RESEARCH NOTES

VOLUME IV

Research Notes

Volume IV

Contents

$Ext^2(-,-)$ for $SL(2,16)$

We shall calculate the values of this functor on irreducible modules, based upon certain results which are definitely "true" but not yet established.* We tabulate our results:

$$Ext^2_{FSL(2,16)}(V_I, V_J)$$

I \ J	0	1	2	3	4	12	23	34	14	13	24	123	124	134	234
0	0	F	F	F	F	0	0	0	0	F	F	0	0	0	0
1	F	F	0	F	0	F	0	0	F	0	0	0	0	0	0
12	0	F	F	0	0	F	0	0	0	0	0	0	0	0	0
13	F	0	0	0	0	0	0	0	0	$F\oplus F$	0	F	0	F	0
123	0	0	0	0	0	0	0	0	0	F	0	F	0	0	0

The last row is done by "inspection" of the resolution. Except clearly $Ext^*(V_{123}, V_J)=0$ if $4 \in J$. Also $Ext^2(V_{123}, V_0)=0$ can be proved schematically:

$$0 = Ext^2(1234, 4) = Ext^2(123, 4 \otimes 4) = Ext^2\left(123, \begin{smallmatrix}0\\1\\0\end{smallmatrix}\right);$$

$$0 = Ext^1\left(123, \begin{smallmatrix}0\\1\end{smallmatrix}\right) \to Ext^2(123, 0) \to Ext^2\left(123, \begin{smallmatrix}0\\1\\0\end{smallmatrix}\right) = 0.$$

By symmetry, we can now fill in the last four columns. (i.e. using the Frobenius automorphism and its powers)

* see p.115 for discussion! All statements are there proved in full

Now let's turn to V_{13}. Using tensor products and symmetry reduces the work considerably.

$$\mathrm{Ext}^2(13,24) = \mathrm{Ext}^2(0,1234)=0$$

$$\mathrm{Ext}^2(13,14) = \mathrm{Ext}^2(134,1)=0$$

Similarly, $\mathrm{Ext}^2(13,J)=0$ if $J=12,\ 23$ or 34 and

$$\mathrm{Ext}^2(13,4) = \mathrm{Ext}^2(134,0)=0$$

$$\mathrm{Ext}^2(13,2) = \mathrm{Ext}^2(123,0)=0.$$

Next, we claim that $\mathrm{Ext}^2(1,13)=0$. First, we have for $i=1,2$

$$\mathrm{Ext}^i\left(\begin{smallmatrix}1\\13\\1\end{smallmatrix},13\right) \cong \mathrm{Ext}^i(4\otimes 4\otimes 3,13) \cong \mathrm{Ext}^i(34,341)=0$$

so

$$0=\mathrm{Ext}^1\left(\begin{smallmatrix}1\\13\\1\end{smallmatrix},13\right) \to \mathrm{Ext}^1\left(\begin{smallmatrix}13\\1\end{smallmatrix},13\right) \to \mathrm{Ext}^2(1,13) \to \mathrm{Ext}^2\left(\begin{smallmatrix}1\\13\\1\end{smallmatrix},13\right)=0$$

so it suffices to prove that $\mathrm{Ext}^1\left(\begin{smallmatrix}13\\1\end{smallmatrix},13\right)=0$. If not, as $\mathrm{Ext}^1(13,13)=0$, there would be a uniserial module $\begin{smallmatrix}13\\1\\13\end{smallmatrix}$. Now $\begin{smallmatrix}13\\1\end{smallmatrix}$, $\begin{smallmatrix}1\\13\end{smallmatrix}$ tensored with 4 stay indecomposable: e.g. $\mathrm{Hom}\left(14,\begin{smallmatrix}1\\13\end{smallmatrix}\otimes 4\right) \cong \mathrm{Hom}\left(\begin{smallmatrix}1\\0\\2\\0\\1\end{smallmatrix},\begin{smallmatrix}1\\13\end{smallmatrix}\right)=0$. Hence, there would be a uniserial module $\begin{smallmatrix}341\\41\\341\end{smallmatrix}$, which does not exist.

Our next assertion is that $\mathrm{Ext}^2(13,13)\cong F\oplus F$. We first prove an upper bound. We have

$$F \cong \mathrm{Ext}^2(123,123) \cong \mathrm{Ext}^2\left(13,1\otimes\begin{smallmatrix}3\\0\\4\\0\\3\end{smallmatrix}\right) = \mathrm{Ext}^2\left(13,\begin{smallmatrix}13\\1\\14\\1\\13\end{smallmatrix}\right).$$

Also, we claim that

$$F \cong \mathrm{Ext}^1\left(13,\begin{smallmatrix}13\\1\\14\\1\end{smallmatrix}\right).$$

This is enough for the upper bound, as the following is exact:

$$\mathrm{Ext}^1\left(13,\begin{smallmatrix}13\\1\\14\\1\\13\end{smallmatrix}\right) \to \mathrm{Ext}^1\left(13,\begin{smallmatrix}13\\1\\14\\1\end{smallmatrix}\right) \to \mathrm{Ext}^2(13,13) \to \mathrm{Ext}^2\left(13,\begin{smallmatrix}13\\1\\14\\1\\13\end{smallmatrix}\right)$$

For the left-hand term is

$$\mathrm{Ext}^1\left(13, \begin{smallmatrix}13\\14\\13\end{smallmatrix}\right) \cong \mathrm{Ext}^1(123, 123) = 0$$

and the right-hand term was just proved to be F. Now we must calculate this $\mathrm{Ext}^1\left(13, \begin{smallmatrix}13\\14\end{smallmatrix}\right)$ to get the upper bound established.

But $\mathrm{Ext}^1\left(13, \begin{smallmatrix}1\\4\\1\end{smallmatrix}\right) = \mathrm{Ext}^1(13, 13 \otimes 3) = \mathrm{Ext}^1(1, 1 \otimes (3 + 3 + 34)) = F \oplus F$

and the following is exact:

$$0 \to \mathrm{Hom}(13,13) = F \to \underbrace{\mathrm{Ext}^1\left(13, \begin{smallmatrix}1\\4\\1\end{smallmatrix}\right)}_{\cong F\oplus F} \to \mathrm{Ext}^1\left(13, \begin{smallmatrix}13\\14\end{smallmatrix}\right) \to \underbrace{\mathrm{Ext}^1\left(13, \begin{smallmatrix}13\\14\\13\end{smallmatrix}\right)}_{=0}.$$

Now, we need a lower bound on $\mathrm{Ext}^2(13,13)$. Considering $\mathrm{Ext}^1(-,13)$ we see there is a module $\begin{smallmatrix}1 & & 3\\ & 13 & \end{smallmatrix}$. Also $\mathrm{Ext}^1\left(13, \begin{smallmatrix}1 & & 3\\ & 13 & \end{smallmatrix}\right) = 0$. For otherwise there is an extension and the "13" is at the top, say above the "1". Tensoring with "4" we get, as on the previous page $\begin{smallmatrix}341\\41\\341\end{smallmatrix} \oplus 34$.

Thus,

$$0 = \mathrm{Ext}^1\left(13, \begin{smallmatrix}1 & & 3\\ & 13 & \end{smallmatrix}\right) \to \mathrm{Ext}^1(13, 1 \oplus 3) \to \mathrm{Ext}^2(13,13)$$

so we're done.

Now $\mathrm{Ext}^1(13,3) = 0$ by symmetry with $\mathrm{Ext}^1(13,1) = 0$. Hence, all that remains to do for V_{13} is to determine $\mathrm{Ext}^2(13,0)$. We shall prove that it is F in a number of steps:

1) $\mathrm{Ext}^2(0, 23) = 0$ (we shall need this later anyway!)

For $\mathrm{Ext}^2\left(\begin{smallmatrix}0\\2\\0\end{smallmatrix}, 23\right) = \mathrm{Ext}^2(1 \otimes 1, 23) \cong \mathrm{Ext}^2(1, 123) = 0$

and

$$0 = \mathrm{Ext}^1\left(\begin{smallmatrix}2\\0\end{smallmatrix}, 23\right) \to \mathrm{Ext}^2(0, 23) \to \mathrm{Ext}^2\left(\begin{smallmatrix}0\\2\\0\end{smallmatrix}, 23\right) = 0$$

is exact.

2) If $\mathrm{Ext}^1(\substack{0\\1}, 13) \neq 0$ then $\mathrm{Ext}^1(1, \substack{0\\3}) \neq 0$.

For $\mathrm{Ext}^1(\substack{0\\1}, 13) \neq 0$ implies $\mathrm{Ext}^1(0, \substack{1\\13}) \neq 0$ as $\mathrm{Ext}^1(0,13)=0$.

But $\mathrm{Ext}^1(0, \substack{1\\13}) = \mathrm{Ext}^1(0, 1 \oplus \substack{0\\3}) \simeq \mathrm{Ext}^1(1, \substack{0\\3})$.

3) If $\mathrm{Ext}^1(\substack{0\\1}, 13) \neq 0$ then there is a uniserial module $\substack{1\\0\\3\\23}$.

For if the hypothesis holds then, by 2), there is a uniserial module $\substack{1\\0\\3}$. Now $\mathrm{Ext}^1(3, 23) \neq 0$ and $\mathrm{Ext}^2(\substack{1\\0}, 23) = 0$, as $\mathrm{Ext}^2(0, 23) = 0$ by 1), $\mathrm{Ext}^2(1, 23) = 0$ from before. Hence, we can "glue." That is,

$$\mathrm{Ext}^1(\substack{1\\0\\3}, 23) \to \mathrm{Ext}^1(3, 23) \to \mathrm{Ext}^2(\substack{1\\0}, 23)$$

is exact, last term is zero, so previous map is onto. Interpret this by extensions.

4) $\mathrm{Ext}^1(\substack{0\\1}, 13) = 0$.

For otherwise 3) applies. A similar argument now gives us that a uniserial module $\substack{1\\0\\3\\23\\123}$ exists. For $\mathrm{Ext}^2(\substack{1\\0\\3}, 123) = 0$, as $\mathrm{Ext}^2(I, 123) = 0$ if $I = 1, 0$ or 3, while $\mathrm{Ext}^1(23, 123) \neq 0$.

5) $\mathrm{Ext}^2(0, 13) \simeq F$.

For, the following is exact:

$$\mathrm{Ext}^1(\substack{0\\1}, 13) \to \mathrm{Ext}^1(1, 13) \to \mathrm{Ext}^2(0,13) \to \mathrm{Ext}^2(\substack{0\\1}, 13)$$

so this is

$$0 \to F \to \mathrm{Ext}^2(0,13) \to \mathrm{Ext}^2(\substack{0\\1}, 13).$$

However, the last term is also zero. For

$$Ext^1(0, 13) \to Ext^2(\begin{smallmatrix}0\\1\end{smallmatrix}, 13) \to Ext^2(\begin{smallmatrix}0\\1\\0\end{smallmatrix}, 13)$$

and $Ext^1(0,13)=0$, $Ext^2(\begin{smallmatrix}0\\1\\0\end{smallmatrix}, 13) = Ext^2(4\otimes 4, 13) = Ext^2(4, 134) = 0$.

Now let's turn to V_{12}. From above results, including $Ext^2(12, 0) = 0$ (see 1) on page 110) all that remain is $Ext^2(12, I)$, $I = 1, 2, 12$. Let's list the steps (We need other useful statements as well?)

a) $Ext^2(12, 2) = F$

Using $\begin{smallmatrix}12\\124\\12\end{smallmatrix} = 12\otimes 3 \otimes 3$ have the exact sequence

$$Hom(2, \begin{smallmatrix}12\\124\end{smallmatrix}) \to Ext^1(2,12) \to Ext^1(2, \begin{smallmatrix}12\\124\\12\end{smallmatrix}) \to Ext^1(2, \begin{smallmatrix}12\\124\end{smallmatrix}) \to Ext^2(2,12) \to Ext^2(2, \begin{smallmatrix}12\\124\\12\end{smallmatrix})$$

which is

$$0 \to F \to F \to F \to Ext^2(2,12) \to 0$$

(which suffices) since

$$Hom(2, \begin{smallmatrix}12\\124\end{smallmatrix}) = 0,$$

$$Ext^1(2, 12) = 0,$$

$$Ext^1(2, \begin{smallmatrix}12\\124\\12\end{smallmatrix}) = Ext^1(2, 123\otimes 3) \cong Ext^1(23, 123) = F,$$

$$Ext^1(2, \begin{smallmatrix}12\\124\end{smallmatrix}) = F \text{ by inspection of } P_{124},$$

$$Ext^2(2, \begin{smallmatrix}12\\124\\12\end{smallmatrix}) = Ext^2(2, 123\otimes 3) = Ext^2(23, 123) = 0.$$

b) $Ext^2(12, 1) = F$

First we have, again using $\begin{smallmatrix}124\\12\end{smallmatrix}$, + $\begin{smallmatrix}12\\124\\12\end{smallmatrix}$ next,

$$Ext^1(1, 124) \to Ext^2(1,12) \to Ext^2(1, \begin{smallmatrix}124\\12\end{smallmatrix}) \to Ext^2(1,124)$$

ie.

$$0 \to Ext^2(1,12) \to Ext^2(1, \begin{smallmatrix}124\\12\end{smallmatrix}) \to 0$$

and also

$$\mathrm{Ext}^1(1,12) \to \mathrm{Ext}^2\left(1, \begin{smallmatrix}124\\12\end{smallmatrix}\right) \to \mathrm{Ext}^2\left(1, \begin{smallmatrix}12\\124\\12\end{smallmatrix}\right) \to \mathrm{Ext}^2(1,12)$$

i.e. $$0 \to X \to \mathrm{Ext}^2(1, 123\otimes 3) \to X$$

$$0 \to X \to F \to X$$

Hence $X \neq 0$ so $X = F$.

c) $\mathrm{Ext}^2(12,12) = F$.

This is a very similar argument to b). Have from $\begin{smallmatrix}12\\412\\12\end{smallmatrix}$,

$$\mathrm{Ext}^1\left(12, \begin{smallmatrix}12\\412\end{smallmatrix}\right) \to \mathrm{Ext}^2(12,12) \to \mathrm{Ext}^2\left(12, \begin{smallmatrix}412\\12\end{smallmatrix}\right) \to \mathrm{Ext}^2\left(12, \begin{smallmatrix}12\\412\end{smallmatrix}\right)$$

i.e. $$0 \to X \to F \to Y$$

By inspection of P_{412}, defining X, Y this way and $\begin{smallmatrix}12\\412\\12\end{smallmatrix} = 12\otimes 3\otimes 3$.

Also,

$$\mathrm{Ext}^2(12, 412) \to \mathrm{Ext}^2\left(12, \begin{smallmatrix}12\\412\end{smallmatrix}\right) \to \mathrm{Ext}^2(12,12)$$

i.e. $$0 \to Y \to X$$

$\therefore$ $X = 0$ yields $Y = 0$ so $0 \to F \to 0$ is exact, a contrad. $\therefore X \neq 0$;

but $0 \to X \to F$ is exact so $X = F$

This completes the argument for V_{12}. This now yields that, after usual trivialities, all that's left: $\mathrm{Ext}^2(0,0)$, $\mathrm{Ext}^2(0,1)$, $\mathrm{Ext}^4(1,1)$.

But $\mathrm{Ext}^2(0,0) = 0$, by restriction to triangular matrices and an eigenvalue argument; for know $\mathrm{Ext}_{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)}(0,0)$ and how $\left\langle\left(\begin{smallmatrix}*&0\\0&*\end{smallmatrix}\right)\right\rangle$ acts on this.

We now conclude with a few easy steps!

i) $\mathrm{Ext}^1(0,0) = \mathrm{Ext}^2(0,0) = \mathrm{Ext}^3(0,0) = 0$

Eigenvalue argument.

ii) $\mathrm{Ext}^2(0,1) \simeq \mathrm{Ext}^2(4,4)$

For $\mathrm{Ext}^2(4,4) \simeq \mathrm{Ext}^2(0, \begin{smallmatrix}0\\1\\0\end{smallmatrix})$. Also,

$$\mathrm{Ext}^1(0,0) \to \mathrm{Ext}^2(0, \begin{smallmatrix}1\\0\end{smallmatrix}) \to \mathrm{Ext}^2(0, \begin{smallmatrix}0\\1\\0\end{smallmatrix}) \to \mathrm{Ext}^2(0,0)$$

so
$$0 \to \mathrm{Ext}^2(0, \begin{smallmatrix}1\\0\end{smallmatrix}) \to \mathrm{Ext}^2(4,4) \to 0$$

is exact. Moreover,

$$\mathrm{Ext}^2(0,0) \to \mathrm{Ext}^2(0, \begin{smallmatrix}1\\0\end{smallmatrix}) \to \mathrm{Ext}^2(0,1) \to \mathrm{Ext}^2(0,0)$$

so
$$0 \to \mathrm{Ext}^2(4,4) \to \mathrm{Ext}^2(0,1) \to 0$$

is exact.

iii) $\mathrm{Ext}^2(4,4) \neq 0$.

$\mathrm{Ext}^2(4,4) \simeq \mathrm{Ext}^2(0, 4^* \otimes 4)$. Now let $A = R/(4)$, $R/(2)$ being F, from a suitable ring*. Consider $SL(2, A)$.

Have
$$1 \to \left\{ \begin{pmatrix} 1+F2 & F2 \\ F2 & 1+F2 \end{pmatrix} \right\} \to SL(2,A) \to SL(2,F) \to 1$$

This does not split, as $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ has order 4 as does each element in its coset. Hence $\mathrm{Ext}^2(1,1) \neq 0$ so (ii) holds.

iv) $\mathrm{Ext}^2(0,1) \simeq F$.

Need only an upper bound. But we use $\begin{smallmatrix}1\\13\\1\end{smallmatrix} = 1 \otimes 2 \otimes 2$

$$\mathrm{Ext}^1(0, \begin{smallmatrix}1\\13\end{smallmatrix}) \to \mathrm{Ext}^2(0,1) \to \mathrm{Ext}^2(0, \begin{smallmatrix}1\\13\\1\end{smallmatrix})$$

is exact. But $\mathrm{Ext}^1(0, \begin{smallmatrix}1\\13\end{smallmatrix}) = 0$ by 4) on p 111 + $\mathrm{Ext}^2(0, \begin{smallmatrix}1\\13\\1\end{smallmatrix}) = \mathrm{Ext}^2(13, 2) \simeq F$.

Thus (iv) holds and the table can be filled in completely.

* Can we use defining relations? Sort of $F \otimes \mathbb{Z}/4\mathbb{Z}$.

Here's a list of what we've used without proof:

$Ext^2(123, 1) = Ext^2(123, 2) = Ext^2(123, 3) = Ext^2(123, 12) = Ext^2(123\ 23) = 0$

$Ext^2(123, 123) \simeq Ext^2(123, 13) \simeq F$.

We would like to have actual proofs.

We can get started, e.g.: $Ext^2(123, 13) \neq 0$. For, briefly, $\begin{smallmatrix}2\\0\\1\end{smallmatrix}$ (uniserial) exists and this gives $\begin{smallmatrix}23\\3\\13\end{smallmatrix}$ uniserial. Also $\begin{smallmatrix}123\\23\\3\end{smallmatrix}$ a uniserial exists. But $\begin{smallmatrix}123\\23\\3\\13\end{smallmatrix}$ does not and the obstruction for this, from the appropriate long exact sequence, lies in $Ext^2(123, 13)$.

Similarly, we get $Ext^2(123, 123) \neq 0$ from the existence of $\begin{smallmatrix}123\\23\end{smallmatrix}$, $\begin{smallmatrix}23\\123\end{smallmatrix}$ and the non existence of $\begin{smallmatrix}123\\23\\123\end{smallmatrix}$.

But now we can get the equalities

$$Ext^2(123, 123) \simeq Ext^2(123, 13) \simeq F$$

when we establish the following:

<u>Lemma</u>. $\dim_F Ext^k(V_{2,\ldots,n}, V_J) \leq 1$, $k \geq 0$.

This is meant for <u>all</u> $SL(2, 2^n)$ and all J. Of course, we can assume $J \subsetneq N$. We use the results on pages 82 and 90. From page 82, with the same notation (so $V_I / B = W_I$) we have an exact sequence

$$\cdots \to \lambda^{-3} \otimes W_N \to \lambda^{-1} \otimes W_N \to W_{\{2,\ldots,n\}} \to 0$$

But, by the Green correspondence,

$$Ext^k(V_{2\ldots n}, V_J) \simeq Ext^k(W_{2\ldots n}, W_J).$$

Now any $\lambda^i \otimes W_N$ has a simple top and bottom (as can tensor with λ^{-i}) and no other repeated composition factors. Since $\dim W_J < 2^n$ we clearly have

$$\dim_F \mathrm{Hom}(\lambda^i \otimes W_N, W_J) \le 1$$

for any i. This establishes the lemma.

Now, working with W_{234} instead of W_{123} we're still in need of dealing with:

$$\mathrm{Ext}^2(234, 2),\ \mathrm{Ext}^2(234, 3),\ \mathrm{Ext}^2(234, 4),\ \mathrm{Ext}^2(234, 23),\ \mathrm{Ext}^2(234, 34)$$

But

$$W_2 = \begin{array}{c} \lambda^2 \\ \lambda^{-2} \end{array}, \qquad W_3 = \begin{array}{c} \lambda^4 \\ \lambda^{-4} \end{array}, \qquad W_4 = \begin{array}{c} \lambda^8 \\ \lambda^{-8} \end{array}$$

$$W_{23} = \begin{array}{c} \lambda^6 \\ \lambda^{-2}\ \lambda^2 \\ \lambda^{-6} \end{array} \qquad W_{34} = \begin{array}{c} \lambda^{12} \\ \lambda^4\ \lambda^{-4} \\ \lambda^{-12} \end{array}$$

do not have λ^5 as a composition factor $\therefore$ all the $\mathrm{Ext}^2(234, -)$ left are zero as desired.

<u>Hence, the table is completely proved</u>

Numbers of maps

Here's an interesting table (& then the evidence).
The integer n refers to $SL(2, 2^n)$.

n	# maps in resol. of $V_\emptyset$	# dee nos in B_0	# maps in resol of all V_I, $I \subsetneq N$
1	1	2	1
2	6	8	10
3	21	26	48
n	$(2^n-1)n$?	$3^n - 1$?	?? $(n-1) + \frac{1}{2} \sum_{\substack{I,J \\ \neq N}} C_{IJ}$??

Evidence.

Table of maps used for $SL(2, 2^2)$. Red circle denotes image of maps in resol of $V_\emptyset$, blue circle of maps in resol of V_I some I, not in resol of $V_\emptyset$.

0	1	0	1	2
2		2	0	0
0	1	0	2	1
			0	0
			1	2

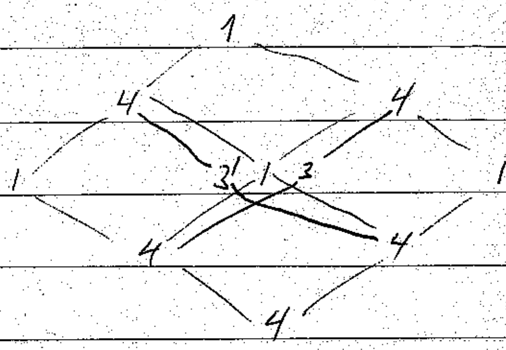
Table of maps in resolution of $V_\emptyset$ over $SL(2,8)$.
Use red circle for images (Use blue circles for images of maps in resolutions of other V_I, $I \neq \emptyset$.)

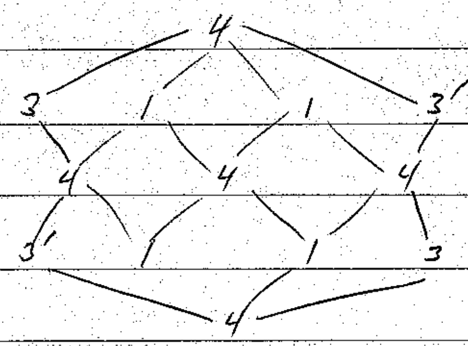
0	1	0		3	13	3		0	1	0
2	12	2		23		23		2	12	2
0	1	0		3	13	3		0	1	0

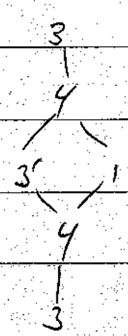
1	13	1		2	12	2		3	23	3
0	3	0		0	1	0		0	2	0
2	23	2		3	13	3		1	12	1
0	3	0		0	1	0		0	2	0
1	13	1		2	12	2		3	23	3

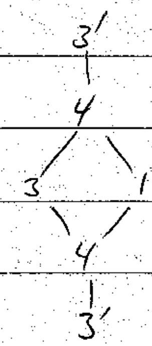
12	23	13
2	3	1
0	0	0
3	1	2
0	0	0
2	3	1
12	23	13

Let's turn to A_6. Here are the projectives

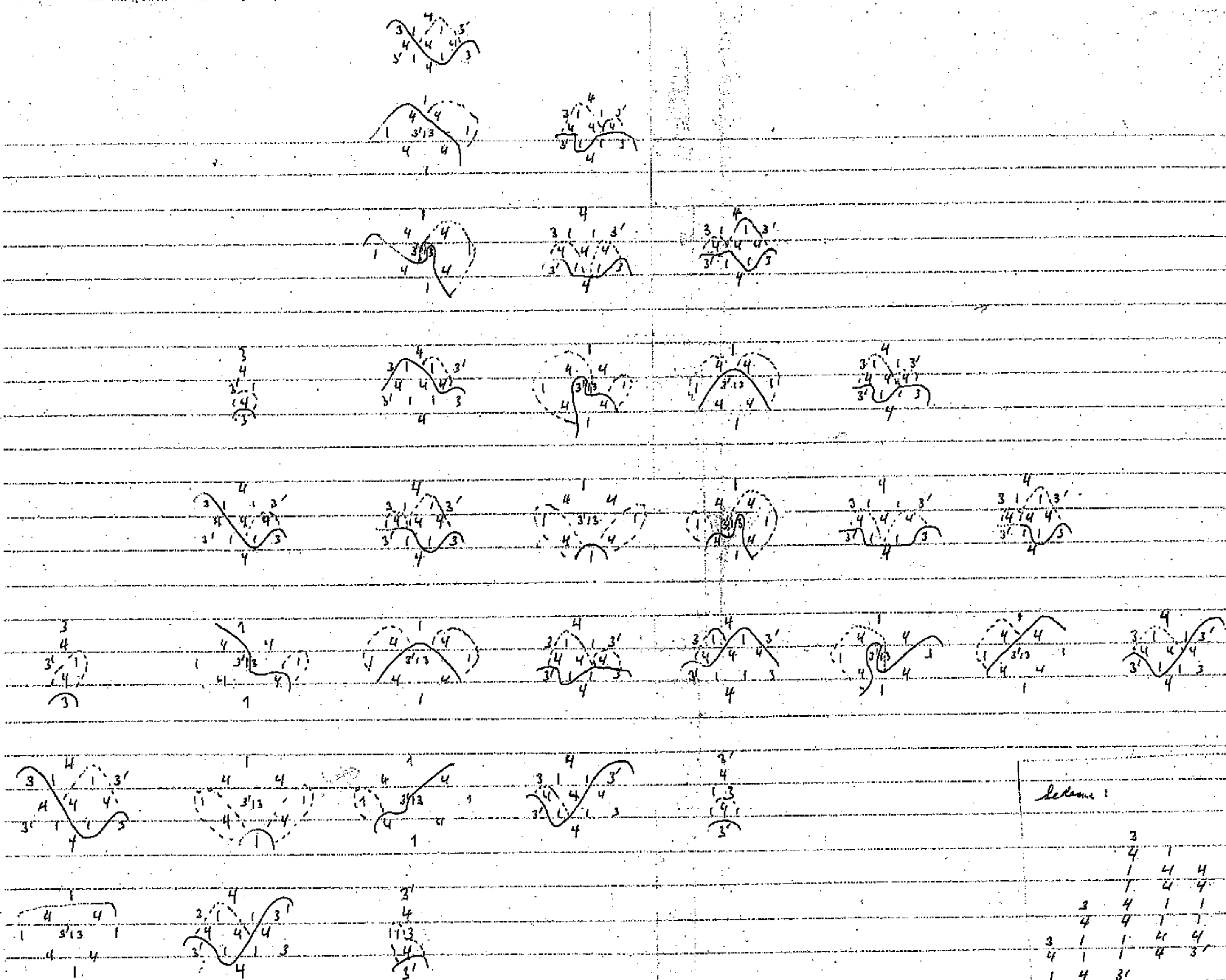








Now let's do the resolution properly.



Schema:

3
4 1
1 4 4
1 4 4 1
3 4 1 1 4 4
4 4 1 1 4 4 1
3 1 1 4 4 1 1 4 3'
4 1 1 4 3'
1 4 3'

Table of images of maps:

1

4 4

1 3′ 1 3 1

4 4

1

4

3 1 1 3′

4 4 4

3′ 1 1 3

4

3

4

3′ 1

4

3

3′

4

1 3

4

3′

Now let's compare this with the normalizer of the Sylow 3-subgroup in $A_6 = Z_3 \times Z_3 . Z_4$.

Projectives:

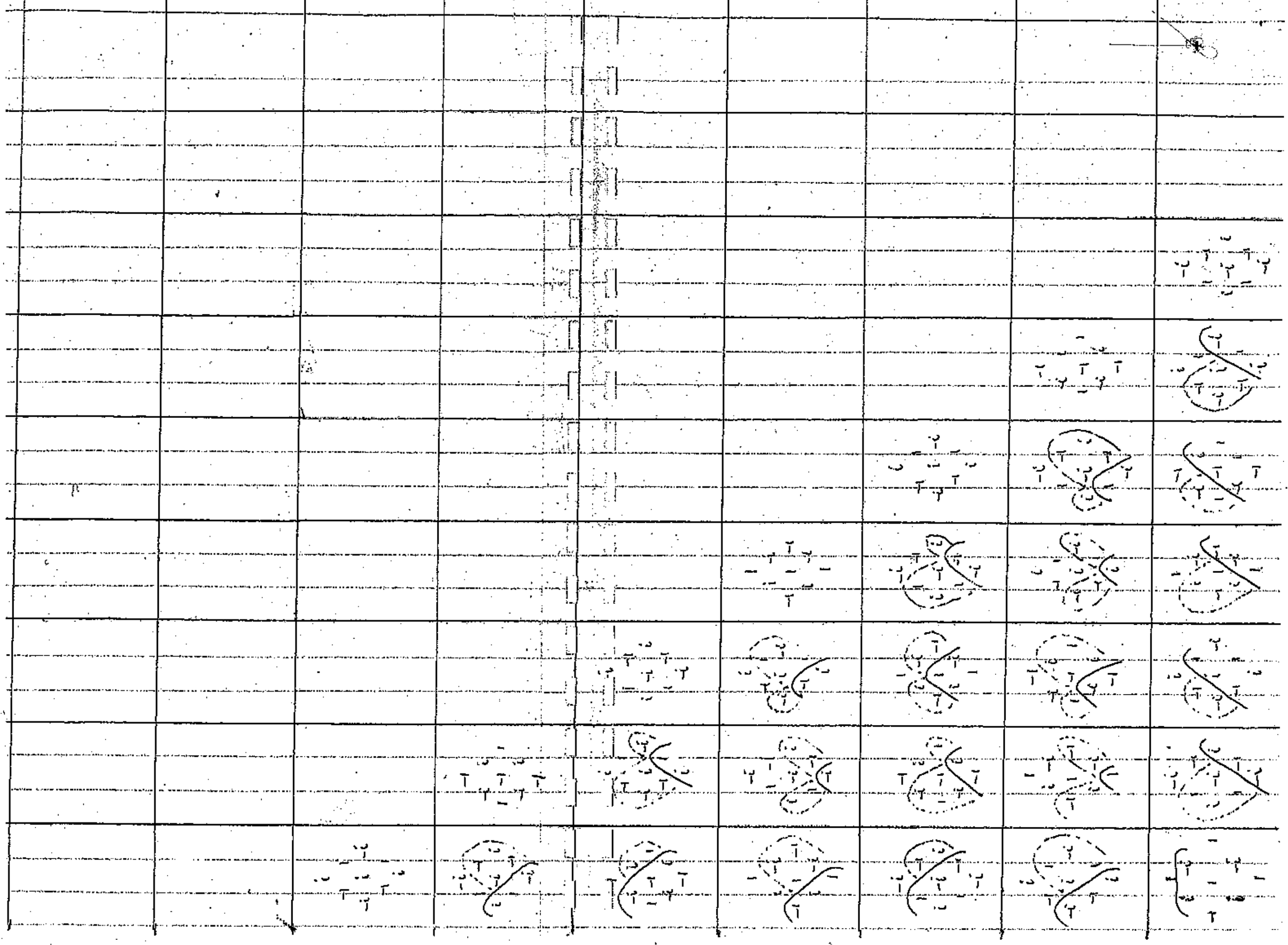
$$
\begin{array}{ccccc}
 & & 1 & & \\
 & i & & -i & \\
-1 & & 1 & & -1 \\
 & i & & -i & \\
 & & 1 & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & -1 & & \\
 & -i & & i & \\
1 & & -1 & & 1 \\
 & -i & & i & \\
 & & -1 & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & i & & \\
 & -1 & & 1 & \\
-i & & i & & -i \\
 & -1 & & 1 & \\
 & & i & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & -i & & \\
 & 1 & & -1 & \\
i & & -i & & i \\
 & 1 & & -1 & \\
 & & -i & &
\end{array}
$$

Get <u>16</u> maps (not the same as for A_6, which has 14).
$\therefore$ <u>no</u> conjecture about # maps the same locally + globally.

Pattern of resol: (details, next page)

-1	i	$-i$	-1	1	$-i$	i	1
$+i$	1	-1	$-i$	$-i$	-1	1	$-i$
$-i$	-1	1	$-i$	i	1	-1	i
-1	i	$-i$	-1	1	$-i$	i	1
1	$-i$	i	1	-1	i	$-i$	-1
$-i$	-1	1	$-i$	i	1	-1	i
i	1	-1	i	$-i$	-1	1	$-i$
1	$-i$	i	1	-1	i	$-i$	-1

reflection
"= complex conj"



Decomposition matrices for $SL(2,2^n)$.

We calculate a few examples, primarily using the Cartan matrices. (Leave out V_N.)

$SL(2,2^2)$

0	1	2
1	0	0
1	1	0
1	0	1
1	1	1

$SL(2,2^3)$

0	1	2	3	12	23	13
1	0	0	0	0	0	0
1	1	1	1	0	0	0
1	0	1	1	1	0	0
1	0	1	0	1	0	0
1	1	0	1	0	1	0
1	0	0	1	0	1	0
1	1	1	0	0	0	1
1	1	0	0	0	0	1

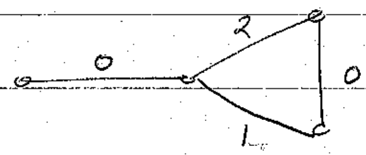
Next $SL(2,2^4)$:

0	4	1	2	3	34	14	12	23	24	13	234	134	124	123
1														
1	1	1	1	1					1	1				
1	1	1	1		1				1					
1	1		1	1			1		1					
1	1		1		1		1		1					
1	1	1		1				1		1				
1		1	1	1		1				1				
1		1		1		1		1		1				
1	1	1			1						1			
1	1				1						1			
1		1	1			1						1		
1		1				1						1		
1			1	1			1						1	
1			1				1						1	
1	1			1				1						1
1				1				1						1

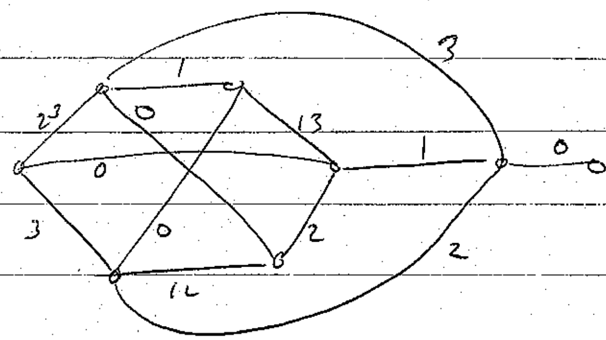
What is the rule?

Perhaps there is a nice graph. Some examples:

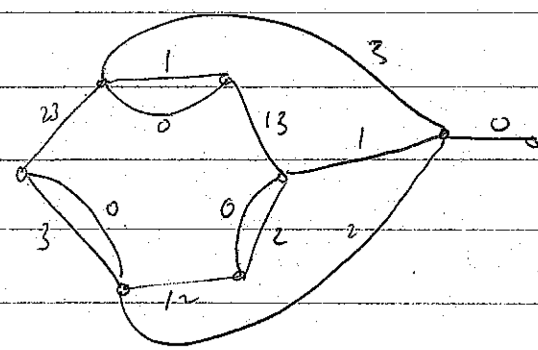
$SL(2, 2^2) \simeq A_5$:



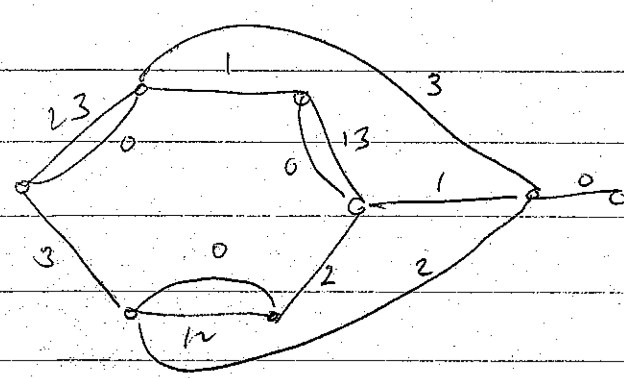
$SL(2, 2^3)$:



or perhaps



or



Seems unlikely because nothing like this happens for $GL(3,2)$.

Lifting

First, the notation. G a finite group, R a discrete valuation ring which is complete with its field of quotients K a splitting field of characteristic zero for G and its residue class field F a splitting field of characteristic p for G. Let B be a p-block of G with defect group D.

We say B is a TI-block if $D^g \cap N(D) = 1$ whenever $g \notin N(D)$. We say an FG-module is liftable if it lifts to an RG-module, and the same for subgroups of G. (Really: RG-lattice.)

Proposition. If B is a TI-block, U is an indec. FG-module in B and V is its correspondent, an $FN(D)$-module, then U is liftable if, and only if, V is liftable.

Proof. Can assume U is not projective without loss. Now $U_{N(D)} = V \oplus R$, R a projective $FN(D)$-module and $V^G = U \oplus S$, S a projective FG-module. Say U lifts, so $U = \overline{M}$, M an RG-lattice. Hence $\overline{M_{N(D)}} = V \oplus R$. and we get a similar relation if V lifts. ∴ all is done once we prove the following result (and apply it to G and to $N(D)$):

Lemma. If M is an RG lattice with $\overline{M} = X \oplus Q$ where Q is projective then X is liftable.

Proof. We have that there is an RG-projective P with $\overline{P} \cong Q$.

Hence there is an isomorphism of P into M which has image which reduced modulo the prime gives Q.
This submodule splits as the module P is injective. Hence, done as X lifts to the complement of the image of P.

Remarks: 1 Should be able to prove using Green's homological result on lifting.

2. Since B is TI have that $N(D)$ contains a Sylow p-subgroup of G and D is strongly closed in it.

Next, we say B is a (0,1) block if all the decomposition numbers of B are either zero or one (Or perhaps a "zero-one" block.)

Proposition. Assume that B is a $(0,1)$-block.
1) If V is an irreducible KG-module in B and U is a modular constituent of V then there is an R-lattice M such that $K\otimes_R M \simeq V$ while $\overline{M}$ has a simple top isomorphic with U.
2) If M_1 and M_2 are as in 1) then $M_1 \simeq M_2$.
3) If M is an RG-lattice with $\overline{M}$ multiplicity free with a simple top then $K\otimes_R M$ is irreducible.

Proof. 1) follows directly from Thompson's lemma since the char χ of V is a summand of each proj indec corr to each modular constituent of V.
Now for 2), let I be the prime ideal for R so we can assume, without any loss of generality, that $M_1 \supseteq M_2 \supset IM_1$.

But then, let $M_1/N_1 \cong U$, $M_1 \supsetneq N_1 \supseteq IM_1$. Since M_2/IM_2 has a simple top this implies that M_2/IM_1 has U as image. Hence, as M_1/IM_1 is multiplicity-free, we have $M_2 + N_1 = M_1$. But N_1/IM_1 is the radical of M_1/IM_1 so $M_2/IM_1 = M_1/IM_1$ so $M_2 = M_1$.

The last thing is to prove 3). Suppose $K\otimes_R M$ is not irreducible. Say $K\otimes M$ has irreducible summands with characters χ_1, χ_2. ∴ as $\bar{M}$ is mult-free get $\chi_1 \neq \chi_2$ and from the decomposition matrix

$$\begin{array}{c|cccc} \chi_1 & o\!-\!\!-\!o & \times \cdots \times & \\ \chi_2 & & \times \cdots \times & o\!-\!\!-\!o \end{array}$$

That is, they have no modular constituent in common. ∴ there is no indec proj with χ_1 and χ_2 both as constituents ∴ $\bar{M}$ is not the image of such, so doesn't have a simple top, a contradiction.

Here's some guesses as to the modules which lift to irreducibles:

Say $n = 4$, here are ones that should lift.

$$\begin{array}{c} 123 \\ 23 \\ 3 \\ 0 \\ 4 \end{array} \qquad \begin{array}{ccc} & 23 & \\ 123 & & 3 \\ & & 0 \\ & & 4 \end{array} \qquad \begin{array}{ccc} & & 3 \\ & 23 & 0 \\ 123 & & 4 \end{array} \qquad \begin{array}{ccc} & 0 & \\ 3 & & 4 \\ 23 & & \\ 123 & & \end{array} \qquad \begin{array}{c} 4 \\ 0 \\ 3 \\ 23 \\ 123 \end{array}$$

$$\begin{array}{ccc} 123 & 3 & \\ 23 & 0 & \\ & 4 & \end{array} \qquad \begin{array}{ccc} 123 & & 4 \\ 23 & 0 & \\ 3 & & \end{array} \qquad \begin{array}{ccc} 123 & & \\ 23 & 0 & \\ 3 & & 4 \end{array}$$

$$\begin{array}{cccc} & 23 & 0 & \\ 123 & 3 & 4 & \end{array} \qquad \begin{array}{cccc} & 23 & 0 & 4 \\ 123 & 3 & & \end{array} \qquad \text{etc...}$$

Isomorphic blocks (cont.)

Usual set-up: $G_0 \triangleleft G$, G/G_0 solvable p'-gp., $G = C(S)G_0$, S a Sylow p-subgroup of G_0, F algebraically closed field of characteristic p. A, A_0 the principal block summands of FG, FG_0 respectively, $\mathcal{B}, \mathcal{B}_0$ the categories of f.g. A & A_0 modules. Let e, e_0 be the unit elements of A, A_0, resp.

Let $T: FG \to FG_0$ be the truncation map, so T is F-linear, $g_0 T = g_0$ if $g_0 \in G$ & $gT = 0$ if $g \in G$, $g \notin G_0$. Let $n = |G : G_0|$.

Corollary. The map

$$a \to n(aT)$$

is an isomorphism of the algebra A onto the algebra A_0.

Lemma 1. $n(eT) = e_0$.

Proof. e "comes from" $\sum_{\chi \in \mathcal{B}} e_\chi = \sum_{\chi \in \mathcal{B}} \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)}\, g$ while there is a similar expression for e_0. The results on restrictions of characters proves this result.

Let ϵ be the map of A_0 to A given by $a_0 \epsilon = a_0 e$, for $a_0 \in A$, so ϵ is a ring homomorphism.

Lemma 2. If $a_0 \in A_0$ then $n(a_0 \epsilon) T = a_0$.

Proof. $(a_0 \epsilon) T = (a_0 \overset{a_0}{T})(eT)$ by inspection, as $a_0 \in FG_0$, so $(a_0 \epsilon) T = a_0 n^{-1} e_0$. So the result holds as $a_0 e_0 = a_0$

We conclude the proof. We have two linear transformations: ϵ ; nT. Their composite is the identity on A_0 and $\dim_F A_0 = \dim_F A$. Hence, ϵ is a one-to-one map of A_0 onto A and nT maps A one-to-one onto A_0. Moreover, the map ϵ and the identity on A_0 are ring homomorphisms so that nT is also!

Modules for M_{11}.

We shall work over an algebraically closed field F of characteristic two. From G. James, J. Alg. v. 27, p 80 we have the decomposition matrix for the principal 2-block

V_1	V_{10}	V_{44}
1		
	1	
	1	
	1	
1	1	
		1
1		1
1	1	1

where V_i is a simple module of dimension i, $V_1 = F$. Hence, the Cartan matrix C_0 for the principal block is

	V_1	V_{10}	V_{44}
V_1	4	2	2
V_{10}	2	5	1
V_{44}	2	1	3

Using Thompson's lemma on indecomposables (from vertices & sources paper) get immediately

$$\mathrm{Ext}^1(V_1, V_{10}) \neq 0, \quad \mathrm{Ext}^1(V_{10}, V_{10}) \neq 0, \quad \mathrm{Ext}^1(V_1, V_{44}) \neq 0.$$

Now let $A_6 = M_{10}'$ and let U be a four-dimensional simple FA_6-module.

<u>LEMMA 1.</u> $U^{M_{11}}$ is a non-split extension of V_{44} by V_{44}.

<u>Proof</u>. First let's deal with the characters. The Brauer characters of M_{11} are - see James - (and calculate)

1	3	5	11	11	← (orders)
1	1	1	1	1	
10	1	0	-1	-1	
44	-1	-1	0	0	
16	-2	1	β	$\bar\beta$	
16	-2	1	$\bar\beta$	β	

Looking at orders of centralizers and fusion get for the character of $U^{M_{11}}$

88	-2	-2	0	0

so U^G has the desired composition series. Along the way, as U is not invariant in M_{10}, get that $U^{M_{10}}$ is a simple eight dimensional module, the only such. (M_{10} has a one-dimensional an eight and a sixteen dimensional simple module & no others)

Let $U^{M_{10}} = W_8$ and let W_1 be the trivial M_{10} module.

Now $W_8^{M_{11}}\big|_{M_{10}} = W_8 \oplus W_8^{x}\big|_{M_{10}\cap M_{10}^{x}}^{M_{10}}$

by Mackey and double transitivity, where $x \in M_{11}$, $x \notin M_{10}$.

But uniqueness of the eight dimensional module means the second summand is $W_8\big|_{M_9}^{M_{10}}$. We claim this does not have W_8 as a direct summand. Shall show in fact that $W_8\big|_{M_9}^{M_{10}}$ is projective.

Picture:

$$\begin{array}{c} M_{10} \\ \Big| \; \mathbb{Z}_2 \\ PSL(2,9) = A_6 = M_{10}' \\ \Big| \\ 1 \end{array}$$

Now $M_9 \cap M_{10}' = Z_3 \times Z_3 . Z_4$ by inspection. Also there are no involutions in $M_{10} - M_{10}'$ so $M_9 = Z_3 \times Z_3 . Q_8$. By our results on nearly projective modules, it suffices to see that the involution in Q_8 acts freely on W_8. But $W_8\big|_{A_6}$ is the sum of the two four-dimensional modules so the result is true as we have already studied & as Erdmann did more.

Hence $U^{M_{11}} \not\cong V_{44} \oplus V_{44}$. For U is a direct summand of $U^{M_{11}}\big|_{M_{10}}$ so if $U^{M_{11}} \cong V_{44} \oplus V_{44}$ then it is a summand of V_{44} so therefore appears twice. This contradicts what we have just proved so the lemma holds.

<u>LEMMA 2.</u> $\mathrm{Ext}^1_{FM_{11}}(V_1, V_{44}) \simeq F$

<u>Proof.</u> We have the exact sequence, in view of Lemma 1,

$$0 \to \mathrm{Ext}^1(V_1, V_{44}) \to \mathrm{Ext}^1(V_1, U^{M_{11}}) \to \mathrm{Ext}^1(V_1, V_{44})$$

and

$$\mathrm{Ext}^1_{FM_{11}}(V_1, U^{M_{11}}) \simeq \mathrm{Ext}^1_{FA_6}(V_1/_{A_6}, U) \simeq F$$

as we already know. Hence, $\dim_F \mathrm{Ext}^1(V_1, V_{44}) \leq 1$ and if it is 0 then $0 \to 0 \to F \to 0$ is exact, a contradiction.

<u>LEMMA 3.</u> The projective cover P_{44} of V_{44} has the following structure:

$$\begin{array}{ccc} & V_{44} & \\ & & V_1 \\ V_{44} & & V_{10} \\ & & V_1 \\ & V_{44} & \end{array}$$

<u>Proof.</u> The structure of $U^{M_{11}}$ gives a direct decomposition of the middle of P_{44} with V_{44} as a summand. $M_{11} = M_{11}'$ so $\mathrm{Ext}^1_{FM_{11}}(V_1, V_1) = 0$ so the above result finishes off this one.

<u>LEMMA 4.</u> $\mathrm{Ext}^1_{M_{11}}(V_1, V_{10}) \simeq F$

<u>Proof.</u> We have, as $\mathrm{Ext}^1_{FM_{11}}(V_1, V_1) = 0$,

$$\begin{aligned} \mathrm{Ext}^1_{FM_{11}}(V_1, V_{10}) &\simeq \mathrm{Ext}^1_{FM_{11}}(V_1, V_1 \oplus V_{10}) \\ &\simeq \mathrm{Ext}^1_{FM_{10}}(V_1/M_{10}, W_1) \\ &= \mathrm{Ext}^1_{FM_{10}}(W_1, W_1) \simeq F \end{aligned}$$

as $M_{10}/M_{10}' \simeq Z_2$.

LEMMA 5. $V_{10}/_{M_{10}}$ is uniserial of the form $\begin{matrix} W_1 \\ W_8 \\ W_1 \end{matrix}$

Proof The composition factors are right as follows from the following Brauer characters of M_{10}:

	1	3	5
W_1	1	1	1
W_8	8	-1	-2

But now

$$\begin{aligned} \mathrm{Hom}_{FM_{10}}(W_8, V_{10}/_{M_{10}}) &\simeq \mathrm{Hom}_{FM_{11}}(W_8^{M_{11}}, V_{10}) \\ &\simeq \mathrm{Hom}_{FM_{11}}(U^{M_{11}}, V_{10}) \\ &= 0 \end{aligned}$$

by Lemma 1. Similarly $\mathrm{Hom}(V_{10}/_{M_{10}}, W_8) = 0$ so the lemma is proved.

LEMMA 6. $\mathrm{Ext}^1_{FM_{11}}(V_{10}, V_{10}) \simeq F$.

Proof. Now $\mathrm{Ext}^1_{FM_{11}}(V_{10}, V_{10}) \neq 0$ as we remarked before the lemmas. Hence, E.T.S: $\dim_F \mathrm{Ext}(V_{10}, V_{10}) \leq 1$. But

$$2 + \dim_F \mathrm{Ext}(V_{10}, V_{10}) = \dim_F \mathrm{Ext}(V_1 \oplus V_{10}, V_1 \oplus V_{10})$$

so it suffices to show the R.H.S at most three. But

$$\mathrm{Ext}_{FM_{11}}(V_1 \oplus V_{10}, V_1 \oplus V_{10}) \simeq \mathrm{Ext}_{FM_{10}}(W_1, W_1 \oplus V_{10}/_{M_{10}})$$

so it suffices to show that

$$\dim_F \mathrm{Ext}(W_1, V_{10}/_{M_{10}}) \leq 2.$$

But we have, by the previous result, the exact sequence

$$0 = Hom_{FM_{10}}\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right) \to Ext^1_{FM_{10}}(W_1, W_1) \to Ext^1_{FM_{10}}(W_1, V_{10}/_{M_{10}}) \to Ext^1\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right)$$
$$\to Ext^2(W_1, W_1)$$

which is

$$0 \to F \to Ext^1_{FM_{10}}(W_1, V_{10}/_{M_{10}}) \to Ext^1\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right) \to 0$$

so it suffices to prove that

$$\dim_F Ext^1\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right) \leq 1.$$

However, we have the exact sequence

$$Hom\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right) \to Hom(W_1, W_1)$$
$$\to Ext^1(W_1, W_8) \to Ext^1\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right) \to Ext^1(W_1, W_1)$$
$$\left(\to Ext^2(W_1, W_8) \quad\right)$$

But

$$Ext^1_{FM_{10}}(W_1, W_8) \simeq Ext^1_{FA_6}(F, U) \simeq F$$

so we have

$$0 \to F \to F \to Ext^1\left(W_1, \begin{matrix} W_1 \\ W_8 \end{matrix}\right) \to F$$

so our claim is valid.

(note: also $Ext^2_{FM_{10}}(W_1, W_8) \simeq Ext^2_{FA_6}(F, U) = 0$, so get result exactly, i.e. don't need $Ext^1(V_1, V_{10}) \neq 0$ by Thompson lemma).

LEMMA 7. $\mathrm{Ext}^n(V_{44}, V_{10}) = 0$, $n \geq 0$.

Proof First,

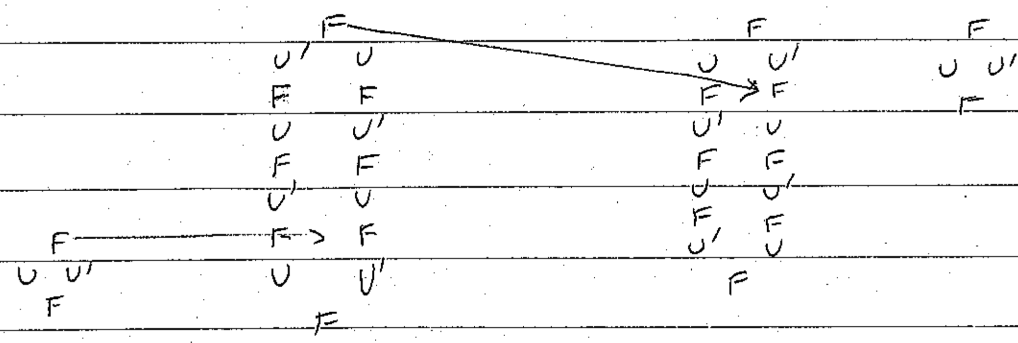
$$\mathrm{Ext}^n_{FM_{11}}(U^{M_{11}}, V_{10}) \cong \mathrm{Ext}^n_{FA_6}(U, V_{10}|_{A_6}).$$

And, schematically,

$$V_{10}|_{A_6} = \begin{smallmatrix} & F & \\ U & & U' \\ & F & \end{smallmatrix}$$

where U' is other four-dimensional. This follows from structure of $V_{10}|_{M_{10}}$ and $U^{M_{10}} = U'^{M_{10}} \cong W_8$.

But $\begin{smallmatrix} & F & \\ U & & U' \\ & F & \end{smallmatrix}$ is periodic, with resolution,



so $\mathrm{Ext}^n(\begin{smallmatrix} & F & \\ U & & U' \\ & F & \end{smallmatrix}, U) = 0$, all $n \geq 0$. Hence, we have shown $\mathrm{Ext}^n_{FM_{11}}(U^{M_{11}}, V_{10}) = 0$. But now we have

$$0 \to \mathrm{Ext}^1(V_{44}, V_{10}) \to \mathrm{Ext}^1(U^{M_{11}}, V_{10})$$

so the result holds for $n=1$ (being obvious if $n=0$). Say $\mathrm{Ext}^k(V_{44}, V_{10}) = 0$. Then we have

$$0 \to \mathrm{Ext}^{k+1}(V_{44}, V_{10}) \to \mathrm{Ext}^{k+1}(U^{M_{11}}, V_{10})$$

so the lemma is proved.

LEMMA 8. There is a uniserial module with composition factors, in order, V_{10}, V_1, V_{10}.

Proof. Suppose not; we shall then derive the structure of the projective covers P_{10}, P_1 of V_{10}, V_1, calculate the first few terms of the minimal projective resolution of V_1 and contradict the known vanishing of the Schur multiplier of M_{11}, or other independent calculation.

We begin by looking at P_{10}. From P_{44} we know that it has an image $\begin{smallmatrix}V_{10}\\V_1\\V_{44}\end{smallmatrix}$ which is uniserial, say P_{44}/X. Look at $X/\mathrm{rad}\,X$. Since $\mathrm{Ext}(V_{44}, V_{10}) = 0$ and there is no uniserial V_{10}, V_1, V_{10} we are lead to two cases:

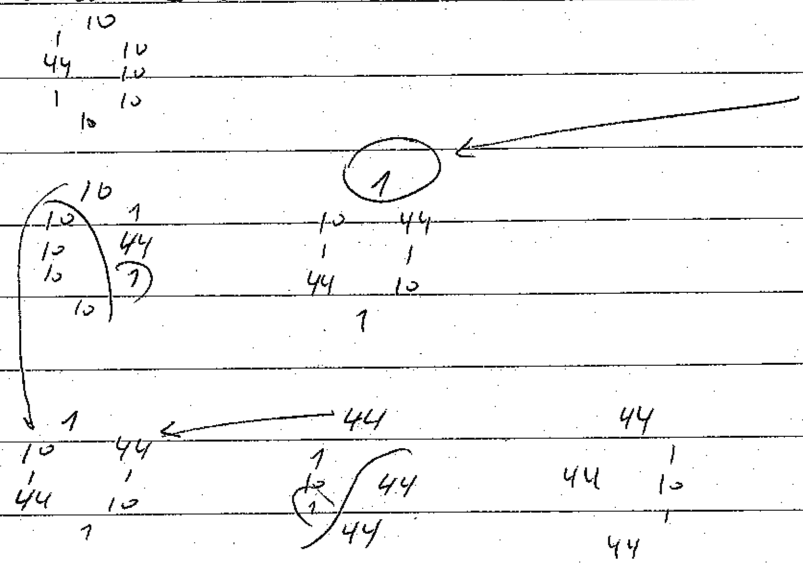
Case 1. P_{10} has an image which is uniserial $\begin{smallmatrix}V_{10}\\V_1\\V_{44}\\V_1\end{smallmatrix}$. Hence, it has a submodule with four V_{10}'s as composition factors. Also, by duality, it has $\begin{smallmatrix}V_1\\V_{44}\\V_1\\V_{10}\end{smallmatrix}$ as submodule so the middle of P_{10} is a direct sum and

$$P_{10}:\quad \begin{array}{ccc} & V_{10} & \\ V_1 & & V_{10} \\ V_{44} & & V_{10} \\ V_1 & & V_{10} \\ & V_{10} & \end{array}$$

as $\mathrm{Ext}(V_{10}, V_{10}) \cong F$. Then the middle of P_1 has $\begin{smallmatrix}V_{10}\\V_1\\V_{44}\end{smallmatrix}$ (from P_{10}) and $\begin{smallmatrix}V_{44}\\V_1\\V_{10}\end{smallmatrix}$ from P_{44} & these two intersect trivially as have different socles, so

$$P_1:\quad \begin{array}{ccc} & V_1 & \\ V_{10} & & V_{44} \\ V_1 & & V_1 \\ V_{44} & & V_{10} \\ & V_1 & \end{array}$$

Now we calculate the resolution



The contradiction.

Case 2 Rad P_{44} / Rad X splits over $X/\mathrm{Rad}\,X$.

A little more analysis now yields that $X/\mathrm{Rad}\,X \simeq V_{10}$. For $X/\mathrm{Rad}\,X$ has no V_1 as summand or else $\mathrm{Ext}(V_{10}, V_1)$ of dimension exceeding one or we're in case 1. If V_{44} appears then we contradict the Cartan matrix.

Now then X is an image of P_{10} and the kernel therefore has V_{10} as its socle and has, by count, composition factors V_1, V_{10}, V_{44}. Since $\mathrm{Ext}^1(V_{10}, V_{44}) = 0$ this kernel has structure as follows: $\begin{smallmatrix} V_{44} \\ V_1 \\ V_{10} \end{smallmatrix}$. Hence, $\mathrm{Ext}^2\left(\begin{smallmatrix} V_{10} \\ V_1 \\ V_{44} \end{smallmatrix}, V_1\right) = 0$ as we have the exact sequence

$$0 \to \begin{smallmatrix} V_{44} \\ V_1 \\ V_{10} \end{smallmatrix} \to P_{10} \to P_{10} \to \begin{smallmatrix} V_{10} \\ V_1 \\ V_{44} \end{smallmatrix} \to 0,$$

We shall contradict this. Note that our calculation also gives $\mathrm{Ext}^1\left(\begin{smallmatrix} V_{10} \\ V_1 \\ V_{44} \end{smallmatrix}, V_1\right) = 0$.

We have the exact sequence

$$\mathrm{Ext}^1\left(\begin{smallmatrix}V_{10}\\V_1\\V_{44}\end{smallmatrix}, V_1\right) \to \mathrm{Ext}^1(V_{44}, V_1)$$

$$\mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\end{smallmatrix}, V_1\right) \to \mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\\V_{44}\end{smallmatrix}, V_1\right)$$

which is, as we know P_{44},

$$0 \to F \to \mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\end{smallmatrix}, V_1\right) \to \mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\\V_{44}\end{smallmatrix}, V_1\right).$$

Hence, it suffices, to reach a contradiction, to show that $\mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\end{smallmatrix}, V_1\right) \simeq F \oplus F$.

But we have the exact sequence

$$\mathrm{Ext}^1(V_1, V_1) \to \mathrm{Ext}^2(V_{10}, V_1) \to \mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\end{smallmatrix}, V_1\right) \to \mathrm{Ext}^2(V_1, V_1)$$

which is

$$0 \to \mathrm{Ext}^2(V_{10}, V_1) \to \mathrm{Ext}^2\left(\begin{smallmatrix}V_{10}\\V_1\end{smallmatrix}, V_1\right) \to 0$$

so it is enough to see that $\mathrm{Ext}^2(V_{10}, V_1) \simeq F \oplus F$.

However,

$$\begin{aligned}\mathrm{Ext}^2(V_1, V_{10}) &\simeq \mathrm{Ext}^2(V_1, V_1 \oplus V_{10})\\ &\simeq \mathrm{Ext}^2_{FM_{10}}(V_1, V_1)\\ &\simeq F \oplus F\end{aligned}$$

as $M_{10}/M_{10}' \simeq Z_2$ & as M_{10} has a multiplier as well of even order.

<u>LEMMA 9.</u> There is no uniserial module with composition factors, in order, V_{10}, V_{10}, V_1

<u>Pf.</u> If there were then the middle T_1 of P_1 would contain $\begin{smallmatrix}V_{10}\\V_{10}\end{smallmatrix}$. $T_1 / \begin{smallmatrix}V_{10}\\V_{10}\end{smallmatrix}$ would have composition factors V_9, V_1, V_{44}, V_{44} so there would also be a submodule of T_1 with such factors & so $\begin{smallmatrix}V_{10}\\V_{10}\end{smallmatrix}$ & that would give a direct sum decomposition of T_1. This other summand would have top & bottom each V_{44} contradicting $\mathrm{Ext}(V_{44}, V_1)$ of dimension one.

LEMMA 10. There is a uniserial module with composition factors in order V_{10}, V_{10}, V_{10}

Proof. It suffices to see that $\mathrm{Ext}\left(\begin{smallmatrix} V_{10} \\ V_{10}\end{smallmatrix}, V_{10}\right) \neq 0$ as $\mathrm{Ext}(V_{10}, V_{10}) \simeq F$, by lemma 6, which we now imitate. We have

$$\mathrm{Ext}\left(\begin{smallmatrix} V_{10} \\ V_{10}\end{smallmatrix}, V_{10}\right) \simeq \mathrm{Ext}\left(\begin{smallmatrix} V_{10} \\ V_{10}\end{smallmatrix}, V_{10} \oplus V_1\right), \text{ by Lemma 9}$$

$$\simeq \mathrm{Ext}_{FM_{10}}\left(\begin{smallmatrix} V_{10} \\ V_{10}\end{smallmatrix}\Big|_{M_{10}}, W_1\right).$$

Let $X = \begin{smallmatrix} V_{10} \\ V_{10}\end{smallmatrix}\Big|_{M_{10}}$ so now we have an exact sequence

$$\mathrm{Ext}_{FM_{10}}(X, W_1) \to \mathrm{Ext}_{FM_{10}}\left(X, \begin{smallmatrix} W_1 \\ W_1\end{smallmatrix}\right) \to \mathrm{Ext}(X, W_1).$$

Hence, to show the lemma, i.e. show $\mathrm{Ext}(X, W_1) \neq 0$, it suffices to prove $\mathrm{Ext}_{FM_{10}}\left(X, \begin{smallmatrix} W_1 \\ W_1\end{smallmatrix}\right) \neq 0$. But

$$\mathrm{Ext}_{FM_{10}}\left(X, \begin{smallmatrix} W_1 \\ W_1\end{smallmatrix}\right) \simeq \mathrm{Ext}_{FA_6}\left(X|_{A_6}, F\right).$$

Hence, it suffices to prove that

$$X|_{A_6} \cong \begin{smallmatrix} & F & \\ U & & U' \\ F & & F \\ U' & & U \\ & F & \end{smallmatrix}$$

where U' is the other four dimensional simple module as then $\mathrm{Ext}_{FA_6}(X|_{A_6}, F) \neq 0$ by inspection.

But this structure of X_{A_6} is easy to obtain, as it has only one F at top & one at bottom + no U or U' at top or bottom (see 1st paragraph of the pf of Lemma 7).

LEMMA 11. The projective cover P_{10} of V_{10} has the following structure:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & V_{10} \arrow[dl, dash] \arrow[dr, dash] & & \\
V_{10} \arrow[d, dash] & & V_1 \arrow[dl, dash] \arrow[dr, dash] & \\
V_{10} \arrow[dr, dash] & V_{10} \arrow[dr, dash] & & V_{44} \arrow[dl, dash] \\
 & V_{10} & V_1 \arrow[l, dash] &
\end{tikzcd}$$

Proof. By the previous two lemmas, the middle T_{10} of P_{10} has two submodules $A = \begin{smallmatrix} V_{10} \\ V_1 \end{smallmatrix}$ and $B = \begin{smallmatrix} V_{10} \\ V_{10} \end{smallmatrix}$. Now $A \cap B = 0$ so T_{10}/B has a submodule isomorphic with A and T_{10}/B has composition factors V_1, V_1, V_{10}, V_{44}. Since $\mathrm{Ext}(V_{10}, V_{44}) = 0$ a little case analysis (using e.g. $\mathrm{Ext}(V_1, V_{44})$ to keep V_{44} from rising to the top of T_{10}) leads to two possible structures for T_{10}/B:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & V_1 \arrow[dl, dash] \arrow[dr, dash] & \\
V_{10} \arrow[dr, dash] & & V_{44} \arrow[dl, dash] \\
 & V_1 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & V_1 \arrow[dl, dash] \\
V_{10} \arrow[dr, dash] & V_{44} \arrow[d, dash] & \\
 & V_1 &
\end{tikzcd}$$

In the first case, we get this quotient appearing (as its dual = itself) as a submodule of T_{10} so we get the required direct decomposition. This leaves the other case to be eliminated.

But now T_{10} has a submodule $C \simeq \begin{tikzcd}[column sep=tiny, row sep=tiny] & V_1 \arrow[dl, dash] \arrow[dr, dash] & & \\ V_{10} & & V_{44} \arrow[dr, dash] & \\ & & & V_1 \end{tikzcd}$.

If $B \cap C = 0$ then

$$T_{10} \simeq \begin{smallmatrix} V_{10} \\ V_{10} \end{smallmatrix} \oplus \begin{tikzcd}[column sep=tiny, row sep=tiny] & V_1 \arrow[dl, dash] \arrow[dr, dash] & & \\ V_{10} & & V_{44} \arrow[dr, dash] & \\ & & & V_1 \end{tikzcd}$$

contradicting $\mathrm{Ext}(V_{10}, V_{10}) \simeq F$. Hence $B \cap C = V_{10}$ so

$$B + C \cong \begin{tikzcd}[column sep=tiny, row sep=tiny] V_{10} \arrow[dr, dash] & & V_1 \arrow[dl, dash] \arrow[dr, dash] & & \\ & V_{10} & & V_{44} \arrow[dr, dash] & \\ & & & & V_1 \end{tikzcd}$$

But then $A \not\subseteq B+C$ as $A \simeq \begin{matrix} V_{10} \\ V_1 \end{matrix}$. From the Cartan matrix it follows that $A \cap (B+C) \neq 0$ so $A \cap (B+C) = V_1$. But then, modulo V_1, we get a direct sum decomposition and T_{10} has at the top V_{10}, V_1 from $B+C$ and V_{10} from A, again a contradiction. The lemma is proved.

<u>LEMMA 12.</u> There is a uniserial module with composition factors V_1, V_{44}, V_1, in that order.

<u>Pf</u> Have the exact sequence

$$\mathrm{Ext}(1,1) \to \mathrm{Ext}(44',1) \to \mathrm{Ext}(44,1) \to \mathrm{Ext}^2(1,1)$$

(using i in place of V_i) and the end terms are zero while $\mathrm{Ext}(44,1) \simeq F$.

We turn to the last projective P_1, the projective cover of V_1. We won't really have a diagram.

<u>LEMMA 13.</u> We have

$$P_1 \cong \begin{tikzcd} & V_1 \arrow[dl, dash] \arrow[dr, dash] & \\ V_{10} \arrow[d, dash] & & V_{44} \arrow[d, dash] \\ V_1 \arrow[d, dash] \arrow[drr, dotted, dash] & & V_1 \arrow[d, dash] \\ V_{44} \arrow[dr, dash] & & V_{10} \arrow[dl, dash] \\ & V_1 & \end{tikzcd}$$

(which really isn't a diagram.)

Proof. Since $\begin{matrix} V_1 \\ V_{10} \\ V_1 \\ V_{44} \end{matrix}$ + $\begin{matrix} V_1 \\ V_{44} \\ V_1 \end{matrix}$ exist we know that P_1 is an extension of $\begin{matrix} V_{10} \\ V_1 \end{matrix}$ by

$$\begin{matrix} & V_1 & \\ V_{10} & & V_{44} \\ V_1 & & V_1 \\ V_{44} & & \end{matrix}$$

now the V_{44} splits over the top V_{10} of $\begin{matrix} V_{10} \\ V_1 \end{matrix}$; but neither V_1 does. Or else T_1, the middle of P_1 has V_1 in its socle of

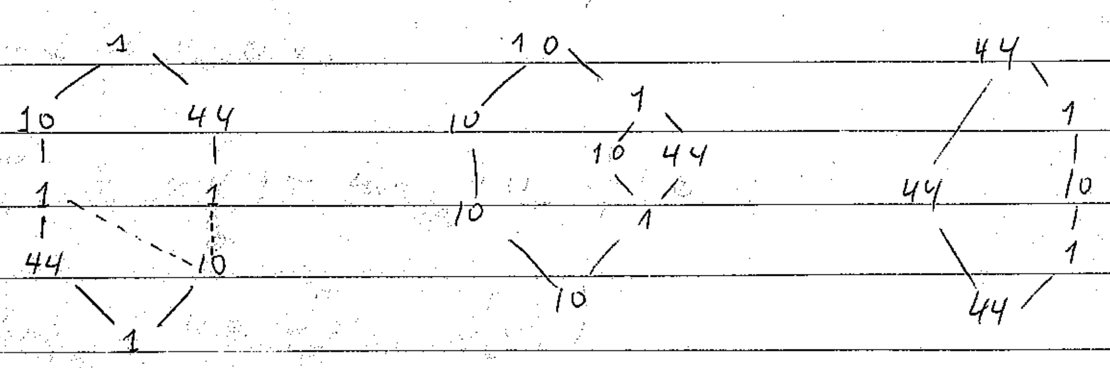
$$T_1 \simeq \begin{matrix} V_{10} \\ V_1 \\ V_{44} \end{matrix} \oplus \begin{matrix} V_{44} \\ V_1 \\ V_{10} \end{matrix}$$

and then there is no

$$\begin{tikzcd} & V_1 \arrow[dl, no head] \arrow[dr, no head] & \\ V_{10} \arrow[dr, no head] & & V_{44} \arrow[dl, no head] \\ & V_1 \arrow[d, no head] & \\ & V_{10} & \end{tikzcd}$$

as we know exists from P_{10}.

<u>SUMMARY</u> The indecomposable projectives in the principal 2-block.



<u>LEMMA 14</u>. $V_{10} \otimes V_{10} \cong V_{10} \oplus$
$$\begin{tikzcd}[column sep=small, row sep=small] & V_{44} \arrow[dl, dash] \arrow[dr, dash] & & V_1 \arrow[dl, dash] \\ V_1 & & V_{44} & \end{tikzcd}$$

<u>Proof</u> First, the composition factors of $V_{10} \otimes V_{10}$ are $V_{10}, V_1, V_1, V_{44}, V_{44}$ by an easy character calculation (see character table on page 134). Since
$$Hom\,(V_{10} \otimes V_{10}, V_1) \simeq Hom\,(V_{10}, V_{10}^* \otimes V_1)$$
$$\simeq Hom\,(V_{10}, V_{10}) \simeq F,$$
and by a similar calculation, V_1 appears in the socle & in the radical quotient of $V_{10} \otimes V_{10}$ one time each.

Also
$$Hom\,(V_1 \oplus V_{10}, V_{10} \otimes V_{10}) \simeq Hom_{FM_{10}}\left(W_1, V_{10} \otimes V_{10}\big|_{M_{10}}\right)$$
$$\simeq Hom_{FM_{10}}\left(V_{10}\big|_{M_{10}}, V_{10}\big|_{M_{10}}\right)$$
$$\simeq Hom_{FM_{10}}\left(\begin{matrix} W_1 \\ W_8 \\ W_1 \end{matrix}, \begin{matrix} W_1 \\ W_8 \\ W_1 \end{matrix}\right)$$
$$\simeq F \oplus F$$
so $V_{10} \subseteq V_{10} \otimes V_{10}$. Likewise V_{10} is an image of $V_{10} \otimes V_{10}$ so V_{10} is a direct summand of $V_{10} \otimes V_{10}$.

Also
$$Hom_{FM_{11}}\left(\begin{matrix} V_{44} \\ V_{44} \end{matrix}, V_{10} \otimes V_{10}\right) \simeq Hom_{FA_0}\left(U, \begin{matrix} F \\ U\,U' \\ F \end{matrix} \otimes \begin{matrix} F \\ U\,U' \\ F \end{matrix}\right)$$
$$\simeq Hom_{FA_0}\left(U \otimes \begin{matrix} F \\ U\,U' \\ F \end{matrix}, \begin{matrix} F \\ U\,U' \\ F \end{matrix}\right)$$

But $U \otimes U{}^{F}_{F}U'$ is easy to deal with, by calculation of its factors, socle + top. It is $\begin{smallmatrix}U\\F\\U'\\\vdots\\U\end{smallmatrix} \oplus (U\otimes U')$ so

$\mathrm{Hom}_{FM_{11}}\left(\frac{V_{44}}{V_{44}}, V_{10}\otimes V_{10}\right) \simeq F$. Thus $V_{44} \subseteq V_{10}\otimes V_{10}$ but $\frac{V_{44}}{V_{44}} \not\subseteq V_{10}\otimes V_{10}$. Similarly, for quotients.

Hence, by looking at submodules of $P_1 \oplus P_{44}$ we can see that either the result is correct or else we have

$$V_{10}\otimes V_{10} \simeq V_{10} \oplus \frac{V_1}{V_{44}} \oplus \frac{V_{44}}{V_1}.$$

But then $\mathrm{Ext}(V_1, V_{10}\otimes V_{10}) \simeq F\oplus F$ so $\mathrm{Ext}(V_{10}, V_{10}) \simeq F\oplus F$, a contradiction.

<u>LEMMA 15</u>. $V_{10}\otimes \frac{V_{44}}{V_{44}}$ is projective.

<u>Proof</u> We have

$$V_{10}\otimes \frac{V_{44}}{V_{44}} \cong V_{10}\otimes U^{M_{11}} \cong \left(V_{10}|_{A_6}\otimes U\right)^{M_{11}}$$

so it suffices to see that $U{}^{F}_{F}U' \otimes U$ is projective. But it is $P_U \oplus U'\otimes U$ (P_U the proj. cover of U in FA_6) as is easy to see — see above work in these notes — so the lemma holds.

Now V_{10} is algebraic as $F_{M_{10}}$ is and $F^{M_{11}} \cong V_1 \oplus V_{10}$. But

<u>LEMMA 16</u> $V_{10}\otimes V_{10}\otimes V_{10} \cong (V_{10}\otimes V_{10}) \oplus V_{10} \oplus V_{10} \oplus P$, where P is projective.

Proof. $V_{10} \otimes V_{10} \simeq (V_{10} \otimes V_{10}) \oplus V_{10} \otimes \left(\begin{smallmatrix} & V_{44} & & V_1 \\ V_1 & & V_{44} & \end{smallmatrix} \right)$.

Hence, by the previous result, it suffices to prove that

$$\mathrm{Hom}\left(V_{10},\ V_{10} \otimes \left(\begin{smallmatrix} & V_{44} & & V_1 \\ V_1 & & V_{44} & \end{smallmatrix} \right)\right) \simeq F \oplus F \oplus F$$

since $\begin{smallmatrix} V_{44} \\ V_{44} \end{smallmatrix} \otimes V_{10}$ contains P_{10} as a summand with multiplicity one, inasmuch as $\mathrm{Hom}\left(V_{10}, \begin{smallmatrix} V_{44} \\ V_{44} \end{smallmatrix} \otimes V_{10}\right) \simeq \mathrm{Hom}\left(V_{10} \otimes V_{10}, \begin{smallmatrix} V_{44} \\ V_{44} \end{smallmatrix}\right) \simeq F$.

But this is

$$\mathrm{Hom}\left(V_{10} \otimes V_{10},\ \begin{smallmatrix} & V_{44} & & V_1 \\ V_1 & & V_{44} & \end{smallmatrix}\right)$$

$$\simeq \mathrm{Hom}\left(V_1 \begin{smallmatrix} V_{44} & & V_1 \\ & V_{44} & \end{smallmatrix},\ \begin{smallmatrix} & V_{44} & & V_1 \\ V_1 & & V_{44} & \end{smallmatrix}\right)$$

$$\simeq F \oplus F \oplus F$$

as desired.

LEMMA 17. V_{44} is algebraic

Pf. The Brauer characters of M_{11} on 11 pts and on pairs are

1	3	5	11	11
11	2	1	0	0
55	1	0	0	0

the latter being easily deduced from the first*. But the latter is the sum, by inspection of the characters of V_1, V_{10}, V_{44}.
Since 55 is odd, V_1 is a summand of the permutation module on 55 pts.
Since $\mathrm{Ext}(V_{10}, V_{44}) = 0$ we get that V_{44} is a summand of a permutation module.

* Look at cycle structure, which is forced from character values as elements have prime orders

<u>LEMMA 18.</u> $V_{44}|_{A_6} \cong \begin{smallmatrix} U \\ F \\ U' \\ F \\ U \end{smallmatrix} \oplus \begin{smallmatrix} U' \\ F \\ U \\ F \\ U' \end{smallmatrix} \oplus Q' \oplus Q''$
where Q', Q'' are the projective irreducible modules of A_6.

<u>Proof</u> First, $V_{44}|_{A_6}$ has the stated composition factors by direct calculation with Brauer characters. Also

$$\mathrm{Hom}_{FA_6}(U, V_{44}) \simeq \mathrm{Hom}_{FM_{11}}\left(\begin{smallmatrix} V_{44} \\ V_{44} \end{smallmatrix}, V_{44}\right) \simeq F$$

so by this, and similar calculations, $V_{44}|_{A_6}$ has U & U' each once in the socle & radical quotient. Also

$$\mathrm{Hom}_{FA_6}(F, V_{44}) \simeq \mathrm{Hom}_{FM_{10}}\left(\begin{smallmatrix} F \\ F \end{smallmatrix}, V_{44}\right)$$
$$\simeq \mathrm{Hom}_{FM_{11}}\left(\frac{V_1 \oplus V_{10}}{V_1 \oplus V_{10}}, V_{44}\right) = 0$$

so there is no F in either place. Thus, $V_{44}|_{A_6} \subseteq P_U \oplus P_{U'} \oplus Q' \oplus Q''$. Now, a case analysis, using Goursat's theorem plus $U \oplus U' \subseteq V_{44}|_{A_6}$ & the composition factors being determined, completes the proof.

<u>LEMMA 19.</u> $\dim_F \mathrm{Hom}\left(\begin{smallmatrix} V_{44} \\ V_{44} \end{smallmatrix}, V_{44} \otimes V_{44}\right) = 22$

<u>Proof</u> We have

$$\mathrm{Hom}_{FM_{11}}\left(\begin{smallmatrix} V_{44} \\ V_{44} \end{smallmatrix}, V_{44} \otimes V_{44}\right) \simeq \mathrm{Hom}_{FA_6}\left(U, V_{44}|_{A_6} \otimes V_{44}|_{A_6}\right)$$

$$\simeq \mathrm{Hom}_{FA_6}\left(U \otimes \left(\begin{smallmatrix} U \\ F \\ U' \\ F \\ U \end{smallmatrix} \oplus \begin{smallmatrix} U' \\ F \\ U \\ F \\ U' \end{smallmatrix} \oplus Q' \oplus Q''\right), \begin{smallmatrix} U \\ F \\ U' \\ F \\ U \end{smallmatrix} \oplus \begin{smallmatrix} U' \\ F \\ U \\ F \\ U' \end{smallmatrix} \oplus Q' \oplus Q''\right)$$

We now shall calculate the left-hand argument.

First, let's deal with $U \otimes \begin{smallmatrix} U \\ F \\ U' \\ F \\ U \end{smallmatrix}$ which is thus

$Q' \oplus Q'' \oplus \dfrac{U \otimes \begin{smallmatrix} U \\ F \end{smallmatrix}}{U \otimes \begin{smallmatrix} F \\ U \end{smallmatrix}}$ } (some extension). But $U \otimes \begin{smallmatrix} U \\ F \end{smallmatrix} = \mathrm{Rad}\, P_U$

and $U \otimes \begin{smallmatrix} F \\ U \end{smallmatrix} = \mathrm{Corad}\, P_U$ (the socle quotient). We assert that $\mathrm{Ext}^1(\mathrm{Rad}\, P_U, \mathrm{Corad}\, P_U) = F$. Indeed, we have

$$0 = \mathrm{Ext}(P_U, \mathrm{Corad}\, P_U) \to \mathrm{Ext}(\mathrm{Rad}\, P_U, \mathrm{Corad}\, P_U) \to \mathrm{Ext}^2(U, \mathrm{Corad}\, P_U) \to \mathrm{Ext}^2(P_U, \mathrm{Corad}\, P_U) = 0$$

and

$$0 = \mathrm{Ext}^2(U, P_U) \to \mathrm{Ext}^2(U, \mathrm{Corad}\, P_U) \to \mathrm{Ext}^3(U,U) \cong F \to \mathrm{Ext}^3(U, P_U) = 0 .$$

But there is an extension

$$0 \to \mathrm{Corad}\, P_U \to P_F \to \mathrm{Rad}\, P_U \to 0$$

which doesn't have U in its socle, as $U \otimes \begin{smallmatrix} U \\ F \\ U' \\ F \\ U \end{smallmatrix}$ does, as is easily seen as usual. Hence, our extension of $\mathrm{Rad}\, P_U$ on top of $\mathrm{Corad}\, P_U$ is the split one so

$$U \otimes \begin{matrix} U \\ F \\ U' \\ F \\ U \end{matrix} \simeq \begin{matrix} F \\ U' \\ F \\ U \\ F \\ U' \\ F \\ U \end{matrix} \oplus \begin{matrix} U \\ F \\ U' \\ F \\ U \\ F \\ U' \\ F \end{matrix} \oplus Q' \oplus Q''$$

Also,

$$U \otimes \begin{matrix} U' \\ F \\ U \\ F \\ U' \end{matrix} \simeq P_U \oplus Q' \oplus Q' \oplus Q'' \oplus Q''$$

while

$$U \otimes Q' \simeq P_{U'} \oplus Q''$$

$$U \otimes Q'' \simeq P_U \oplus Q'$$

so we can compute the dimension term by term.

<u>LEMMA 20</u>. $V_{44} \otimes V_{44}$ has composition factors V_1, with multiplicity 28, V_{10} with multiplicity 24, V_{44} with multiplicity 35, V_{16}' with multiplicity 4 and V_{16}'' with multiplicity 4. (Here V_{16}', V_{16}'' are the other irreducibles, as above).

<u>Proof</u> Direct calculation with Brauer characters.

Let's get some structure of $V_{10} \otimes V_{44}$. We begin by looking at M_{10} some more. The irreducible modules - as we assumed early and is easily gotten from the character table of A_6 and the action - are W_1, W_8 and W_{16}, the latter being projective.

Now $P_U^{M_{10}}$ is the projective cover of W_8, by calculating $\mathrm{Hom}_{FM_{10}}(\ , P_U^{M_{10}})$ easily. This gives its composition factors. Also

$$\mathrm{Ext}_{FM_{10}}(W_8, W_8) \simeq \mathrm{Ext}_{FA_6}(U \oplus U', U) = 0$$

and $\begin{matrix} W_8 \\ W_1 \end{matrix}$ exists and is unique by a similar calculation. Another one shows $\mathrm{Ext}\left(\begin{matrix} W_8 \\ W_1 \end{matrix}, W_8\right) = 0$ and this readily gives the projective cover of W_8 to be uniserial:

$$\begin{matrix} W_8 \\ W_1 \\ W_1 \\ W_8 \\ W_1 \\ W_1 \\ W_8 \\ W_1 \\ W_1 \\ W_8 \\ W_1 \\ W_1 \\ W_8 \end{matrix}$$

Also $V_{44}/_{A_6}$ has been done above so we get

$$V_{44}/_{M_{10}} = \begin{matrix} W_8 \\ W_1 \\ W_1 \\ W_8 \\ W_1 \\ W_1 \\ W_8 \end{matrix} \oplus W_{16}.$$

Hence, if $0 \to W_{44} \to P_{44} \to C_{44} \to 0$ is exact, then $C_{44}/_{M_{10}} \equiv \begin{matrix} W_8 \\ W_1 \\ W_1 \\ W_8 \\ W_1 \\ W_1 \end{matrix}$ where the congruence means modulo projectives.

This yields, by calculating resolutions a bit and taking cocycles modulo coboundaries that $\mathrm{Ext}^1(C_{44}/_{M_{10}}, V_{44}/_{M_{10}}) \simeq F \oplus F$. Also, the structure of $V_{44}/_{M_{10}}$ yields $\mathrm{Hom}_{FM_{10}}(V_{44}/_{M_{10}}, V_{44}/_{M_{10}})$ has dimension four over F. But, if $X = \left(V_{44}/_{M_{10}}\right)^{M_{11}}$ then the following sequence is exact:

$$\mathrm{Hom}(P_{44}, X) \to \mathrm{Hom}(V_{44}, X) \to \mathrm{Ext}(C_{44}, X) \to 0$$

so our calculation of dimensions yield that the image of $\mathrm{Hom}(P_{44}, X)$ in $\mathrm{Hom}(V_{44}, X)$ is two-dimensional, so $P_{44} \oplus P_{44} \subseteq X$. But $X = (V_{44}|_{M_{10}})^{M_{11}} \cong F_{M_{10}}{}^{M_{11}} \otimes V_{44} \simeq (V_1 \oplus V_{10}) \otimes V_{44}$ so $P_{44} \oplus P_{44} \subseteq V_{10} \otimes V_{44}$. We almost have the next result

LEMMA 21. $V_{10} \otimes V_{44} \simeq Y \oplus P_{44} \oplus P_{44} \oplus P_{16}' \oplus P_{16}''$ where Y has no projective summand and has composition factors V_1, with multiplicity two, V_{10}, with multiplicity three and V_{44} with multiplicity two.

Here P_{16}', P_{16}'' are the irreducible projectives of M_{11}. We have $X = (V_{44}|_{M_{10}})^{M_{11}} \supseteq W_{16}{}^{M_{11}} \geq P_{16}' \oplus P_{16}''$ the latter by an easy character calculation. It remains only to calculate the composition factors of $V_{10} \otimes V_{44}$ as then there is no room for any more projectives. Can calculate* $V_{10} \otimes \frac{V_{44}}{V_{44}}$ and divide in half if you like.

At this point it seems hard to proceed.

---

* $V_{10} \otimes \frac{V_{44}}{V_{44}} = \left(\begin{smallmatrix} & F & \\ U & & U' \\ & F & \end{smallmatrix} \otimes U \right)^{M_{11}} = (P_U \oplus Q' \oplus Q'')^{M_{11}}$.

$\mathrm{Hom}(V_1, V_{10} \otimes \frac{V_{44}}{V_{44}}) = 0$, $\mathrm{Hom}(V_{10}, V_{10} \otimes \frac{V_{44}}{V_{44}}) \simeq \mathrm{Hom}_{FA_6}(\begin{smallmatrix} & F & \\ U & & U' \\ & F & \end{smallmatrix}, P_U \oplus Q' \oplus Q'') = F$,

$\mathrm{Hom}(V_{44}, V_{10} \otimes \frac{V_{44}}{V_{44}}) \simeq \mathrm{Hom}_{FA_6}(\begin{smallmatrix} U \\ F \\ U' \\ F \end{smallmatrix} \oplus \begin{smallmatrix} U' \\ F \\ U \\ F \end{smallmatrix} \oplus Q' \oplus Q'', P_U \oplus Q' \oplus Q'') \simeq F \oplus F \oplus F \oplus F \oplus F$

so $V_{10} \otimes \frac{V_{44}}{V_{44}} \simeq P_{10} + 5 P_{44} + 2(P_{16}' + P_{16}'')$.

Quadratic fours groups

The following result is hopefully useful in dealing with modules in characteristic two of groups of odd type.

THEOREM. Let V be a faithful FG-module where F is of characteristic two. If E is a fours subgroup of G which acts quadratically on V and X is any p-subgroup, for an odd prime p, normalized by E then

$$\bigcap_{t \in E^{\#}} [X, t] = 1.$$

Proof. Let V_0 be an EX composition factor of V. If E is not faithfully represented on V_0 then there is $e \in E^{\#}$ in the kernel of the representation of EX on V_0. Hence $[X, e]$ is in this kernel and so is $\bigcap_{t \in E^{\#}} [X, t]$. If this holds for all V_0 then, since X has odd order, $\bigcap_{t \in E^{\#}} [X, t]$ acts trivially on V so this intersection is 1 as desired.

Hence, we can assume that there is a composition factor V_0 with E acting faithfully. Hence, it suffices to establish the

LEMMA. If $G = EX$, where X is a normal p-subgroup for an odd prime p and E is a fours subgroup and V is an irreducible FG-module on which E acts faithfully then V_E contains a free summand.

Proof. First, suppose that V_X is reducible. It follows, from Clifford theory, since EX/X is a 2-group and F has characteristic two, that V is an induced module: there is an E_0X module W, where $1 \leq E_0 < E$ with $W^G \equiv V$. If $E_0 = 1$ then V_E is free while if $|E_0| = 2$ then W_{E_0} is not trivial so V_E certainly has a free summand.

Hence, we may assume that V_X is irreducible. Since V is irreducible, it follows that E does not centralize X - or else $E \leq O_2(G)$. Hence, as X is a p-group, there is a normal subgroup Y of G, Y of index p in X with E not centralizing X/Y. If V_Y is irreducible then we can apply an induction to EY and get the required conclusion. Hence $V_Y = W_1 \oplus \ldots \oplus W_p$, p conjugate irreducible modules, the only other possibility.

Let $\langle t \rangle = C_E(X/Y)$. We assert that t leaves each W_i invariant and acts non-trivially on each W_i, $1 \leq i \leq p$. Since t has order two, it leaves fixed one of the p modules, say $W = W_1$. Now X/Y permutes these transitively and commutes with t so our first claim is true, and moreover, if t were trivial on any W_i it would also be trivial on all of them, a contradiction.

Hence, to get the required free module, it suffices to see that if $u \in E - \langle t \rangle$ then $W_i u \neq W_i$ for some i. But u inverts X/Y so u cannot be in the kernel of the permutation action of the p-modules. The proof is complete.

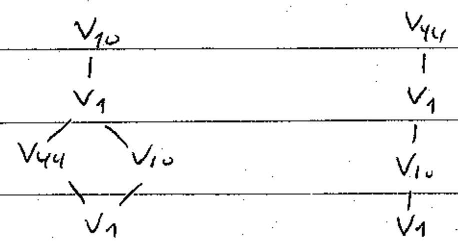
We have one other curious result:

PROPOSITION. If $G = EX$ where E is a fours group and X is a normal subgroup of odd order and if V is an irreducible FG-module for a field F of characteristic two then any odd-dimensional indecomposable summand of V_E is isomorphic with F.

Proof. By Berger's theorem (or in this case our work), V is algebraic so that so is V_E. Hence, so is any summand W of V_E. But the odd-dimensional FE indecomposable modules are the $\Omega^n F$, $n \in \mathbb{Z}$ (see D.L. Johnson's quick proof). If $n \neq 0$ none of them is algebraic, for $\Omega^{kj} F$ is a summand of $(\Omega^j F)^k$.

<u>Addendum to "Modules for M_{11}"</u>

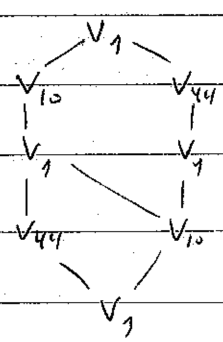
Here's a quick way to get P_1 once one has P_{10} and P_{44}. One has the modules



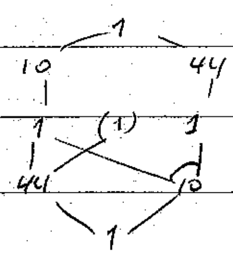
as submodules of the radical of P_1. Moreover, their intersection contains V_1 and can be at most the module $\begin{smallmatrix} V_{10} \\ V_1 \end{smallmatrix}$. A look at the composition factors of P_1 which is given by James shows that the latter is the intersection and that the sum of these two submodules is the whole radical of P_1.

The way to represent this fact seems to be as follows:

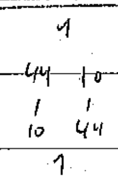
$P_1 \cong$

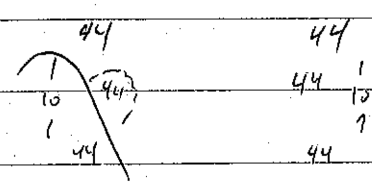


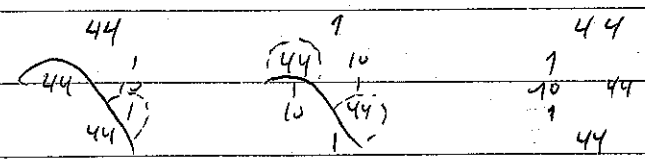
And we must remember this diagram is not "true". Perhaps use:

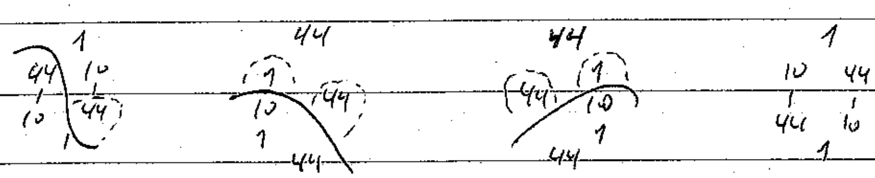


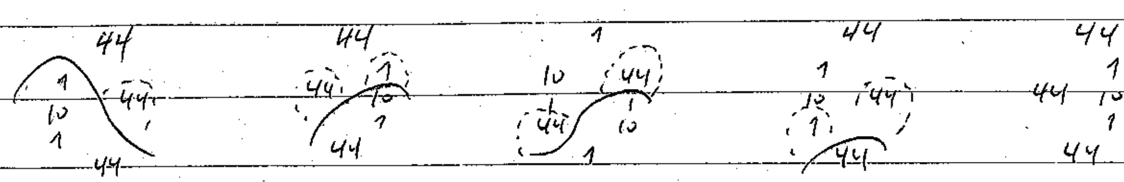
V_{44} resolution:

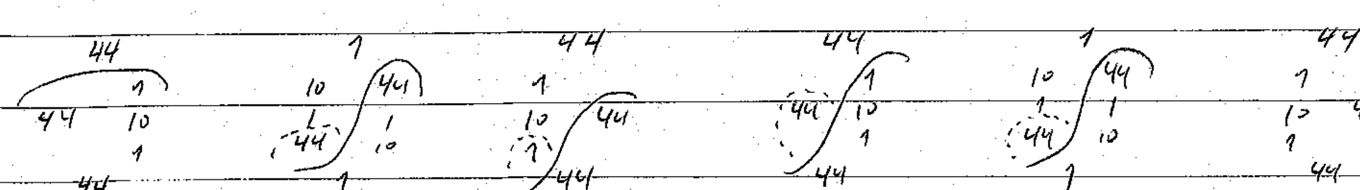




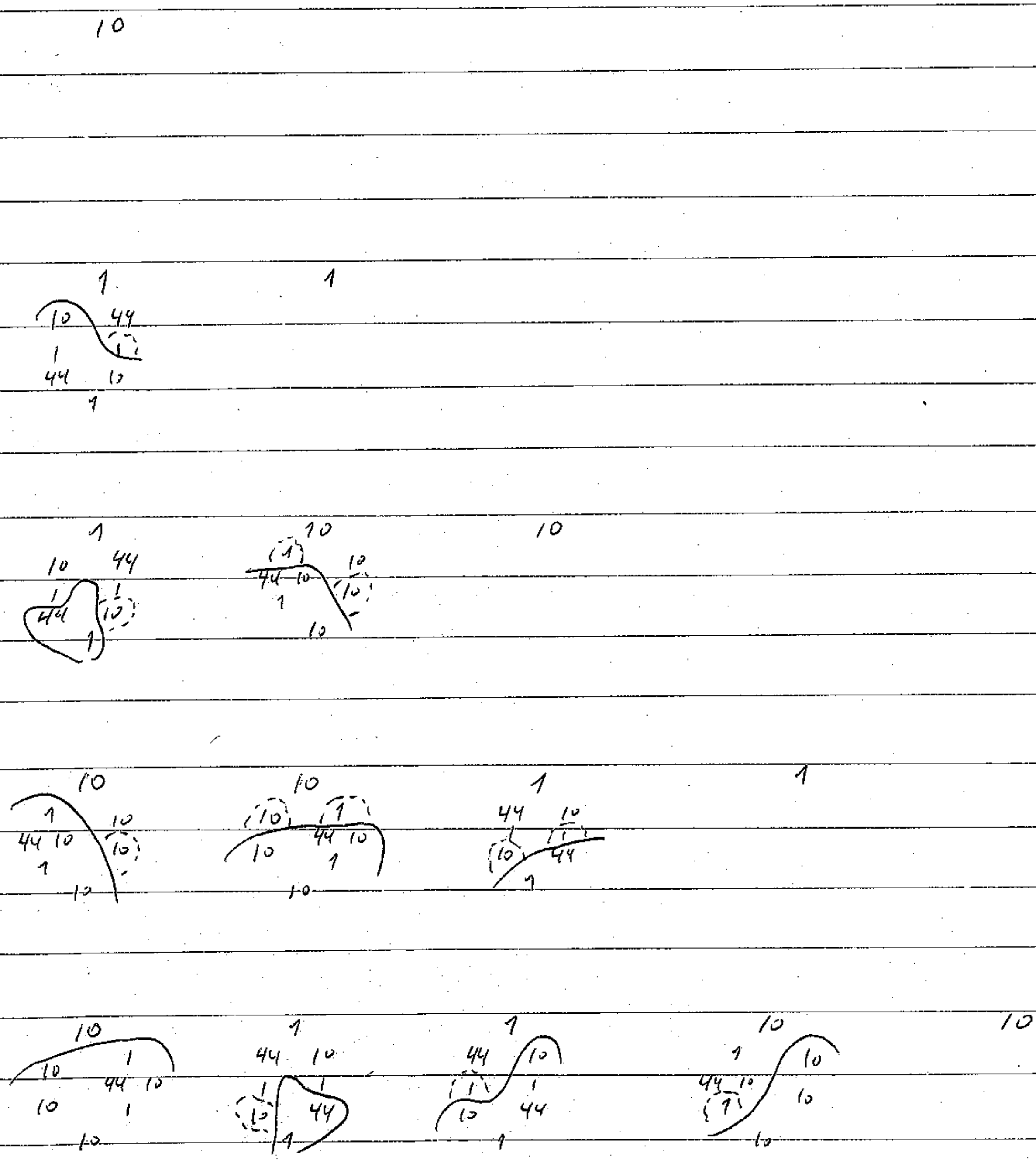




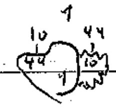




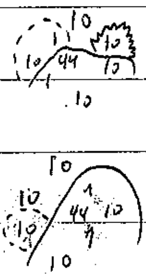
V_{10} resolution :



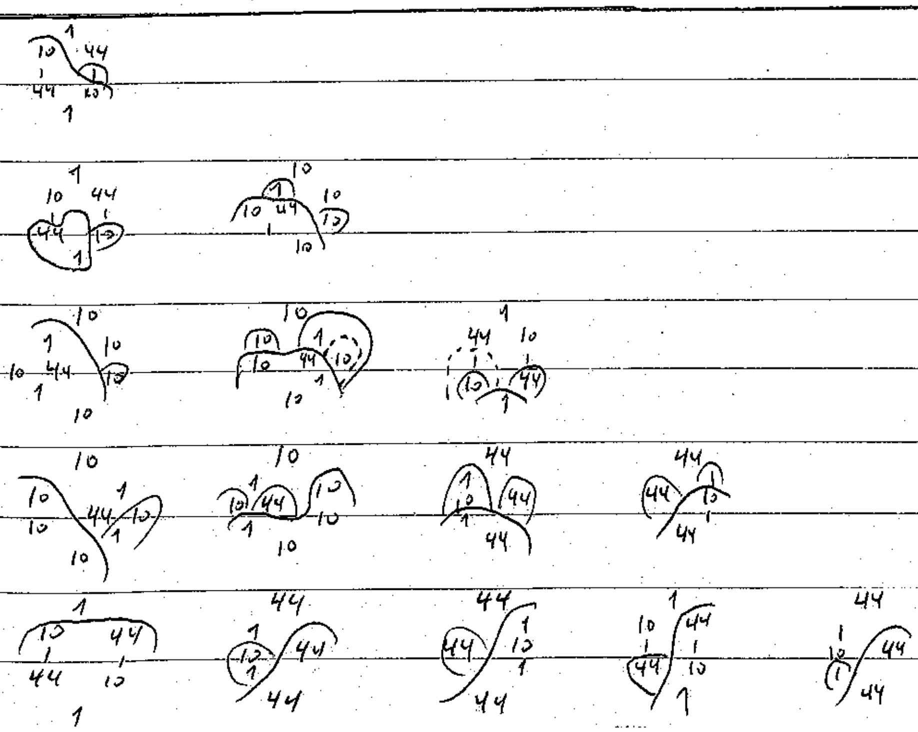
The resolution for V_1 seems hard and seems to need three dimensions to describe. Here's a try:



$z=1$



$z=0$



<u>Cyclic periodic modules</u>

Often is seems that there are maps $P \xrightarrow{\varphi} Q$ of indecomposable projectives with $P\varphi \subseteq \operatorname{Rad} Q$, $P\varphi \not\subseteq \operatorname{Rad}^2 Q$ and $P\varphi$ periodic. The next result was motivated by this, in the case of a p-group and a generator of it.

<u>Theorem</u> If F is a field, G is a finite group and $g \in G$ then

$$(1-g)FG$$

is a periodic module

<u>Proof</u>. Let $FG/\langle g\rangle$ be the permutation module so it suffices to show there is an exact sequence

$$0 \to (1-g)FG \to FG \to FG/\langle g\rangle \to 0$$

as the right hand term is periodic, being induced from a cyclic subgroup.

There is a homomorphism

$$FG \to FG\langle g\rangle \to 0$$

which sends each $x \in G$ to $\langle g\rangle x$. Clearly $(1-g)FG$ is contained in the kernel.

But suppose that $\sum \alpha_x x$ is in the kernel so $\sum_x \alpha_x \langle g\rangle x = 0$. Hence, if $C = \langle g\rangle t$ then $\sum_{y\in C} \alpha_y = 0$. If g has order n then $\sum_{i=0}^{n-1} \alpha_{g^i t} = 0$. Thus $\sum_{i=0}^{n-1} \alpha_{g^i t}\, g^i$ has augmentation zero in $F\langle g\rangle$ so it lies in $(1-g)F\langle g\rangle$. Hence $\sum_{i=0}^{n-1} \alpha_{g^i t}\, g^i t$ lies in $(1-g).F\langle g\rangle.\, t \subseteq (1-g)FG$. That is, $\sum_{y\in C} \alpha_y\, y \in (1-g)FG$ for each coset C and the result is proved.

The period is now easily seen to be 1 or 2. Indeed,

<u>LEMMA.</u> There is an exact sequence
$$0 \to F\,G/\langle g\rangle \to FG \to (1-g)FG \to 0.$$

<u>Proof.</u> map $\langle g\rangle t$, a coset of $\langle g\rangle$ in G to the sum of its elements in FG, so if g has order n then
$$\langle g\rangle t \to (1+g+\dots+g^{n-1})\,t.$$
map $FG \to (1-g)FG$ by left multiplication by $(1-g)$.
Clearly this is onto while the composition of the two maps is zero.
The exact sequence of the previous argument shows the dimensions are such that the sequence of this lemma must be exact.

In summary, we have an exact sequence:
$$0 \to (1-g)FG \to FG \to FG \to (1-g)FG \to 0.$$
The maps in order: 1, left mult by $(1+g+\dots+g^{n-1})$, left mult by $1-g$.

Modules for M_{11}: a fundamental correction and other material

C. Ringel has pointed out that our structure for projectives leads to a "wild" case for modules, which is very surprising. We have found an error. What is true is

$$\mathrm{Ext}^2_{FM_{10}}(W_1, W_1) \simeq F$$

not 0, as is used in Lemma 6, or $F \oplus F$ as in Lemma 8. The reason is the extension trap for A_6. We have $1 \to A_6 \to M_{10} \to Z_2 \to 1$ and $1 \to Z_2 \to \hat{A}_6 \to A_6 \to 1$ but these cannot be put together. Remember! The use of this in Lemma 6 makes no difference. However, the proof of Lemma 8 collapses. Lemmas 9 and 10 are still O.K. Lemma 8 should now read:

LEMMA 8. Either there is a uniserial $\begin{smallmatrix} V_{10} \\ V_1 \\ V_{10} \end{smallmatrix}$ or we have the structure in 2) below.

Now before Lemmas 11, 12 and 13 we should have: Assume we're in the case that $\begin{smallmatrix} V_{10} \\ V_1 \\ V_{10} \end{smallmatrix}$ exists. Lemmas 14-21 are still O.K. The reference in the proof of Lemma 14, at the end, to P_1, P_{44} is still O.K. just by what we still have at this point about P_1, P_{44}, quite easily.

We shall now show, first, that $\begin{smallmatrix} V_{10} \\ V_1 \\ V_{10} \end{smallmatrix}$ does not exist and then shall study the remaining case which is what really occurs. We continue numbering the lemmas consecutively, though this should follow Lemma 13.

LEMMA 22. The module V_{10} is periodic

Before proving this let's draw the desired conclusion:

LEMMA 23. There does not exist a uniserial module $\begin{smallmatrix}V_{10}\\V_1\\V_{10}\end{smallmatrix}$.

Proof. If there were then we have deduced the structures of P_1, P_{10}, P_{44} and we have above calculated the minimal resolution of V_{10}, contradicting Lemma 22.

Proof (of Lemma 22). It suffices to show that the restriction $V_{10}|_S$, of V_{10} to a Sylow 2-subgroup S, is periodic. Hence, we examine the action of S on the eleven points. Now $|M_{10}:M_{10}'|=2$ and $M_{10}' \simeq PSL(2,9)$ so $M_9 \cap M_{10}' \simeq Z_3 \times Z_3 . Z_4$ the Borel subgroup. But $|M_9| = 2^3.3^2$, there are no involutions in $M_{10}-M_{10}'$ so $M_9 = Z_3 \times Z_3 . Q$, Q a quaternion group of order eight. $\therefore M_8 = Q$ which acts faithfully and so regularly. Let $S = QT$, $|T|=2$. Hence $V_1 \oplus V_{10}|_S \simeq F \oplus F_T^S \oplus F_Q^S$ so $V_{10}|_S$ is certainly periodic inasmuch as F_T and F_Q certainly are.

Hence we have: $0 \to X \to P_{10} \to \begin{smallmatrix}V_{10}\\V_1\\V_{44}\end{smallmatrix} \to 0$ and $X/\mathrm{rad}\,X \simeq V_{10}$ with $\begin{smallmatrix}V_1\\V_{44}\end{smallmatrix}$ splitting over that V_{10}, i.e. over $X/\mathrm{rad}\,X$. That is $\mathrm{rad}\,P_{10}/\mathrm{rad}\,X$ splits over $X/\mathrm{rad}\,X$.

<u>LEMMA 24.</u> The projective module P_{10} has the following structure:

$$\begin{tikzcd}[column sep=small, row sep=small]
& 10 \arrow[dl, dash] \arrow[dr, dash] & \\
1 \arrow[d, dash] \arrow[ddrr, dash] & & 10 \arrow[d, dash] \arrow[ddll, dash] \\
44 \arrow[d, dash] & & 10 \arrow[d, dash] \\
1 \arrow[dr, dash] & & 10 \arrow[dl, dash] \\
& 10 &
\end{tikzcd}$$

<u>Proof.</u> We have $0 \to \begin{smallmatrix} 44 \\ 1 \\ 10 \end{smallmatrix} \to P_{10} \to P_{10} \to \begin{smallmatrix} 10 \\ 1 \\ 44 \end{smallmatrix} \to 0$ exact, so the submodule X has "top" V_{10} and socle V_{10}. Since $\exists$ $\begin{smallmatrix} 10 \\ 10 \\ 10 \end{smallmatrix}$ uniserial we must have this at the top of X as $0 \to \begin{smallmatrix} 44 \\ 1 \end{smallmatrix} \to P_{10} \to X \to 0$. Also $\begin{smallmatrix} 10 \\ 1 \end{smallmatrix}$ is at the top of P_{10} and so of X so get structure of X:

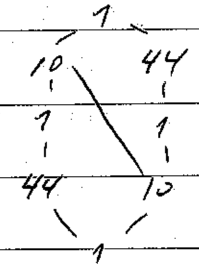
$$X: \quad \begin{tikzcd}[column sep=small, row sep=small]
& 10 \arrow[ddl, dash] \arrow[dr, dash] & \\
& & 10 \arrow[d, dash] \\
1 \arrow[dr, dash] & & 10 \arrow[dl, dash] \\
& 10 &
\end{tikzcd}$$

Now the extension $0 \to X \to Y \to 44 \to 0$ contained in P_{10} does not split so since $Ext(V_{44}, V_{10}) = 0$ get the structure of Y:

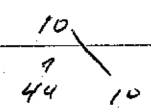
$$Y: \quad \begin{tikzcd}[column sep=small, row sep=small]
& & 10 \arrow[d, dash] \arrow[ddll, dash] \\
44 \arrow[d, dash] & & 10 \arrow[d, dash] \\
1 \arrow[dr, dash] & & 10 \arrow[dl, dash] \\
& 10 &
\end{tikzcd}$$

Hence, the middle M of P_{10} is an extension of $\begin{tikzcd}[column sep=tiny, row sep=tiny] 44 \arrow[d, dash] & 10 \arrow[d, dash] \arrow[dl, dash] \\ 1 & 10 \arrow[d, dash] \\ & 10 \end{tikzcd}$ by 1. But here, by duality, get structure of M/V_1 and so get easily $\begin{tikzcd}[column sep=tiny, row sep=tiny] 44 \arrow[d, dash] \arrow[dr, dash] & \\ 1 & 10 \end{tikzcd} \subseteq M$. Also $\begin{tikzcd}[column sep=tiny, row sep=tiny] & 10 \arrow[d, dash] \arrow[ddl, dash] \\ & 10 \arrow[d, dash] \\ 1 & 10 \end{tikzcd} \subseteq M$ so get complete answer.

<u>LEMMA 25.</u> The projective module P_1 has the following structure:



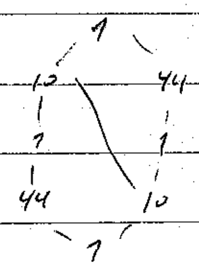
<u>Proof.</u> We have, from P_{44}, that $\begin{smallmatrix}44\\1\\10\end{smallmatrix} \subseteq M_1$, the middle of P_1. Also, factoring out $\begin{smallmatrix}10\\10\\10\end{smallmatrix}$ in P_{10} we get that



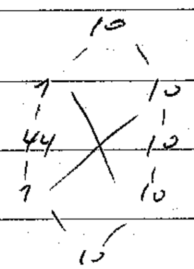
$\subseteq M_1$ so the proof is complete.

<u>Summary</u>

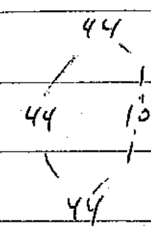
P_1 :



P_{10} :



P_{44} :



This is the answer Ringel suggested.

Here's a better proof of Lemma 10 shown to us by Ringel, which follows lines we have used. We have $\begin{smallmatrix}1\\10\\1\\44\end{smallmatrix}$ from P_{44} so let X be a submodule of P_1 which has only V_{10} at its top and such that we have a commutative diagram

$$\begin{array}{ccc} P_1 & \longrightarrow & \begin{smallmatrix}1\\10\\1\\44\end{smallmatrix} \\ \cup\vert & & \cup\vert \\ X & \longrightarrow & \begin{smallmatrix}10\\1\\44\end{smallmatrix} \end{array}$$

Since P_1 has socle V_1 and $X \subseteq P_1$ it follows that X has V_1 at least twice as a composition factor and since $P_{10} \to X \to 0$, say with kernel Y, it has V_1 exactly twice. Similarly V_{44} exactly once. And from P_1 it has V_{10} once or twice.

But $Y \subseteq P_{10}$ and it has the composition factors "left over" by X so it has only V_{10} as composition factor with multiplicity three or four. But $\mathrm{Ext}^1(V_{10}, V_{10}) \cong F$ so the uniserial module $\begin{smallmatrix}V_{10}\\V_{10}\\V_{10}\end{smallmatrix}$ exists, as desired.

It may be possible now to pursue this line and get at the structure of P_{10} and then P_1 more easily than the route we have used above.

LEMMA 26. The module V_{10} has vertex the quaternion subgroup Q, source F and Green correspondent the irreducible two-dimensional module for $FN(Q)$.

Proof. We keep the notation of the proof of Lemma 22, so $S = TQ$. Then we saw that

$$V_{10}|_S \cong F_T^{\,S} \oplus F_Q^{\,S}.$$

Since there is a summand of $V_{10}|_S$ which has the same vertex as V_{10}, we need only eliminate the possibility of vertex T and source F to get the first two parts of the lemma.

So assume V_{10} has vertex T. The two projective modules of $N(T)/T$ are the $FN(T)$-modules with vertex T and these have dimension eight each. Thus, $V_{10}|_{N(T)}$ is the direct sum of an eight-dimensional module with vertex T and a two-dimensional module with vertex contained in T, clearly impossible.

Next, $N(Q)/Q \cong \Sigma_3$ so the two projective $F\Sigma_3$-modules are the only candidates for the Green correspondent of V_{10}. But how does Q act on the eleven letters? Regularly plus three fixed points. And the Z_3 in $N(Q)$ must permute these three transitively as $M_8 = Q$. Thus V_{10} restricted to Q times the Z_3 has an eight-dimensional summand with the Q acting regularly and a two-dimensional summand where the Z_3 acts. The lemma is now proved, by inspection of the projective $F\Sigma_3$-modules

LEMMA 27. The vertex of V_{44} is a fours group E and its source is F.

Proof. Let Π be the set of eleven letters on which M_{11} acts. Thus $F\Pi \simeq V_1 \oplus V_{10}$. The action of M_{11} on two-element subsets of Π gives a module isomorphic with $F\Pi \wedge F\Pi$, which has F as a summand, and the other composition factors being V_{10} and V_{44}. Hence $F\Pi \wedge F\Pi \simeq V_1 \oplus V_{10} \oplus V_{44}$.
Hence $V_{10} \wedge V_{10} \simeq V_1 \oplus V_{44}$.

But $F\Pi|_S = F \oplus F_T^S \oplus F_Q^S$ so

$$V_{10} \wedge V_{10}|_S \simeq (F_Q^S \wedge F_Q^S) \oplus (F_T^S \wedge F_T^S) \oplus (F_T^S \otimes F_Q^S).$$

This gives

$$V_{44}|_S \simeq (F_T^S \wedge F_T^S) \oplus (F_T^S \otimes F_Q^S)$$

as $F_Q^S \wedge F_Q^S \simeq F$ as $\dim_F F_Q^S = 2$. But $F_T^S \otimes F_Q^S$ is free, by Mackey's theorem — or, by a dimension count and an easy "Hom" calculation to determine the socle. But $F_T^S \wedge F_T^S$ is isomorphic with the permutation module of S on two-element subsets of the space S/T. Let $\{\alpha, \beta\}$ be such a set. Hence, as the stabilizer of $\alpha \in S/T$ has order two, the stabilizer of $\{\alpha,\beta\}$ has order 1, 2 or 4. But V_{44} is not a periodic module so one of these stabilizers cannot be cyclic, since $V_{44}|_S$ has a summand with the same vertex. This proves the result.

It remains to get the actual Green correspondent of V_{44}.

LEMMA 28. The Green correspondent of V_{44} is the two-dimensional irreducible $F[N(E)]$-module.

Proof. Let U_1, U_2 be the irreducible $F[N(E)]$-modules of dimensions one and two. Let Q_1 and $Q_2 = U_2$ be the corresponding projective covers as $F[N(E)/E]$-modules. Then since V_{44} has vertex E and source F exactly one of Q_1 and Q_2 is a summand of $V_{44}|_{N(E)}$ and this is its Green correspondent.

But $V_1 \oplus V_{44}$ is a permutation module, with $Z_3 \times Z_3 . S$ as stabilizer (since $M_9 = Z_3 \times Z_3 . Q$) so $V_1 \oplus V_{44}|_{N(E)}$ is a sum of permutation modules with stabilizers having a normal Sylow 3-subgroup. Hence, it suffices to show that if $H \leq N(E)$ and H has a normal Sylow 3-subgroup then $F_H^{N(E)}$ has neither Q_1 nor Q_2 as summands or has Q_2 and not Q_1 or has both Q_1 and Q_2 (for the latter case is then out by the first paragraph).

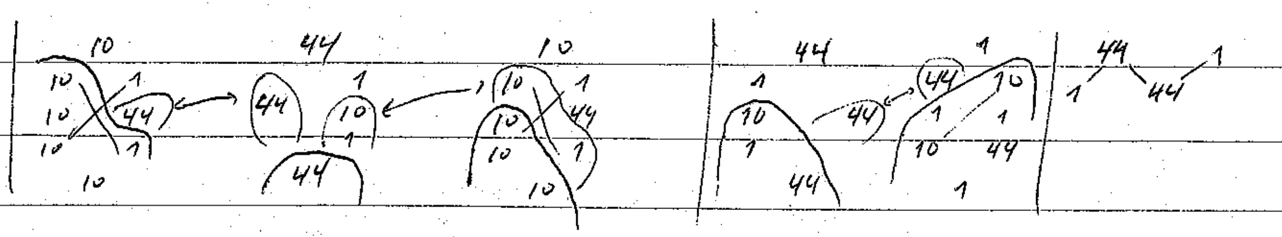
But Q_1 and Q_2 have vertex E so we need only look at subgroups $H \geq E$, since E is normal in $N(E)$. This leaves $H = E$ and $H = D$ to consider, the latter being the dihedral Sylow 2-subgroup of $N(E)$. But $F_E^{N(E)} \cong Q_1 \oplus Q_2 \oplus Q_2$ and $F_D^{N(E)} \cong U_1 \oplus Q_2$, so the proof is complete.

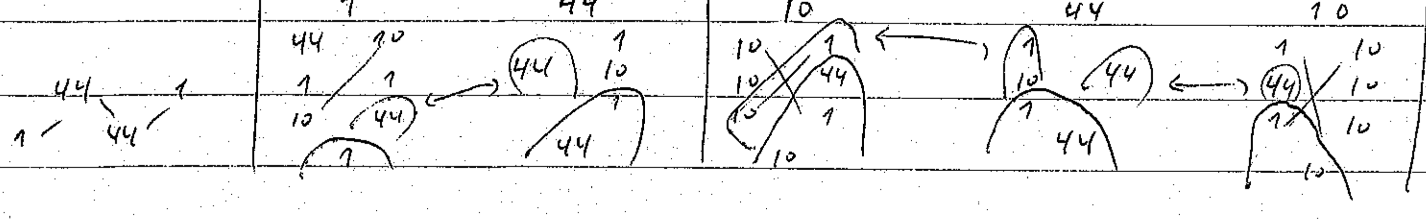
Now we continue to investigate the Green correspondence for M_{11}.

<u>LEMMA 29.</u> The module $\begin{smallmatrix} & 44 & & 1 \\ 1 & & 44 & \end{smallmatrix}$ has vertex Q and source F.

Proof. Since this module is a summand of $V_{10} \otimes V_{10}$ it has vertex contained in Q. Hence, to get the vertex we need only show that it can be no smaller. The statement on the source will then follow, since $F_Q^G = V_{10} \oplus \cdots$ implies $F_Q^G \otimes F_Q^G = (V_{10} \otimes V_{10}) \oplus \cdots$ so the module in question is a summand of a permutation module on the cosets in M_{11} of a subgroup of Q.

But if the vertex is $Z(Q)$ then since the trivial $FZ(Q)$ module has period one and since there are two modules with that vertex, our module – call it M now – can have period at most two. Similarly if the vertex of M were a $Z_4 \subseteq Q$ then since $N(Z_4)$ is a Sylow 2-subgroup, and inducing indecomposable to $N(Z_4)$ gets indecomposables, get again that M has period at most two. Hence it suffices to show the period is four by a direct calculation:





As an application, we can now completely decompose $F_Q^{M_{11}}$.

LEMMA 30. The following are the indecomposable summands — and their multiplicities — of $F_Q^{M_{11}}$:

V_{10}	2
$\begin{smallmatrix} & 44 & & 1 \\ 1 & & 44 & \end{smallmatrix}$	1
P_{10}	1
P_{44}	5
P_{16}	2
P_{16}'	2

Proof. The last two are easy: $\mathrm{Hom}_{FM_{11}}(F_Q^{M_{11}}, V_{16}) = \mathrm{Hom}_{FQ}(F, V_{16}|_Q)$ which is two-dimensional since V_{16} is free as a Q-module. Also $F_Q^{N(Q)}$ contains as a summand the Green correspondent of V_{10} twice so the first two multiplicities are at least as large as stated. But the dimensions of all the thirteen modules above add up to $\dim_F(F_Q^{M_{11}})$; hence, it suffices to show that $\dim_F \mathrm{Hom}_{FM_{11}}(F_Q^{M_{11}}, V_{10}) = 3$ and $\dim_F \mathrm{Hom}_F(F_Q^{M_{11}}, V_{44}) = 6$. That is, $\dim_F \mathrm{Hom}_{FQ}(F, V_{10}) = 3$ — which is true, as we know — and $\dim_F \mathrm{Hom}_{FQ}(F, V_{44}) = 6$. That is, $\dim_F \mathrm{Hom}_{FQ}(F, V_{10} \wedge V_{10}) = 7$ as $V_{10} \wedge V_{10} = V_1 \oplus V_{44}$. But $V_{10}|_Q = F \oplus F \oplus FQ$ so it suffices to see that $\dim_F \mathrm{Hom}_{FQ}(F, FQ \wedge FQ) = 4$. But it is easy to see that if Q acts by right multiplication on its two-element subsets then there are exactly four orbits. The proof is complete.

Trivial source ring

Let G be a group and F a field of characteristic p. Let R be the subgroup of the Green ring spanned by the isomorphism classes of indecomposable FG-modules which have trivial sources.

LEMMA. An indecomposable FG-module U has a trivial source if, and only if it is a summand of a permutation module.

Proof. One half of this is trivial. On the other hand, say $U \mid F_H{}^G$, with the obvious notation, and let the p-group Q be the vertex of U. Then $F_H{}^G{}_Q$, by Mackey's theorem, is a direct sum of permutation modules for Q, each absolutely indecomposable since each has an irreducible socle. Hence one of these is the source of U inasmuch as $U_Q \mid F_H{}^G{}_Q$. But this cannot be $F.Q/R$ for any proper subgroup of Q! The proof is complete.

PROPOSITION. R is a ring.

Pf. This is immediate from the lemma, applying Mackey's tensor product theorem to two permutation modules.

THEOREM.(*) R is semi-simple.

Pf. We mimic straight-forwardly Green's work in his paper on transfer in volume 1 of the J. of Algebra, 1964. Because of Green's ideas and the lemma we need only prove the following: Let Q be a p-subgroup of G, let $L = N(Q)$; let W be the ring of modules with trivial source contained in Q and W' the ideal of W where the source is properly contained in Q; it suffices to show that W/W' is semi-simple.

Hence, it suffices to show that $W/W' \simeq K^0(F[L/Q])$ since this is isomorphic with a ring of functions with values in the complex numbers. But W splits over W', the complement being the ring of projective $F[L/Q]$-modules, since $F_Q^{\;L}$ is the direct sum of these & since these all have source F and vertex Q. For $F_Q^{\;L}$ is just $|L:Q|\,F$.

COROLLARY. The ring of inductively generated indecomposable FM_{11}-modules is semi-simple.

Immediate, where F and M_{11} are now as in those sections dealing with M_{11}.

(*) Added later: Here's a reference. Andreas Dress, Modules ... Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 44 (1975) 101–109. See Th 3.

Uniserial modules for $Z_p \times Z_p$.

Let $P = \langle x, y \rangle \simeq Z_p \times Z_p$ and let F be a field of characteristic p. Let V be an n-dimensional uniserial FP-module with submodules

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = 0.$$

We shall classify all modules. In particular $n \leq p$. Also the method clearly generalizes to $Z_p \times \cdots \times Z_p$.

LEMMA. If $g \in P$ then V is cyclic as $F\langle g \rangle$-module if, and only if g acts non-trivially on V_0/V_2.

Proof. If V is cyclic for g then the $F\langle g \rangle$-submodules of V must be the FP-submodules. Hence, g is not trivial on V/V_2.

Conversely, suppose g is not trivial on V/V_2 so $V(g-1) + V_2 = V_1$. It suffices to prove, by induction, that g is not trivial on V_i/V_{i+2}, i.e. $V_i(g-1) + V_{i+2} = V_{i+1}$. But suppose it holds for V_i/V_{i+2} and let $\overline{V_i} = V_i/V_{i+3}$. If g is trivial on $V_{i+1}/V_{i+3} = \overline{V_{i+1}}$, then $\overline{V_i}(g-1)$ and $\overline{V_{i+2}}$ are FP-modules which form a direct sum, a contradiction.

COROLLARY. The module V is cyclic for all but at most one of the cyclic subgroups of P of order p. Hence, $n \leq p$.

Pf. For at most one can act trivially on V/V_2.

Now suppose V is cyclic as $F\langle x\rangle$-module, so with respect to a suitable basis x is represented by

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and so y, as $xy = yx$ is represented by

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ & 1 & \alpha_1 & \alpha_2 & & \alpha_{n-2} \\ & & 1 & \alpha_1 & & \\ & & & & \ddots & \end{pmatrix}$$

These representations all make sense and no two are equivalent, giving us a set of isomorphism classes parametrized by F^{n-1} (the affine space).

If V is not cyclic as $F\langle x\rangle$-module then it must be as $F\langle y\rangle$-module. Here we get canonical forms:

$$y:\ \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$x:\ \begin{pmatrix} 1 & 0 & \beta_2' & \beta_3 & \beta_4 & \cdots & \beta_{n-2} & \beta_{n-1} \\ & 1 & 0 & \beta_2 & \beta_3 & & & \beta_{n-2} \\ & & 1 & 0 & \beta_2 & & & \\ & & & 1 & 0 & & & \\ & & & & & \ddots & & \end{pmatrix}$$

Here the parameter space is the affine space F^{n-2}.

On A_4 as a subgroup

Suppose $A_4 \subset G$ being the normalizer of a Sylow 2-subgroup which is the centralizer of each of its non-identity elements. Let F be an algebraically closed field of characteristic two.

Prop. There are at least three isomorphism classes of simple FG modules in the principal 2-block.

Pf Since G is not 2-nilpotent there is more than just F so we can assume there are just two: F, V. Let $1, \omega, \bar{\omega}$ be the linear characters of A_4 over F identified with the corresponding modules. We have a table of Green correspondents:

$$1 \longleftrightarrow F$$
$$\omega \longleftrightarrow X$$
$$\bar{\omega} \longleftrightarrow X^*.$$

The projective cover of 1 looks as follows: $\begin{array}{ccc} & 1 & \\ \omega & & \bar{\omega} \\ & 1 & \end{array}$.

Inducing we get

$$\begin{array}{ccc} & F & \\ X & & X^* \\ & F & \end{array} \quad + \ldots$$

the dots being other projectives. The first summand is P_F, the cover of F. For its composition factors are in the principal 2-block so it suffices to see that P_V – the cover of V – is not a summand (For P_F is a summand at most once: $\mathrm{Hom}_{FG}\left(F, \left(\omega \, {}_1^1 \, \bar{\omega}\right)^G\right) \simeq \mathrm{Hom}_{FA_4}\left(1, \omega \, {}_1^1 \, \bar{\omega}\right) \simeq F$)

But $P_V \leq \begin{array}{ccc} & F & \\ X & & X^* \\ & F & \end{array} \Rightarrow P_V \leq X \oplus X^*$ as $P_V / \mathrm{rad}\, P_V \simeq \mathrm{soc}\, P_V \simeq V$.

But X and X^* have no projective summands.

Now $\overset{\omega}{\underset{\omega}{\bar{\omega}}}1$ induces to $V^* \overset{V}{\underset{V}{\ }} F + \cdots$ (projectives for dots). Also $Hom_{FG}(F, V^* \overset{V}{\underset{V}{\ }} F) \subseteq Hom_{FG}(F, V^* \overset{V}{\underset{V}{\ }} F + \cdots)$
$\cong Hom_{FA_4}(1, \overset{\omega}{\underset{\omega}{\bar{\omega}}}1) = 0$ so we deduce that $P_F \nmid V^* \overset{V}{\underset{V}{\ }} F$ so

$$m P_V \cong V^* \overset{V}{\underset{V}{\ }} F$$

Hence if X has F as composition factor with multiplicity a and V multiplicity b then the Cartan matrix

$$C = \begin{array}{cc} & \begin{array}{cc} F & V \end{array} \\ \begin{array}{c} P_F \\ P_V \end{array} & \begin{pmatrix} 2a+2 & 2b \\ \frac{3a+1}{m} & \frac{3b}{m} \end{pmatrix} \end{array}$$

so taking determinants, $4 = \frac{4b}{m}$ and $b = m$. But C is symmetric so $\frac{3a+1}{m} = 2b$ and $3a+1 = 2b^2$ and $3a = 2b^2 - 1$.
But $2b^2 - 1 \not\equiv 0 \ (3)$, by testing the residue classes. A contradiction.

Character correspondences

One often studies the relationships between character values for a group and a subgroup; often deep facts about modules underlie the surface. We investigate a situation we do not understand. This is motivated by a paper of Kawanaka on representations of the unitary groups. These ideas go back further – see Kawanaka's paper.

We fix some notation. Let σ be an automorphism of order n of a group G, let U be the subgroup of fixed points, let $E = \langle\sigma\rangle G$ be the semi-direct product. We now let

$ccl_E(\sigma G)$ = the set of E conjugacy classes in σG

ccl_G^E = the set of G classes normal in E, i.e. σ-invariant

ccl_U = the U classes.

We shall introduce some maps.

Lemma If $e \in \sigma G$ then e^n lies in an element of ccl_G^E which depends only on the E-class of e.

Proof. Since σ permutes the G-classes, i.e. ccl_G, and $e = \sigma g$ for some $g \in G$ it follows that $(e^n)^G$ is σ-invariant. Also $(e^{g\sigma^i})^n = (e^n)^{g\sigma^i}$ so the rest holds too.

Hence, we have a function $L : ccl_E(\sigma G) \to ccl_G^E$. Also any element of U is σ-invariant so u^G is σ-invariant since u is in u^G, is σ-invariant and σ permutes the G-classes.

Hence, we have a map $R : ccl_U \to ccl_G^E$. The next result is clear.

<u>Lemma</u> R is one-to-one if, and only if, two elements of U which are G-conjugate are U-conjugate. R is onto provided every σ-invariant G-class intersects U.

Now we want to consider a very special situation:

<u>Special situation</u>

1) R is one-to-one and onto;

2) L is one-to-one and onto;

3) If $x \in \sigma G$, $u \in U$ and $L(x^E) = R(u^U)$ then
$$|C_G(x)| = |C_U(u)|.$$

<u>Remarks</u>: 1. The last means, since $C_E(x) = \langle x, C_G(x) \rangle$, $|C_E(x)| = n\,|C_U(u)|$.

2. It might be enough to look at the case where all we have is that L and R are one-to-one and have the same images and delete the third hypothesis entirely, in that this might be enough for the conjecture to be true.

3. Here are some examples of the special situation:

a) $G = GL(d, q^2)$, $n = 2$, σ the unitary automorphism, $U = GU(n, q)$ — see Kawanaka.

b) $n = p$, G of order prime to p, a prime. This is the Glauberman situation in his ~~auto~~ character correspondence.

c) Let $G = Z_p \times \cdots \times Z_p$, p times for a prime p and let σ be the "wreathing" automorphism of order p. This is a special situation by inspection.

<u>Conjecture</u> If φ is an irreducible character of U then there is a sign ϵ_φ and an irreducible character χ_φ of E such that whenever $e \in \sigma G$ and $u \in U$ lie in corresponding classes then

$$\chi_\varphi(e) = \epsilon_\varphi \, \varphi(u).$$

<u>Remarks</u> 1. This is true in example a) if q is odd as Kawanaka shows, by a <u>very</u> indirect and hard proof!

2. This is true in c) by inspection.

3. This is true in b) by Glauberman – or Brauer's deep theorems, as is easily seen. Though one needs to apply the results carefully. For the correspondence there is between values on u and on σu – while $(\sigma u)^p = u^p$. To get the right form – for us – after using a field automorphism.

4. Let's look at the unitary problem in characteristic two. Consider $G = A_5 \simeq SL(2, 2^2)$, $U = \Sigma_3 \simeq SU(2, 2^2)$, $\sigma = (12)$, $E = \Sigma_5$. Then the classes of U have representatives 1, $(12)(34)$, (345), and $ccl_E(\sigma G)$ has representatives (12), (1234), $(12)(345)$ which correspond in that order. Let's look at the character tables:

U:

1	(12)(34)	(345)
1	1	1
1	−1	1
2	0	−1

Σ_5

1	(12)	(1234)	(12)(345)	....
1	1	1	1	
1				
4	2	0	−1	
4				
5	+1	−1	1	
5				

5. The conjecture implies a lot more about the characters, as the above example suggests. Let φ, ϵ_φ, χ_φ be as in the conjecture so χ_φ does not vanish on σG — $\chi_\varphi(\sigma) = \epsilon_\varphi \varphi(1)$ after all. If λ is a linear character of E/G and non-principal then $\lambda(\sigma) \neq 1$ so $\lambda \chi_\varphi \neq \chi_\varphi$. Hypothesis three of the special situation and orthogonality relations imply then that the $\lambda \chi_\varphi$ as φ varies & λ varies, give all the characters of E not vanishing on σG. Conversely, if we suitably strengthened the conjecture then hypothesis 3) would follow from it.

We turn now to some "homological" descriptions of the situation (Along the lines of a paper of Glauberman – Math. Z.)

Prop R is one-to-one if, and only if, for each $u \in U$, $H^1(\langle\sigma\rangle, C_G(u)) = 0$.

Pf. First, suppose that R is one-to-one. Let $\langle\tau\rangle$ and $\langle\sigma\rangle$ be complements to $C_G(u)$ in $\langle\sigma\rangle C_G(u)$. Hence, it suffices to show that $\langle\sigma\rangle, \langle\tau\rangle$ are conjugate by an element of $C_G(u)$. But $\sigma^n = \tau^n = 1$ and can assume, without loss, that $\sigma \in \sigma G$, so by hypothesis there is $g \in G$ with $\tau^g = \sigma$. $\therefore$ u^g, u centralize σ so since R is one-to-one, $g = cv$ $c \in C(u)$, $v \in U$. $\therefore \tau^{cv} = \sigma$ $\therefore \tau^c = \sigma$ and half the proof is complete.

Now assume the cohomology set vanishes. Let $u, v \in U$, $u^g = v$. $\therefore$ have to show that u, v are U-conjugate. But $\langle\sigma\rangle$ and $\langle\sigma^g\rangle$ are complements to $C_G(v)$ in $\langle\sigma\rangle C_G(v)$, for $\sigma^g \in C_{\langle\sigma\rangle G}(u^g)$ $= C_{\langle\sigma\rangle G}(v) = \langle\sigma\rangle C_G(v)$. $\therefore \sigma^{gc} = \sigma$ when $c \in C_G(v)$. $\therefore u^{gc} = v^c = v$ so done.

Prop R is one-to-one and onto if, and only if, for every g in G either $|C_E(g) : C_G(g)| < n$ or the extension $C_E(g)/C_G(g)$ splits and all complements are conjugate.

Pf If these conditions hold then R is one-to-one, by the previous result. To see that it is onto let g lie in a class of cl_G^E. Thus $C_E(g)$ covers E/G, ~~as~~ g^G is a class of E. Hence by assumption there is τ in E of order n, $\tau \in \sigma G$ so that $C_E(g) = \langle\tau\rangle C_G(g)$ a semi-direct product. Hence, $\langle\tau\rangle$ and $\langle\sigma\rangle$ are conjugate by our assumptions applied to the identity element, so τ is conjugate to σ by an element of G and so g is conjugate in G to an element centralizing σ. This proves half the proposition.

Now suppose that R is a bijection. Suppose that $g \in G$ and $|C_E(g) : C_G(g)| = n$. It suffices to establish the results on the extension. Since $C_E(g)$ covers E/G it follows that g is an element of a class in cl_G^E. Hence, since R is onto, can assume $g = u \in \mathcal{U}$. $\therefore \langle\sigma\rangle$ is the desired complement and the conjugacy holds by the previous result,

<u>Prop</u> The image of L contains the image of R if, and only if, every element of U is a σ-norm.

That is, an "H^0" result, every fixed point is a norm. A σ-norm is an element of the form $g^{\sigma^{n-1}} g^{\sigma^{n-2}} \cdots g^{\sigma} g$ for some g in G.

<u>Pf.</u> Suppose that the image of L contains the image of R. Then if u is in $\mathcal{U}$ there is an element of u^G which is the n-th power of an element of σG. Hence, by conjugation so is u,

$$\begin{aligned} u &= (\sigma g)^n \\ &= \sigma^n g^{\sigma^{n-1}} g^{\sigma^{n-2}} \cdots g^{\sigma} g \end{aligned}$$

as desired, since $\sigma^n = 1$.

Next, suppose that every element u of U is a σ-norm.

Let $g \in G$ be in a class which is in the image of R. To see that it's in a class which is in the image of L let g be conjugate to u in U; that's possible by assumption. Then

$$u = x^{\sigma^{n-1}} \cdots x$$

for some $x \in G$ so $(\sigma x)^n = u$ and the proof is complete.

Prop If the image of L contains the image of R then L is one-to-one if, and only if for any u in U and $x, y \in \sigma G \cap C_E(u)$ with $x^n = y^n = u$ we have x, y conjugate in $C_E(u)$.

Seems to be about a map $H^1(\langle\sigma\rangle, C_G(u)/\langle u\rangle) \to H^2(\langle\sigma\rangle, \langle u\rangle)$.

Pf. L one-to-one implies there is $g \in G \ni x^g = y$ so $(x^n)^g = y^n$ so $g \in C_G(u)$. Conversely, suppose $x, y \in \sigma G$ and x^n and y^n are conjugate in G. $\therefore$ by assumption, can assume $x^n = y^n = u \in U$ so the proof is complete.

Projective extension generators

Let F be an algebraically closed field of prime characteristic p and let G be a finite group.

Def. The FG-modules $U_1, U_2, \ldots, U_d$ are projective extension generators – or generators – provided that for any FG-module U there is a projective FG-module P and a series of submodules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = P \oplus U$$

such that every V_i / V_{i-1} is isomorphic with one of the U_j.

Question If $U_1, \ldots, U_d$ are generators and each U_i is indecomposable is $d \geq e$, where e is the number of non-projective simple FG-modules, up to isomorphism.

Using a technique we used before but then where the Green correspondence was much stronger we have

Prop If $U_1, \ldots, U_d$ are generators and S is a simple FG-module then there are i, j such that

$$\mathrm{Hom}_{FG}(S, U_i) \neq 0, \quad \mathrm{Hom}_{FG}(U_j, S) \neq 0$$

Pf. Apply the definition to S to get a series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = S \oplus P.$$

Choose i maximal such that V_i has zero projection on S. Then S is isomorphic with a submodule of V_{i+1}/V_i, by Goursat's Theorem. Also S is not the intersection here, or we are done. Thus $S \oplus P/V_{i+1}$ has S in its socle so we are done.

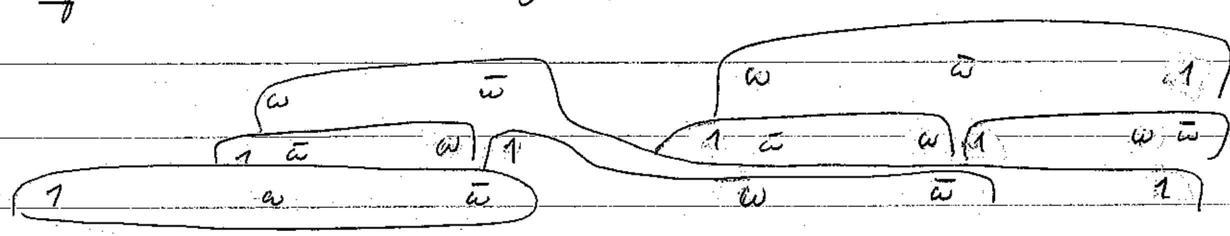
An example $G = A_4$, $p = 2$. Say $U_1, \dots, U_d$ are indecomposable generators. Then $\exists i \ni \dim_F U_i$ is odd. But we have the easy

Lemma If $U_1, \dots, U_d$ are generators and V is an invertible FG-module then $V_1, \dots, V_d$ are generators where V_i is the non-projective part of $V \otimes U_i$, $1 \le i \le d$.

Hence by inspection of A_4 modules can assume $U_1 = F \overset{def}{=} 1$. Hard to go further. If we drop indecomposability we have

Prop (J. G. Thompson) $1 \oplus \omega \oplus \bar{\omega}$ is a generator

Pf Put 1 – and similarly $\omega, \bar{\omega}$ – as follows:



On the Brauer correspondence.

We're really continuing the section beginning on page 66, but this is independent. Let F be algebraically closed of prime characteristic p. Let G be a finite group, H and Q subgroups with Q a p-subgroup and $Q\,C_G(Q) \leq H \leq N(Q)$. Let b be a block of H regarded as an $F[H\times H]$-module.

Theorem There exists a unique block B of FG such that b is isomorphic with a summand of the restriction to $H\times H$ of B; moreover, in a direct decomposition of this restriction there is at most one such summand.

Now $FG = FH + FHtH + \cdots$ over double cosets. Hence, we need only establish the

Proposition If $t \in G-H$ then no summand of the $F[H\times H]$-module $FHtH$ is isomorphic with b.

We need several results from Green's paper "Blocks of modular representations," Math. Zeit. 79 (1962) 100-115. First, from equation (2.31).

Lemma 1 If $t \in G-H$ then the $F[H\times H]$-module $FHtH$ is relatively projective for the subgroup $\{(h^{t^{-1}}, h) \mid h \in H\cap H^t\}$.

And from Lemma 2.3d (using the usual diagonal map δ)

<u>Lemma 2</u> A vertex of the $F[H\times H]$-module b has the form
$$(P_1\cap P_2)\delta$$
where P_1 and P_2 are Sylow p-subgroups of H.

Now suppose the proposition is false. Then, by Lemma 1, there is $t\in G-H$ such that b — as $F[H\times H]$-module — is relatively H_t projective, where
$$H_t = \{(h^{t^{-1}}, h) \mid h\in H\cap H^t\}.$$
Now then by Lemma 2, $(P_1\cap P_2)\delta$ is conjugate in $H\times H$ with a subgroup of H_t. But $Q\leq P_1\cap P_2$ since Q is a normal p-subgroup of H so $Q\delta$ is $H\times H$-conjugate with a subgroup of H_t. Hence, there exist $h_1, h_2\in H$ such that for all $x\in Q$,
$$(x,x)^{(h_1,h_2)} \in H_t.$$
That is,
$$x^{h_1} = (x^{h_2})^{t^{-1}},\quad x^{h_2}\in H\cap H^t$$
ie
$$\begin{cases} x^{h_2 t^{-1}} \in H. \\ h_1^{h_1} t h_2^{-1} \in C(Q) \end{cases}$$
But $C(Q)\leq H$ so $t\in H$, a contradiction

Now the obvious questions arise: Prove the first main theorem, Nagao's version of the second and so on ... using these methods. We begin by observing that the proposition above really proves the following:

<u>Proposition</u> If $t \in G - H$ and U is an indecomposable $F[H \times H]$-module which is a summand of the $F[H \times H]$-module $FHtH$ then the vertex of U contains no conjugate in $H \times H$ of $Q\delta$.

This is the right form!

<u>First main Theorem</u> If B is a block of G with defect group D (+) then there exists a unique block b of $N(D)$ with defect group D such that $b^G = B$.

<u>Proof.</u> Let U be the Green correspondent (*) in $N(D) \times N(D)$ for B. To establish the existence part of the theorem, it's enough to see that U is a block, i.e. $U / FN(D)$ where $FN(D)$ is the $F[N(D) \times N(D)]$ module. But $U / B_{N(D) \times N(D)}$ so $U / FN(D)tF(D)$ for some $t \in G$. But U also has vertex δD as U is the Green correspondent of B and since B has δD as its vertex. The proposition

(*) $N(D) \times N(D) \geq N_{G \times G}(\delta D)$ so theory applies.

+ i.e. for us, $D\delta$ is a vertex of the $F[G \times G]$-module B.

above now yields that $t \in H$.

The uniqueness remains. Suppose that b_1, b_2 are blocks of $N(D)$ with defect groups D, i.e. vertex δD as $F[N(D)\times N(D)]$-modules. Suppose $b_i^G = B$, $i=1,2$. That is $b_i \mid B_{N(D)\times N(D)}$. $\therefore b_i$ is a summand of $B|_{N(D)\times N(D)}$ with vertex δD so $b_1 = b_2$ by the Green correspondence.

Remarks. 1 also get Srinivasan's result: $B \mid b_i^{G\times G}$.

2. Get b^G defined more generally. That is, if R is defect group of b then get b^G defined if $C_G(R) \leq H$.

3 Can define b^G of course whenever the theorem works. Is this the same as with Brauer's definition.

Toroidal Characters

Let G be a group of Lie type and characteristic p with the usual subgroups B, H, U, N and $W = N/H$. And let T be a torus, σ the Steinberg character.

Prop If λ is a linear character of T then there is a generalized character η such that

$$\lambda^G = \sigma\,\eta$$

Proof T has order prime to p so λ^G is "projective" so we can apply our results on tensor products and projectives — or quote Feit.

Question: Can we take η as the character that Deligne + Lusztig construct corresponding to λ? The degrees are correct.

In the case of the principal series we have the following:

Prop If λ is a linear character of H, $\tilde{\lambda}$ its extension to B then

$$\lambda^G = \sigma \cdot \tilde{\lambda}^G$$

Proof. It suffices to see that $\sigma_B = (1_H)^B$. For then

$$\sigma\,\tilde{\lambda}^G = (\sigma_B \cdot \tilde{\lambda})^G = \left(\left(1_H\right)^B \tilde{\lambda}\right)^G = \left(\left(1_H\,\tilde{\lambda}_H\right)^B\right)^G = \left(\lambda_H{}^B\right)^G = \lambda^G.$$

But let S be the Steinberg module over an algebraically closed field F of characteristic p. Thus S_B is projective and as $\dim_F S_B = |U|$, S_B is an indecomposable projective FB-module. To establish our assertion on σ_B it suffices to see that $S_B = (F_A)^B$. But as S_B is as stated, it suffices to see that $\mathrm{Hom}_{FB}(F, S_B) \neq 0$. That is, $\mathrm{Hom}_{FG}(F^G, S) \neq 0$. But S is a summand of $(F_B)^G$ so the proposition is proved.

<u>Example</u>: $SL(2, 2^n)$

	1	a	b	c
	1	1	1	1
	2^n	0	1	-1
	2^n+1	1	$\lambda+\bar\lambda$	0
	2^n-1	-1	0	$-\mu-\bar\mu$
λ^G	$(2^n+1)2^n$	0	$\lambda+\bar\lambda$	0
μ^G	$(2^n-1)2^n$	0	0	$\mu+\bar\mu$

Works for the other tori here too.

Free Summands

Let V be a faithful FG-module for a finite group G and field F. We give a new proof of the following result of Bryant and Kovacs:

<u>Theorem</u> The polynomial algebra $P(V)$ on V as an FG-module contains a free summand.

Note that this implies that so does $T(V)$ the tensor algebra on V.

<u>Pf</u> Let L be the quotient field of $P(V)$, K the fixed field of G so L/K is a Galois extension with G as Galois group. Hence, there is $f \in L$ such that all the elements

$$f\sigma$$

$\sigma \in G$, are linearly independent over K. Now $f = g/h$, $g, h \in P(V)$ so $f \prod_{\sigma \in G} h\sigma \in P(V)$ and this product - call it p - also has the property that all the $p\sigma$ are linearly independent over K, since $\prod h\sigma \in K$. Hence, all these images $p\sigma$ are certainly linearly independent over F, $F \subseteq K$!!

Periodicity of Weyl Modules

(Joint work with Laci Kovacs)

Let $G = SL(2,q)$ and let V_n be the $n+1$-dimensional FG-module of homogeneous polynomials of degree n in two variables, where F is algebraically closed of characteristic p, $p \mid q$. Thus $V_0 \simeq F$, V_1 is the standard module.

Theorem The sequence of modules

$$V_1, V_2, \ldots, V_n, \ldots$$

is, apart from projective summands, periodic of period $q(q-1)$. That is, if $V_n = W_n \oplus P_n$ where P_n is projective and W_n has no projective summand then the sequence

$$W_1, W_2, \ldots, W_n, \ldots$$

is periodic of period $q(q-1)$.

This is due to Glover in the case $q = p$. He also has determined the structure of all the modules.

Lemma 1 V_q is the Steinberg module

Pf Let x, y be the two variables. We are asserting that V_q, $q = p^e$ is the tensor product of the e conjugates of V_p. But V_p has basis $x^{p-1}, x^{p-2}y, \ldots, y^{p-1}$ and applying the Frobenius, the first conjugate is isomorphic with the subspace of V_{p^2-p}

with basis $x^{p^2-p}, x^{p^2-2p}y^p, \dots, y^{p^2-p}$. And so on for the other conjugates. Hence, as $p^e = q$ it suffices to see that every element of V_q can be written as sum of a products of an element of V_p, an element from the above subspace of V_{p^2-p} and so on. Schematically, we want

$$V_q = \{x^{p-1}, x^{p-2}y, \dots, y^{p-1}\}\{x^{p^2-p}, x^{p^2-2p}y^p, \dots, y^{p^2-p}\}\dots\{x^{p^e-p^{e-1}}, \dots, y^{p^e-p^{e-1}}\}$$

But consider $x^i y^j$, $i+j = q-1$. Express

$$i = a_0 + a_1 p + \dots + a_{e-1} p^{e-1}$$

where the a's are between 0 and $p-1$. Then

$$x^i = x^{a_0} x^{a_1 p} \dots x^{a_{e-1} p^{e-1}}.$$

Hence, it suffices to prove that

$$y^j = y^{b_0} y^{b_1 p} \cdots y^{b_{e-1} p^{e-1}}$$

when $a_k + b_k = p-1$ for all k. But

$$(p-1) + (p-1)p + \dots + (p-1)p^{e-1} = (p-1)\frac{p^e-1}{p-1}$$

so our claim is valid.

__Lemma 2__ V_{kq} is projective for all $k \geq 1$.

Pf We just proved this for $k=1$. Then, by the Wall sequence,

$$0 \to V_{q-1} \otimes V_q \to V_q \otimes V_{q+1} \to V_{2q} \to 0$$

is exact, so V_{2q} is also projective. If V_{kq} is projective then

$$0 \to V_{q-1} \otimes V_{kq} \to V_q \otimes V_{kq+1} \to V_{(k+1)q} \to 0$$

is exact so the lemma is established.

__Lemma 3__ $V_{q(q-1)+1} \simeq V_1 \oplus \text{projective}.$

Pf We assert that there is an exact sequence

$$0 \to V_{q^2-2q} \to V_{q(q-1)+1} \to V_1 \oplus V_q \to 0$$

This will prove the result. Since the dimensions add up we need only produce a monomorphism of V_{q^2-2q} into the kernel of a map of $V_{q(q-1)+1}$ onto $V_1 \oplus V_q$.

First observe that $x^q y - xy^q$ is fixed by $SL(2,q)$. (For, an easy calculation shows it is sent by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to

$$(a^q d - bc^q)\, x^q y + (-ad^q + b^q c)\, xy^q.$$

But $a^q = a$ as $a \in GF(q)$ and so on .. and $ad - bc = 1$.) We map $v \in V_{q^2-2q}$ to $v(x^q y - xy^q)$. This is a homomorphism and it is a monomorphism since the polynomial algebra in x and y is an integral domain.

Next, let $\mathcal{F}_{q(q-1)+1}$ be the polynomial functions over $GF(q)$ in x, y homogeneous of degree $q(q-1)$. The natural map of $V_{q(q-1)+1}$ onto $\mathcal{F}_{q(q-1)+1}$ has the image of V_{q^2-2q} in its kernel as $x^q y - xy^q$ is in the kernel. Here we must assume we're working over $GF(q)$, prove the result there and then go up to F. Hence, it suffices to get the structure of $\mathcal{F}_{q(q-1)+1}$ to be $V_1 \oplus V_q$.

But when dealing with functions the only relations are $x^q = x$, $y^q = y$ (by dimension count over all fields) so we have to see that every polynomial $x^i y^j$ $i+j = q(q-1)$ is congruent to some $x^{q-1-k} y^k$ or $x^{q-1} y^{q-1}$ and that all these occur - so we get the Steinberg module (the first of these) and a trivial module $(x^{q-1} y^{q-1})$.

First, suppose that neither i nor j is divisible by $q-1$ so

$$i \equiv k,\ j \equiv l \quad (\text{modulo } q-1)$$

where $0 \le k, l < q-1$. $\therefore$ $i+j \equiv k+l \ (q-1)$ and $i+j \equiv 0 \ (q-1)$

so $k+l = q-1$.

Say $q-1$ divides i and so it also dividing j so $i+j = q(q-1)$.

We're considering $x^{s(q-1)} y^{t(q-1)}$, $s+t=q$. $\therefore$ easy to see that for functions we get $x^{q-1}y^{q-1}$ or x^{q-1} or y^{q-1}

Also easy to see everything needed does arise.

<u>Theorem</u> For all n, V_n and $V_{n+q(q-1)}$ are congruent modulo projectives.

<u>Pf</u> We have the exact sequence

$$0 \to V_{n-1} \otimes V_{q(q-1)} \to V_n \otimes V_{q(q-1)+1} \to V_{n+q(q-1)} \to 0$$

i.e.

$$0 \to \text{projective} \to V_n \otimes (V_1 \oplus \text{projective}) \to V_{n+q(q-1)} \to 0$$

and the result is established.

<u>Remark</u>. If $q \nmid n$ then probably get only one non-projective indecomposable summand of V_n; for look at restriction to Sylow p-subgroup - get matrices à la Glover's thesis. Restrictions are nested so count fixed points.

<u>Examples</u> (<u>not</u> <u>entirely</u> <u>proved</u>)

Using Wall sequence, get structure of V_n for $SL(2, 2^\infty)$:

<table>
<tr><th>V_1</th><th>V_2</th><th>V_3</th><th>V_4</th><th>V_5</th><th>V_6</th><th>V_7</th><th>V_8</th><th>V_9</th></tr>
<tr><td>0</td><td>1</td><td>0
2</td><td>12</td><td>2
0
3</td><td>1
13</td><td>0
3
23</td><td>123</td><td>23
3
0
4</td></tr>
</table>

<table>
<tr><th>V_{10}</th><th>V_{11}</th><th>V_{12}</th><th>V_{13}</th><th>V_{14}</th><th>V_{15}</th><th>V_{16}</th></tr>
<tr><td>13
1
14</td><td>3
/
0 — 2
/ /
4 — 24</td><td>12
124</td><td>2 — 24
/ /
0 — 4
/
34</td><td>1
14
134</td><td>0
4
34
234</td><td>1234</td></tr>
</table>

∴ dropping down to $SL(2,4)$:

<table>
<tr><th>V_1</th><th>V_2</th><th>V_3</th><th>V_4</th><th>V_5</th><th>V_6</th><th>V_7</th><th>V_8</th></tr>
<tr><td>0</td><td>1</td><td>0
2</td><td>12</td><td>2
0
1</td><td>1
0
2
0</td><td>0
1 ⊕ 12</td><td>P_2</td></tr>
</table>

<table>
<tr><th>V_9</th><th>V_{10}</th><th>V_{11}</th><th>V_{12}</th><th>V_{13}</th></tr>
<tr><td>1
0 ⊕ 12
2</td><td>0
2 ⊕ 12
0
1</td><td>2   1
0   0
1   2
\ /
0</td><td>1
0
12 ⊕ 2
0
1</td><td>2
4
0 ⊕ 1 ⊕ 12
0
2</td></tr>
</table>