \\ \hline \frac{q-1}{2} \end{array}$$

Hence have section of $\dfrac{1}{\frac{q-1}{2} \oplus \frac{q-1}{2} \oplus \left(\frac{q-1}{2}\right)' \oplus \left(\frac{q-1}{2}\right)'}$

so middle of P_1 is completely reducible, contradicting simplicity of $L_2(q)$.

Let's look at $L_2(q)$, $q \equiv 3\,(4)$. The character table — from Lambert's Oxford tables — is as follows:

$\lvert C(x)\rvert$	1	q	q	$q-1/2$	$q+1$	$q+1/2$
Class	1			a^i $(1 \le i \le \frac{q-3}{4})$	$b^{\frac{q+1}{4}}$	b^j $(1 \le j \le \frac{q-3}{4})$
Order	1	p	p	odd	2	
χ_0	1	1	1	1	1	1
χ	q	0	0	1	-1	-1
χ'	$(q-1)/2$	$(-1+\sqrt{-q})/2$	$(-1-\sqrt{-q})/2$	0	$(-1)^{\frac{q+3}{4}}$	$(-1)^{j+1}$
χ''	$(q-1)/2$	$(-1-\sqrt{-q})/2$	$(-1+\sqrt{-q})/2$	0	$(-1)^{\frac{q+3}{4}}$	$(-1)^{j+1}$
$(1 \le k \le \frac{q-3}{4})\ \chi_k$	$q-1$	-1	-1	0	$-2(-1)^k$	$-(\mu^{jk}+\mu^{-jk})$
$(1 \le l \le \frac{q-3}{4})\ \chi_l$	$q+1$	1	1	$\lambda^{il}+\lambda^{-il}$	0	0
				(λ = prim $\frac{q-1}{2}$ root)		(μ = prim $(q+1)/4$ root)

What are the 2-blocks? Have $|G|_2 \mid \chi_l(1)$ so these are each a 2-block of defect zero. From Brauer, J of alg III, p 243 get $B_0(2)$ to consist of $\chi_0, \chi, \chi', \chi'', \chi_k$ for certain k.

Let's reformulate the character table to make the block structure clear. We begin by rewriting the table in terms of linear characters.

Classes	1	u	v	$Z^{\#}_{q-1/2}$	$Z^{\#}_{q+1/2}$
χ_0	1	1	1	1	1
χ	q	0	0	1	-1
χ'	$\frac{1}{2}(q-1)$	$-\frac{1}{2}+\frac{\sqrt{-q}}{2}$	$-\frac{1}{2}-\frac{\sqrt{-q}}{2}$	0	$-\varepsilon$
χ''	$\frac{1}{2}(q-1)$	$-\frac{1}{2}-\frac{\sqrt{-q}}{2}$	$-\frac{1}{2}+\frac{\sqrt{-q}}{2}$	0	$-\varepsilon$
χ_μ	$q-1$	-1	-1	0	$-(\mu+\bar{\mu})$
χ_λ	$q+1$	1	1	$\lambda+\bar{\lambda}$	0

$\Big\}$ ε = char of order 2 of $Z_{\frac{q+1}{2}}$

where μ runs over linear characters of $Z_{q+1/2}$ of order > 2, one from each complex conjugate pair and where λ runs over non-principal linear characters of $Z_{\frac{q-1}{2}}$ one from each complex conjugate pair.
The χ_μ in $B_0(2)$ are those for which μ has order a power of two — by formula (7.3) on page 243 of Brauer's paper.

Get three modular irreducibles at most in the block by Brauer's relations 7.3 on elements of odd order. But χ_0, χ', χ'' irreducible mod 2 as usual using subgroups. Hence we calculate

D_0, C_0:

		χ_0	χ'	χ''
	χ_0	1	0	0
	χ	1	1	1
D_0:	χ'	0	1	0
	χ''	0	0	1
$\binom{2^{n-2}-1}{\text{rows}}$	χ_μ	0	1	1

		χ_0	χ'	χ''
	χ_0	2	1	1
C_0:	χ'	1	$2^{n-2}+1$	2^{n-2}
	χ''	1	2^{n-2}	$2^{n-2}+1$

We next note that $\chi'\chi'' = \chi_0 + \sum_\lambda \chi_\lambda$. Indeed, we calculate

$\chi'\chi''$	$\frac{1}{4}(q^2-2q+1)$	$\frac{1}{4}(q+1)$	$\frac{1}{4}(q+1)$	0	1
$\chi_0+\sum_\lambda \chi_\lambda$	$1+\left(\frac{q-3}{4}\right)(q+1)$	$1+\frac{q-3}{4}$	$1+\frac{q-3}{4}$	$1+\underbrace{\sum_\lambda(\lambda(\ )+\bar\lambda(\ ))}_{=-1}$	1

Some more calculations:

$\sum_\mu \chi_\mu$	$\frac{q^2-4q+3}{4}$	$-\frac{q-3}{4}$	$-\frac{q-3}{4}$	0	$1+\varepsilon$
$\chi''+\sum_\mu \chi_\mu$	$\frac{q^2-2q+1}{4}$	$-\frac{q-1}{4}-\frac{\sqrt{-q}}{2}$	$-\frac{q-1}{4}+\frac{\sqrt{-q}}{2}$	0	1
χ'^2	$(q-1)^2/4$	$\frac{1}{4}-\frac{q}{4}-\frac{\sqrt{-q}}{2}$	$\frac{1}{4}-\frac{q}{4}-\frac{\sqrt{-q}}{4}$	0	1

Hence, $\chi'^2 = \chi'' + \sum_\mu \chi_\mu$. Thus $\underline{\text{mod } 2}$, χ'^2 has $B_0(2)$ contribution with χ' as constituent $2^{n-2}-1$ times and χ'' 2^{n-2} times. Would like that part to be uniserial.

[[Next, we wish to determine the action of $Z_{2^{n-1}}$ on the $\frac{1}{2}(q-1)$ dimensional irreducibles mod 2. But lets look at the $D_{q-1} = Z_{\frac{q-1}{2}} \cdot Z_2$. The Brauer character on $Z_{\frac{q-1}{2}}$ is regular – we know all indec modules over char 2 for this group so we get all J_2's and exactly one J_1, corresponding to pairs $\lambda, \bar\lambda$ and to the principal character of $Z_{\frac{q-1}{2}}$. This implies that under $Z_{2^{n-1}}$ the $\frac{q-1}{2}$ dimensional module has at most one Jordan block of size less than 2^{n-1}. Or else would get too many fixed points. But $\frac{q-1}{2} = \frac{q\pm 1}{2} - 1$ so under $Z_{2^{n-1}}$ get $*\, J_{2^{n-1}} + J_{2^{n-1}-1}$.

Now modulo projectives of $Z_{2^{n-1}}$, $J_1 \otimes J_1 \equiv J_{2^{n-1}-1} \otimes J_{2^{n-1}-1}$ as $0 \to J_{2^{n-1}-1} \to J_{2^{n-1}} \to J_1 \to 0$ is exact and can tensor with $J_{2^{n-1}-1}$. Hence, action of $Z_{2^{n-1}}$ on $V' \otimes V'$ is $J_1 +$ free. $\therefore$ B_0 part or other block part is free other is $J_1 +$ free. But

$$\dim B_0 \text{ part} = (2^{n-2}-1)\dim V' + 2^{n-2}\dim V''$$
$$\equiv (2^{n-2}-1)(-1) + 2^{n-2}(-1) \quad (\text{modulo } 2^{n-1})$$
$$= 1.$$

Thus B_0 part is $J_1 +$ free and rest is free as $Z_{2^{n-1}}$ module.]]

Let e_0 be the idempotent corresponding to B_0, e_0 in center of modular group algebra $\therefore$ B_0 part of $V' \otimes V'$ is $e_0(V' \otimes V')$.

<u>Lemma</u> The "middle" of $P_{V'}$ is isomorphic to $1 \oplus e_0(V' \otimes V')$.

<u>Proof</u> We know P_1 has Loewy series $1;\ V' \oplus V'';\ 1$.
Thus $P_1 \otimes V'$ has a series with factors $V';\ (V' \oplus V'') \otimes V';\ V'$.
Hence, $e_0(P_1 \otimes V')$ has series with factors $V';\ e_0(V' \otimes V') + 1;\ V'$.
But $e_0(P_1 \otimes V')$ is projective as it is a summand of projective $P_1 \otimes V'$.
But by dimension counting, have "C" after all, yet we have $P_{V'}$
$\therefore$ done

<u>Corollary</u> $Ext^1(V', V') = Ext^1(V'', V'') = 0$.

<u>Proof</u> Enough to prove first. But by Lemma enough to show that $Hom(V' \otimes V', V') = 0$. But $Hom(V' \otimes V', V') \cong Hom(V', V'' \otimes V')$, as $V'' \cong (V')^*$. However $V'' \otimes V' = 1 +$ projective not in B_0, so we have result.

(cont'd p 45)

<u>A diversion back to A_5:</u>

Have irreducibles I_1, I_2, I_2', I_4. Projectives are

$$P_1:\ \begin{array}{ccc} & 1 & \\ 2 & & 2' \\ | & & | \\ 2' & & 2 \\ & 1 & \end{array} \qquad P_2:\ \begin{array}{c} 2 \\ \left.\begin{array}{c} 1 \\ 2' \end{array}\right\} M_2 \\ 1 \\ 2 \end{array} \qquad P_2':\ \begin{array}{c} 2' \\ | \\ \left.\begin{array}{c} 1 \\ 2 \end{array}\right\} M_2' \\ | \\ 2' \end{array} \qquad P_4 = I_4.$$

Here $I_2 \otimes I_2 \simeq M_2$, $I_2' \otimes I_2' \simeq M_2'$, $I_2 \otimes I_2' \simeq I_4$ (see p1)

Next, what is I_2^3? Have $I_2^3 = M_2 \otimes I_2$.

But I_2^3 sum of $I_2' \otimes I_2$ and an extension of $I_1 \otimes I_2$ by $I_1 \otimes I_2$

Thus $I_2^3 = 2I_2 + I_4$. Similarly $I_2'^3 = 2I_2' + I_4$.

<u>Prop</u> Subring of Green ring $/\mathbb{Z}$ generated by irreducibles has rank at most nine.

<u>Pf</u> Let $x = [I_2]$, $y = [I_2']$ in Green ring. Then $x^3 = 2x + xy$, $y^3 = 2y + xy$ $\therefore$ subring is spanned by $1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2$. Done.

We calculate these monomials – i.e. find modules representing them.

Now $I_2^2 I_2' \cong I_2 \otimes I_2 \otimes I_2' = M_2 \otimes I_2'$ $\left(\begin{smallmatrix} 1 \\ 2' \\ 1 \end{smallmatrix} \otimes 2' \right)$

$\therefore$ is extension of I_2' by M_2' by I_2'. But

$\mathrm{Hom}(M_2 \otimes I_2', I_1) \simeq \mathrm{Hom}(M_2, (I_2')^*) = \mathrm{Hom}(M_2, I_2') = 0$

Also $\mathrm{Hom}(I_1, M_2 \otimes I_2') = 0$. $\therefore$ get $I_2^2 I_2' \simeq P_2'$.

Similarly $I_2'^2 I_2 \simeq P_2$.

next, $I_2^4 = (2I_2 + I_4)\, I_2 = 2I_2^2 + I_2 I_2' I_2$
$= 2M_2 + P_2'$.

similarly, $I_2'^4 = 2M_2' + P_2$. Also

$$I_2^3 I_2' = (2I_2 + I_4)\, I_2' = 2I_4 + P_2$$

and $I_2'^3 I_2 = 2I_4 + P_2'$. Finally,

$$I_2^2 I_2'^2 = I_4 I_4 .$$

But $I_4 I_4 = P_4 P_4$ so is projective. Also $I_4 = I_4^*$ so I_4^2 has 1 at top and one at bottom ("is" matrices of trace zero and scalars) so $I_4^2 = P_1 + ?$ But $\dim I_4^2 = 16$, $\dim P_1 = 12$ so $I_4^2 = P_1 + I_4$.

<u>Prop</u> Irreducibly generated subring of integral Green ring has $\mathbb{Z}$-rank nine.

<u>Pf</u> $I_1, I_2, I_2', M_2, I_4, I_2', P_2, P_2', P_1$ are in the ring. ∴ done by previous proposition.

Let's next show this subring is semi simple (must be by Conlon). Suffices to construct nine homomorphisms into $\mathbb{C}$. (Tensor with $\mathbb{Q}$ so have algebra, a comm alg.) That is, have to solve over $\mathbb{C}$

$$x^3 = 2x + xy$$
$$y^3 = 2y + xy$$

One solution is $x = y = 0$, as $1 \to 1$. Rest are – up to symmetries and alg conjugates $(2,2)$, $(-1,-1)$, $(\sqrt{2}, 0)$, $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$
(1) (1) (4) (2)

Division to $L_2(7)$. Let's try the same thing here.

Recall irreducibles $I_1, I_3, I_3^*, I_8 = P_8$.

$$P_1 : \begin{array}{c} 1 \\ \hline 3+3^* \\ \hline 1 \end{array} \qquad P_3 : \begin{array}{c} 3 \\ \hline 1\oplus \begin{smallmatrix} 3^* \\ 3 \\ 3^* \end{smallmatrix} \\ \hline 3 \end{array} \qquad P_{3^*} : \begin{array}{c} 3^* \\ \hline 1+ \begin{smallmatrix} 3 \\ 3^* \\ 3 \end{smallmatrix} \\ \hline 3^* \end{array}$$

Also know $3 \otimes 3 = \begin{smallmatrix} 3^* \\ 3 \\ 3^* \end{smallmatrix}$, $3^* \otimes 3^* = \begin{smallmatrix} 3 \\ 3^* \\ 3 \end{smallmatrix}$, $3 \otimes 3^* = 1 \oplus 8$.

Now

$$3 \otimes 3 \otimes 3^* = 3 \otimes (1 \oplus 8) = 3 \oplus 3 \otimes 8.$$

But $3 \otimes 8$ is projective, of dimension 24. Also

$$\mathrm{Hom}(3 \otimes 8, 1) \simeq \mathrm{Hom}(8, 3^*) = 0$$

$$\mathrm{Hom}(3 \otimes 8, 3) \simeq \mathrm{Hom}(8, 3^* \otimes 3) \text{ of dim } 1$$

$$\mathrm{Hom}(3 \otimes 8, 3^* \simeq \mathrm{Hom}(8, 3^* \otimes 3^*) = 0$$

Thus, $3 \otimes 8 = P_3 \oplus a P_8$ so $a = 1$ and $3 \otimes 8 = P_3 \oplus 8$.

Deduce that

$$3 \otimes 3 \otimes 3^* \cong 3 \oplus 8 \oplus P_3.$$

Similarly,

$$3 \otimes 3^* \otimes 3^* \cong 3^* \oplus 8 \oplus P_{3^*}.$$

Next, we are after $3 \otimes 3 \otimes 3$. First we need to calculate $\begin{smallmatrix} 3^* \\ 3 \end{smallmatrix} \otimes 3$.

Now $\begin{smallmatrix} 3^* \\ 3 \end{smallmatrix} \otimes 3$ is an extension of $3 \otimes 3$ by $3^* \otimes 3 \simeq 1 \oplus 8$

so $\begin{smallmatrix} 3^* \\ 3 \end{smallmatrix} \otimes 3 \simeq 8 \oplus \begin{array}{c} 1 \\ \hline 3^* \\ 3 \\ 3^* \end{array}$ where double bars means "some" extension.

Now $\mathrm{Hom}(1, \begin{smallmatrix} 3^* \\ 3 \end{smallmatrix} \otimes 3) \simeq \mathrm{Hom}(3^*, \begin{smallmatrix} 3^* \\ 3 \end{smallmatrix}) = 0$. Hence $\begin{smallmatrix} 3^* \\ 3 \end{smallmatrix} \simeq 8 \oplus$

submodule of P_{3^*}. Hence, get

$$\substack{3^*\\ 3} \otimes 3 \simeq 8 \oplus \frac{\substack{3^*\\ 3 \oplus 1}}{3^*}.$$

But now

$$3 \otimes 3 \otimes 3 \simeq \substack{3^*\\ 3\\ 3^*} \otimes 3 \simeq 8 \oplus \frac{\substack{3^*\\ 3 \otimes 3}}{1} \simeq 8 \oplus 8 \oplus \frac{\substack{3^*\\ 3 \oplus 1\\ 3^*}}{1}$$

But

$$\begin{aligned} \mathrm{Hom}(3^*, 3 \otimes 3 \otimes 3) &\simeq \mathrm{Hom}(3^* \otimes 3^*, 3 \otimes 3) \\ &\simeq \mathrm{Hom}\left(\substack{3\\ 3^*\\ 3}, \substack{3^*\\ 3\\ 3^*}\right) \text{ of dim } 1. \end{aligned}$$

Hence there is embedding of $3 \otimes 3 \otimes 3$ in $P_8 \oplus P_8 \oplus P_1 \oplus P_{3^*}$ containing P_8's, the bottoms of P_1 and P_{3^*} and projecting onto the obvious submodule of P_{3^*}. Picture

$$8 \oplus 8 \oplus \frac{\substack{1\\ \overline{3 \oplus 3^*}}}{1} \oplus \frac{\substack{\underline{3^*}\\ 3\\ 3^*\\ 3 \oplus 1}}{3^*}$$

Next, claim $\frac{3 \oplus 1}{3^*}$ is contained in the embedded image.
Need to see $\dim \mathrm{Hom}\left(\substack{3\\ 3^*}, 3 \otimes 3 \otimes 3\right) \geq 1$, $\dim \mathrm{Hom}\left(\substack{1\\ 3^*}, 3 \otimes 3 \otimes 3\right) \geq 2$, the latter as $\mathrm{Hom}(1, 3 \otimes 3 \otimes 3)$ is of dimension 1. But

$$\begin{aligned} \mathrm{Hom}\left(\substack{3\\ 3^*}, 3 \otimes 3 \otimes 3\right) &\simeq \mathrm{Hom}\left(\substack{3\\ 3^*} \otimes 3^*, \substack{3^*\\ 3\\ 3^*}\right) \\ &\simeq \mathrm{Hom}\left(8 \oplus \frac{\substack{3\\ 3^* \oplus 1}}{3}, \substack{3^*\\ 3\\ 3^*}\right) \quad \text{(by 1st line this page)} \end{aligned}$$

$\therefore$ this is of dimension 1. Next,

$$\begin{aligned} \mathrm{Hom}\left(\substack{1\\ 3^*}, 3 \otimes 3 \otimes 3\right) &\simeq \mathrm{Hom}\left(\substack{1\\ 3^*} \otimes 3^*, 3 \otimes 3\right) \\ &\simeq \mathrm{Hom}\left(\substack{3^*\\ \overline{\overline{3}}\\ 3^*\\ 3}, \substack{3^*\\ 3\\ 3^*}\right) \end{aligned}$$

But claim $\substack{1\\ 3^*} \otimes 3^* \simeq \substack{3^*\\ 3\\ 3^*\\ 3}$. E.T.S. $\mathrm{Hom}\left(3^*, \substack{1\\ 3^*} \otimes 3^*\right) = 0$.
But it is isomorphic to $\mathrm{Hom}\left(1 \oplus 8, \substack{1\\ 3^*}\right) = 0$. Thus,

$$\mathrm{Hom}\left(\substack{1\\ 3^*}, 3 \otimes 3 \otimes 3\right) \simeq \mathrm{Hom}\left(\substack{3^*\\ 3\\ 3^*\\ 3}, \substack{3^*\\ 3\\ 3^*}\right) \text{ of dim } 2.$$

Now claim

$$3\otimes 3\otimes 3 \simeq 8 \oplus 8 \oplus \begin{smallmatrix}3^*\\ 1\end{smallmatrix} = \begin{smallmatrix}3^*\\ 3\oplus 1\\ 3^*\end{smallmatrix}$$

To see this, need only show $\mathrm{Hom}(3\otimes 3\otimes 3, 1)$ of dim 1, which is clear and $\begin{smallmatrix}3^*\\ 1\end{smallmatrix} \not\subseteq 3\otimes 3\otimes 3$, i.e. $\dim \mathrm{Hom}(\begin{smallmatrix}3^*\\ 1\end{smallmatrix}, 3\otimes 3\otimes 3) = 1$, as $3^* \subseteq 3\otimes 3\otimes 3$ already. But

$$\mathrm{Hom}(\begin{smallmatrix}3^*\\ 1\end{smallmatrix}, 3\otimes 3\otimes 3) \simeq \mathrm{Hom}(\begin{smallmatrix}3^*\\ 1\end{smallmatrix}\otimes 3^*, 3\otimes 3)$$

But

$$\left(\begin{smallmatrix}3^*\\ 1\end{smallmatrix}\otimes 3^*\right)^* \simeq \begin{smallmatrix}1\\ 3\end{smallmatrix}\otimes 3 \simeq \begin{smallmatrix}3\\ 3^*\\ 3\\ 3^*\end{smallmatrix}$$ (as at bottom of p40)

so

$$\mathrm{Hom}(\begin{smallmatrix}3^*\\ 1\end{smallmatrix}, 3\otimes 3\otimes 3) \simeq \mathrm{Hom}\left(\begin{smallmatrix}3\\ 3^*\\ 3\\ 3^*\end{smallmatrix}, \begin{smallmatrix}3^*\\ 3\\ 3^*\end{smallmatrix}\right)$$

of dimension 1. ∴ our claim holds. Similarly get

$$3^*\otimes 3^*\otimes 3^* \simeq 8\oplus 8\oplus \begin{smallmatrix}3\\ 1\end{smallmatrix} \oplus \begin{smallmatrix}3\\ 3^*\oplus 1\\ 3\end{smallmatrix}$$

Let's list all our results so far:

$$3\otimes 3 \simeq \begin{smallmatrix}3^*\\ 3\\ 3^*\end{smallmatrix},\quad 3^*\otimes 3^* \simeq \begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix},\quad 3\otimes 3^* \simeq 1\oplus 8.$$

$$\begin{smallmatrix}3^*\\ 3\end{smallmatrix}\otimes 3 \simeq 8 \oplus \begin{smallmatrix}3\\ 3^*\oplus 1\\ 3\end{smallmatrix}$$

$$\begin{smallmatrix}1\\ 3^*\end{smallmatrix}\otimes 3^* \simeq \begin{smallmatrix}3^*\\ 3\\ 3^*\\ 3\end{smallmatrix}$$

also can dualize + use auts as well.

$$3\otimes 8 \simeq P_3 \oplus 8.$$

$$3\otimes 3\otimes 3 \simeq 8\oplus 8\oplus \begin{smallmatrix}3^*\\ 1\end{smallmatrix} = \begin{smallmatrix}3^*\\ 3\oplus 1\\ 3^*\end{smallmatrix},\quad 3\otimes 3\otimes 3^* \simeq 3\oplus 8\oplus P_3$$

Next,

$$3 \otimes 3 \otimes 3 \otimes 3^* \cong 3 \otimes (3 \oplus 8 \oplus P_3) \cong \begin{smallmatrix} 3^* \\ 3 \\ 3^* \end{smallmatrix} \oplus P_3 \oplus 8 \oplus (3 \otimes P_3)$$

To get $3 \otimes P_3$, need dimensions of $\mathrm{Hom}(3 \otimes P_3, 1)$, $\mathrm{Hom}(3 \otimes P_3, 3)$ and $\mathrm{Hom}(3 \otimes P_3, 3^*)$ as $3 \otimes P_3$ has dimension 48 and is projective.

But

$$\mathrm{Hom}(3 \otimes P_3, 1) \cong \mathrm{Hom}(P_3, 3^*) = 0$$

$$\mathrm{Hom}(3 \otimes P_3, 3) \cong \mathrm{Hom}(P_3, 3^* \otimes 3) \cong \mathrm{Hom}(P_3, 1 \oplus 8) = 0$$

$$\mathrm{Hom}(3 \otimes P_3, 3^*) \cong \mathrm{Hom}\left(P_3, \begin{smallmatrix} 3 \\ 3^* \\ 3 \end{smallmatrix}\right) \cong \mathrm{Hom}\left(\begin{smallmatrix} & 3 & \\ 1 \oplus & \begin{smallmatrix} 3^* \\ 3 \\ 3^* \end{smallmatrix} & \\ & 3 & \end{smallmatrix}, \begin{smallmatrix} 3 \\ 3^* \\ 3 \end{smallmatrix}\right)$$

Hence,

$$3 \otimes P_3 \cong P_{3^*} \oplus P_{3^*} \oplus 8 \oplus 8.$$

Thus

$$\boxed{\begin{aligned} 3 \otimes 3 \otimes 3 \otimes 3^* &\cong \begin{smallmatrix} 3^* \\ 3 \\ 3^* \end{smallmatrix} \oplus 8 \oplus 8 \oplus 8 \oplus P_3 \oplus P_{3^*} \oplus P_{3^*} \\ 3 \otimes 3^* \otimes 3^* \otimes 3^* &\cong \begin{smallmatrix} 3 \\ 3^* \\ 3 \end{smallmatrix} \oplus 8 \oplus 8 \oplus 8 \oplus P_3 \oplus P_3 \oplus P_{3^*} \end{aligned}}$$

Similarly

Next

$$3 \otimes 3 \otimes 3^* \otimes 3^* \cong (1 \oplus 8) \otimes (1 \oplus 8)$$
$$\cong 1 \oplus 8 \oplus 8 \oplus (8 \otimes 8)$$

Calculate $8 \otimes 8$ by same method as $3 \otimes 8$ (could use Brauer characters though)

$\mathrm{Hom}(8 \otimes 8, 1)$ of dimension 1

$\mathrm{Hom}(8 \otimes 8, 3) \cong \mathrm{Hom}(8, P_3 \oplus 8)$ of dim 1

$\mathrm{Hom}(8 \otimes 8, 3^*) \cong \mathrm{Hom}(8, P_{3^*} \oplus 8)$ of dim 1

Hence,

$$8 \otimes 8 \cong P_1 \oplus P_3 \oplus P_{3^*} \oplus 8 \oplus 8 \oplus 8.$$

Deduce that

$$\boxed{3 \otimes 3 \otimes 3^* \otimes 3^* \simeq 1 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus P_1 \oplus P_3 \oplus P_{3^*}}$$

next, let's try $3 \otimes 3 \otimes 3$ again, by a different method. A sketch

If A is an indecomposable module let $H(A)$ be the indecomposable derived by resolution, à la Heller. Similarly, for any module modulo projectives, written "$\equiv$." Also $H(A \otimes B) \equiv H(A) \otimes B$ by tensoring the resolution as usual over the field.

Let $A = \dfrac{3^*}{1 \oplus 3}$ so $H(A) = \begin{matrix} 3^* \\ 3 \\ 3^* \end{matrix} \cong 3 \otimes 3$.

$\therefore$ want $H\left(\dfrac{3^*}{1 \oplus 3} \otimes 3\right)$. $\therefore$ must calculate $\dfrac{3^*}{1 \oplus 3} \otimes 3$. It is

$$\frac{3^*}{1 \oplus 3} \otimes 3 \cong 8 \oplus \frac{1}{3 \oplus \begin{matrix} 3^* \\ 3 \\ 3^* \end{matrix}}$$

above techniques should lead to $\dfrac{3^*}{1 \oplus 3} \otimes 3 \simeq 8 \oplus \begin{matrix} 1 \\ 3 \end{matrix} = \dfrac{\begin{matrix} 3^* \\ 1 \oplus 3 \end{matrix}}{3^*}$

We resolve this:

$$\frac{\begin{matrix} 1 \\ 3 \oplus 3^* \end{matrix}}{1} \oplus \frac{\begin{matrix} 3^* \\ 1 \oplus \begin{matrix} 3 \\ 3^* \\ 3 \end{matrix} \end{matrix}}{3} \longrightarrow \begin{matrix} 1 \\ 3 \end{matrix} = \frac{\begin{matrix} 3^* \\ 1 \oplus 3 \end{matrix}}{3^*}$$

Kernel is $\begin{matrix} 3^* \\ 1 \end{matrix} = \dfrac{\begin{matrix} 3^* \\ 3 \oplus 1 \end{matrix}}{3^*}$ as expected!

Better description

$$3 \otimes 3 \otimes 3 \cong \begin{matrix} 3^* \\ 1 \end{matrix} = \begin{matrix} 3^* \\ 3 \\ 3^* \end{matrix} = \begin{matrix} 1 \\ 3^* \end{matrix} \oplus \text{proj.}$$

Hence have $\begin{array}{cc} & 3^* \\ 1 & 3 \\ & 3^* \end{array}$ and $\begin{array}{c} 1 \\ 3^* \end{array}$ joined at 3^*. Let's tensor again and then join to guess $3\otimes 3\otimes 3\otimes 3$. Now

$$\left(\begin{array}{cc} & 3^* \\ 1 & 3 \\ & 3^* \end{array}\right)^* \cong \begin{array}{cc} & 3 \\ 1 & 3^* \\ 3 & \end{array}$$

so

$$0 \to \begin{array}{cc} & 3 \\ 1 & 3^* \\ 3 & \end{array} \to P_3 \to \begin{array}{c} 3 \\ 3^* \end{array} \to 0$$

is exact. Now

$$\begin{array}{c} 3 \\ 3^* \end{array} \otimes 3^* \simeq \begin{array}{cc} & 3 \\ 1 & 3^* \\ 3 & \end{array} + \text{proj.} \quad (\text{see p 41})$$

Thus, as following is exact,

$$0 \to \begin{array}{c} 3^* \; 3 \\ 1 \end{array} = \begin{array}{cc} 3 & \\ 3^* & 1 \\ 3 & \end{array} \to P_1 \oplus P_3 \to \begin{array}{cc} & 3 \\ 1 & 3^* \\ 3 & \end{array} \to 0$$

get $\left(\begin{array}{cc} & 3^* \\ 1 & 3 \\ & 3^* \end{array}\right)^* \otimes 3^* \simeq \begin{array}{c} 3^* \; 3 \\ 1 \end{array} = \begin{array}{cc} 3 & \\ 3^* & 1 \\ 3 & \end{array} + \text{proj}$

Thus $\begin{array}{cc} & 3^* \\ 1 & 3 \\ & 3^* \end{array} \otimes 3 \simeq \begin{array}{c} 1 \\ 3 \; 3^* \end{array} = \begin{array}{cc} 3^* & \\ 3 & 1 \\ 3^* & \end{array} + \text{proj}$

Hence, guess

$$3\otimes 3\otimes 3\otimes 3 \simeq \begin{array}{c} 1 = 1 \\ 3 \quad 3^* \end{array} = \begin{array}{c} 3^* \\ 3 \\ 3^* \end{array} = \begin{array}{c} 3^* \quad 3 \\ 1 = 1 \end{array} + \text{proj}.$$

$$\boxed{3\otimes 3\otimes 3\otimes 3 \simeq \begin{array}{c} 3 \quad 3^* \\ 1 = 1 \end{array} = \begin{array}{c} 3^* \\ 3 \\ 3^* \end{array} \quad \begin{array}{c} 1 = 1 \\ 3^* \quad 3 \end{array} + \text{proj}}$$

Seems pointless to continue.

Back to $L_2(q)$, $q \equiv 3\ (4)$. Write $V = V'$, $V^* = V''$.

$$\begin{array}{ll} & V \\ P_V: & 1 \oplus e_0(V\otimes V) \\ & V \end{array}$$

Let F be the field so $1 \cong F$, as modules.
Let $P = P_V$, M maximal submodule,
$K = {}^{1}_{V}$, $N = M/K$. Hence, have
exact sequences

$$0 \to M \to P \to V \to 0$$
$$0 \to K \to M \to N \to 0$$
$$0 \to V \to K \to 1 \to 0$$

Hence have

$$0 \to \overset{\cong F}{Hom(V,P)} \to Hom(P,P) \to Hom(M,P) \to \overset{=0}{Ext(V,P)} \to$$
$$0 \to Hom(M,M) \to Hom(M,P) \to \overset{=0}{Hom(M,V)} \to Ext(M,M) \to 0$$

Thus, $\dim Hom(M,M) = \dim Hom(P,P) - 1 = c_{V,V} - 1$, $\underline{Ext(M,M) = 0}$

also,

$$0 \to Hom(N,N) \to Hom(M,N) \to \overset{=0}{Hom(K,N)} \to$$
$$0 \to \overset{=0}{Hom(M,K)} \to Hom(M,M) \to Hom(M,N) \to Ext(M,K) \to \overset{=0}{Ext(M,M)} \to$$

Thus, $\dim Hom(N,N) = \dim Hom(M,N)$ so

$$\boxed{\dim_F Hom(N,N) = -1 + c_{V,V} + \dim Ext(M,K)}$$

But $0 \to Hom(N,K) \to \overset{=0}{Hom(M,K)} \to \overset{\cong F}{Hom(K,K)} \to Ext(N,K) \to Ext(M,K) \to Ext(K,K)$

Claim: $Ext(K,K) = 0$. This will imply

$$\boxed{\dim_F Hom(N,N) = -2 + c_{V,V} + \dim Ext(N,K)}$$

Lemma $Ext(K,K) = 0$.

Pf Have first

$$0 \to \underset{\cong 0}{Hom(1,V)} \to \underset{\cong 0}{Hom(1,K)} \to \underset{\cong F}{Hom(1,1)} \to \underset{\cong F}{Ext(1,V)} \to Ext(1,K) \to \underset{=0}{Ext(1,1)}$$

so that $Ext(1, K) = 0$. Next $Ext(V, K) = 0$. Let $0 \to K \to E \to V \to 0$.
Thus, cannot have K the unique maximal submodule as there is no $\begin{matrix} V \\ 1 \\ V \end{matrix}$.
Hence, from other maximal submodule get $0 \to V \to X \to V \to 0$ which also splits $\therefore Ext(V, K) = 0$ as claimed. Finally,

$$0 \to \underset{=0}{Hom(1, K)} \to \underset{\neq 0}{Hom(K, K)} \to \underset{\simeq F}{Hom(V, K)} \to \underset{=0}{Ext(1, K)} \to Ext(K, K) \to \underset{=0}{Ext(V, K)}$$

so lemma is proved.

Next, have two sequences

$$0 \to \overset{=0}{Hom(N, 1)} \to Hom(N, K) \to \overset{=0}{Hom(N, V)} \to Ext(N, 1) \to Ext(N, K)$$

$$0 \to \overset{=0}{Hom(N, 1)} \to \overset{=0}{Hom(N, K^*)} \to Hom(N, V^*) \to Ext(N, 1)$$

Thus $\dim Ext(N, K) \geq \dim Ext(N, 1) \geq \dim Hom(N, V^*)$

As a consequence

$$\boxed{\dim_F Hom(N, N) \geq (C_{V,V} - 1) + (\dim_F Hom(N, V^*) - 1)}$$

Hence

$$\boxed{\begin{aligned} \dim_F Hom(N, N) \leq C_{V,V} - 1 &\Rightarrow \dim Hom_F(N, V^*) = 1 \\ &\Rightarrow \dim Ext(V, V^*) = 1 \Rightarrow \text{done.} \end{aligned}}$$

However, $N = e_0(V \otimes V)$ and

$$\begin{aligned} Hom(V \otimes V, V \otimes V) &\simeq Hom(V \otimes V \otimes V^* \otimes V^*, 1) \\ &\simeq Hom((V \otimes V^*) \otimes (V \otimes V^*), 1) \end{aligned}$$

From the character table, p 34, and the projectivity of the χ_λ we have

$$V \otimes V^* \simeq F \oplus \sum_\lambda V_\lambda$$

Hence

$$(V \otimes V^*) \otimes (V \otimes V^*) \simeq F \oplus 2\sum_\lambda V_\lambda \oplus \sum_{\lambda_1, \lambda_2} V_{\lambda_1} \otimes V_{\lambda_2}$$

Thus,

$$\begin{aligned} \dim_F \mathrm{Hom}((V\otimes V^*)\otimes(V\otimes V^*), F) &= 1 + \sum_{\lambda_1,\lambda_2} \dim_F \mathrm{Hom}(V_{\lambda_1}\otimes V_{\lambda_2}, F) \\ &= 1 + \sum_{\lambda_1,\lambda_2} \dim_F \mathrm{Hom}(V_{\lambda_1}, V_{\lambda_2}) \\ &= 1 + \sum_\lambda 1 = 1 + \tfrac{1}{2}\left(\tfrac{q+1}{2} - 2\right) \\ &= \tfrac{q+1}{4}. \end{aligned}$$

That is,

$$\boxed{\dim_F \mathrm{Hom}(V\otimes V, V\otimes V) = \frac{q+1}{4}}$$

Now $(\chi')^2 = \chi'' + \sum_\mu \chi_\mu$ so $V\otimes V = e_0(V\otimes V) + (1-e_0)(V\otimes V)$ where $(1-e_0)(V\otimes V)$ "comes" from the other $\chi_\mu \notin B_0(2)$.
(This seems to indicate we shall have to study somewhat the structure of the other blocks!) Let's assume the following: Each χ_μ is irred mod 2, the blocks correspond to classes of characters of $O(C(Z_{2n-1}))$ so indecomposables in such a block are uniserial with same composition factor, any two of the same length are isomorphic, etc....

This implies that

$$V \otimes V = e_0(V \otimes V) + \sum e_B(V \otimes V)$$

second sum over all blocks B with $Z_{2^{n-1}}$ as defect group.
Now $e_B(V \otimes V)$ is direct sum of uniserial modules
each made up of V_B — which is the V_μ reduced modulo 2,
The number is the number of characters in the block
Hence, clearly

$$\dim \operatorname{Hom}(e_B(V \otimes V), e_B(V \otimes V)) \geq \# \chi_\mu \text{ in } B$$

with the minimum exactly when $e_B(V \otimes V)$ is uniserial.
Thus,

$$\frac{q+1}{4} \geq \dim \operatorname{Hom}(e_0(V \otimes V), e_0(V \otimes V)) + \#(\chi_\mu \notin B_0)$$

So

$$\dim \operatorname{Hom}(e_0(V \otimes V), e_0(V \otimes V)) \leq q+1/4 - \left\{\frac{1}{2}\left(\frac{q+1}{2} - 2\right) - \frac{1}{2}(2^{n-1}-2)\right\}$$
$$= \underline{2^{n-2}}.$$

But $C_{V,V} = 2^{n-2}+1$ (see page 34). Hence, as $N \cong e_0(V \otimes V)$
have

$$\dim_F \operatorname{Hom}(N, N) \leq C_{V,V} - 1.$$

Thus, all done, modulo statements on other blocks
That is, get $e_0(V \otimes V)$ uniserial just as desired.

Let's find out what the 2-blocks are. Only remaining blocks partition the χ_μ, $\chi_\mu \in B_0(2)$. Let $a \in Z^{\#}_{q+1/2}$ so we want to know when $\omega_\mu(a) \equiv \omega_{\mu'}(a)$ modulo a prime divisor of two, as the ω_μ agree elsewhere on G! But, if $|a| > 2$,

$$\omega_\mu(a) = \frac{|G|}{|C(a)|}\,\frac{\chi_\mu(a)}{\chi_\mu(1)} = \frac{\frac{1}{2}(q-1)q(q+1)}{\frac{1}{2}(q+1)}\,\frac{(-\mu(a)-\bar{\mu}(a))}{(q-1)}$$

$$= -q\,(\mu(a)+\bar{\mu}(a))$$

$$\equiv \mu(a)+\bar{\mu}(a)$$

as $-q \equiv 1$ (modulo 2). (Need only consider $|a|>2$ by theory.)

Let $A = O(Z_{q+1/2})$ so $\chi_\mu, \chi_{\mu'}$ are in same block if, and only if, the $\omega_\mu, \omega_{\mu'}$ are congruent there. But A is abelian, a subgroup of index 2 in a dihedral group and $\mu+\bar{\mu}$ is the restriction to the elements of odd order in that dihedral group of a character of defect zero. Hence $\chi_\mu, \chi_{\mu'}$ are in the same block if, and only if, $\mu|A = \mu'|A$.

Let B be such a block so $|B| = 2^{n-1}$, the defect group is $Z_{2^{n-1}}$. All the $\chi_\mu \in B$ agree on elements of odd order so there is a unique modular irred $\varphi_B \in B$. Thus $\chi_\mu \doteq m\varphi_B$. But $\sum_{\chi_\mu \in B} m^2 = C$ and $C = 2^{n-1}$ by general theory $\therefore m = 1$.

$\therefore$ rest is now easy. But to get uniseriality of corresponding projective indecomposable, seem to have to quote Janusz!

(cont'd on p 55)

A_6 in characteristic three

$A_6 \simeq PSL(2,9)$ so can use theory for $SL(2,9)$.
Have basic reps and Steinberg's theorem. Easily deduce
Brauer characters:

1	2	4	5	5
1	1	1	1	1
3	-1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
3	-1	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
4	0	-2	-1	-1
9	1	1	-1	-1

Get decomposition numbers for $B_0(3)$ – using char table p8

$$\begin{array}{c|cccc} & 1 & 3 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 & 1 \\ 8 & 1 & 1 & 0 & 1 \\ 8 & 1 & 0 & 1 & 1 \\ 10 & 0 & 1 & 1 & 1 \end{array} \qquad C = \begin{pmatrix} 5 & 1 & 1 & 4 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 4 & 2 & 2 & 5 \end{pmatrix}$$

The two "threes" are conjugate under an automorphism.
Call them $3, 3'$. By usual arguments, as all are self-dual, get

$$P_3 = \begin{array}{c} 3 \\ \hline 1 \oplus 3' \oplus 4 \\ \hline 3 \end{array}, \quad P_{3'} = \begin{array}{c} 3' \\ \hline 1 \oplus 3 \oplus 4 \\ \hline 3' \end{array}$$

Hence, deduce that the middle M_1 of P_1 has $3 \oplus 3'$ as a direct summand. Guess

$$P_1 = \begin{array}{c} 1 \\ \hline 3 \oplus 3' \oplus \begin{smallmatrix}4\\|\\4\\|\\4\\|\\4\end{smallmatrix} \\ \hline 1 \end{array}, \qquad P_4 = \begin{array}{c} 4 \\ \hline \begin{smallmatrix}4\\|\\4\\|\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\3'\end{smallmatrix} \oplus \begin{smallmatrix}3'\\3\end{smallmatrix} \\ \hline 4 \end{array}$$

Now six-dim perm rep over $\mathbb{Z}$ has 2 dims of Hom so same is true mod three. Only one invariant vector so module is $\begin{smallmatrix}1\\4\\1\end{smallmatrix}$ or $\begin{smallmatrix}1\\1\end{smallmatrix} \oplus 4$. But A_6 is simple so is $\begin{smallmatrix}1\\4\\1\end{smallmatrix}$. Best thing now is to build up tensor products. Have

$$\boxed{3 \otimes 3' = 9}$$

$$3 \otimes 3 \doteq 1 \oplus 1 \oplus 3 \oplus 4$$

But $\mathrm{Hom}(3 \otimes 3, 1) \simeq \mathrm{Hom}(3,3) \simeq F$, $\mathrm{Hom}(1, 3 \otimes 3) \simeq F$.

Why carry on? Perhaps come back to!

$\underline{L_3(3)}$ - structure mod 3.

	1	2	4	8	8	6	3	3'	13	13^5	13^{-1}	13^{-5}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	12	$\boxed{4}^{(1)}$	0	0	0	$1^{(4)}$	$\boxed{3}^{(5)}$	0	-1	-1	-1	-1
χ_3	27	3	-1	-1	$-1^{(2)}$	0	0	0	1	1	1	1
χ_4	16	0	0	0	0	0	-2	$1^{(3)}$	α	β	γ	δ
χ_5	16	0	0	0	0	0	-2	1	δ	α	β	γ
χ_6	16	0	0	0	0	0	-2	1	γ	δ	α	β
χ_7	16	0	0	0	0	0	-2	1	β	γ	δ	α
χ_8	26	-2	0	$\sqrt{2}i$	$-\sqrt{2}i^{(6)}$	1	-1	-1	0	0	0	0
χ_9	26	-2	0	$-\sqrt{2}i$	$\sqrt{2}i$	1	-1	-1	0	0	0	0
χ_{10}	26	2	2	0	0	-1	-1	-1	0	0	0	0
χ_{11}	13	-3	1	-1	-1	0	4	1	0	0	0	0
χ_{12}	39	-1	-1	1	1	-1	3	0	0	0	0	0

(1) $1 - a + b = 0$, $1 - \frac{a^2}{12} + \frac{b^2}{27} = 0 \Rightarrow a = \frac{-8 \pm 48}{10}$, $a \in \mathbb{Z} \Rightarrow a = 4$, $b = 3$. $(\#(2', 2' = 13) = 0)$

(2) Steinberg char. on torus

(3) $\chi(3') \equiv 1\ (3)$ and $|C(3')| = 9$, $C(3') \in \mathbb{Z}$.

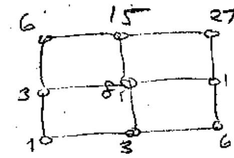
(4) $\chi(6) \equiv \chi(2)\ (3)$ and 6 col + 13 col orthog

(5) $\chi(3) \equiv \chi(1)\ (3)$, $\chi(3) \equiv \chi(6)\ (2)$ and $|C(3)| = 54$.

(6) $\chi(8) \equiv \chi(1)\ (2)$, $|C(8)| = 8$, $8 \not\sim 8^{-1}$ etc.

$$\alpha = \lambda + \lambda^3 + \lambda^9,\quad \beta = \lambda^2 + \lambda^5 + \lambda^6,\quad \gamma = \lambda^4 + \lambda^{10} + \lambda^{12},\quad \delta = \lambda^7 + \lambda^8 + \lambda^{11}$$

$(m_1+1)(m_2+1)\left(\frac{m_1+m_2+2}{2}\right)$



Lie theory

Brauer characters

	1	2	4	8	8	13	13^5	13^{-1}	13^{-5}
φ_1	1	1	1	1	1	1	1	1	1
φ_2	3	-1	1	$-1+\sqrt{2}i$	$-1-\sqrt{2}i$	α	β	γ	δ
φ_3	3	-1	1	$-1-\sqrt{2}i$	$-1+\sqrt{2}i$	γ	δ	α	β
φ_4	7	-1	-1	1	1	$1+\beta+\delta$	$1+\alpha+\gamma$	$1+\beta+\delta$	$1+\alpha+\gamma$
φ_5	6	2	0	$-\sqrt{2}i$	$\sqrt{2}i$	$\beta+\gamma$	$\gamma+\delta$	$\delta+\alpha$	$\alpha+\beta$
φ_6	6	2	0	$\sqrt{2}i$	$-\sqrt{2}i$	$\alpha+\delta$	$\alpha+\beta$	$\beta+\gamma$	$\gamma+\delta$
φ_7	15	-1	-1	-1	-1	$-1+\alpha$	$-1+\beta$	$-1+\gamma$	$-1+\delta$
φ_8	15	-1	-1	-1	-1	$-1+\gamma$	$-1+\delta$	$-1+\alpha$	$-1+\beta$
φ_9	27	3	-1	-1	-1	1	1	1	1

$\Big\}$ get last from dec of χ_4, χ_6

Dec nos:

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
χ_1	1	0	0	0	0	0	0	0	0
χ_2	0	0	0	0	1	1	0	0	0
χ_3	0	0	0	0	0	0	0	0	1
χ_4	1	0	0	0	0	0	1	0	0
χ_5	0	0	1	1	0	1	0	0	0
χ_6	1	0	0	0	0	0	0	1	0
χ_7	0	1	0	1	1	0	0	0	0
χ_8	1	1	0	1	0	0	0	1	0
χ_9	1	0	1	1	0	0	1	0	0
χ_{10}	1	1	1	1	1	1	0	0	0
χ_{11}	0	1	1	1	0	0	0	0	0
χ_{12}	2	0	0	1	0	0	1	1	0

Cartan numbers:

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
φ_1	10	2	2	5	1	1	4	4	0
φ_2	2	4	2	4	2	1	0	1	0
φ_3	2	2	4	4	1	2	1	0	0
φ_4	5	4	4	7	2	2	2	2	0
φ_5	1	2	1	2	3	2	0	0	0
φ_6	1	1	2	2	2	3	0	0	0
φ_7	4	0	1	2	0	0	3	1	0
φ_8	4	1	0	2	0	0	1	3	0
φ_9	0	0	0	0	0	0	0	0	1

Difficult to guess module structure.

Now we turn to $q \equiv 1$ (modulo 4). We begin with the ordinary character table.

Classes:	1	u	v	$Z^{\#}_{\frac{q-1}{2}}$	$Z^{\#}_{\frac{q+1}{2}}$
χ_0	1	1	1	1	1
χ	q	0	0	1	-1
χ'	$q+1/2$	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$	ε	0
χ''	$q+1/2$	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$	ε	0
χ_μ	$q-1$	-1	-1	0	$-(\mu+\bar{\mu})$
χ_λ	$q+1$	1	1	$\lambda+\bar{\lambda}$	0

ε is char of order 2 of $Z_{\frac{q+1}{2}}$

where μ runs over non-principal linear characters of $Z_{\frac{q+1}{2}}$, one from each complex-conjugate pair, and where λ runs over linear characters of order > 2 of $Z_{\frac{q-1}{2}}$, one from each complex-conjugate pair.

Again we use Brauer's paper (referred to on p 33) to determine the principal 2-block. It consists of $\chi_0, \chi, \chi', \chi''$ and the χ_λ for which λ has a power of two. As before get there are three modular irreducibles in $B_0(2)$. Now $\chi_0 + \chi$ and $\chi' + \chi''$ agree on elements of odd order so trivial module is in $\bar{\chi}'$ and $\bar{\chi}''$. By minimal classes for $N(S_2)$ get modular irreducibles $\varphi_0, \varphi', \varphi''$ and

$$\chi' \doteq \varphi_0 + \varphi'$$
$$\chi'' \doteq \varphi_0 + \varphi''$$

so that we have for D_0, C_0:

D_0:

	φ_0	φ'	φ''
χ_0	1	0	0
χ	1	1	1
χ'	1	1	0
χ''	1	0	1
$(2^{n-2}-1$ rows$)$ χ_λ	2	1	1

C_0:

	φ_0	φ'	φ''
φ_0	2^n	2^{n-1}	2^{n-1}
φ'	2^{n-1}	$2^{n-2}+1$	2^{n-2}
φ''	2^{n-1}	2^{n-2}	$2^{n-2}+1$

We now turn to calculating $\varphi'^2, \varphi''^2, \varphi'\varphi''$. It seems worthwhile to determine all other blocks now. Have $B_0(2)$ and clearly all the χ_μ are each a 2-blocks of defect zero. Hence, it remains to see how the χ_λ are partitioned into blocks.

[[Thus, it's enough to check congruences for ω_λ on elements $a \in Z_{\frac{q-1}{2}}$, a of odd order. But

$$\omega_\lambda(a) = \frac{|G|}{|C(a)|}\,\frac{\chi_\lambda(a)}{\chi_\lambda(1)} = \frac{\frac{1}{2}(q-1)q(q+1)}{\frac{1}{2}(q-1)}\cdot\frac{\lambda(a)+\bar\lambda(a)}{q+1} = q(\lambda(a)+\bar\lambda(a))$$

Hence, $\omega_\lambda(a) \equiv \lambda(a)+\bar\lambda(a)$ (modulo 2)]] Use argument as in the second paragraph on page 49. Rest is also just as in that case down to desired uniseriality.

				$O(Z_{\frac{q-1}{2}}{}^{\#})$	
φ'	$\frac{q-1}{2}$	$\frac{-1+\sqrt{q}}{2}$	$\frac{-1-\sqrt{q}}{2}$	0	-1
φ''	$\frac{q-1}{2}$	$\frac{-1-\sqrt{q}}{2}$	$\frac{-1+\sqrt{q}}{2}$	0	-1
$\varphi'\varphi''$	$\frac{(q-1)^2}{4}$	$-\frac{q-1}{4}$	$-\frac{q-1}{4}$	0	1
$\sum_\mu \chi_\mu$	$\frac{q-1}{4}\cdot q-1$	$\frac{q-1}{4}(-1)$	$\frac{q-1}{4}(-1)$	0	1

Hence, $\varphi'\varphi'' = \sum \chi_\mu$ so $V'\otimes V''$ is projective, where V', V'' are modules affording φ', φ'', respectively.

				$O(Z_{\frac{q-1}{2}}^{\#})$	
$(\varphi')^2$	$\frac{(q-1)^2}{4}$	$\frac{q+1}{4}-\frac{\sqrt{q}}{2}$	$\frac{q+1}{4}+\frac{\sqrt{q}}{2}$	0	1
$\sum_\lambda \chi_\lambda$	$\frac{q-5}{4}(q+1)$	$\frac{q-5}{4}$	$\frac{q-5}{4}$	-2	0
$\chi_0+\chi''+\sum_\lambda \chi_\lambda$	$\frac{q^2-2q+1}{4}$	$\frac{q-5}{4}+\frac{3}{2}-\frac{\sqrt{q}}{2}$	$\frac{q-5}{4}+\frac{3}{2}-\frac{\sqrt{q}}{2}$	0	1

Hence have relation. Deduce directly that $V'\otimes V'$ has composition factors with multiplicities as follows:

$$V_0 : \quad 1+1+(2^{n-2}-1)2 = 2^{n-1}$$
$$V' : \quad 2^{n-2}-1$$
$$V'' : \quad 1+(2^{n-2}-1) = 2^{n-2}$$

plus others from blocks with cyclic defect groups >1.

Note that the principal part of $V'\otimes V'$ and $P_{V'}$ differ in composition factors only in that $P_{V'}$ has two more V''s. That is $M_{V'}$ (obvious middle part of $P_{V'}$) and $e_0(V'\otimes V')$ have same composition factors, where e_0 is principal block idempotent.

<u>Proposition</u>: $M_{V'} \simeq e_0(V'\otimes V')$.

<u>Proof</u>. Let H be the subgroup of order $\frac{q-1}{2}\cdot q$ and let $\pi = 1_H^G$, $G=L_2(q)$ so π is a permutation representation. Its character is $\chi_0+\chi$ so considering π as a module over the splitting field F, it has composition factors V_0, V_0, V', V''. As $q+1$ is even, there is a series

$$0 \subset V_0 \subset M \subset \pi$$

with $\pi/M \simeq V_0$ so M/V_0 has composition factors

V' and V''. (In fact V_0 is the socle of Π, M is the first term of the descending Loewy series of Π and $M/V_0 \simeq V' \oplus V''$.)

Hence, $\Pi \otimes V'$ has a series with factors V'; $V' \otimes V'$, $V'' \otimes V'$ or vice versa; V'. But, we claim that $\Pi \otimes V'$ is projective! Indeed,

$$\begin{aligned}\Pi \otimes V' &= (1_H)^G \otimes V' \\ &\simeq (1_H \otimes V/H)^G \\ &\simeq (V/H)^G\end{aligned}$$

and V/H is projective as H is Frobenius, a'la Hall-Higman. Thus, so is the induced module.

But $\Pi \otimes V'$ has V' as a submodule so $P_{V'}$ is a summand. Consideration of the principal part of $\Pi \otimes V'$ and the remark on comp. factors preceding the statement of the proposition now yields the result!

<u>Lemma</u> $\dim_F \mathrm{Hom}(V' \otimes V', V'' \otimes V'') = \frac{q-1}{4}$.

<u>Proof</u> First, as V', V'' are self-dual, by inspection, we have

$$\begin{aligned}\mathrm{Hom}(V' \otimes V', V'' \otimes V'') &\simeq \mathrm{Hom}(V' \otimes V'', V' \otimes V'') \\ &\simeq \mathrm{Hom}(\oplus V_\mu, \oplus V_\mu) \\ &\simeq \tfrac{q-1}{4} F\end{aligned}$$

(bar here means Brauer)

as the number of V_μ - proj. ind. affording $\bar{\chi}_\mu$ - is $\frac{q-1}{4}$.

<u>Lemma</u> $\dim_F \operatorname{Hom}(e_0(V'\otimes V'), e_0(V''\otimes V'')) \leq 2^{n-2}$.

<u>Proof</u> Now $V'\otimes V' = e_0(V'\otimes V') + (1-e_0)(V'\otimes V')$ and the latter summand is a direct sum of uniserial modules with a single repeated composition factor. The total number of composition factors from one such block is the number of ordinary irreducibles in the block. Hence, as the same holds for $V''\otimes V''$ we have

$$\dim \operatorname{Hom}((1-e_0)(V'\otimes V'), (1-e_0)(V'\otimes V') \geq \#(\chi_\lambda \notin B_0)$$

and the minimum is achieved when each block contributes a single uniserial module to $(1-e_0)(V'\otimes V')$, $(1-e_0)(V''\otimes V'')$. Hence, by the previous lemma,

$$\tfrac{q-1}{4} \geq \dim \operatorname{Hom}(e_0(V'\otimes V'), e_0(V''\otimes V'')) + \#(\chi_\lambda \notin B_0)$$

so

$$\dim \operatorname{Hom}(e_0(V'\otimes V'), e_0(V''\otimes V'') \leq \tfrac{q-1}{4} - \left\{\tfrac{1}{2}\left(\tfrac{q-1}{2}-2\right) - \tfrac{1}{2}\left(2^{n-1}-2\right)\right\} = 2^{n-2}.$$

We can restate this as follows:

<u>Proposition</u> $\dim_F \operatorname{Hom}(M_{V'}, M_{V''}) \leq \dim_F (P_{V'}, P_{V''})$

For $\dim_F (P_{V'}, P_{V''})$ is the Cartan number for φ', φ'', and $M_{V'} \simeq e_0(V'\otimes V')$, $M_{V''} \simeq e_0(V''\otimes V'')$.

<u>Denote</u>: $M' = M_{V'}$, $M'' = M_{V''}$, $P' = P_{V'}$, $P'' = P_{V''}$.

Proposition $\dim_F \operatorname{Hom}(M', M'') = \dim \operatorname{Hom}_F(P', P'')$.

By the previous page this has as a consequence:

Corollary $V' \otimes V'$ is the direct sum of $e_0(V' \otimes V')$ and the sum of the indecomposable projectives, one for each block with non-identity cyclic defect group.

Pf (of Prop) Let N', N'' be maximal submodules of P', P'' resp. Hence,

$$0 \to \overset{=0}{\operatorname{Hom}(V', P'')} \to \operatorname{Hom}(P', P'') \to \operatorname{Hom}(N', P'') \to \overset{=0}{\operatorname{Ext}(V', P'')}$$

so $\operatorname{Hom}(P', P'') \simeq \operatorname{Hom}(N', P'')$. But

$$0 \to \operatorname{Hom}(N', N'') \to \operatorname{Hom}(N', P'') \to \operatorname{Hom}(N', V'') \to$$

so $\operatorname{Hom}(N', V'') \simeq \operatorname{Hom}(M', V'') \simeq \operatorname{Hom}(e_0(V' \otimes V'), V'')$ as $\operatorname{Hom}(V', V'') = 0$. But $\operatorname{Hom}(V' \otimes V', V'') = \operatorname{Hom}(V', V' \otimes V'') = 0$. Thus, $\operatorname{Hom}(N', N'') \simeq \operatorname{Hom}(N', P'')$ so $\operatorname{Hom}(N', N'') \simeq \operatorname{Hom}(P', P'')$. Also we have

$$0 \to \operatorname{Hom}(M', N'') \to \operatorname{Hom}(N', N'') \to \operatorname{Hom}(V', N'') \to$$

and $\operatorname{Hom}(V', N'') = 0$ in a similar way as $\operatorname{Hom}(N', V'') = 0$ was just shown. Thus $\operatorname{Hom}(M', N'') \simeq \operatorname{Hom}(N', N'')$. Finally,

$$0 \to \underset{0''}{\operatorname{Hom}(M', V'')} \to \operatorname{Hom}(M', N'') \to \operatorname{Hom}(M', M'') \to \underset{\substack{0'' \\ \text{as above}}}{\operatorname{Ext}(M', V'')}$$

Thus, the result is proved.

<u>But</u> <u>doesn't</u> <u>seem</u> <u>to</u> <u>lead</u> <u>directly</u> <u>to</u> <u>structure</u> <u>of</u> <u>projectives</u>!

Proposition There is no uniserial module with successive composition factors V', F and V'.

Pf. Suppose otherwise so that the map $Ext(V', V') \to Ext(\frac{V'}{F}, V')$ is not onto. Hence, the map $Ext(V' \otimes V', F) \to Ext(\frac{V'}{F} \otimes V', F)$ is not onto. Thus, $Ext(e_0(V' \otimes V'), F) \to Ext(e_0(\frac{V'}{F} \otimes V'), F)$ is not onto.

But $Ext(V', F) \simeq F$ as $M_{V'} \simeq e_0(V' \otimes V')$ (see below) / so $\frac{V'}{F} \subseteq \Pi$ the permutation module constructed before. Hence $e_0(\frac{V'}{F} \otimes V') \simeq N_{V'}$. Hence, there is a module $\frac{e_0(V' \otimes V')}{\frac{V'}{F}}$ with first terms of upper Loewy series being F, V'. Take direct sum of this with same for V'', factor out diagonal F, get submodule of P_F with F occurring with multiplicity $1 + 2(2^{n-1}) = 2^n + 1 > 2^n = C_{F,F}$.

Let's gather up some information:

Lemma $Ext(F, V') \simeq Ext(F, V'') \simeq F$.

Proof $Ext(F, V') \simeq Ext(V', F)$ by duality

$\simeq Hom(N', F)$, where $N' = Rad(P_{V'} = P')$

$\simeq Hom(M', F)$, as $V' \subset Rad\, M'$

$\simeq Hom(V' \otimes V', F)$ as $M' \equiv e_0(V' \otimes V')$

$\simeq Hom(V', V')$, by duality

$\simeq F$ by absolute irreducibility.

We let Π be as before, let $Y = \Pi / F$ and chose X so that $\Pi / X \simeq F$. That is

$$0 \subset F \subset X \subset \Pi$$

where $X/F \simeq V' \oplus V''$

<u>Lemma</u>. The upper and lower Loewy series of Π are given by $0 \subset F \subset X \subset \Pi$. Moreover, $X \simeq Y^*$ and $Y \simeq P_F / \mathrm{Rad}\,\mathrm{Rad}(P_F)$.

<u>Proof</u> For first part enough to show, by what we know by automorphisms and duality, that $\mathrm{Hom}(\Pi, V') = 0$. But $\mathrm{Hom}_{FG}(KG \otimes_{KH} F, V') \simeq \mathrm{Hom}_{FH}(F, V'|H) = 0$, as $V'|H$ is irreducible. Now last statement follows from $\mathrm{Rad}\, P_F / \mathrm{Rad}\,\mathrm{Rad}\, P_F \simeq V' \oplus V''$ as clearly $\mathrm{Ext}(F,F) = 0$ and by previous lemma. Thus Y is only module of its structure. Similarly for X so thus $X \simeq Y^*$.

<u>Lemma</u> $\mathrm{Ext}(F, Y) \simeq F$

<u>Proof</u> The exact sequence $0 \to V' \oplus V'' \to Y \to F \to 0$ gives the exact sequence

$$\mathrm{Hom}(F,Y) \to \mathrm{Hom}(F,F) \to \mathrm{Ext}(F, V' \oplus V'') \to \mathrm{Ext}(F,Y) \to \mathrm{Ext}(F,F)$$

which is

$$0 \to F \to F \oplus F \to \mathrm{Ext}(F,Y) \to 0$$

so done.

<u>Lemma</u> $\mathrm{Ext}(F, \Pi) \simeq 0$ or F

<u>Proof</u> The exact sequence $0 \to F \to \Pi \to Y \to 0$ yields the exact sequence

$$\mathrm{Ext}(F,F) \to \mathrm{Ext}(F,\Pi) \to \mathrm{Ext}(F,Y) \to$$

which is

$$0 \to \mathrm{Ext}(F,\Pi) \to F$$

[<u>Rk</u>: This gives $\mathrm{Ext}(\Pi,\Pi)$ of dimension 0, 1 or 2 for exact sequence $0 \to F \to \Pi \to Y \to 0$ yields

$$\underset{\simeq F}{\mathrm{Hom}(Y,\Pi)} \to \underset{\simeq F\oplus F}{\mathrm{Hom}(\Pi,\Pi)} \to \underset{\simeq F}{\mathrm{Hom}(F,\Pi)} \to \underset{\text{see below}}{\mathrm{Ext}(Y,\Pi)} \to \mathrm{Ext}(\Pi,\Pi) \to \underset{0 \text{ or } F}{\mathrm{Ext}(F,\Pi)}$$]

<u>Lemma</u> $\mathrm{Ext}(Y,\Pi) \simeq \mathrm{Ext}(\Pi, Y) \simeq 0$ or F

<u>Proof</u> $\mathrm{Ext}(Y,\Pi) \simeq \mathrm{Ext}(F, Y^* \otimes \Pi) \simeq \mathrm{Ext}(F, X \otimes \Pi)$
$\simeq \mathrm{Ext}(F, F\otimes\Pi) \simeq \mathrm{Ext}(F,\Pi)$ as $(V'\otimes V'')\otimes\Pi$ <u>is projective</u>! (by proof of $M_{V'} \simeq e_0(V'\otimes V')$)

Similarly, $\mathrm{Ext}(\Pi, Y) \simeq \mathrm{Ext}(X\otimes\Pi, F) \simeq \mathrm{Ext}(\Pi, F) \simeq \mathrm{Ext}(F,\Pi)$ as $\Pi^* \simeq \Pi$ by its definition.

<u>Lemma</u> $\mathrm{Ext}(Y,Y) \simeq 0$ or F

<u>Proof</u> The exact sequence $0 \to F \to \Pi \to Y \to 0$ yields

$$0 \to \mathrm{Hom}(Y,Y) \to \mathrm{Hom}(\Pi,Y) \to \mathrm{Hom}(F,Y) \to \mathrm{Ext}(Y,Y) \to \underset{\simeq 0 \text{ or } F}{\mathrm{Ext}(\Pi,Y)}$$

$$\ddot{0}$$

<u>Lemma</u> $\mathrm{Ext}(Y, V') = \mathrm{Ext}(Y, V'') = 0$

<u>Proof</u> The exact sequence $0 \to V' \oplus V'' \to Y \to F \to 0$ yields

$$0 \to \mathrm{Hom}(Y, V' \oplus V'') \to \underset{\simeq F}{\mathrm{Hom}(Y,Y)} \xrightarrow[\text{clearly onto}]{} \underset{\simeq F}{\mathrm{Hom}(Y,F)} \to \mathrm{Ext}(Y, V' \oplus V'') \to \underset{0 \text{ or } F}{\mathrm{Ext}(Y,Y)}$$

Hence, $\mathrm{Ext}(Y, V' \oplus V'') \simeq 0$ or F. But $\mathrm{Ext}(Y, V') \simeq \mathrm{Ext}(Y, V'')$ and $\mathrm{Ext}(Y, V' \oplus V'') \simeq \mathrm{Ext}(Y, V') \oplus \mathrm{Ext}(Y, V'')$ so done.

<u>Lemma</u> $\mathrm{Ext}^2(F, F) \simeq F$, $\mathrm{Ext}^2(F, V') = \mathrm{Ext}^2(F, V'') = 0$.

<u>Proof</u> Consider the minimal projective resolution of F

$$P_3 \to P_2 \to P_1 \to P_0 \to F$$

We claim that $P_2 \simeq P_F$; this is enough as $\mathrm{Ext}^2(F, F) \simeq \mathrm{Hom}(P_2, F)$, by minimality of the resolution and irreducibility of F. Now $P_0 \simeq P_F$. Since $P_F / \mathrm{Rad}\,\mathrm{Rad}(P_F) \simeq V' \oplus V''$ we have that $P_1 \simeq P_{V'} \oplus P_{V''}$. Moreover, $\mathrm{Ker}(P_1 \to P_0) \supseteq V' \oplus V''$, or else P_0 would have $P_{V'}$ or $P_{V''}$ as submodule and therefore summand! But

$$\dim P_0 = 2^n + 2^{n-1}\left(\tfrac{q-1}{2}\right) + 2^{n-1}\left(\tfrac{q-1}{2}\right) = 2^n + 2^{n-1}(q-1)$$

$$\dim P_{V'} = 2^{n-1} + (2^{n-2}+1)\left(\tfrac{q-1}{2}\right) + 2^{n-2}\left(\tfrac{q-1}{2}\right) = 2^{n-1} + (2^{n-1}+1)\left(\tfrac{q-1}{2}\right)$$

so

$$\dim \mathrm{Ker}(P_2 \to P_1) = \dim P_{V'} + \dim P_{V''} - \dim P_0 + 1$$

$$= 1 + q - 1 = q.$$

Thus, have $\mathrm{Ker}(P_2 \to P_1) / V' \oplus V'' \simeq F$. Hence,

$\mathrm{Ker}(P_2 \to P_1) / \mathrm{Rad}\,\mathrm{Ker} \simeq F$, $F \oplus V'$ or $F \oplus V''$ (not $F \oplus V' \oplus V''$ as $F \notin P_{V'} \oplus P_{V''}$). But latter cases yield $\mathrm{Ext}^2(F, V') \neq 0$, $\mathrm{Ext}^2(F, V'') = 0$ (or vice versa) which is nonsense by automorphism. Hence done.

<u>Lemma</u> $\mathrm{Ext}(V', V') = \mathrm{Ext}(V'', V'') = 0$.

<u>Proof</u>. Have exact sequence $0 \to V' \oplus V'' \to Y \to F \to 0$ so

$$\underbrace{\mathrm{Ext}(Y, V')}_{0} \to \mathrm{Ext}(V' \oplus V'', V') \to \underbrace{\mathrm{Ext}^2(F, V')}_{0}$$

<u>Prop.</u> $P_{V'}$, $P_{V''}$ uniserial just as desired.

<u>Proof</u>. By previous lemma, $M_{V'}$ ($\simeq e_0(V' \otimes V')$) has radical quotient just F (as we know $\mathrm{Ext}(V', V'') = 0$ as $V' \otimes V''$ projective).
Now there is no uniserial V', F, V' by previous proposition.
Keep on going!

Picture - as now P_F easy by counting:

$$P_F: \quad \begin{array}{ccc} & F & \\ \hline V' & & V'' \\ F & & F \\ V'' & \oplus & V' \\ \vdots & & \vdots \\ V' & & V'' \\ \hline & F & \end{array} \qquad P_{V'}: \quad \begin{array}{c} V' \\ \hline F \\ V'' \\ F \\ V' \\ \vdots \\ \hline V' \end{array} \qquad P_{V''}: \quad \begin{array}{c} V'' \\ \hline F \\ V' \\ F \\ V'' \\ \vdots \\ \hline V'' \end{array}$$

<u>Second block for A_7</u> (p30 all over again)

The modules are V, V^* and $V \wedge V$. $C = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.
Consider $V \otimes V$. Have $V \otimes V \doteq V \oplus V\wedge V \oplus V \wedge V$ by character table on page 14. Also $V \otimes V \,|\, A_6$ is uniserial with factors $1, 4, 1, 4_c, 1, 4, 1$, by results on page 17. Hence, as $V \otimes V \to V \wedge V \to 0$ is exact get $V \otimes V$ uniserial with factors $V\wedge V, V, V \wedge V$. Similarly for $V^* \otimes V^* \simeq (V \otimes V)^*$. $\therefore$ $Ext(V, V\wedge V) \neq 0$.

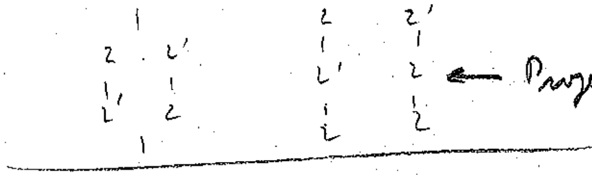
Also, $Ext_*(V, V^*) \simeq Ext^*(V \otimes V, F) = 0$ as $V \otimes V$ has no factor in principle block. Hence, consider P_V. Next term of Loewy series is $V \wedge V$ or $V \wedge V \oplus V \wedge V$, by what we have proved. In latter case this forces next factor in Loewy series to be V^* as $Ext(V^*, V) \neq 0$, a contradiction. $\therefore$ P_V starts out $V, V \wedge V$. Same shows no $V \wedge V$ next again violate $Ext(V^*, V) \neq 0$ so P_V uniserial: $V, V\wedge V, V^*, V \wedge V, V$. Thus P_{V^*} is similar.

Now we turn to $P_{V \wedge V}$. Have uniserial $V \wedge V, V^*, V\wedge V, V$ and $V\wedge V, V, V \wedge V, V^*$ we can "glue". Get $M_{V\wedge V} = \begin{array}{c} V^* \\ V\wedge V \\ V \end{array} \oplus \begin{array}{c} V \\ V\wedge V \\ V^* \end{array}$

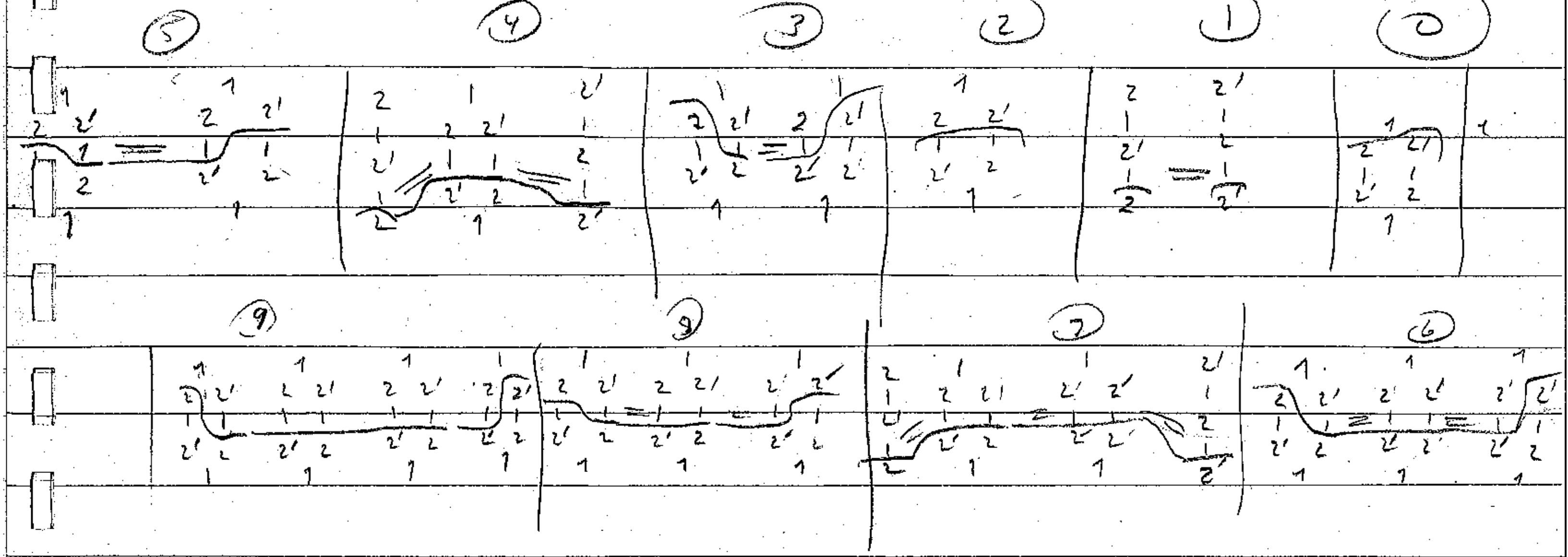
Picture:

$$P_{V\wedge V} \quad \begin{array}{c} \overline{\quad V\wedge V \quad} \\ V \qquad V^* \\ V\wedge V \oplus V\wedge V \\ V^* \qquad V \\ \overline{\quad V \wedge V \quad} \end{array} \qquad P_V: \begin{array}{c} V \\ V\wedge V \\ V^* \\ V \wedge V \\ V \end{array} \qquad P_{V^*}: \begin{array}{c} V^* \\ V\wedge V \\ V \\ V\wedge V \\ V^* \end{array}$$

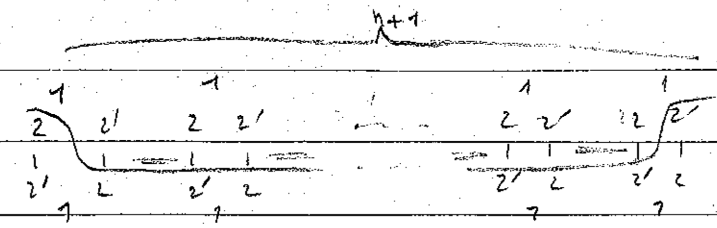
A5


← Projection

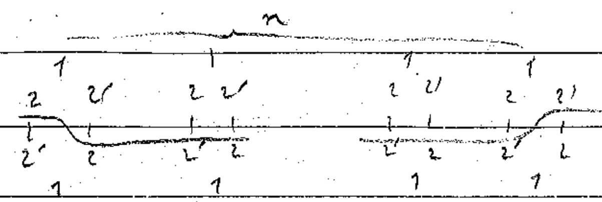
Resolution



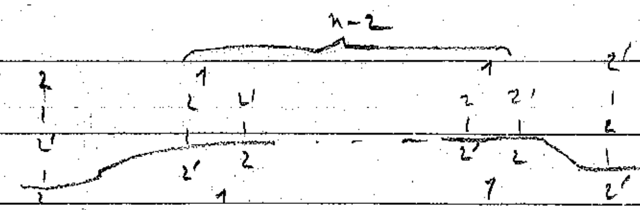
P_{3n}:



P_{3n-1}:



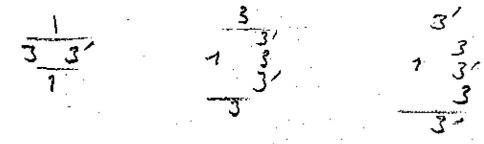
P_{3n-2}:

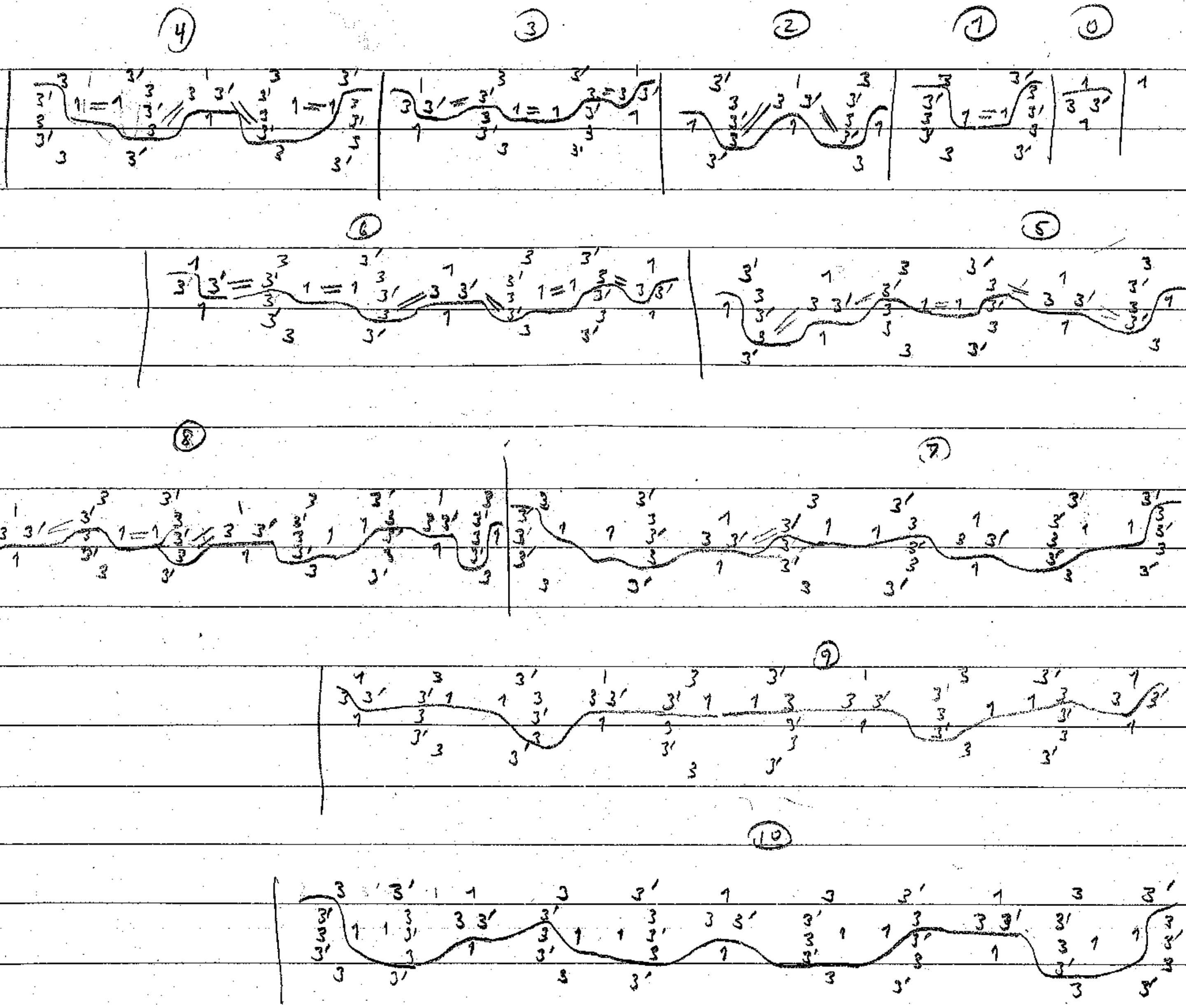


Apparent pattern: $\dim_F H^n(G, X)$

$X \backslash n=$	0	1	2	3	4	5	6	7	8	9
F	1	0	1	2	1	2	3	2	3	4
2	0	1	0	0	1	0	0	1	0	0
2'	0	1	0	0	1	0	0	1	0	0

$L_2(7)$

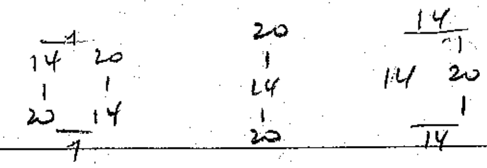


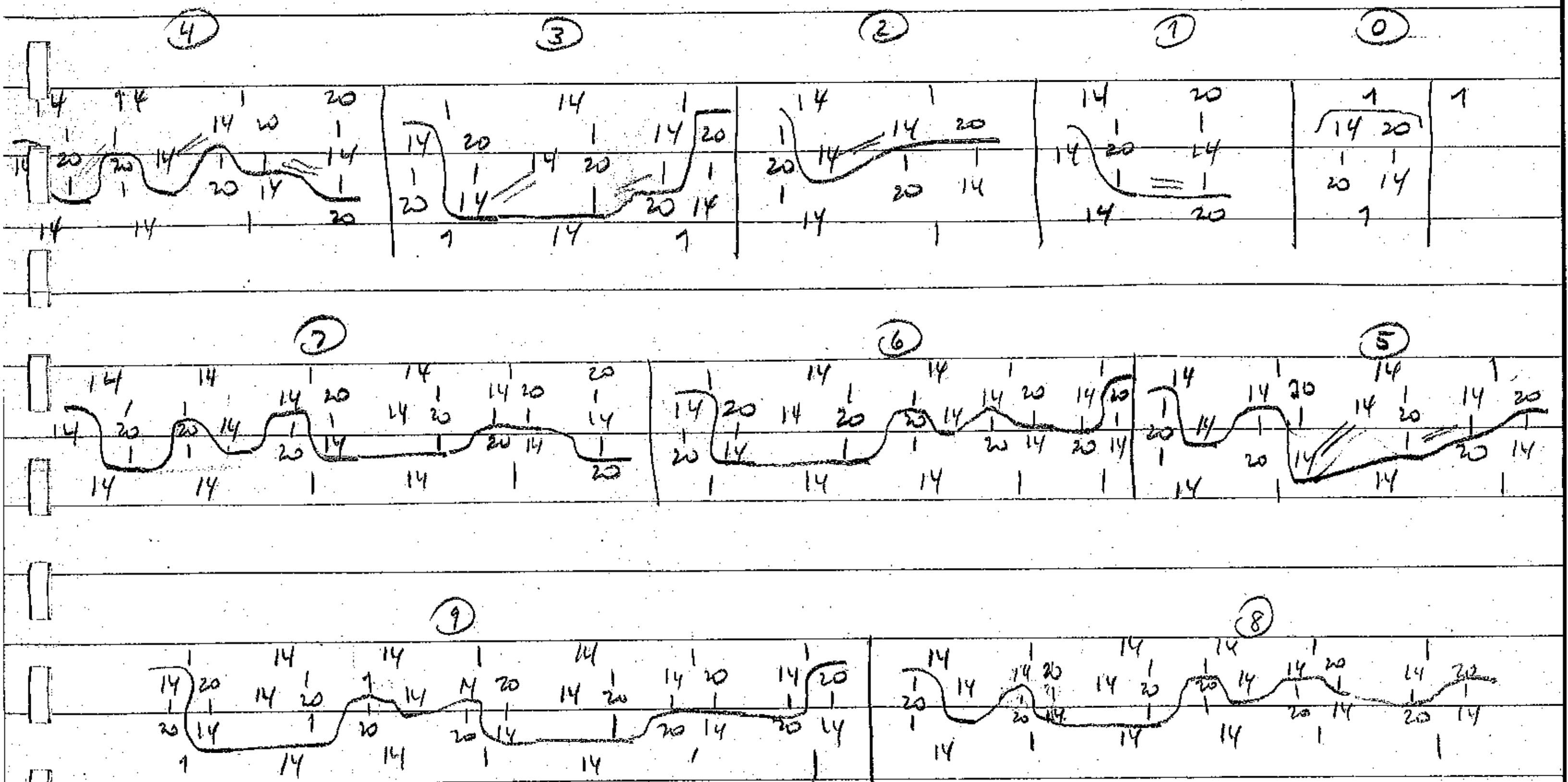


apparent pattern : $\dim_F H^n(G, X)$

$X \backslash n$	0	1	2	3	4	5	6	7	8	9	10
F	1	0	1	2	1	2	3	2	3	4	3
3	0	1	1	1	2	2	2	3	3	3	4
3′	0	1	1	1	2	2	2	3	3	3	4

A_9





apparent pattern $\dim_F H^n(G, X)$:

$X \backslash n$	0	1	2	3	4	5	6	7	8	9
F	1	0	1	2	1	2	3	2	3	4
14	0	1	1	1	2	2	2	3	3	3
20	0	1	0	0	1	0	0	1	0	0

A construction

Consider the following situation:

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r, shift left, "u_2"] & P_{-1} \arrow[l, shift left, "d_{-1}"] \arrow[r, shift left, "u_{-1}"] & P_0 \arrow[l, shift left, "d_0"] \arrow[r, shift left, "u_0"] & P_1 \arrow[l, shift left, "d_1"] \arrow[r, shift left, "u_1"] & P_2 \arrow[l, shift left, "d_2"] \arrow[r] & {} \end{tikzcd}$$

where the P_i are R-modules for some ring R, the u_i and d_i are R-homomorphisms, $\{P_i, d_i\}$ and $\{P_i, u_i\}$ are both acyclic complexes and satisfy the following conditions:

1) $du + ud = 0$
2) $\mathrm{Ker}(du) = \mathrm{Ker}\, d + \mathrm{Ker}\, u = \mathrm{Ker}\, ud$

} with appropriate indexing

Now consider the following situation

$$\begin{tikzcd}[column sep=small] & & & 0 & & & & \\ & & & P_0/\mathrm{Ker}\, d_0 + \mathrm{Ker}\, u_0 \arrow[u] & & & & = Q_{-1} \\ & & & P_0 \arrow[u] & & & & = Q_0 \\ & & P_{-1} \arrow[ur, "u_{-1}"] & \oplus & P_1 \arrow[ul, "d_1"'] & & & = Q_1 \\ & P_{-2} \arrow[ur, "u_{-2}"] & \oplus & P_0 \arrow[u, "d_0 + u_0"] & \oplus & P_2 \arrow[ul, "d_2"'] & & = Q_2 \\ P_{-3} \arrow[ur, "u_{-3}"] & & P_{-1} \arrow[ur, "d_{-1}+u_{-1}"] & & P_1 \arrow[ur, "d_1+u_1"] & & P_3 \arrow[ul, "d_3"'] & = Q_3 \\ \vdots & & \vdots & & \vdots & & \vdots & \end{tikzcd}$$

→ Rk (MacLane) Have single complex from a double complex!

Claim: This is an acyclic complex of R-modules and maps.

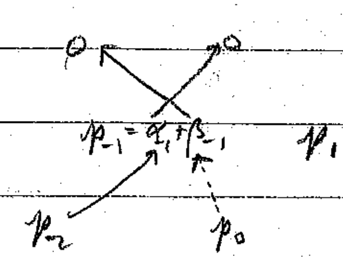
First, let's see we have a differential. This is immediate from the equations:

$$d^2 = 0, \quad du + ud = 0, \quad u^2 = 0$$

Next, exactness. At Q_{-1} clear as map is onto.
at Q_0 also clear as $\operatorname{Ker} d_0 + \operatorname{Ker} u_0 = \operatorname{Im} d_{-1} + \operatorname{Im} u_{-1}$.
Consider Q_1, say $p_{-1} \in P_{-1}$, $p_1 \in P_1$ and $p_{-1} u_0 + p_1 d_0 = 0$.
(If either summand is zero so are both and exactness of d, u finish the question.) The picture is now:

$$\begin{tikzcd}[column sep=small]
 & P_{-1} \arrow[d, dash] & & & P_0 \arrow[d, dash] & & & P_1 \arrow[d, dash] & & P_0 \\
 & \operatorname{Ker} d_{-1} + \operatorname{Ker} u_{-1} \arrow[dl, dash] \arrow[dr, dash] & & & \operatorname{Ker} d_0 + \operatorname{Ker} u_0 \arrow[dl, dash] \arrow[dr, dash] & & & \operatorname{Ker} d_1 + \operatorname{Ker} u_1 \arrow[dl, dash] \arrow[dr, dash] & ; & P_{-1} \oplus P_1 \\
\operatorname{Ker} d_{-1} \arrow[dr, dash] & & \operatorname{Ker} u_{-1} \arrow[dl, dash] & \operatorname{Ker} d_0 \arrow[dr, dash] & & \operatorname{Ker} u_0 \arrow[dl, dash] & \operatorname{Ker} d_1 \arrow[dr, dash] & & \operatorname{Ker} u_1 \arrow[dl, dash] & P_2 \oplus P_0 \oplus P_2 \\
 & \operatorname{Ker} d_{-1} \cap \operatorname{Ker} u_{-1} \arrow[d, dash] & & & \operatorname{Ker} d_0 \cap \operatorname{Ker} u_0 \arrow[d, dash] & & & \operatorname{Ker} d_1 \cap \operatorname{Ker} u_1 \arrow[d, dash] & & \\
 & 0 & & & 0 & & & 0 & &
\end{tikzcd}$$

Suppose $p_{-1} + p_1 \in \operatorname{Ker}(Q_1 \to Q_0)$ where $p_{-1} \in P_{-1}$, $p_1 \in P_1$.
Thus $p_{-1} u_{-1} + p_1 d_1 = 0$. $\therefore$ $p_1 u_{-1} d_0 = 0$. Hence,
$p_{-1} = \alpha_{-1} + \beta_{-1}$ where $\alpha_{-1} \in \operatorname{Ker} u_{-1}$, $\beta_{-1} \in \operatorname{Ker} d_{-1}$.
Let $p_{-2} \in P_{-2}$ be such that $p_{-2} u_{-2} = \alpha_{-1}$.
Choose $p_0 \in P_0$ with $p_0 d_0 = \beta_{-1}$. Next



$$\begin{aligned}(p_0 u_0 - p_1) d_1 &= p_0 u_0 d_0 - p_1 d_1 \\ &= -p_0 d_0 u_{-1} - p_1 d_1 \\ &= -\beta_1 u_1 - p_1 d_1\end{aligned}$$

But $0 = p_1 u_{-1} + p_1 d_1 = \alpha_{-1} u_{-1} + \beta_{-1} u_1 + p_1 d_1$ so
$$(p_0 u_0 - p_1) d_1 = \alpha_{-1} u_{-1} = 0 \quad (\text{as } \alpha_{-1} \in \operatorname{Ker} u_{-1}).$$
Thus $p_1 - p_0 u_0 = p_2 d_2$. Hence,
$$\begin{aligned}p_{-2} + p_0 + p_2 &\to p_{-2} u_{-2} + p_0 d_0 + p_0 u_0 + p_2 d_2 \\ &= \alpha_{-1} + \beta_{-1} + p_1 = p_{-1} + p_1 \text{ as desired!}\end{aligned}$$

Next, consider $\underline{Q_2}$, and suppose $p_{-2}+p_0+p_2 \to 0$

That is,

$$\underbrace{(p_{-2}u_{-2}+p_0d_0)}_{\in P_{-1}}+\underbrace{(p_0u_0+p_2d_2)}_{\in P_1}=0$$

Thus,

$$p_{-2}u_{-2}+p_0d_0 = p_0u_0+p_2d_2=0.$$

Hence, by previous argument, there are p_{-3}, p_{-1}, p_1 such that

p_{-2} p_0 p_2

p_{-3} p_{-1} p_1 p_3

$$p_{-2}+p_0=(p_{-3}u_{-3}+p_{-1}d_{-1})+(p_{-1}u_{-1}+p_1d_1).$$

Hence, enough to show there is $p_3\in P_3$ such that we have $p_2-p_1u_1=p_3d_3$. But $(p_2-p_1u_1)d_2 = p_2d_2-p_1u_1d_2$
$= -p_0u_0-p_1u_1d_2 = -(p_{-1}u_{-1}+p_1d_1)u_0-p_1u_1d_2 = -p_1d_1u_0-p_1u_1d_2=0$

We next describe the general case, Q_n. Suppose

$$p_{-n}+p_{-n+2}+\dots+p_{n-2}+p_n\to 0$$

Thus, in particular we have $p_{n-2}u_{n-2}+p_nd_n=0$

By previous case of the obvious induction, (i.e. prev case for shifted sequence)

p_{-n} p_{-n+2} $\cdots$ p_{n-2} p_n

there exist $p_{-n-1},\dots,p_{n-1}$ such that

$$p_{-n-1}+\dots+p_{n-1}\to p_{-n}+\dots+p_{n-2}+p_{n-1}u_{n-1}.$$

Hence, suffices to show that $(p_n-p_{n-1}u_{n-1})d_n=0$ as then let p_{n+1} be chosen with $p_{n+1}d_{n+1}=p_n-p_{n-1}u_{n-1}$ and

$$p_{-n-1}+\dots+p_{n+1}\to p_{-n}+\dots+p_n.$$

But $(p_n-p_{n-1}u_{n-1})d_n = p_nd_n-p_{n-1}u_{n-1}d_n = -p_{n-2}u_{n-2}-p_{n-1}u_{n-1}d_n$

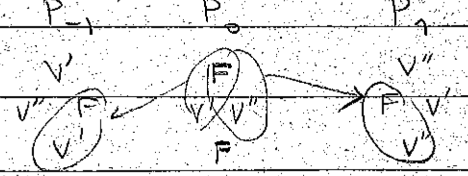
(as $p_{-n}+\dots+p_n\to 0$ and component in P_{n-1} is $p_{n-2}u_{n-2}+p_nd_n$)

$= (p_{n-3}u_{n-3}+p_{n-1}d_{n-1})u_{n-2}+p_{n-1}u_{n-1}d_n$ (by our induction)

$= p_{n-1}d_{n-1}u_{n-2}+p_{n-1}u_{n-1}d_n=0$, Q.E.D.

We do an example, namely $L_2(q)$, $q \equiv 3$ (modulo 8).

First,



have d_0, u_0 pictured. Define u_{-1}. Then define d_1 so $d_0 u_{-1} + u_0 d_1 = 0$ - can do as $d_0 u_{-1}$, $u_0 d_1$ will differ (multiplicatively) by an element of $F^{\#}$. Then give d_{-1}, u_1 and then choose u_{-2}, d_2 suitably. Continue in this way.

Resolution is (dim)

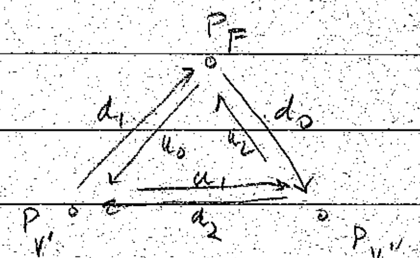
$$
\begin{array}{cr}
F & 1 \\
\uparrow & \\
P_F & 0 \\
P_{V'} \oplus P_{V''} & 1 \\
P_{V''} \oplus P_F \oplus P_{V'} & 2 \\
P_F \oplus P_{V'} \oplus P_{V''} \oplus P_F & 3
\end{array}
$$

etc.

A little inspection also reveals that the preimage of the socle of Q_i is the radical of Q_{i+1} (Q_j being in dimension j) so resolution is minimal.

All Ext^n are as guessed.

Can we put all this in a triangle? Note that splitting field for B_0 here is $GF(4)$. (If $x = -\frac{1}{2} + \frac{\sqrt{-q}}{2}$ then $x^2 + x = -\frac{q+1}{4} = -2^{n-2}|q+1|_{2'}$, so as $n = 2$, $x^2 + x + 1 = 0$, $x \in GF(4)$, $x \notin GF(2)$.) Suppose we define d, u so getting acyclicly both ways and so du and ud differ by a scalar:



Thus
$$u_0 d_1 = d_0 u_2 \alpha_0 \qquad \alpha_i \in GF(4)^{\#}$$
$$u_1 d_2 = d_1 u_0 \alpha_1$$
$$u_2 d_0 = d_2 u_1 \alpha_2$$
Suppose now $d_i \to d_i \delta_i = d_i'$, $u_i \to u_i \pi_i = u_i'$, $1 \le i \le 3$. Get new α_i, call them α_i' so
$$u_0' d_1' = d_0' u_2' \alpha_i' \quad \text{etc}$$
which is
$$\alpha_0' = \pi_0 \delta_1 \delta_0^{-1} \pi_2^{-1} \alpha_0$$
$$\alpha_1' = \pi_1 \delta_2 \delta_1^{-1} \pi_0^{-1} \alpha_1$$
$$\alpha_2' = \pi_2 \delta_0 \delta_2^{-1} \pi_1^{-1} \alpha_2$$
so
$$\alpha_1' \alpha_2' \alpha_3' = \alpha_0 \alpha_1 \alpha_2$$
Hence, if not all $\alpha_i = 1$ and also $\alpha_1 \alpha_2 \alpha_3 \neq 1$ then no hope!

Seems a tricky question.

But can resolve this? — with a helpful conversation with O'Nan —
Given a triangle with invariants $\alpha_1, \alpha_2, \alpha_3$ apply the non-identity automorphism of $F = GF(4)$ getting a new triangle with invariants $\bar\alpha_1, \bar\alpha_2, \bar\alpha_3$. But new triangle is

$$\begin{tikzcd} & P_{\bar F} = P_F \arrow[dl, shift right, "u_0"'] \arrow[dr, "d_0"] & \\ P_{\bar V'} \arrow[ur, "d_1"] \arrow[rr, shift left, "u_1"] & & P_{\bar V''} \arrow[ul, shift left, "u_2"] \arrow[ll, shift left, "d_2"] \end{tikzcd}$$

Have $\bar V' = V''$ so really have

$$\begin{tikzcd} & P_F \arrow[dl, shift right, "\bar u_0"'] \arrow[dr, "\bar d_0"] & \\ P_{V''} \arrow[ur, "\bar d_1"] \arrow[rr, shift left, "\bar u_1"] & & P_{V'} \arrow[ul, shift left, "\bar u_2"] \arrow[ll, shift left, "\bar d_2"] \end{tikzcd}$$

so

$$\begin{tikzcd} & P_F \arrow[dl, shift right, "\bar d_0"'] \arrow[dr, "\bar u_0"] & \\ P_{V'} \arrow[ur, "\bar u_2"] \arrow[rr, shift left, "\bar d_2"] & & P_{V''} \arrow[ul, shift left, "\bar d_1"] \arrow[ll, shift left, "\bar u_1"] \end{tikzcd}$$

$d'_1 = \bar u_2,\ d'_2 = \bar u_1,\ d'_0 = \bar u_0$
$u'_1 = \bar d_2,\ u'_2 = \bar d_1,\ u'_0 = \bar d_0$

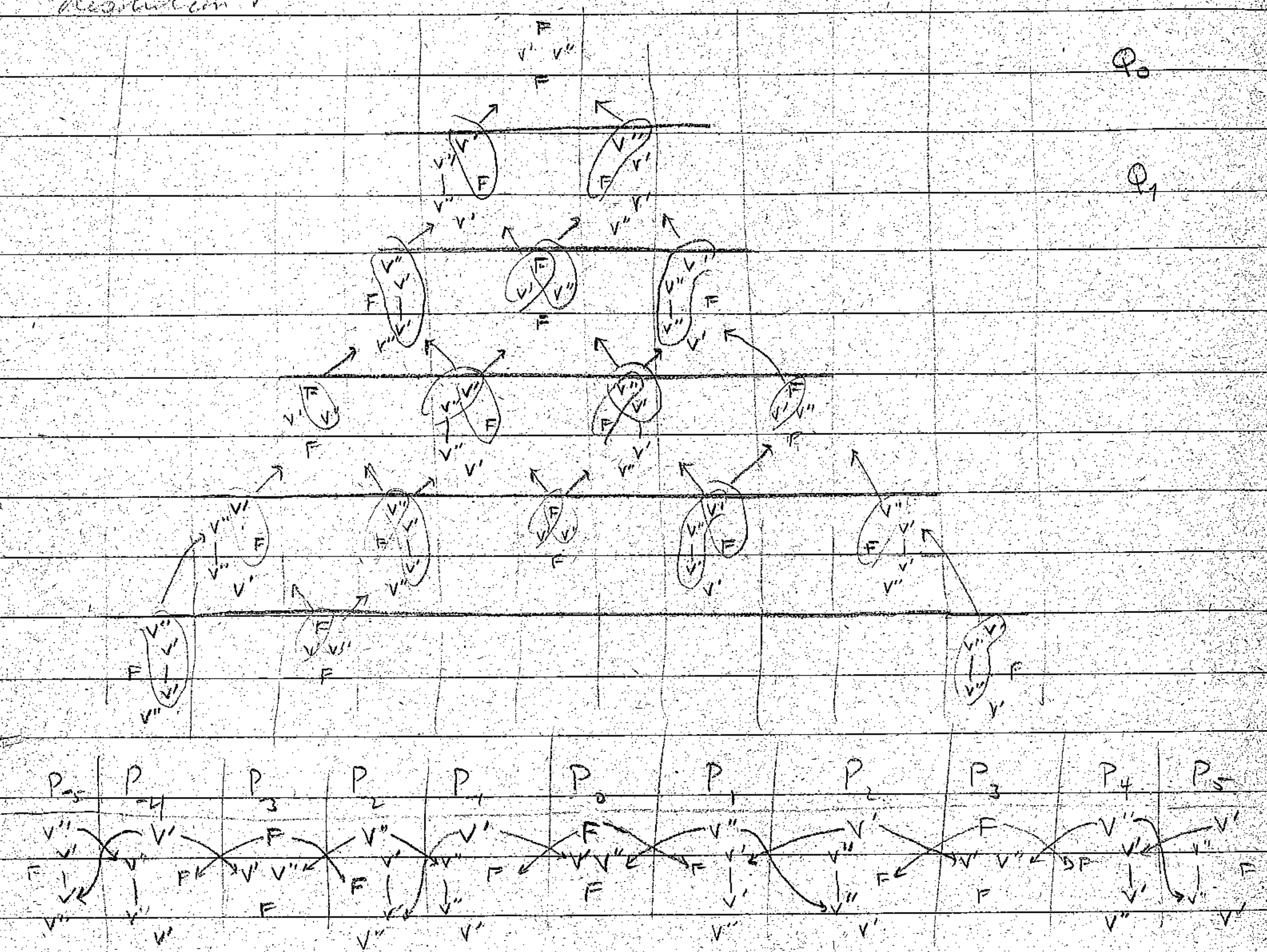
$$\left.\begin{aligned} u'_0 d'_1 &= d'_0 u'_2 \alpha'_0 \\ u'_1 d'_2 &= d'_1 u'_0 \alpha'_1 \\ u'_2 d'_0 &= d'_2 u'_1 \alpha'_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \bar d_0 \bar u_2 &= \bar u_0 \bar d_1 \alpha'_0 \\ \bar d_2 \bar u_1 &= \bar u_2 \bar d_0 \alpha'_1 \\ \bar d_1 \bar u_0 &= \bar u_1 \bar d_2 \alpha'_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha'_0 &= \bar\alpha_0^{-1} = \alpha_0 \\ \alpha'_1 &= \bar\alpha_2^{-1} = \alpha_2 \\ \alpha'_2 &= \bar\alpha_1^{-1} = \alpha_1 \end{aligned}$$

Hence $\alpha_0 \alpha_1 \alpha_2 = \alpha'_1 \alpha'_2 \alpha'_3$. This leads nowhere!!

Also dualizing leads nowhere, also easy to check!

Consider situation next for $q \equiv 7 \pmod{8}$

Resolution:

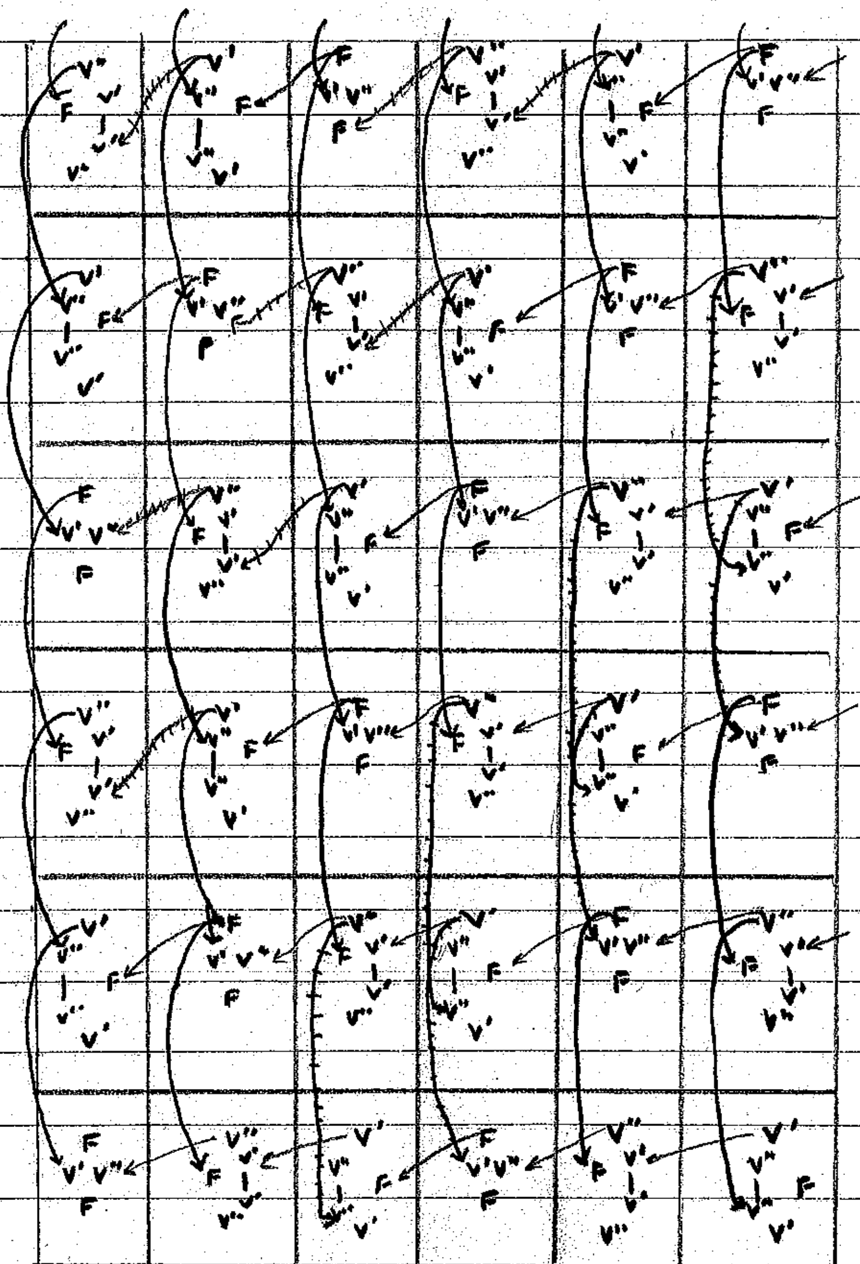


Now can assume $F = GF(2)$ (as here, $x = -\frac{1}{2} + \frac{\sqrt{-7}}{2}$ yields $x^2 + x = 0$, as $n > 2$, so $x = 0, 1$). Thus have maps d, u with $d^2 = u^2 = 0$, $du + ud = 0$. <u>But</u> <u>exactness</u> for d <u>fails</u> at P_{-2}, for u at P_2 etc., modulo 6.

Let's picture the double complex : $\xleftarrow{d}$ $\downarrow u$

(rotate previous page)

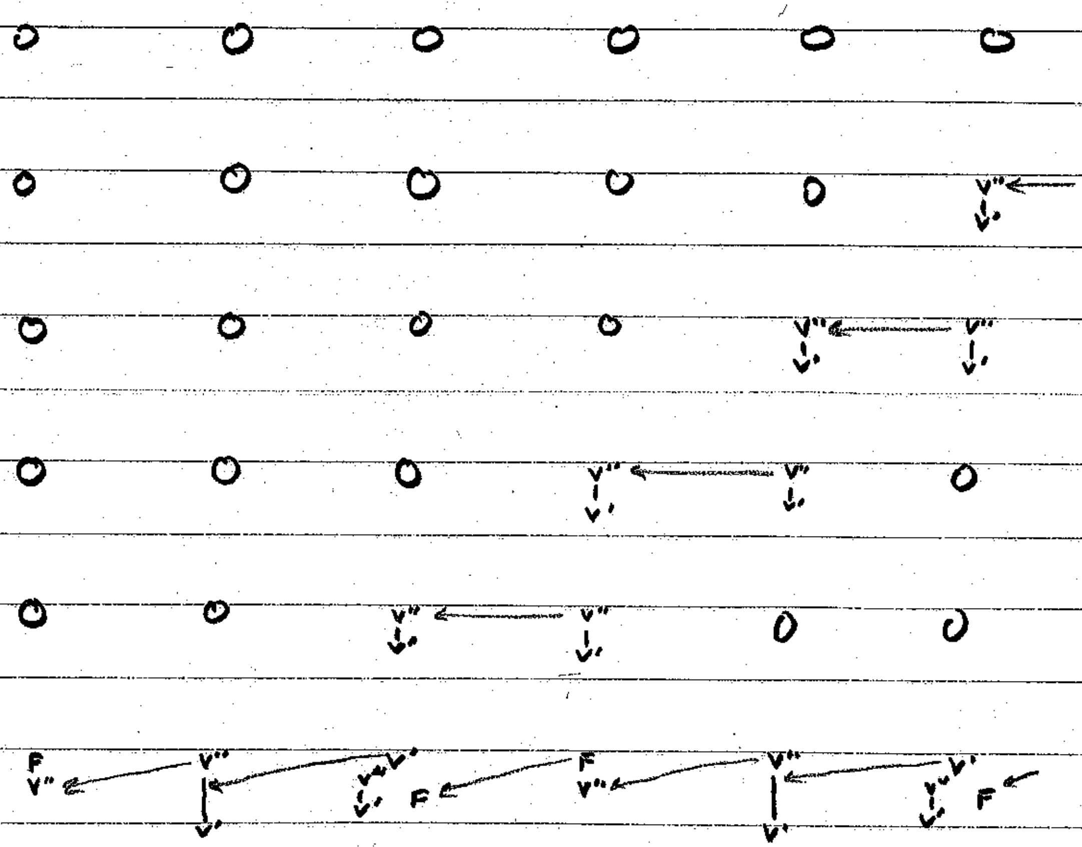
$P_{ij} = P_{j-i}$.



Note: 1) $u\downarrow$ is exact at $P_{p,q}$ if $q \geq p$.

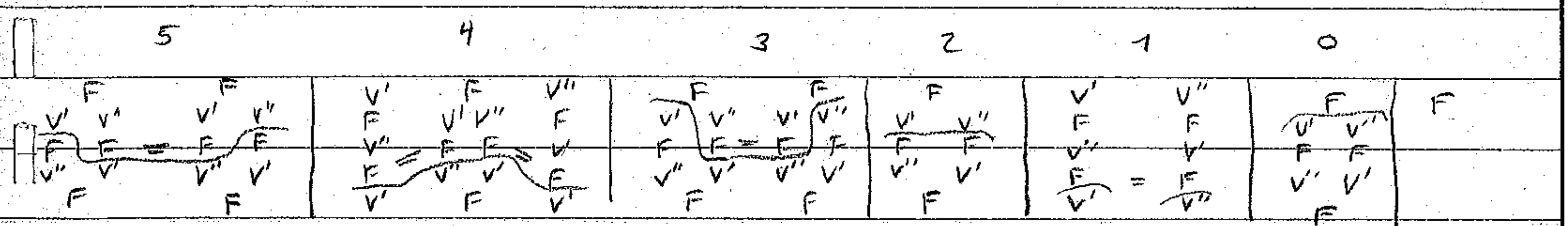
2) $d \leftarrow$ is exact at $P_{p,q}$ if $q \leq p$

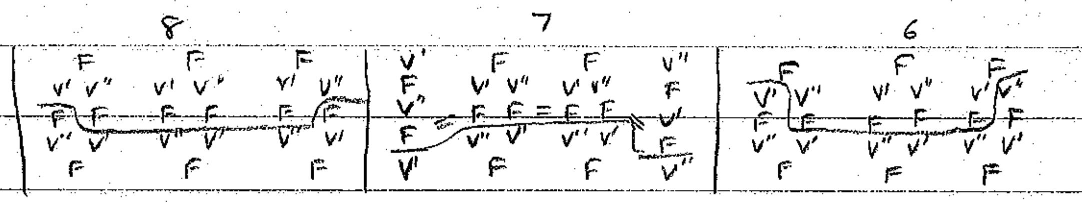
Following a suggestion of Liulevicius we calculate the column homology and induced differential:

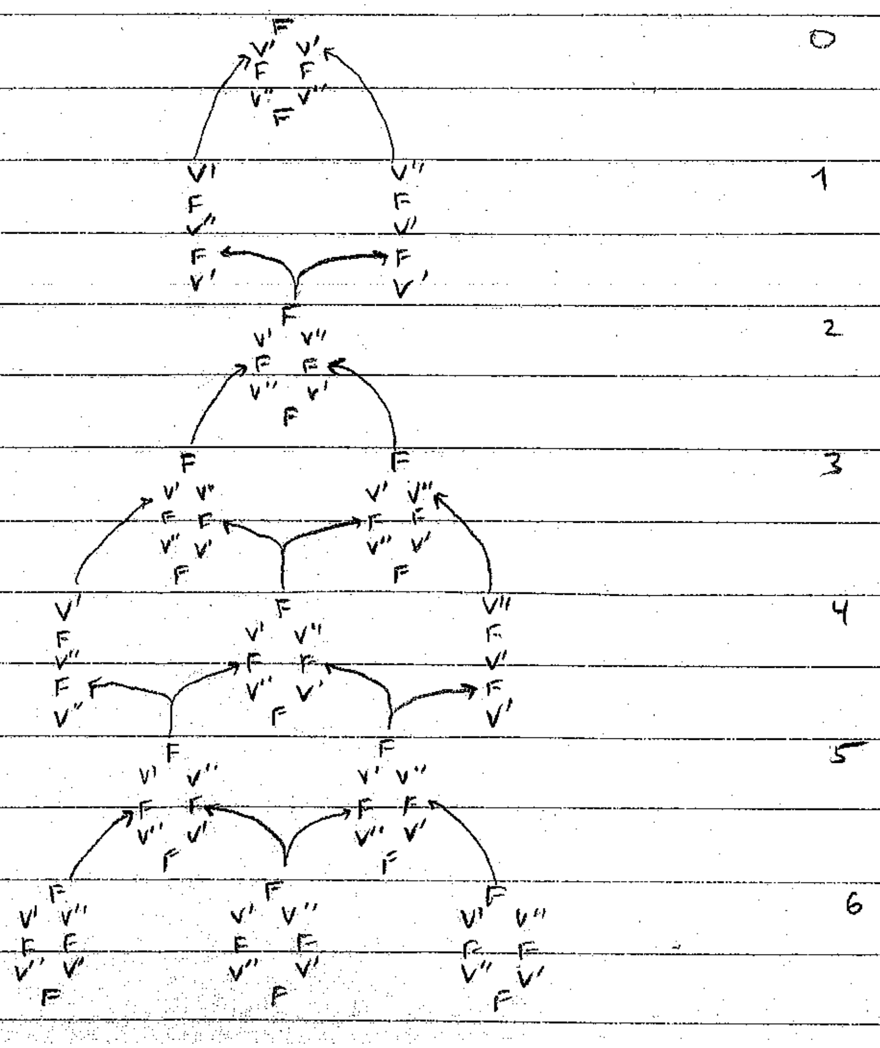


Hence, taking row homology now get all terms 0 except 0,0 which is F so by spectral sequences get result we're after!! That is, have E_2 term of spectral sequence converging to homology of our "resolution" complex, i.e. single complex associated to this double complex.

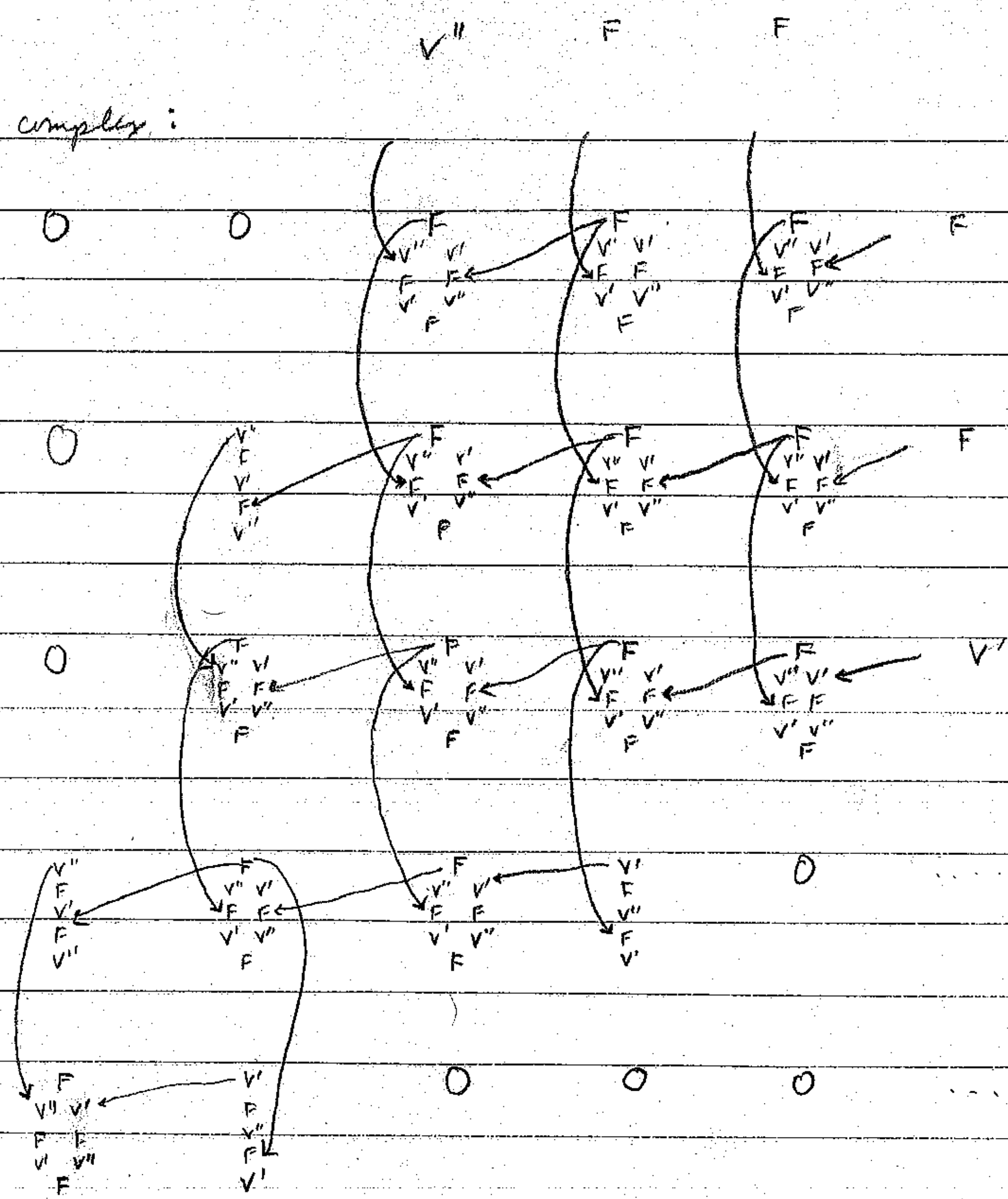
Let's try the same technique of spectral sequences for $q \equiv 5\ (8)$



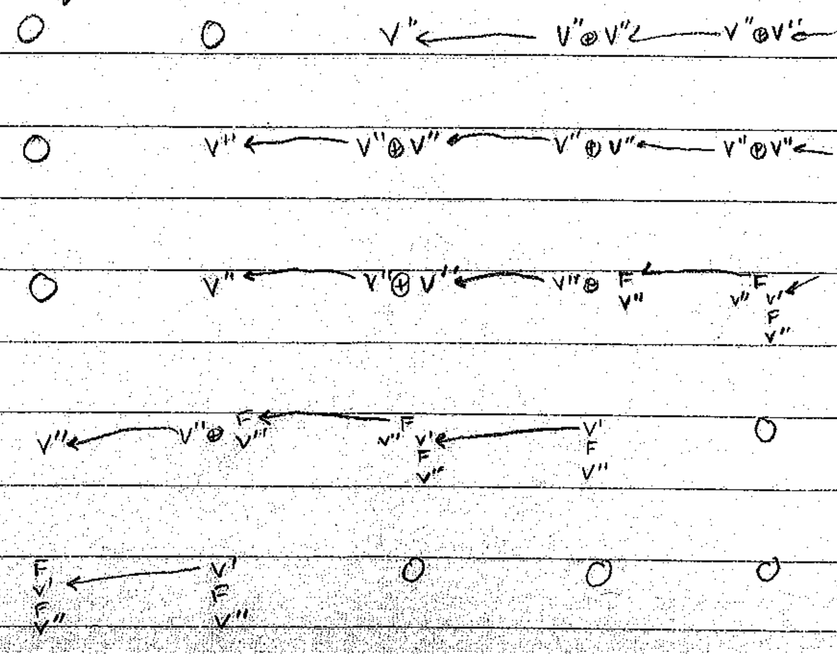




Double complex:

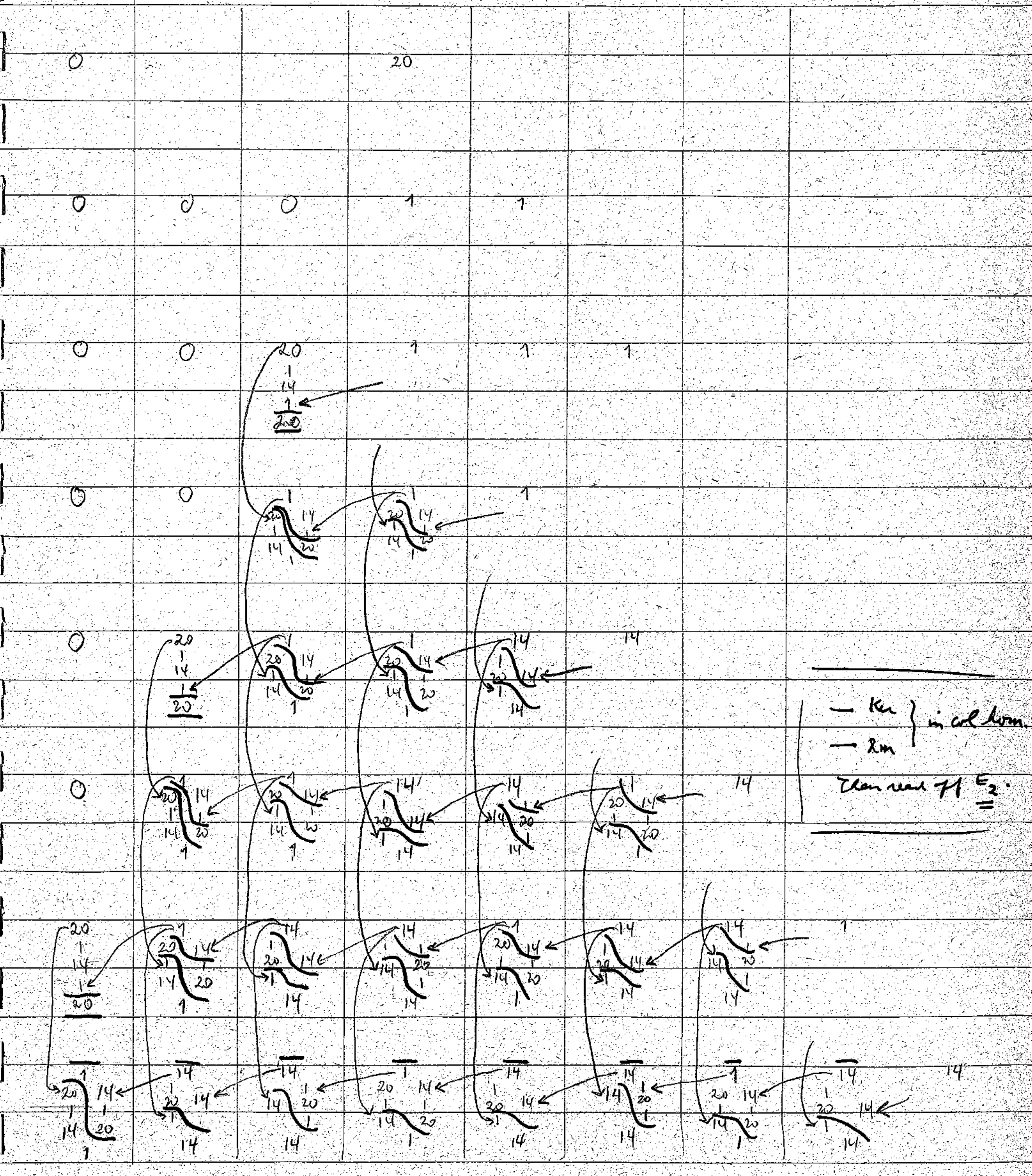


Column homology and induced differential:



{ Hence, get desired acyclicity

Let's use results from page 69 to construct the double complex for A_9:



— Ker } in col hom.
— Im

Then read off E_2.

General theorem seems to be the following:

Theorem If G is a simple group with dihedral Sylow 2-subgroups and F is a splitting field of characteristic two for G Then there is a first quadrant double complex $\{P_{ij}\}$ of $F[G]$-modules with the following properties:

1) Each non-zero term P_{ij} is an indecomposable $F[G]$ projective module

2) There is an augmentation for the associated single complex so that the resulting complex is "the" minimal projective resolution for F.

Also, in all cases the simple spectral sequence argument seems to work to show the single complex is a resolution. Minimality will have to be demonstrated by "inspection".

Related Research Problems

(1) What is the relation between the Cartan matrix mod p of $SL(2,p)$ and that of $SL(2,p^n)$. Generalize to algebraic groups

Remarks: Say $p=2$. Now $SL(2,2) \simeq \Sigma_3$ and $C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.
Also $SL(2,4) \simeq A_5$ and $C = \begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
Let's calculate the case $SL(2,8)$. Let's look at $B_0(2)$

	1	2	$Z_7^\#$	$Z_9^\#$	
ordinary	1	1	1	1	
	7	-1	0	$-(\lambda+\lambda^8)$	$\lambda \in \mathbb{C}^*$ $\langle\lambda\rangle \simeq Z_9$
	7	-1	0	$-(\lambda^2+\lambda^7)$	
	7	-1	0	$-(\lambda^4+\lambda^5)$	
	7	-1	0	$-(\lambda^3+\lambda^6$	
	9	1	$\mu+\mu^6$	0	$\mu \in \mathbb{C}^*$ $\langle\mu\rangle \simeq Z_7$
	9	1	$\mu^2+\mu^5$	0	
	9	1	$\mu^4+\mu^3$	0	
modular	1	$-$	1	1	
	2	$-$	$\mu+\mu^6$	$\lambda+\lambda^8$	
	2	$-$	$\mu^2+\mu^5$	$\lambda^2+\lambda^7$	
using tensor product theorem	2	$-$	$\mu^4+\mu^3$	$\lambda^4+\lambda^5$	
	4	$-$	$\mu+\mu^3+\mu^4+\mu^6$	$\lambda+\lambda^8+\lambda^3+\lambda^6$	
	4	$-$	$\mu^2+\mu^3+\mu^4+\mu^5$	$\lambda^4+\lambda^5+\lambda^3+\lambda^6$	
	4	$-$	$\mu+\mu^2+\mu^5+\mu^6$	$\lambda^2+\lambda^7+\lambda^3+\lambda^6$	

$$\underline{D_0} \quad \begin{array}{c|ccccccc} & 1 & 2 & 2 & 2 & 4 & 4 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 7 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 7 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 7 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 9 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 9 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 9 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{array} \qquad \underline{C} \quad \begin{pmatrix} 8 & 4 & 4 & 4 & 2 & 2 & 2 & 0 \\ 4 & 4 & 2 & 2 & 0 & 2 & 1 & 0 \\ 4 & 2 & 4 & 2 & 2 & 1 & 0 & 0 \\ 4 & 2 & 2 & 4 & 1 & 0 & 2 & 0 \\ 2 & 0 & 2 & 1 & 2 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Probably relevant for odd p is Srinivasan, P.L.M.S. v64 p101.

(2) Is there a "quasi-induction" for discrete series - i.e. generalized blocks and other characters.
References: Feit-Thompson on Loosely embedded, Dade Reynolds
Processes used: Induction, lifting, block truncation.

(3) What is cohomology of $L(G)$ when G is a p-group, L is the Lazard functor?

(4) If $Z_2 \lhd G$ and have projectives for G/Z_2 in char 2 construct those for G (Cartan invariants double)
What if $O_p(G) \neq 1$, have $G/O_p(G)$?

⑤ How do the Quillen results on cohomology growth fit in with the Golod-Šafarevič theorem.

For example, if P is a p-group, $\dim H^1(P, Z_p) = d = \#$ generators, $\dim H^2(P, Z_p)$ roughly between $\frac{d^2}{2}$ and $\frac{d^2}{4}$ dimensions.

But Quillen proves — or at least indicates that

$\dim H^n(G, Z_p) \sim \text{const.}\,(p\text{-rank of } G - 1)^n$.

⑥ What about growth of $\dim H^n(G, V)$, V irreducible FG-module, F splitting field of char. p. The $H^n(G, V)$ are a module for $H^*(G, F)$ so can use Quillen's estimate as this module is finitely generated by Evens.

Remark of Swan: Using Shapiro's Lemma get indecomposables of all intermediate rates of growth.

Remark: Perhaps also for irreducibles but not so easily.

For example, say $q \equiv 1\ (4)$, use old notation used for $PSL(2, q)$ discussion. Have $F \otimes_{ZH} ZG = \Pi$, $|H| = q\frac{(q-1)}{2}$.

Also $0 \to F \to \Pi \to Y \to 0$ so Π has $H^n(G, \Pi) \cong F$ by Shapiro. But $H^n(G, F)$ grows much faster.

Also have an exact sequence: $0 \to Y \to P_{V'} \oplus P_{V''} \to P_F \to F \to 0$

so — as projectives are injective — $\mathrm{Ext}^n(F, Y) \cong \mathrm{Ext}^{n+2}(F, F)$

so $H^n(G, Y)$ also grows rapidly!

<u>Theorem</u> Let G be a normal subgroup of the group G_0 with G_0/G a solvable p'-group and the centralizer of a Sylow p-subgroup of G covering G/G_0. Let F be a splitting field of characteristic p for G_0 and all its subgroups. It follows that restriction of $F[G_0]$-modules to $F[G]$-modules induces an isomorphism of the category of $F[G_0]$-modules in $B_0(G_0, p)$ onto the category of $F[G]$-modules in $B_0(G, p)$.

By "in $B_0(G,p)$" we mean each composition factor is so contained." As a consequence of this, projectives go to projectives, indecomposables to indecomposables, irreducibles to irreducibles and all $F[G]$-maps are, in fact, $F[G_0]$-maps. Of course, much of this will be proved along the way!

<u>Lemma 1</u> Under the above hypotheses, each irreducible character χ in $B_0(G_0, p)$ has an irreducible restriction to G and this restriction defines a one-to-one correspondence between characters in $B_0(G_0, p)$ and $B_0(G, p)$.

<u>Proof</u> This is clear if $G/G_0 = 1$ so we may assume that $|G/G_0| = q$, a prime. Let χ be an irreducible character in $B_0(G_0, p)$. Choose $g \in G - G_0$ of full defect so the number of conjugates k of g is relatively prime to p. Since $\chi \in B_0(G_0, p)$ we have

$$k \frac{\chi(g)}{\chi(1)} \equiv k \quad (\mathfrak{g})$$

where $\mathfrak{g}$ is a prime divisor of p. Since $k \notin \mathfrak{g}$ we have $\chi(g) \neq 0$. But $\chi|G$ is either irreducible or splits into the sum of q distinct irreducible characters, in which case any one of them induces χ implying that χ vanishes off G. Thus, χ is irreducible.

Suppose $\chi'|G = \chi|G$. Then there is a linear character λ of G_0/G such that $\chi' = \chi\lambda$. If $\chi' \in B_0(G_0, p)$ then we assert that $\chi' = \chi$. Indeed, such an inclusion implies

$$k \frac{\chi(g)\lambda(g)}{\chi(1)} \equiv k, \quad k \frac{\chi(g)}{\chi(1)} \equiv k\,\lambda(g)^{-1} \quad (\mathfrak{g})$$

so $k\,\lambda(g) \equiv k \; (\mathfrak{g})$, or $k(\lambda(g)-1) \equiv 0 \; (\mathfrak{g})$, $\lambda(g)-1 \in \mathfrak{g}$. But if $\lambda(g)$ is a primitive q-th root of unity, and $p \neq q$ so this is a contradiction. Hence $\lambda(g) = 1$ so $\lambda = 1$ and $\chi' = \chi$.

It remains now to see that $\chi|G$ is in $B_0(G, p)$ and that each character in $B_0(G, p)$ is a constituent of the restriction of a character of $B_0(G_0, p)$. The first assertion follows from the arguments of Fong, T.A.M.S. (98) '61, 265-266, as the $\mathfrak{F}_i$ containing $B_0(G,p)$ consist just of $B_0(G,p)$, or from Lemma 1, p 155 of Brauer, J. of Alg., Paper I. For the second assertion see Brauer, J of alg, Paper II, p 310-11.

(All summarized in Am J. I very nicely.)

<u>Lemma 2</u> Under the above hypotheses, the same holds for the modular irreducible representations in $B_0(G_0, p)$ and $B_0(G, p)$, respectively.

[<u>Remark</u>: We can proceed by induction on $|G_0|$; the initial case $|G| = 1$ is trivial. Hence, we may assume $O_p(G_0) = 1$. In particular, if x is a p-element of G and $x \neq 1$ then $C(x) \subseteq G_0$. But, by Sylow's theorem, induction applies to $C(x)$. In particular, the <u>numbers</u> of ordinary and modular characters in $B_0(C(x))$ and $B_0(C_G(x))$ coincide. Applying Brauer's second and third main theorems we deduce that the <u>number</u> of modular irreducible characters in $B_0(G_0)$ and $B_0(G)$ is the same – using also the fact that there is no fusion of p-elements of G in G_0. (As $G_0 = C(P)G$ so if $x, y \in P$, $x^{g_0} = y$ some $g_0 \in G_0$ then $x^g = y$ some $g \in G$.) However, can prove the result without using these considerations]

<u>Proof</u>. Let V be an irreducible $F[G]$-module in $B_0(G, p)$. We claim that $V^{g_0} \cong V$ for all $g_0 \in G$. Let φ be the Brauer character of V; it suffices to see that $\varphi^{g_0} = \varphi$. But φ is an integral linear combination of the restrictions to the p'-elements of the irreducible characters χ in $B_0(G, p)$. But $\chi^{g_0} = \chi$ as χ is the restriction of a character χ_0 of G_0. Hence our claim is valid.

Since G_0/G is cyclic – or since it is of p'-order and F is of characteristic two – V extends to an irreducible module V_1

for $F[G_0]$. Let $\lambda_1, \ldots, \lambda_q$ be the q linear characters of G_0/G in F and let $F = F^{\lambda_1}, F^{\lambda_2}, \ldots F^{\lambda_q}$ be the corresponding modules. We assert that $V_i = V_0 \otimes F^{\lambda_i}$, $1 \le i \le n$, are distinct irreducible $F[G_0]$-modules, each restricts to V and no other $F[G_0]$ irreducible has V as a constituent upon restriction to G. First, these are clearly irreducible, as "scalars don't change irreducibility." Also each V_i restricts to V. Now $\mathrm{Hom}_{FG_0}(V \otimes_{FG} FG_0, V_i)$ is isomorphic to $\mathrm{Hom}_{FG}(V, V_i|G)$ so V_i is an image of $V \otimes_{FG} FG_0$. But, if W is any $F[G_0]$ irreducible and V is a constituent of $W|G$ – and therefore a component by Clifford's theorem – then $\mathrm{Hom}_{FG}(V, W|G) \neq 0$ so $\mathrm{Hom}_{FG_0}(V \otimes_{FG} FG_0, W) \neq 0$. Hence, once we know all the V_i are distinct, we have $V \otimes_{FG} FG_0 \simeq \bigoplus_{i=1}^{q} V_i$ and our claims in this paragraph hold. But

$$\begin{aligned} V \otimes_{FG} FG_0 &\simeq (F \otimes_F V_1|G) \otimes_{FG} FG_0 \\ &\simeq (F \otimes_{FG} FG_0) \otimes_F V_1 \\ &\simeq F[G_0/G] \otimes_F V_1 \\ &\simeq \bigoplus_{i=1}^{n} (F_1^{\lambda_i} \otimes V_1) \end{aligned}$$

so, if all the V_i are not distinct then $\dim_F \mathrm{Hom}_{FG_0}(V \otimes_{FG} VG_0, V_i) > 1$ for some i, and $\dim_F \mathrm{Hom}_{FG}(V, V_i|G) > 1$, a contradiction.

Next, at least one of the V_i lies in $B_0(G_0, p)$. Indeed, φ is a modular constituent of some $\chi \in B_0(G, p)$ $\chi = \chi_0|G$ for $\chi_0 \in B_0(G_0, p)$ so there is a modular constituent φ_0 of χ_0 such that φ is a component of $\varphi_0|G$. The results already achieved establish our claim here.

Hence, to conclude, we need only see that we cannot have $V_i, V_j \in B_0(G_0, p)$, $i \neq j$. Let φ_i, φ_j be their Brauer characters and let λ be an ordinary linear character so that $\varphi_i \lambda = \varphi_j$. Choose an irreducible character χ_i such that φ_i is a constituent of χ_i. Thus $\chi_i \in B_0(G_0, p)$. Hence, the same holds for φ_j and $\chi_j = \chi_i \lambda$. But, as we have seen, χ_i does not vanish outside G so $\chi_i \neq \chi_j$ but $\chi_i \mid G = \chi_j \mid G$, a contradiction to Lemma 1.

<u>Lemma 3</u> Under the above hypotheses and with respect to the one-to-one correspondences established, the decomposition matrices and Cartan matrices for $B_0(G_0, p)$ and $B_0(G, p)$ coincide.

<u>Proof</u> This is clear because the processes of restriction and reduction modulo p commute give the result for the decomposition matrices.

We now get the result we actually will apply!

<u>Lemma 4</u>. If Q_0 is an indecomposable $F[G_0]$ projective in $B_0(G_0, p)$ then $Q_0 \mid G = Q$ is an indecomposable $F[G]$ projective in $B_0(G, p)$. Moreover, this defines a one-to-one correspondence between such modules. Finally, if Q_0' and $Q' = Q_0' \mid G$ are similarly defined, we have $\mathrm{Hom}_{FG_0}(Q_0, Q_0') = \mathrm{Hom}_{FG}(Q, Q')$, actual equality

Proof. Now Q is certainly projective. Let V_0 be the irreducible socle of Q_0 so $V_0 \mid G = V$ is irreducible and in $B_0(G,p)$ and so the indecomposable projective corresponding to V is a summand of Q. But the result on Cartan matrices gives an equality. Since each V is the restriction of a V_0, the one-to-one correspondence holds. Now, certainly $\mathrm{Hom}_{FG_0}(Q_0, Q_0') \subseteq \mathrm{Hom}_{FG}(Q, Q')$. But, the Cartan matrices gives the dimensions over F of these "Homs" and they are equal.

Lemma 5 Every $F[G]$-module in $B_0(G,p)$ is the restriction of an $F[G_0]$-module in $B_0(G,p)$.

Proof. Let V be an $F[G]$-module in $B_0(G,p)$ and let $V \subseteq P$ a projective $F[G]$-module in $B_0(G,p)$. Choose an $F[G]$-homomorphism of P into a projective Q in $B_0(G,p)$ with kernel V. But then, by Lemma 4, the kernel is an $F[G_0]$-module. (Preceding lemma's last statement immediately holds for all projectives in the principal blocks — not just the indecomposables.)

Now we can finish the result. Let f be an $F[G]$ homomorphism between V and W, which are $F[G]$-modules in $B_0(G,p)$. Let V_0 and W_0 be $F[G_0]$-modules in $B_0(G_0,p)$ with $V_0 \mid G = V$, $W_0 \mid G = W$. It suffices to show that

$f \in \mathrm{Hom}_{FG_0}(V_0, W_0)$. For then if $V = W$ and f is the identity map, we deduce that $V_0 = W_0$. (Underlying vector space is the same by choice – just need to know G_0 acts the same way.) All parts of the theorem then hold.

First, suppose V is projective. Thus V_0 is also projective as $|G_0 : G|$ is prime to p.⊛ Embed W_0 in a projective Q_0 so W is embedded in the projective $Q = Q_0|G$ (all in B_0's as usual). We have a commutative diagram, columns being the identities:

$$\begin{tikzcd}
V_0 \arrow[d] \arrow[rr, bend left, "fi"] & W_0 \arrow[r, "i"] \arrow[d] & Q_0 \arrow[d] \\
V \arrow[r, "f"] \arrow[rr, bend right, "fi"'] & W \arrow[r, "i"] & Q
\end{tikzcd}$$

It remains to see that f is a $F[G_0]$-map. But, if $v_0 \in V_0$, $g_0 \in G_0$,
$(v_0 g_0) f = (v_0 f) g_0 \iff (v_0 g_0) f i = (v_0 f) g_0 i \iff (v_0 (fi)) g_0 = ((v_0 f) i) g_0$
since i and fi are G_0-maps. Hence this case is O.K.

In general, choose a projective Q and epimorphism $Q \xrightarrow{\varepsilon} V$. We now have, by the above, the commutative diagram

$$\begin{tikzcd}
Q_0 \arrow[r, "\varepsilon"] \arrow[d] \arrow[rr, bend left, "\varepsilon f"] & V_0 \arrow[d] & W_0 \arrow[d] \\
Q \arrow[r, "\varepsilon"] \arrow[rr, bend right, "\varepsilon f"'] & V \arrow[r, "f"] & W
\end{tikzcd}$$

If $v_0 \in V_0$, $g_0 \in G_0$ choose $q_0 \in Q_0$ with $q_0 \varepsilon = v_0$. Hence,
$$(v_0 f) g_0 = ((q_0 \varepsilon) f) g_0 = (q_0 (\varepsilon f)) g_0 = (q_0 g_0)(\varepsilon f) = ((q_0 g_0)\varepsilon) f$$
$$= ((q_0 \varepsilon) g_0) f = (v_0 g_0) f$$
and the theorem is proved.

⊛ Or could choose, by above V_0 as projective, proving less but all we need for argument to follow.

We now turn to the projectives for $\underline{PGL(2,q)}$, $q \equiv -1$ (modulo 4).

Lemma 1 There are exactly two irreducible $F[G]$-modules in $B_0(G,2)$, namely F and V of dimensions 1 and $q-1$ where $V|PSL = V' \oplus V''$.

Proof. By Brauer's Am J Lemma, an irreducible in $B_0(G,2)$ restricts to a module of PSL with composition factors in $B_0(PSL,2)$. We apply Clifford's Theorem, using the fact that V' and V'' are conjugate in G – see the characters – and deduce the result.

Lemma 2 Let B be the "Borel subgroup" of G of order $(q-1)q$. Then $F \otimes_{FB} FG$ is uniserial with composition factors F, V and F in that order.

Proof Let $G_0 = PSL(2,q)$, B_0 be the Borel subgroup of G_0. Hence $F \otimes_{FB} FG$ restricted to G_0 is $F \otimes_{FB_0} FG_0$ – permutation inspection – and the latter has known structure. Lemma 1 yields the result. Or: Get comp factors F, F, V from restrictions. Then F at top and bottom by permutation arguments, $\mathrm{Hom}_{FG}(F \otimes_{FB} FG, F) \simeq \mathrm{Hom}_{FB}(F,F)$ and $\mathrm{Hom}_{FG}(F \otimes_{FB} FG, V) \simeq \mathrm{Hom}_{FB}(F, V|B)$ yield the result too.

Lemma 3 $M_V \simeq \begin{smallmatrix}F\\F\end{smallmatrix} \oplus U$, where U is uniserial, all its composition factors being V, their number being $2^{n-1}-1$, $2^n = |PSL(2,q)|_2$.

Proof. If we induce $P_{V'}$ of G_0 to G we certainly get a projective of G. Also, by above, we clearly get the reduction

of this induced module to G_0 is $P_{V'} \oplus P_{V''}$. Thus, $P_{V'} \otimes_{FG_0} FG$ has an irreducible socle isomorphic to V so it is P_V.
Can now describe structure of P_V. We take $P_{V'}$ and then let G_0 act on $P_{V'}$, via conjugation in G by an outside involution, form the sum and draw the involution flip — all easily described by matrices. Rest clear by inspection!

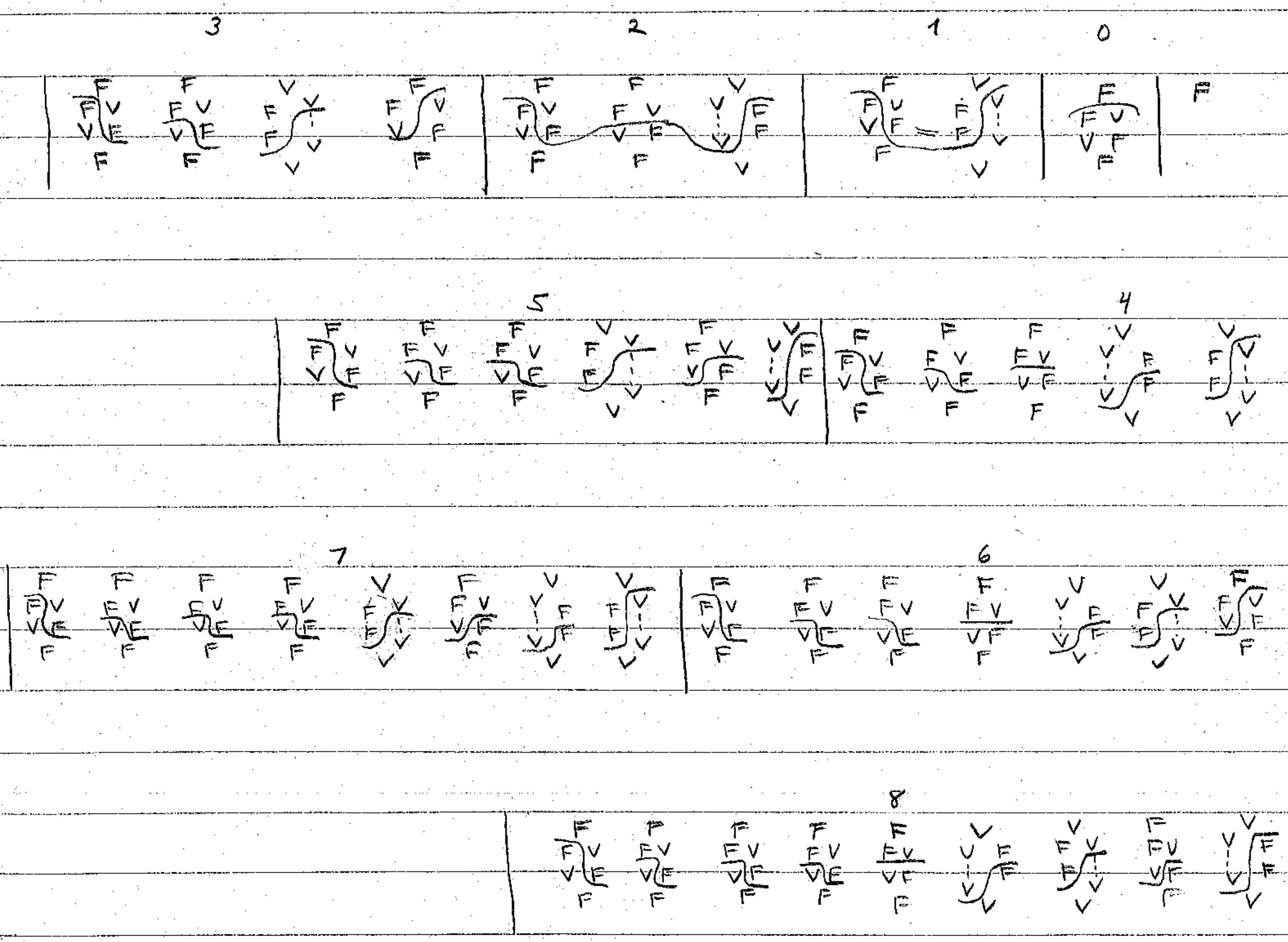
<u>Lemma 4</u> $M_F \simeq \begin{smallmatrix}F\\V\end{smallmatrix} \oplus \begin{smallmatrix}V\\F\end{smallmatrix}$.

<u>Proof</u> Know $\begin{smallmatrix}F\\V\\F\end{smallmatrix}$ exists by Lemma 2. Know $\begin{smallmatrix}F\\F\\V\end{smallmatrix}$ exists by Lemma 4. Hence $\begin{smallmatrix} & F & \\ F & & V \\ V & & F\end{smallmatrix}$ exists. But $P_F \otimes_{FG_0} FG$ is projective and has composition factors F, F, V, V, F, F.
Thus we are done.

<u>Picture</u>: $P_F \simeq \begin{array}{ccc} & \overline{F} & \\ F & & V \\ V & & F \\ & \overline{F} & \end{array}$

$$P_V \simeq \begin{array}{ccc} & \overline{V} & \\ F & & \left.\begin{array}{c} V \\ \vdots \\ V \end{array}\right\} 2^{n-2}-1 \\ F & & \\ & \overline{V} & \end{array}$$

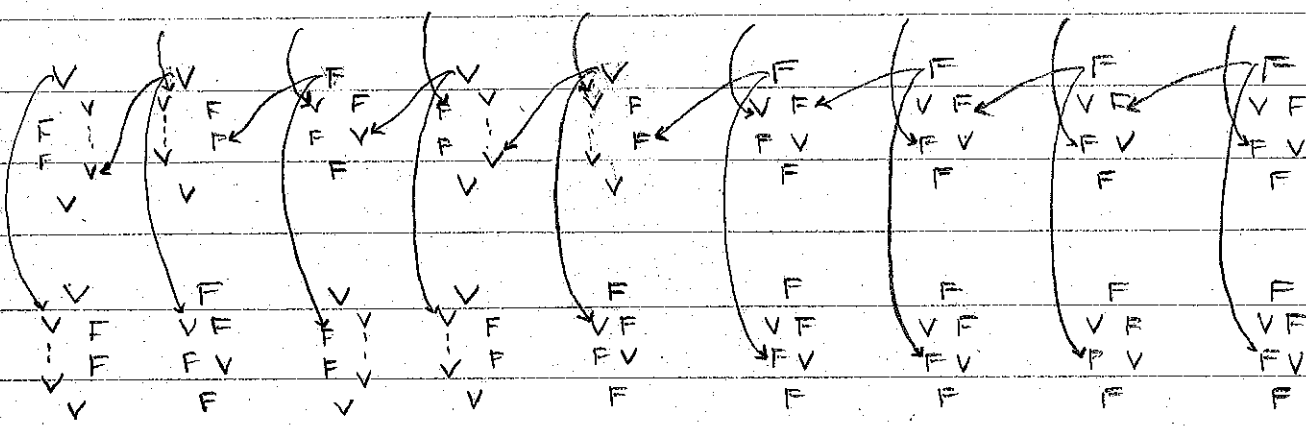
We now go on to the actual resolution, calculation as usual:



Pattern for double complex:

F									
V	V								
V	F	V							
F	V	V	F						
V	V	F	V	V					
V	F	V	V	F	F				
F	V	V	F	F	F	F			
V	V	F	F	F	F	F	F		
V	F	F	F	F	F	F	F	F	
F	F	F	F	F	F	F	F	F	F

As before we are dealing with an infinite row, maps to the left, where the downward maps correspond to a maps to the right. The picture, putting downward maps below:



Let's compute column homology, enter the results in a table.
Then we write in induced row differentials

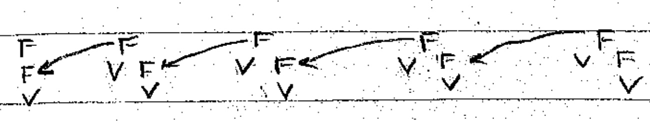
$0 \quad 0 \quad 0 \quad 0 \quad 0$

$0 \quad 0 \quad 0 \quad 0 \quad V$

$0 \quad 0 \quad 0 \quad V \leftarrow V\oplus V$

$0 \quad 0 \quad V \leftarrow V\oplus V \leftarrow V\oplus V$

$0 \quad V \leftarrow V\oplus V \leftarrow V\oplus V \leftarrow V\oplus V$



Hence get resolution.
Minimality too as usual since all images in rows and columns are in radicals

Next, of course, is $PGL(2,q) = G$ for $q \equiv 1$ (modulo 4). Proceed just as before, again get exactly two irreducibles F, V in principal 2-block. Again induced indecomposable projectives are such by inspection of their socles. Then examination of P_V yields

$$P_V \simeq \left.\begin{matrix} V \\ F \\ F \\ V \\ F \\ F \\ | \\ V \end{matrix}\right\} 2^n + 2^{n-1} + 1$$

In fact, P_V clearly has a series with factor $V, \begin{smallmatrix}F\\F\end{smallmatrix}, V, \begin{smallmatrix}F\\F\end{smallmatrix}$, etc.... Comes down to knowing that $\begin{smallmatrix}F\\V'\end{smallmatrix} \otimes \begin{smallmatrix}F\\V''\end{smallmatrix}$ with G acting – i.e. $\begin{smallmatrix}F\\V'\end{smallmatrix} \otimes_{FG_0} FG$ – is uniserial. To see this need only that $\mathrm{Hom}_{FG}(F, \begin{smallmatrix}F\\V'\end{smallmatrix} \otimes_{FG_0} FG) = 0$ i.e. that $\mathrm{Hom}_{FG}((\begin{smallmatrix}F\\V'\end{smallmatrix} \otimes_{FG_0} FG)^*, F) = 0$, i.e. $\mathrm{Hom}_{FG}((\begin{smallmatrix}F\\V'\end{smallmatrix})^* \otimes_{FG_0} FG, F) = 0$ (shall explain this last step below) i.e. $\mathrm{Hom}_{FG_0}(\begin{smallmatrix}V'\\F\end{smallmatrix}, F) = 0$, which is true.

⌈Used the fact that induction and the taking of duals commute. We use matrices to see this. Let R be a representation of G_0. The dual corresponds to $\bar{R}^{-1}$. Just examine the monomial type form for the induced representation. Indeed, let the g_i be cosets of subgroup H of group G, R a rep of H. Hence, if $g \in G$,

$g_i g = h g_j$

$$R^G : g \to \begin{array}{c} \\ i \end{array}\begin{pmatrix} & \overset{j}{R(h)} & \end{pmatrix}$$

Now then $g g_j^{-1} = g_i^{-1} h$ so $g_j g^{-1} = h^{-1} g_i$ so

$$R^G;\ g^{-1} \to \begin{array}{c} \\ j \end{array}\left(\begin{array}{c} i \\ \boxed{R(h^{-1})} \end{array} \right)$$

Hence, as $(R^G)^*(g) = {}^t R(g)^{-1} = {}^t R(g^{-1}) = i \left(\begin{array}{c} j \\ {}^t R(h^{-1}) \end{array} \right)$

But

$$R^{*G}:\ g \to i \left(\begin{array}{c} j \\ R^*(h) \end{array} \right) = i \left(\begin{array}{c} j \\ {}^t R(h)^{-1} \end{array} \right).$$

[Remark of Swan: For groups, $X \otimes_{FH} FG \simeq \mathrm{Hom}_{FH}(FG, X)$
so use second def — reference his coho. dim 1 paper.
Perhaps also see D. G. Higman, Produced and Induced Modules]

We turn to P_F

Lemma 1 $\mathrm{Ext}^2(F, V) \simeq \mathrm{Ext}^3(F, V) \simeq 0$.

Proof $\mathrm{Ext}^n_{F(PGL)}(F, V) \simeq \mathrm{Ext}^n_{F(PSL)}(F, V')$. Recall proj resol there:

$$\to P_F \oplus P_F \to P_F \to P_{V'} \oplus P_{V''} \to P_F \to F$$

Lemma 2 $\mathrm{Ext}^2(F, F) \simeq F \oplus F$

Proof (Could do locally, à la Cartan-Eilenberg) But

$\mathrm{Ext}^2_{FG}(F, F) \simeq F \otimes \mathrm{Ext}_{Z_2 G}(Z_2, Z_2) \simeq F \otimes H^2(G, Z_2)$

But $H^2(G, Z_2) \simeq (H^2(G, \mathbb{Z}) \oplus H^3(G, \mathbb{Z})) \otimes Z_2$

$\simeq (H^1(G, C^*) \oplus H^2(G, C^*)) \otimes Z_2$

$\simeq Z_2 \oplus Z_2$.

Lemma 3 $\mathcal{E}xt^2(F, {F \atop V}) \simeq F \oplus F$

Proof Have $0 \to V \to {F \atop V} \to F \to 0$ exact so

$$\underset{0}{\mathcal{E}xt^2(F, V)} \to \mathcal{E}xt^2(F, {F \atop V}) \to \underset{= F\oplus F}{\mathcal{E}xt^2(F, F)} \to \underset{0}{\mathcal{E}xt^3(F, V)}.$$

Now let $0 \to L \to P_V \to {V \atop F} \to 0$ so L is uniserial with factors $F, V, F, F, V, F, F, \ldots V, F, F, V$.

Thus, by dimension shifting, $\mathcal{E}xt(L, F) \simeq \mathcal{E}xt^2({V \atop F}, F)$

But $\mathcal{E}xt^2({V \atop F}, F) \simeq \mathcal{E}xt^2(F, {F \atop V})$ so

Lemma 4 $\mathcal{E}xt^1(L, F) \simeq F \oplus F$.

This gives, if $0 \to L_0 \to L \to F \to 0$

Lemma 5 There is an element of $\mathcal{E}xt^1(L, F)$ restricting to a non-trivial element of $\mathcal{E}xt^1(L_0, F)$

Proof Have

$$0 \to \mathrm{Hom}(F, F) \to \mathrm{Hom}(L, F) \to \overset{=0}{\mathrm{Hom}(L_0, F)}$$
$$\to \underset{\simeq F}{\mathcal{E}xt(F, F)} \to \underset{\simeq F\oplus F}{\mathcal{E}xt(L, F)} \to \mathcal{E}xt(L_0, F).$$

Lemma 6 There is an extension of F by L which is uniserial.

Proof Take an extension $0 \to F \to E \to L \to 0$ which restricted to L_0 is non-trivial. That is $0 \to F \to E_0 \to L_0 \to 0$.

But then $F \subseteq \mathrm{Rad}\, E_0$ so $E_0 / \mathrm{Rad}\, E_0 \simeq V \Rightarrow E_0$ uniserial.

But also $E_0 = \operatorname{Rad} E$, $E/E_0 \simeq F$ so claim is O.K.

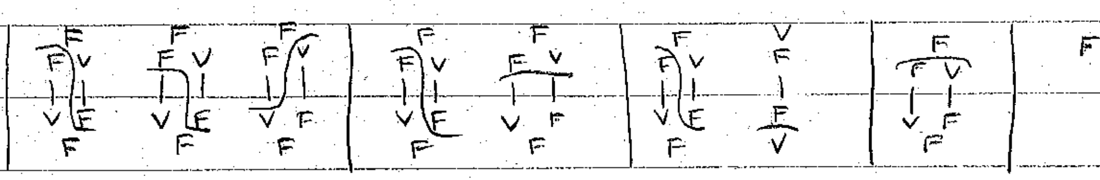
<u>Lemma 7</u> Have

$$P_F \simeq \begin{array}{cc} \multicolumn{2}{c}{F} \\ \hline F & V \\ F & F \\ V & F \\ \vdots & \vdots \\ F & V \\ F & F \\ V & F \\ \hline \multicolumn{2}{c}{F} \end{array}$$

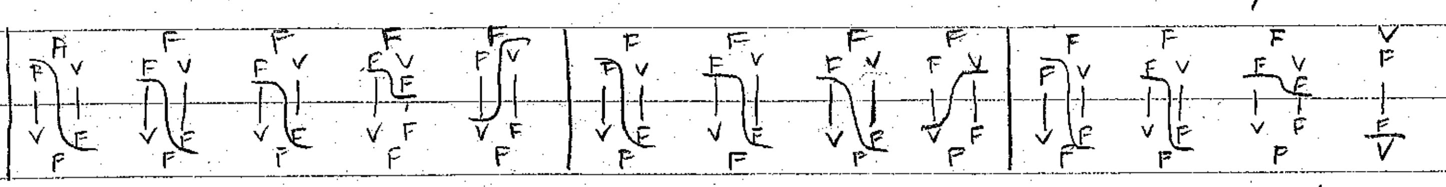
<u>Proof</u> as get P_F from inducing from PSL have right number of composition factors. Now glue the radical of P_V and the module E just constructed to get P_F/F.

Let's go after the resolution:

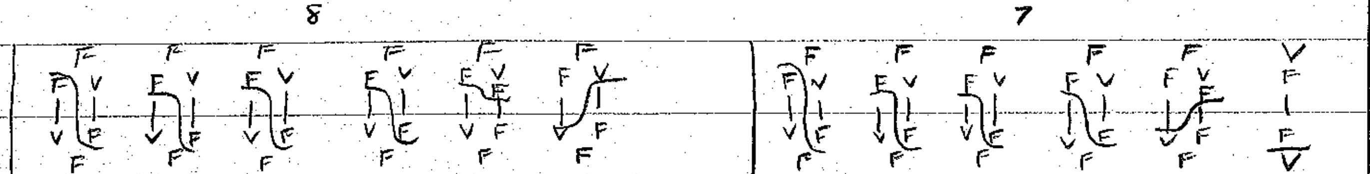
3 2 1 0



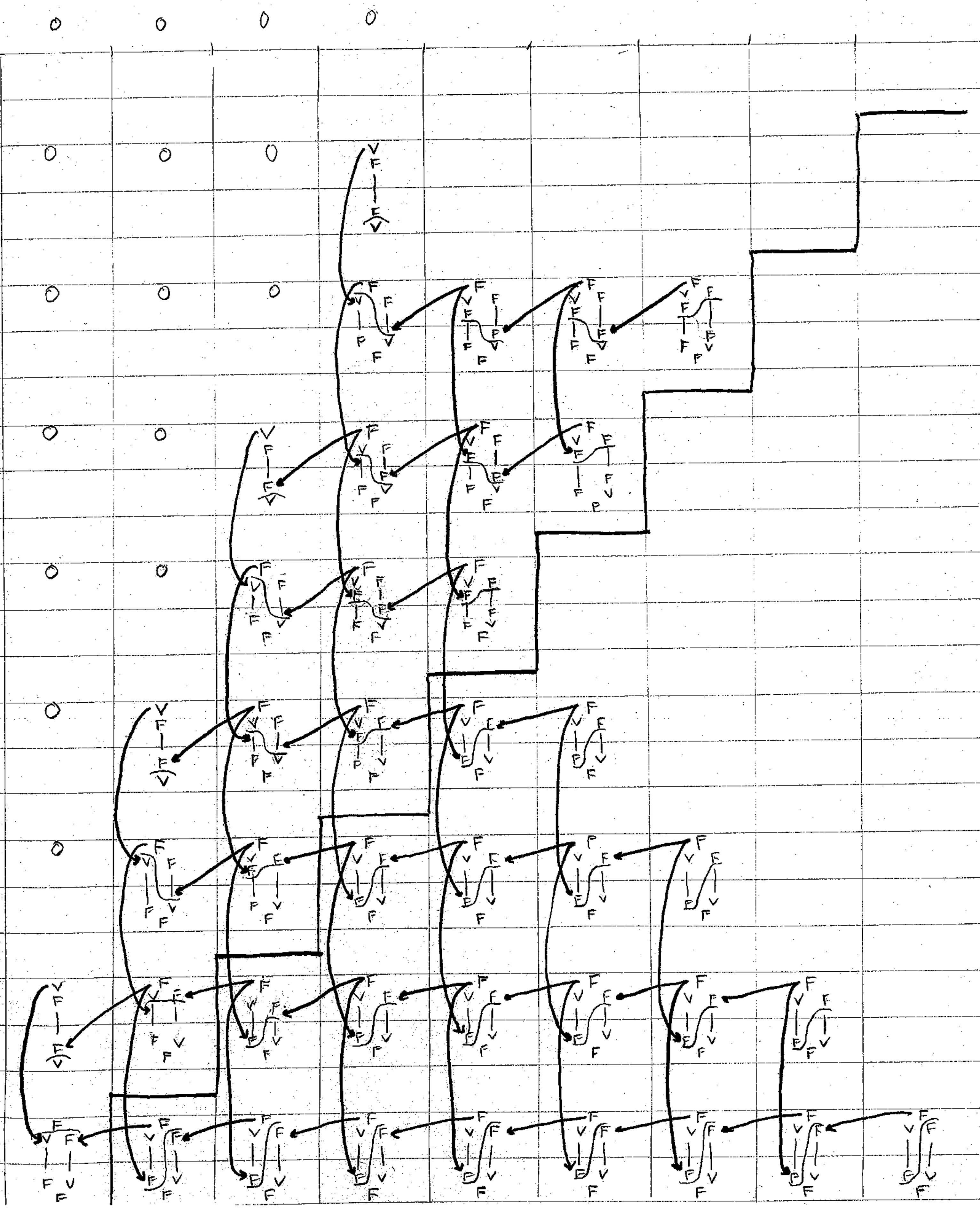
6 5 4



8 7

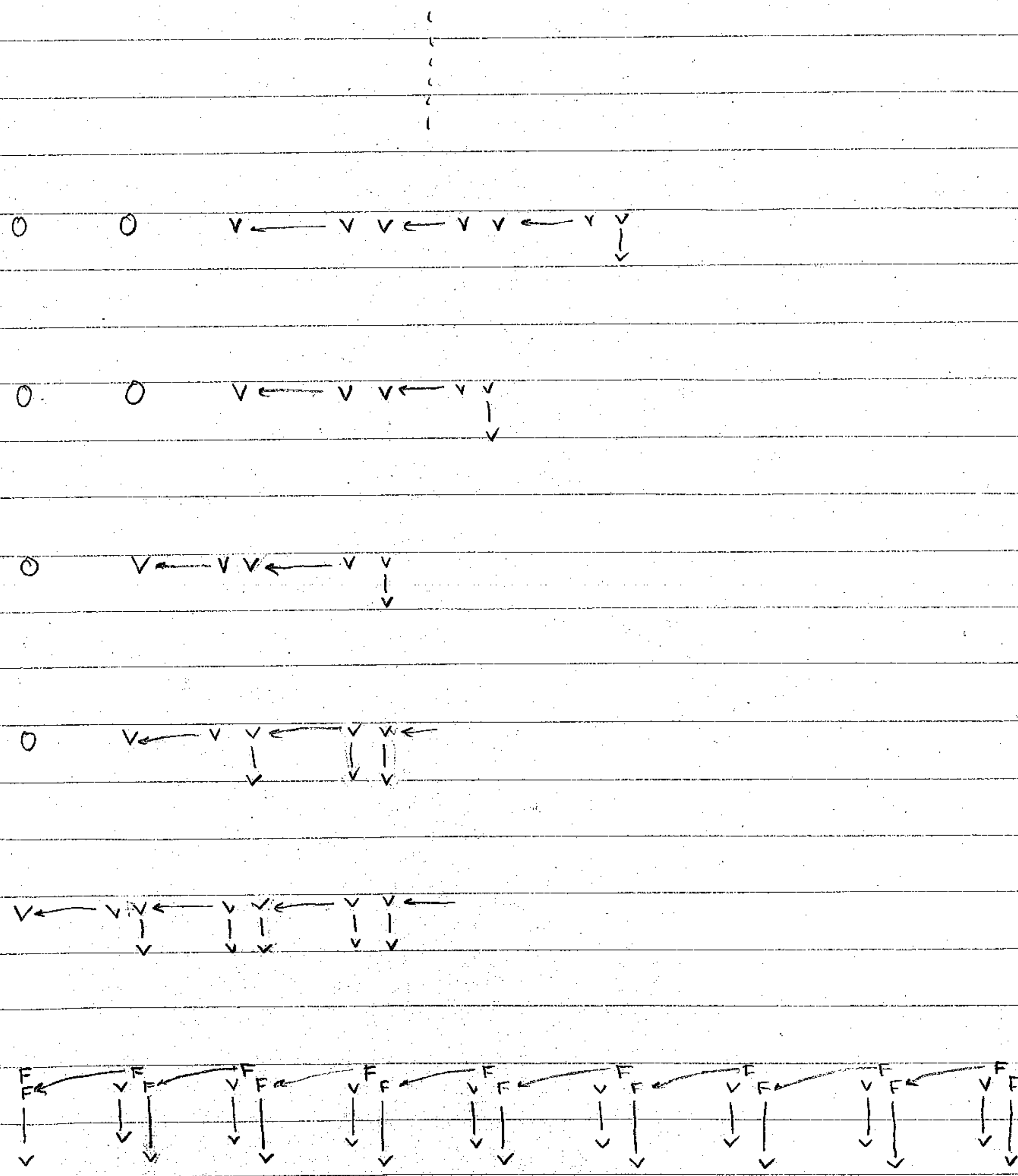


Redo this right in the double complex!



Column homology and induced differentials:

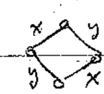
Hence, done by inspection!

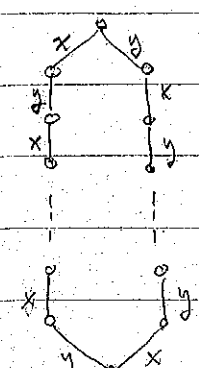


At last we turn to projectives and resolutions for D_{2^n} where $D_{2^n} = \langle x, y \mid x^2 = y^2 = (xy)^{2^{n-1}} = 1 \rangle$. Let $G = D_{2^n}$, $F = GF(2)$. If $X \subseteq D_{2^n}$, X a subset we also denote by X the sum of the corresponding elements of $F[G]$. If $X, Y \subseteq G$ then XY denotes the sum of elements in XY.

Consider the following array of elements of $F[G]$:

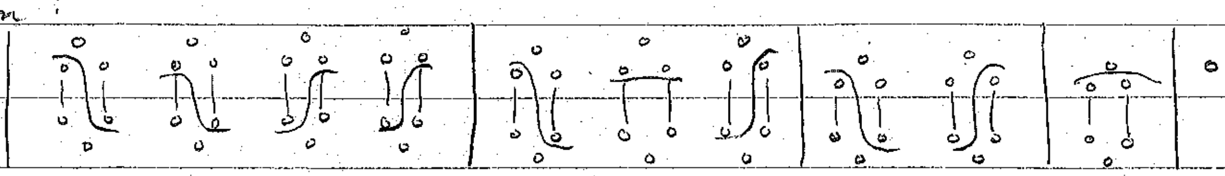
$$
\begin{array}{cc}
\multicolumn{2}{c}{1} \\
\langle x\rangle & \langle y\rangle \\
\langle x\rangle\langle y\rangle & \langle y\rangle\langle x\rangle \\
\langle x\rangle\langle y\rangle\langle x\rangle & \langle y\rangle\langle x\rangle\langle y\rangle \\
\vdots & \vdots \\
\langle x\rangle\langle y\rangle\cdots\langle x\rangle & \langle y\rangle\langle x\rangle\cdots\langle y\rangle \quad (2^{n-1}-1 \text{ factors}) \\
\multicolumn{2}{c}{G}
\end{array}
$$

We claim these are a basis of $F[G]$ and the "sheaving" of x and y are as follows: (If $n = 2$ )

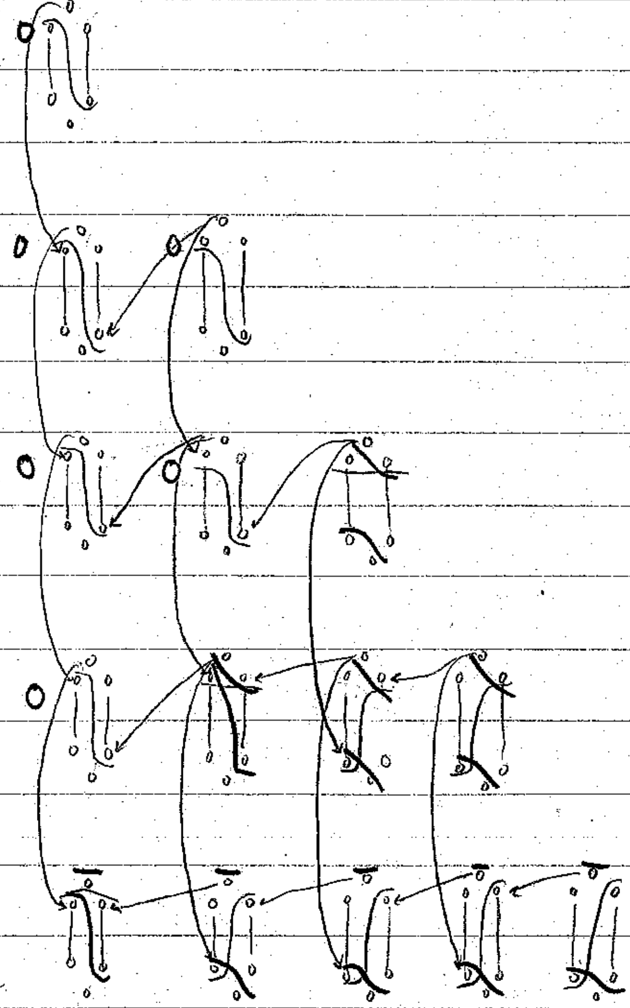


Enough to show spanning as rest is easy. Go by induction as every element is a product of alternating x's and y's.

Resolution :



Complex : (Column homology in red)



Now taking row homology with respect to induced differentials get desired results.

$$P: \quad \begin{tikzcd} \langle x \rangle \arrow[d, dash] & \langle y \rangle \arrow[d, dash] \\ \langle x \rangle - \langle x \rangle & \langle y \rangle - \langle y \rangle \end{tikzcd}$$

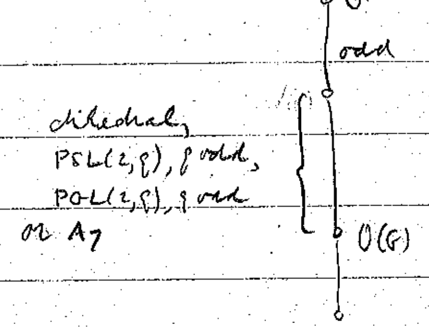
Let's describe this explicitly:

Let L_X be *left* multiplication by the sum X – where P is a right module

$$\begin{tikzcd}
P \arrow[d, "L_{\langle x\rangle}"'] & & & & \\
P \arrow[d, "L_{\langle x\rangle}"'] & P \arrow[l, "L_{\langle y\rangle - \langle y\rangle}"] \arrow[d, "L_{\langle x\rangle}"'] & & & \\
P \arrow[d, "L_{\langle x\rangle}"'] & P \arrow[l, "L_{\langle y\rangle - \langle y\rangle}"] \arrow[d, "L_{\langle x\rangle}"'] & P \arrow[l, "L_{\langle y\rangle - \langle y\rangle}"] \arrow[d, "L_{\langle x\rangle - \langle x\rangle}"'] & & \\
P \arrow[d, "L_{\langle x\rangle}"'] & P \arrow[l, "L_{\langle y\rangle - \langle y\rangle}"] \arrow[d, "L_{\langle x\rangle - \langle x\rangle}"'] & P \arrow[l, "L_{\langle y\rangle}"] \arrow[d, "L_{\langle x\rangle - \langle x\rangle}"'] & P \arrow[l, "L_{\langle y\rangle}"] \arrow[d, "L_{\langle x\rangle - \langle x\rangle}"'] & \\
P & P \arrow[l, "L_{\langle y\rangle}"] & P \arrow[l, "L_{\langle y\rangle}"] & P \arrow[l, "L_{\langle y\rangle}"] & P \arrow[l, "L_{\langle y\rangle}"]
\end{tikzcd}$$

Pattern is clear.

Now generic group G with dihedral Sylow 2-subgroup looks as follows



Apply Block isomorphism to reduce to $G/O(G)$ and then Block isomorphism theorem to get down to middle part.

Result:

<u>Main Theorem</u> Let G be a group with dihedral Sylow 2-subgroups and let F be a splitting field of characteristic two. There is a first quadrant double complex $\{P_{ij}\}$ such that

1) Each non-zero P_{ij} is an indecomposable $F[G]$-projective module.

2) The associated single complex has an augmentation and this augmented complex is the minimal projective resolution for F over $F[G]$.