RESEARCH NOTES

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Rank one: the Steinberg module

We assume the rank one situation, as in our paper in the Journal of Pure and Applied Algebra. We present a different proof that the Steinberg module is simple and projective.

First,

$$\left(F_B{}^G\right)_B \cong F \oplus F_H{}^B$$

so

$$F^G \cong F \oplus S,$$

which defines S, yields that

$$S_B \cong F_H{}^B$$

and S is projective.

Hence, it suffices to prove that $\mathrm{Hom}_{FG}(S, S) \simeq F$. That is, it suffices to see that

$$\mathrm{Hom}_{FG}(F \oplus S, F \oplus S) \simeq F \oplus F$$

But

$$\begin{aligned} \mathrm{Hom}_{FG}(F_B{}^G, F_B{}^G) &\simeq \mathrm{Hom}_{FB}(F_B{}^G{}_B, F) \\ &\simeq \mathrm{Hom}_{FB}(F \oplus F_H{}^B, F) \\ &\simeq F \oplus F \end{aligned}$$

since $F_H{}^B$ is the indecomposable projective module corresponding to F, by dimension counting.

Rank one: induction and restriction matrices

We continue with the rank one situation and all standard notation. Let λ be a typical linear character of H, $\tilde{\lambda}$ the "lift" of this to B so the $\tilde{\lambda}$ correspond one-to-one with the simple FB-modules and we shall use "$\tilde{\lambda}$" to denote the module as well. Also use "$\doteq$" to denote the relation of having the same composition factors.

We now let R, J be the restriction and induction matrices, that is,

$$(V_i)_B \doteq \sum r_{i\lambda}\,\tilde{\lambda}, \qquad R = (r_{i\lambda})$$
$$\tilde{\lambda}^G \doteq \sum j_{\lambda k}\,V_k, \qquad J = (j_{\lambda k}).$$

Then we have the

Theorem $RJ - I = AC$
$$(RJ - I)^2 + (RJ - I) = (RR')\,C.$$

Proof The first equation yields the second, as $ACA + A = M$ so $ACAC + AC = MC$ and clearly $M = RR'$. As for the first,

$$(V_i)_B{}^G \doteq \sum r_{i\lambda}\,j_{\lambda k}\,V_k$$

and

$$\begin{aligned}(V_i)_B{}^G &\simeq \left((V_i)_B \otimes F\right)^G \simeq V_i \otimes (F_B)^G \\ &\simeq V_i \otimes (F \oplus S) \\ &\simeq V_i \oplus (V_i \otimes S) \\ &\doteq V_i \oplus \sum a_{il}\,c_{lk}\,V_k\end{aligned}$$

which establishes the theorem.

Here's an example, $G = SL(2,4)$, $p=2$.

$$\begin{array}{c|c} & \begin{matrix} 1 & \omega & \bar\omega \end{matrix} \\ \begin{matrix} V_0 \\ V_1 \\ V_2 \\ V_{12} \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \end{array} = R$$

$$RR' = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 6 \end{pmatrix}$$

$$J = \begin{array}{c|c} & \begin{matrix} V_0 & V_1 & V_2 & V_{12} \end{matrix} \\ \begin{matrix} 1 \\ \omega \\ \bar\omega \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$RJ = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{pmatrix} \qquad RJ - I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 4 & 2 & 2 & 1 \end{pmatrix}$$

$$(RJ-I)^2 = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 6 & 5 & 4 & 2 \\ 6 & 4 & 5 & 2 \\ 12 & 8 & 8 & 5 \end{pmatrix} \qquad (RJ-I) + (RJ-I)^2 = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 8 & 6 & 6 & 2 \\ 8 & 6 & 6 & 2 \\ 16 & 10 & 10 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad RR'C = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 8 & 6 & 6 & 2 \\ 8 & 6 & 6 & 2 \\ 16 & 10 & 10 & 6 \end{pmatrix}$$

Seems difficult to actually get C from these matrices.

Rank one: the Suzuki groups

Now we specialize to $Sz(q)$, $2q = r^2 = 2^{2n+2}$, $r = 2^{n+1}$.
The elements of H are parametrized by the elements of $GF(q)^{\#}$, the element $h(K)$ being given by

$$\begin{pmatrix} K^{r/2+1} & & & \\ & K^{r/2} & & \\ & & K^{-r/2} & \\ & & & K^{-r/2-1} \end{pmatrix}$$

in Suzuki's definition of the groups in his Annals paper. (This follows readily from his description, as

$$\zeta_1 = (\zeta_1^{r})^{r/2} = K^{\frac{(1+r)}{2}r} = K^{\frac{r+2q}{2}} = K^{r/2+1}$$
$$\zeta_2 = (\zeta_2^{r})^{r/2} = K^{r/2}\ .)$$

Let this representation give the module V_1 so we get $2n+1$ four dimensional modules: $V_1, V_2, \dots, V_{2n+1}$. Their tensor product being the Steinberg module, the usual tensor product result for simple modules well known to hold here.

<u>Lemma</u> $V_1 \otimes V_1 \doteq 4V_0 \oplus V_2 \oplus 2V_{n+2}$.

<u>Proof</u>. Since $V_1 \otimes V_1$ has dimension sixteen and since we have the tensor product theorem on simple modules, either $V_1 \otimes V_1$ is simple or it has only one and four dimensional composition factors.

Calculating the character of $V_1 \otimes V_1$ on $h(K)$ we get

$$K^{n+2} + K^{n+1} + K^{1} + K^{0}$$
$$+ K^{n+1} + K^{n} + K^{0} + K^{-1}$$
$$+ K^{1} + K^{0} + K^{-n} + K^{-n-1}$$
$$+ K^{0} + K^{-1} + K^{-n-1} + K^{-n-2}$$

This is four times the character of V_0 on $h(K)$ plus the character of V_2 on $h(K)$ plus

$$2\left(K^{n+1} + K^{1} + K^{-1} + K^{-n-1}\right).$$

But the value of the character of V_{n+2} on $h(K)$ is

$$\left(K^{2^{n+1}}\right)^{(\frac{n}{2}+1)} + \left(K^{2^{n+1}}\right)^{\frac{n}{2}} + \left(K^{-2^{n+1}}\right)^{\frac{n}{2}+1} + K^{-(2^{n+1})\frac{n}{2}+1}$$

$$= K^{2^{2n+1}+2^{n+1}} + K^{2^{2n+1}} + \cdots$$
$$= K^{n+1} + K + K^{-1} + K^{-(n+1)}$$

Hence, our claim is consistent, in that the character of $V_1 \otimes V_1$ on H is as claimed.

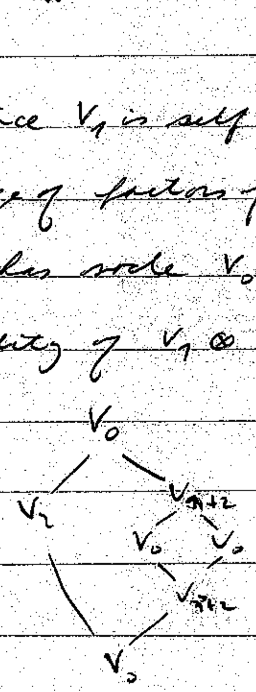
Now since the Suzuki group is inside the 4×4 symplectic group each matrix is symplectic so the module V_1 is self-dual. Hence $\mathrm{Hom}_{FG}(F, V_1 \otimes V_1) \neq 0$ so $V_1 \otimes V_1$ has a trivial composition factor. ∴ it is a module whose composition factors are of dimension 1 or 4. The latter have no eigenvalue K^0 so $V_1 \otimes V_1$ has V_0 as factor with multiplicity 4 and three other 4-dimensional factors.

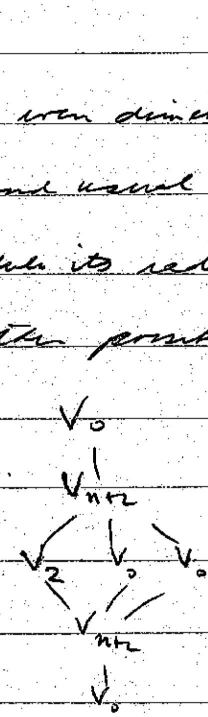
But looking for consecutive powers of K in the character can eliminate all undesired combinations of the four dimensional representations.

<u>Proposition</u> $V_1 \otimes V_1$ is uniserial with composition factors, in order, being

$$V_0,\ V_{n+2},\ V_0,\ V_2,\ V_0,\ V_{n+2},\ V_0$$

<u>Proof</u> Since V_1 is self-dual and even dimensional and using knowledge of factors of $V_1 \otimes V_1$ and usual "Hom tricks" get that $V_1 \otimes V_1$ has socle V_0 and modulo its radical it's also V_0. ∴ using self-duality of $V_1 \otimes V_1$ only other possibilities for $V_1 \otimes V_1$ are as follows:





These both contradict the Cline-Parshall-Scott calculations of Ext^1.

Now the problem is to decompose $V_I \otimes V_N$, $N = \{1, \dots, 2n+1\}$, $I \subseteq N$ which is $P_{N-I} \oplus$?.

Looking at examples can pull out certain $V_J \otimes V_N$.

Seems hard! Now $|G| = q^5 - q^2(q^2-q+1)$. Adding up all dimensions of $V_I \otimes V_N$ get - arriving at $|G|$

$$\sum_{k=0}^{2n+1} \underset{\substack{\uparrow \\ \text{dim simple}}}{2^{2k}} \underset{\substack{\uparrow \\ \#\text{ of distinct}}}{\binom{2n+1}{k}} 2^{2k+4(2n+1-k)} = \sum_{k=0}^{2n+1} 2^{4(2n+1)} \binom{2n+1}{k} = 2^{4(2n+1)} 2^{2n+1} = \underline{\underline{q^5}}$$

Close!

<u>Rank one</u> : <u>Simply generated</u> <u>modules</u> <u>for $SL(2,q)$</u>.

Let $S_1, S_2, \ldots, S_p$ be the polynomial modules of $SL(2,q)$, $q=p^n$, of dimensions $1, 2, \ldots, p$ respectively. If $\mathcal{D}$ is the Galois group of $GF(q)$ and $f: \mathcal{D} \to \{1, 2, \ldots, p\}$, setting

$$S_f = \bigotimes_{\varphi \in \mathcal{D}} S^{\varphi}_{f(\varphi)}$$

and letting f range over all such functions we get all the simple modules. We shall prove a similar result for projectives, presumably what Jeyakumar proved (J. Alg v. 30) but much more easily. And we shall also establish that there are only finitely many simply generated modules. Our methods are the ones we have developed plus the ideas of D. E. Wall — see his notes.

The key result is the following:

<u>Lemma</u> For $1 \leq i < p$ there exists a module U_i having the following properties:

1) S_i is a homomorphic image of U_i;

2) $\dim U_i = 2p$;

3) S_p "divides" U_i, in that there are modules $V_i, W_i \ni U_i \oplus V_i \simeq S_p \otimes W_i$;

4) S_p is a composition factor of U_1.

In fact, we get a lot more information on the structure of U_i. Let $V^{(p)}$ denote the "first" conjugate of the module V, i.e. apply the Frobenius. Then we have the following additional information:

a) If $n>1$ then U_i is uniserial with composition factors, in order, S_i, $S_{p-i}\otimes S_2^{(p)}$, S_i;

b) If $n=1$ then U_{p-1} is uniserial with composition factors, in order, S_{p-1}, $S_2^{(p)}$, S_{p-1}; (slightly different if $p=2$ see d))

c) If $n=1$, $1<i<p-1$, then U_i has the structure $\begin{array}{ccc} & S_i & \\ S_{p-i-1} & & S_{p-i+1} \\ & S_i & \end{array}$

d) If $n=1$ then $U_1 \simeq \begin{array}{c} S_1 \\ S_{p-2} \\ S_1 \end{array} \oplus S_p$. (not indecomposable!)

From the lemma we get the result we want. Let $\underline{U_p = S_p}$.

<u>Theorem</u> With the above notation,

$$P_f \cong \bigotimes_{\varphi\in\mathcal{J}} U_{f(\varphi)}^{\varphi}$$

is the indecomposable projective module corresponding to S_f, except in the case that $f(\varphi)=1$ for all φ, in which case the tensor product is the direct sum of the Steinberg module and the given answer.

Pf (of Theorem) Part 3) of the lemma shows that $\bigotimes_{\varphi} S_p^{\varphi}$ "divides" P_f; as this is the Steinberg module dividing P_f we get that P_f is projective. Similarly, part 1) of the lemma and the description of S_f yields that S_f is a homomorphic image of P_f. Part 4) also guarantees that the Steinberg module is a summand in the exceptional case. Hence, it suffices to establish

$$\sum_f \dim S_f \dim P_f = |SL(2,p^n)| + p^n.$$

But the sum on the left is just

$$\left(1.2p + 2.2p + \dots + (p-1).2p + p.p\right)^n$$

as each term in the expansion corresponds to exactly one term of the sum.
But this is just

$$\left(2p + 4p + \dots + (2p-2)p + 2p.p - p^2\right)^n$$
$$= \left(2p\,(1+2+\dots+p) - p^2\right)^n$$
$$= \left(2p\,\frac{p(p+1)}{2} - p^2\right)^n$$
$$= (p^3)^n = p^{3n} = (p^n-1)(p^n)(p^n+1) + p^n$$

as desired.

To prove the lemma we begin by constructing U_{p-1}. We claim that $S_2 \otimes S_p$ will do. We need only see that S_{p-1} is a homomorphic image. But we shall also verify that U_{p-1} has the detailed structure stated. From the Wall exact sequence (see his notes, or Glover's thesis) we have the exact sequence

$$(*) \qquad 0 \to S_{p-1} \to S_2 \otimes S_p \to S_{p+1} \to 0$$

where S_{p+1} is the polynomial module of dimension $p+1$. Also

$$\mathrm{Hom}_{FG}(S_2 \otimes S_p, S_{p-1}) = \mathrm{Hom}_{FG}(S_p, S_2 \otimes S_{p-1})$$
$$\simeq \mathrm{Hom}_{FG}(S_p, S_{p-2} \oplus S_p) \qquad \text{(see Wall or Glover)}$$
$$\simeq F$$

And $S_{p+1} \supseteq S_2^{(p)}$ given by x^p, y^p. Now

$$\mathrm{Hom}_{FG}(S_2^{(p)}, S_2 \otimes S_p) = \mathrm{Hom}_{FG}(S_2 \otimes S_2^{(p)}, S_p)$$

If $n > 1$ then $S_2 \otimes S_2^{(p)}$ is simple so this is zero. Also if $n = 1$, $p > 3$. In these cases this implies that S_{p-1} in $(*)$ is the radical. For otherwise $S_2^{(p)}$ would be in the socle of $S_2 \otimes S_p$. Hence, $S_2 \otimes S_p$ has two S_{p-1}'s as composition factors and all is proved. (Count dims.)

The cases $n=1$, $p=2,3$ have to be dealt with case by case.

Note $SL(2,2) \cong \Sigma_3$, $SL(2,3) = Q_8 \cdot Z_3$.

Also note that we can now assume $p>2$ as we've done in that case — so now U_{p-1} is as stated.

Next, we consider $S_2 \otimes U_{p-1}$. We show that there is U_{p-2} with all the desired properties such that $S_2 \otimes U_{p-1} \cong U_{p-2} \oplus S_p \oplus S_p$.

First,

$$\operatorname{Hom}_{FG}(S_p, S_2 \otimes U_{p-1}) \cong \operatorname{Hom}_{FG}(S_2 \otimes S_p, U_{p-1})$$
$$\cong \operatorname{Hom}_{FG}(U_{p-1}, U_{p-1}) \cong F \oplus F.$$

Now $U_{p-1} \doteq S_{p-1} \oplus S_{p-1} \oplus S_2^{(p)}$ so as $S_2 \otimes S_{p-1} \cong S_{p-2} \oplus S_p$, get that remaining composition factors other than S_p of $S_2 \otimes U_{p-1}$ are S_{p-2}, S_{p-2} and $S_2 \otimes S_2^{(p)}$ (which is $S_1 \oplus S_3$ if $n=1$). Also

$$\operatorname{Hom}_{FG}(S_{p-2}, S_2 \otimes U_{p-1}) \cong \operatorname{Hom}_{FG}(S_{p-3} \oplus S_{p-1}, U_{p-1}) = F$$

and similarly S_{p-2} is a quotient of $S_2 \otimes U_{p-1}$. Also

$$\operatorname{Hom}_{FG}(S_2 \otimes S_2^{(p)}, S_2 \otimes U_{p-1}) \cong \operatorname{Hom}((S_1 \otimes S_2^{(p)}) \oplus (S_3 \otimes S_2^{(p)}), U_{p-1}),$$

which is clearly zero if $n>1$. (The case of $n=1$ requires a little more argument, which we omit.) This shows U_{p-2} is as desired too.

Next, suppose that $p-1>j>1$, U_j is as desired. We claim that we can take U_{j-1} such that $U_j \otimes S_2 \cong U_{j+1} \oplus U_{j-1}$ (downwards induction!) We skip over the case of $n=1$.

(Note: Here could quote Brauer tree!) But $S_2 \otimes U_j$ has the correct composition factors. Also

$$\mathrm{Hom}_{FG}(U_{j+1}, S_2 \otimes U_j) \simeq \mathrm{Hom}(S_2 \otimes U_{j+1}, U_j) \simeq F \oplus F$$

$\therefore U_{j+1} \subseteq S_2 \otimes U_j$. Also U_{j+1} is a homomorphic image. Hence, by composition factors, get U_{j+1} is a summand of $S_2 \otimes U_{j+1}$.

We now finish up as above by a few calculations:

$$\mathrm{Hom}(S_{j-1}, S_2 \otimes U_j) \simeq \mathrm{Hom}(S_{j-1} \otimes S_j, U_j) \simeq F$$

$$\mathrm{Hom}(S_2^{(p)} \otimes S_{p-j+1}, S_2 \otimes U_j) \simeq \mathrm{Hom}(S_2^{(p)} \otimes (S_{p-j+2} \oplus S_{p-j}), U_j) = 0$$

For our next theorem, we require the next result for $n > 1$:

<u>Lemma</u> $S_2 \otimes U_1 \simeq U_2 \oplus (S_2^{(p)} \otimes S_p)$

<u>Pf</u> U_1 has composition factors $S_1, S_{p-1} \otimes S_2^{(p)}, S_1$ so $S_2 \otimes U_1$ has composition factors $S_2, S_{p-2} \otimes S_2^{(p)}, S_p \otimes S_2^{(p)}, S_2$.

$\therefore$ E.T.S. U_2 is a summand. Proceed as above:

$$\mathrm{Hom}(U_2, S_2 \otimes U_1) \simeq \mathrm{Hom}(S_2 \otimes U_2, U_1)$$
$$\simeq \mathrm{Hom}(U_1 \oplus U_3, U_1) \simeq F \oplus F.$$

Now this gives the

<u>Theorem</u> There are only finitely many simply generated modules.

Of course, for $n=1$ this is immediate.

Pf Suffices to show that every simply generated module occurs as a summand of a "power product"

$$S_2^{a_1} \; S_2^{(p)\,a_2} \; S_2^{(p^2)\,a_3} \; \cdots \; S_2^{(p^{n-1})\,a_n}$$

where $0 \leq a_i \leq 2p-2$. For all simple modules appear in such a way. S_2 and its conjugates generate these $\therefore$ really enough just to deal with S_2^{2p-1}. But this follows readily from all the above calculations and the last lemma; i.e. every summand of S_2^{2p-1} can be expressed by "lower terms" i.e. sum of the exponents less - pretty much as in the case for that we've done.

A_7 in characteristic three

From Fong's notes we have the decomposition matrix:

$$
D_0 = \begin{array}{|cccc}
1 & 13 & 10 & 10 \\
\hline
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
$$

so that the Cartan matrix is

$$
C_0 = \begin{pmatrix} 7 & 4 & 2 & 2 \\ 4 & 3 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}
$$

The modular characters are thus as follows:

$$
\begin{array}{cc|cccccc}
 & & 1 & 2 & 3 & 4 & 7 & 7 \\
\hline
\varphi_1 & \multirow{4}{*}{B_0} & 1 & 1 & 1 & 1 & 1 & 1 \\
\varphi_{13} & & 13 & 1 & -2 & -1 & -1 & -1 \\
\varphi_{10} & & 10 & -2 & 0 & 0 & \frac{-1+\sqrt{-7}}{2} & \frac{-1-\sqrt{-7}}{2} \\
\bar{\varphi}_{10} = \varphi_{10}^* & & 10 & -2 & 0 & 0 & \frac{-1-\sqrt{-7}}{2} & \frac{-1+\sqrt{-7}}{2} \\
\varphi_6 & \multirow{2}{*}{B_1} & 6 & 2 & 1 & 0 & -1 & -1 \\
\varphi_{15} & & 15 & -1 & 0 & -1 & 1 & 1
\end{array}
$$

The modular characters of A_6 are (recall!)

	1	2	5	5	4
ψ_1	1	1	1	1	1
ψ_4	4	0	-1	-1	-2
ψ_3	3	-1	$\frac{1+\sqrt5}{2}$	$\frac{1-\sqrt5}{2}$	1
ψ_3'	3	-1	$\frac{1-\sqrt5}{2}$	$\frac{1+\sqrt5}{2}$	1
ψ_9	9	1	-1	-1	1

Inducing up to A_7:

	1	2	5	4	7	7
$\psi_1^{A_7}$	7	3	2	1	0	0
$\psi_4^{A_7}$	28	0	-2	-2	0	0
$\psi_3'^{A_7} = \psi_3^{A_7}$	21	-3	1	1	0	0
$\psi_9^{A_7}$	63	3	1	1	0	0

Hence, $\psi_3^{A_7} = \varphi_1 + \varphi_{10} + \varphi_{10}^*$. Using "$W$" for A_6-modules, "V" for A_7,

$$\mathrm{Hom}_{FA_7}(W_3^{A_7}, V_1) \simeq \mathrm{Hom}_{FA_6}(W_3, W_1) = 0$$

$\therefore$ $W_3^{A_7}$ is a uniserial module, with V_1 in the middle. Since V_{10}, V_{10}^* are interchanged by an automorphism of A_7 – from Σ_7 – we get

__Lemma__ There exist uniserial modules as follows: $\begin{matrix} V_{10} \\ V_1 \\ V_{10}^* \end{matrix}$, $\begin{matrix} V_{10}^* \\ V_1 \\ V_{10} \end{matrix}$

<u>Lemma</u> $\mathrm{Ext}_{FA_7}(V_{10}, V_{13}) = \mathrm{Ext}(V_{10}^*, V_{13}) = 0$.

Pf.
$$\mathrm{Ext}^n_{FA_7}(V_{10}, V_{13}) \cong \mathrm{Ext}^n_{FA_7}(V_{10}, V_{13} \oplus V_{15})$$
$$\cong \mathrm{Ext}^n_{FA_7}(V_{10}, W_4^{A_7})$$
(from the characters as V_{13}, V_{15} are in different blocks)
$$\cong \mathrm{Ext}^n_{FA_6}(V_{10}|_{A_6}, W_4).$$
When $n=0$ we deduce, as $V_{10} \not\cong V_{13}$, $\mathrm{Hom}(V_{10}|_{A_6}, W_4) = 0$ and similarly $\mathrm{Hom}(W_4, V_{10}|_{A_6}) = 0$. Also, from the characters, $\varphi_{10}|_{A_6} = \psi_3 + \psi_3' + \psi_4$. Hence, $V_{10}|_{A_6}$ is either $\begin{matrix} W_3 \\ W_4 \\ W_3' \end{matrix}$ or $\begin{matrix} W_3' \\ W_4 \\ W_3 \end{matrix}$. Hence, from what we know about A_6, applying Heller's Ω operator we get $\begin{matrix} W_1 \\ W_4 \\ W_3 \end{matrix}$ or $\begin{matrix} W_1 \\ W_4 \\ W_3' \end{matrix}$ so, in either case, $\mathrm{Ext}^1(V_{10}|_{A_6}, W_4) = 0$. This proves the result.

<u>Lemma</u> P_{10} and P_{10}^* have the following diagrams:

$$\begin{tikzcd}[column sep=small] & V_{10} \arrow[d, dash] & \\ & V_1 \arrow[dl, dash] \arrow[dr, dash] & \\ V_{10}^* \arrow[dr, dash] & & V_{13} \arrow[dl, dash] \\ & V_1 \arrow[d, dash] & \\ & V_{10} & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & V_{10}^* \arrow[d, dash] & \\ & V_1 \arrow[dl, dash] \arrow[dr, dash] & \\ V_{10} \arrow[dr, dash] & & V_{13} \arrow[dl, dash] \\ & V_1 \arrow[d, dash] & \\ & V_{10}^* & \end{tikzcd}$$

Pf Use above lemmas, the Cartan matrix and usual type arguments.

<u>*Lemma*</u> $\mathrm{Ext}^1(V_1, V_{13}) \simeq \mathrm{Ext}^1(V_{13}, V_1) \simeq F \oplus F$.

<u>*Pf*</u> $\mathrm{Ext}^1_{FA_7}(V_1, V_{13}) \simeq \mathrm{Ext}^1_{FA_7}(V_1, V_{13} \oplus V_{15}) \simeq \mathrm{Ext}^1_{FA_6}(W_1, W_1) \simeq F \oplus F$

Here's a <u>guess</u> for the structure of P_{13}:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & V_{13} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & V_1 \arrow[dl, dash] \arrow[dr, dash] & & V_1 \arrow[dl, dash] \arrow[dr, dash] & \\
V_{10} \arrow[dr, dash] & & V_{13} \arrow[dl, dash] \arrow[dr, dash] & & V_{10}^* \arrow[dl, dash] \\
 & V_1 \arrow[dr, dash] & & V_1 \arrow[dl, dash] & \\
 & & V_{13} & &
\end{tikzcd}
$$

Free actions on products of spheres

First, we prove a general result:

Theorem If G is a finite group then there is a torus on which G acts freely.

Let $\langle x\rangle$ be any cyclic subgroup of G. Since $\langle x\rangle$ can act freely on a circle, by rotation, there is a 2-dimensional real inner product space W with $\langle x\rangle$ acting and fixing no non-zero vector. Here, action means orthogonal action.

Let W^G be the induced module so, if T is a transversal to $\langle x\rangle$ in G then

$$W^G = \bigoplus_{t\in T} (W\otimes t)$$

a direct sum of vector spaces. We make W^G an inner product space by demanding that

$$(w\otimes t_1, w'\otimes t_2) = 0 \qquad \text{if } t_1 \neq t_2$$
$$(w\otimes t, w'\otimes t) = (w, w')$$

It's now easy to check that W^G is an inner product space and G acts orthogonally, e.g. if $g\in G$, $tg = ht'$, $h\in\langle x\rangle$

$$(w, w') = (w\otimes t, w'\otimes t)$$
$$((w\otimes t)g, (w'\otimes t)g) = (wh\otimes t', w'h\otimes t')$$
$$= (wh, w'h)$$
$$= (w, w') = (w\otimes t, w'\otimes t).$$

Also G permutes the summands $W \otimes t$. Since $W \otimes t$ is an inner product space and the summation $\oplus W \otimes t$ is orthogonal G acts on the product of the spheres $S(W \otimes t)$, each S^1 the circle. Also x fixes no point of this product as x fixes no point of $S(W)$.

Now, for each $x \in G$, $x \neq 1$ construct such an action and take the product of all the tori so constructed. This completes the proof.

Here's another argument, actually proving a stronger result. Express $G \cong F/R$, where F is a free group. Let $H = F/R'$ so, assuming F is finitely generated, R/R' is finitely generated free abelian and self-centralizing in $F/R' = H$. Moreover, H is torsion-free. Hence, by a result in J. Wolf's book, H acts by Euclidean transformations of some Euclidean space E such that R/R' acts as translations (which span - i.e. have maximal rank). Hence $E/(R/R')$ is the desired torus.

Using our linear techniques, however, we can also prove the following:

<u>Theorem</u> (Elliott Stein) If G has rank one then G acts freely on a product of two spheres.

Here, the rank of G is the maximal rank of all abelian subgroups of G.

We shall observe the

Lemma If G is a finite group of rank one then G has a cyclic normal subgroup N of odd order such that G/N has at most one involution.

Hence, by Wall + company, G/N acts on a sphere freely. We can get, by induction, as N is normal in G, an inner product space, with G acting and no element of $N^{\#}$ fixing any non-zero vector. For this will be true on each summand of the induced module since N is normal. Using the sphere in this space crossed with the Wall sphere, we are done. (like an argument in Ron Lewis' paper TAMS 132 p538.)

Pf (of Lemma). $G/O(G)$ has at most one involution by the Brauer-Suzuki theorem. If $G = O(G)$ we're done by setting $N = 1$. Otherwise let $Z/O(G)$ be the subgroup of order two of $G/O(G)$. Hence, every Sylow subgroup of Z is cyclic so that Z is metacyclic. Let $N = Z'$ so that N is cyclic, of odd order, normal in G and Z/N has exactly one involution so that G/N also has that property.

Note that our proof does rely on deep results applied to G/N and Stein proceeds directly using surgery.

We now turn our attention to Gene Lewis' conjecture: If G acts freely on $S^n \times S^n$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \leq G$ then G has an involution in its center. Recall Milnor's theorem that if G acts freely on S^n and $\mathbb{Z}_2 \leq G$ then G has an involution in its center. Now in proving Milnor's theorem by induction on $|G|$, we could apply the Brauer-Suzuki theorem and deduce that D_{2p}, p odd is a minimal counterexample. Of course, this is not how the proof goes, but let's try the same for Lewis' conjecture.

Prop If G is a minimal counterexample to Lewis' conjecture then one of the following holds:

a) $G \cong D_{2p} \times D_{2q}$ for odd primes p and q.

b) G contains a normal fours group V, a 3-element y normalizing but not centralizing V and $G = V\langle y \rangle$.

Pf. Suppose that G contains a fours subgroup W and $C_W(O(G)) = 1$. Then we can apply Lemma 5.34 of Thompson's N-group paper to $W \cdot O(G)$ and deduce that $D_{2p} \times D_{2q} \leq G$ and the proposition holds. Hence, we may now assume that if W is a fours subgroup of G then $C_W(O(G)) \neq 1$.

Now let z be an involution of G such that $\bar{z} \in \bar{G} = G/O(G)$ is in the center of a Sylow 2-subgroup of $\bar{G}$. If $\bar{z} \notin Z(\bar{G})$ then there is a 2-subgroup $\bar{T}$ of $\bar{G}$ containing $\bar{z}$ and an element $\bar{y}$ of odd order in $N_{\bar{G}}(\bar{T})$ with $\bar{z}^{\bar{y}} \neq \bar{z}$. Moreover, we can assume $\bar{z} \in Z(\bar{T})$ so $\langle \bar{z}, \bar{z}^{\bar{y}} \rangle$ is a fours subgroup and is normalized by $\bar{y}$ since

G, and hence $\bar G$, has 2-rank 2 by Conner's theorem ($Z_2 \times Z_2 \times Z_2 \not\subseteq G$). Moreover, since the three involutions of the fours subgroup $\langle \bar z, \bar z \bar y \rangle$ are conjugate we deduce that its preimage in G is the direct product of $Z_2 \times Z_2$ and $O(G)$, by the previous paragraph. This leads to case b), so we may now assume that if $\bar z$ is an involution in the center of a Sylow 2-subgroup $\bar S$ of $\bar G$ then $\bar z \in Z(\bar G)$.

Hence, if $Z(\bar S)$ has 2-rank exceeding one then there is a fours subgroup of $Z(\bar G)$. But some involution in there must centralize $O(G)$, by the first paragraph, so we deduce that $Z(G) \neq 1$. Thus, we may now assume that $Z(\bar S)$ is cyclic.

Thus, we may choose a non-central involution $\bar t$ of $\bar S$. Let $\bar u$ be an $\bar S$ conjugate of $\bar t$ other than $\bar u$ so $\langle \bar t, \bar u \rangle$ is a dihedral group of order 2^k, $k \geq 2$. Since $\bar G$ has 2-rank 2 and $\bar z \in Z(\bar S)$ we deduce that there is a fours subgroup $\langle \bar z, \bar s \rangle$ of $\bar G$ with $\bar s$ conjugate to $\bar s \bar z$. We may assume that $z \notin C(O(G))$, or we get $z \in Z(G)$, so letting $\langle z, s \rangle$ be a fours subgroup pre-image, $C_{\langle z,s \rangle}(O(G))$ is $\langle s \rangle$ or $\langle sz \rangle$ as it is not 1. This contradicts the conjugacy of $\bar s$ and $\bar s \bar z$ as $s \in C(O(G))$ implies $sz \in C(O(G))$ and conversely.

This establishes the proposition.

Henceforth we assume that $G = V\langle y \rangle$ *as in b). We assume that* G *acts freely on* $S^n \times S^n$.

<u>Lemma 1</u> There is a complex of free $\mathbb{Z}G$-modules

$$C_{2n} \to C_{2n-1} \to \cdots \to C_0$$

which has the following homology, as $\mathbb{Z}G$-modules:

a) $\mathbb{Z}$ in dimensions 0 and $2n$;

b) $\mathbb{Z} \oplus \mathbb{Z}$ in dimension n;

c) 0 elsewhere.

<u>Proof</u>. We just take the singular chain complex. No non-identity element of G fixes any element of its natural basis, i.e. maps of the standard k-simplex into $S^n \times S^n$, since G acts freely so moves some point in the image of the map. The homology, as abelian groups is now right. Also in dimension $2n$ it is $\mathbb{Z}$ as G can't invert $\mathbb{Z}$ since $G = O^2(G)$. In dimension 0 it's also clear by inspection. Now n is odd by a result of Rene Lewis (TAMS 132 p 538) so by the Lefschetz formula, the trace of g on the middle homology is the sum of the other traces, that is 2. Hence, g acts trivially there and so does G, thus proving the lemma. (e.g. go over to $\mathbb{C}$ by tensoring.)

<u>It may happen!?</u> Say $G = A_4$, F is algebraically closed of char 2. Have

$$0 \to \Omega^{3k}(F) \to C_{3k-1} \to \cdots \to C_0 \to F$$

$$0 \to \Omega^{3k-1}(F)^* \to \Omega^{3k}(F) \to F \oplus F \to 0$$

$$0 \to F \to C_{6k-2} \to \cdots \to C_{3k} \to \Omega^{3k-1}(F)^*$$

so suggests A_4 acts on $S^{3k-1} \times S^{3k-1}$, where also need $3k-1$ odd. (Which is also forced, it seems likely, from char 3 considerations).

This suggests that $n \equiv 2$ (modulo 3), so as n is odd, probably $n \equiv 5$ (modulo 6). This is also suggested by the spectral sequence of the covering

$$H^p(A_4, H^q(S^n \times S^n, Z_2)) \Rightarrow H^*(S^n \times S^n / A_4, Z_2)$$

The only obvious way the differentials can kill off all the cohomology above dimension $2n$ forces - it seems likely - that $n \equiv 2$ (mod 3). For example, if $n = 5$, the E^2 term - just listing dimensions

1	0	1	2	1	2	3	2	3	4	3	4	5	4	5	6	—
0	0															
0	0															
0	0															
0	0															
2	0	2	4	2	4	6	4	6	8	6	8	10	8	10	12	
0	0															
0	0															
0	0															
0	0															
1	0	1	2	1	2	3	2	3	4	3	4	5	4	5	6	

(The middle is the sum of the ends.)

<u>Prop</u> Under case d), $n \equiv 5 \pmod{6}$.

<u>Pf</u> We just use Corollary 2.7 in Gene Lewis' paper, TAMS 132, p 535. We have

$$0 \to H^n(G,\mathbb{Z}) \to H^{n+1}(G,\mathbb{Z}) \to \mathbb{Z}/|G|\mathbb{Z} \oplus \mathbb{Z}/|G|\mathbb{Z} \to H^{n+1}(G,\mathbb{Z}) \to H^n(G,\mathbb{Z}) \to 0$$

so taking the 2-part,

$$0 \to H^n(G,\mathbb{Z})_2 \to H^{n+1}(G,\mathbb{Z})_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_4 \to H^{n+1}(G,\mathbb{Z})_2 \to H^n(G,\mathbb{Z})_2 \to 0$$

But $H^n(G,\mathbb{Z})_2$ is elementary, $n > 0$. Let tabulate

$\dim_{GF(2)} H^n(G,A)_2$	$\underline{\underline{n}}$										
	1	2	3	4	5	6	7	8	9	10	11
$\underline{\underline{A}}$ $\mathbb{Z}_2$	0	1	2	2	2	3	2	3	4	3	4
$\mathbb{Z}$	0	0	1	1	0	2	1	1	2	2	1

All that matters in the Lewis sequence is the difference of the dimensions, which must be two. This gives the result.

On Morita equivalence of blocks

Cliff's work on Fong's theory has as an easy consequence - not stated as such in his work - the

Theorem If G is a p-nilpotent group, F is an algebraically closed field of characteristic p, B is a p-block of G with defect group D and b is the corresponding block of $N(D)$ then the algebras

$$B, \; b, \; FD$$

are Morita equivalent.

Let's just do a special case to see what's going on. Let $G = PK$, P a Sylow p-subgroup of G and K the normal p-complement. Suppose $D = P$. Let V be the simple B module so V_K is also simple. We'll prove that B and FP are Morita equivalent. To an FP-module U we associate

$$U \otimes V$$

where we regard U as FG-module, with K acting trivially. Now $V^*_K \otimes V_K$ has F as constituent with multiplicity one, by character theory. Since $p \nmid \dim V$ it follows that

$$V^* \otimes V \cong F \oplus \ldots$$

where the other summand has no trivial composition factor. Hence if U_1 is another FP-module then

$$\begin{aligned} \mathrm{Hom}_{FG}(U \otimes V, U_1 \otimes V) &\cong \mathrm{Hom}_{FG}(U \otimes U_1^*, V^* \otimes V) \\ &\cong \mathrm{Hom}_{FG}(U \otimes U_1^*, F \oplus \ldots) \\ &\cong \mathrm{Hom}_{FG}(U \otimes U_1^*, F) \\ &\cong \mathrm{Hom}_{FG}(U, U_1) \cong \mathrm{Hom}_{FP}(U, U_1) \end{aligned}$$

Hence, the maps of U to U_1 give rise exactly to the maps of $U \otimes V$ to $U_1 \otimes V$.

Now $FP \otimes V$ has V as composition factor with multiplicity $|P|$, is projective, so by standard results on the Cartan matrix, $FP \otimes V$ is the indecomposable projective in B. It's endomorphism algebra is the same as that of FP, by the above arguments. This proves our claim.

Vertices and defect groups

As usual G is a finite group, F is an algebraically closed field of prime characteristic p and B is a p-block of G with defect group D. We write B_H when we consider B as an FH-module where $H \leq G \times G$.

We're interested in a new proof of the

Theorem If U is an FG-module in B then U is relatively D-projective.

Usual argument: Can assume U is indecomposable and has vertex Q; need to show $Q \leq_G D$. Let V be the Green correspondent of U so V is an indecomposable $FN(Q)$-module with vertex Q. Let V lie in the p-block b of $N(Q)$ and let d be the defect group of b so $Q \leq d$. By Nagao's theorem, $b^G = B$ so $d \leq_G D$ and we're done.

We shall give a more direct module-theoretic approach.

Lemma $B_{\delta G}$ is relatively δD-projective.

Proof. Let U be an indecomposable summand of $B_{\delta G}$; it suffices to show that there is $x \in G$ such that U is relatively $\delta(D^x)$-projective. But $B_{G\times G}$ is relatively δD-projective so $B_{G\times G} \mid V^{G\times G}$ where V is a $F[\delta D]$-module. Thus, by Mackey's theorem, U is relatively $(\delta D)^{(g_1, g_2)} \cap \delta G$ -

projective for some $g_1, g_2 \in G$. But $(d,d)^{(g_1,g_2)} \in \delta G$ implies $d^{g_1} = d^{g_2}$ so $(\delta D)^{(g_1,g_2)} \cap \delta G \leq \delta(D^{g_1})$.
Thus $x = g_1$ works.

Now $G \simeq \delta G$ so we can regard B as a FG-module via $\beta \cdot g = g^{-1}\beta g$ for $\beta \in B$, $g \in G$. Hence B_G - this module is relatively D-projective. Let U be as in the theorem. Then $U \otimes B$, the FG-module, is also relatively D-projective; for $B \mid X^G$ where X is an FD-module yields $U \otimes B \mid U \otimes X^G \simeq (U_D \otimes X)^G$. Thus, the theorem is nearly proved once we establish the

<u>Lemma</u> $U \mid U \otimes B$.

<u>Proof</u>. There is an F linear map of $U \otimes B$ to U sending $u \otimes \beta$ to $u\beta$; for $u \in U$ an FG-module and $B \subseteq FG$. This is an FG-homomorphism:

$$\begin{array}{ccc} u \otimes \beta & \longrightarrow & u\beta \\ \downarrow & & \downarrow \\ ug \otimes g^{-1}\beta g & \longrightarrow & u\beta g \end{array}$$

It is a split epimorphism in fact. For let ϵ be the identity of B and map $u \to u \otimes \epsilon$. This is an FG homomorphism of U to $U \otimes B$:

$$\begin{array}{ccc} u & \longrightarrow & u \otimes \epsilon \\ \downarrow & & \downarrow \\ & & ug \otimes g^{-1}\epsilon g \\ ug & \longrightarrow & ug \otimes \epsilon \end{array}$$

The composite sends $u \to u \otimes \epsilon \to u\epsilon = u$ so our claim + the lemma hold

Permutation modules

We apply module-theoretic methods to permutation modules. This suggests the query: Is this in Scott's work? Is this a new argument? ⊛

Let X be a transitive G-set and let F be a field of prime characteristic p. Then the FG-module FX has a unique F at the "top" and at the "bottom." For $\sum_{x \in X} x$ and its multiples are the only fixed points and the module FX is self-dual. (Or use the module form of Frobenius reciprocity.)

Theorem. There is an indecomposable summand U of FX with the following properties:

a) U has a submodule and a quotient module isomorphic with F;

b) a Sylow p-subgroup Q of a point stabilizer is a vertex of U;

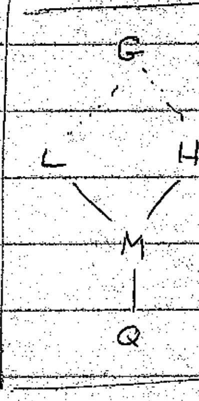
c) F is the source of U;

d) The Green correspondent of U is the FL-module, $L = N(Q)$, which is $F[L/Q]$-module, regarded as FL-module, which is the projective cover of F.

The argument depends on Burry's idea of applying the Green correspondence twice to analyze restriction + induction and on the Feit cohomology.

⊛ Scott says a) is in his Archiv paper and the rest is known to him too.

<u>Proof</u> Let H be a point stabilizer so $FX \simeq (F_H)^G$, the induced module. Let Q be a Sylow p-subgroup of H, $L = N_G(Q)$, $M = N_H(Q)$.



1) $(F_M)^G \simeq FX + \dots$, where the dots represent FG-modules with vertices conjugate to proper subgroups of Q.

Indeed, F_M is the Green correspondent of the FH-module F_H so $F_M{}^H = F + \dots$ with the same meaning for the dots. Now induce these modules up to G and apply transitivity of induction.

We get an immediate consequence:

2) Can assume $M = H$; that is, it suffices to prove the result for $(F_M)^G$.

Our next step is

3) $F_M{}^L$ is a projective $F[L/Q]$-module and in a decomposition into indecomposable modules has the projective cover V of F occurring with multiplicity one.

For M/Q is a p'-group and $F_M{}^L$ is a permutation module. The summands of $(F_M)^L$ have vertex Q. So it's enough to prove the following

4) The Green correspondent U of V has a submodule and a quotient module isomorphic with F.

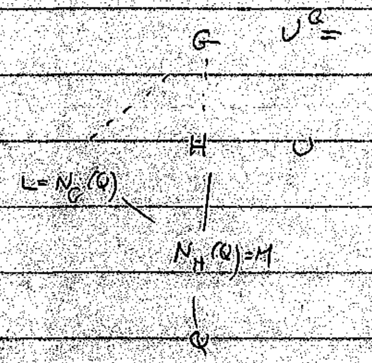
Since U is self-dual, being the Green correspondent of the self-dual module V, it suffices to produce a submodule of U isomorphic with F. Hence, it suffices to see that $H^0(G, U) \neq 0$ or $H^0(G, \mathfrak{X}, U) \neq 0$, with the usual Green correspondence notation and Feit cohomology. Hence, it's enough to show that $H^0(L, \mathcal{Y}, V) \neq 0$ – by Feit's theorem on the Green correspondence.

But* $H^0(L, V) \simeq F$ by the structure of V. Hence, we only need that $C_V(P)\, N_{P,H} = 0$ whenever $P \in \mathcal{Y}$. But if $P \in \mathcal{Y}$ then $P \not\supseteq Q$ since either $|P| < |Q|$ or $P \neq Q$ and P and Q are G-conjugate. $\therefore$ $P \subsetneq PQ$ so we need only show

$$C_V(P)\, N_{P,PQ} = 0.$$

Choose a "cross section" of P in PQ consisting of elements of Q. Each of these acts trivially on V and the number of these is $|P:PQ|$ which is a positive power of p. Hence, since F has characteristic p, all is proved.

Now let's look a bit deeper and compare with Scott's work. First, let's see what Burry's approach gives. Here's the picture



* This argument known to L. Scott.

Here U is an indecomposable FH-module with vertex Q. Let V be an FM-module which is the Green correspondent of U. Thus V also has vertex Q so V_Q is a direct sum of conjugates of a common source of U and V. Thus, $(V^L)_Q$ also has this property so V^L is a direct sum of indecomposable FL-modules all with the same source. We now get a one-to-one correspondence, respecting multiplicities, between isomorphism classes of indecomposable summands of V^L and indecomposable summands of U^G which have vertex Q.

This correspondence, when $U = F_H$, $V = F_M$ is part of Scott's generalization of Brauer's First Main Theorem. However, he goes much further than this and gets all non-projective summands of $F_H{}^G$.

On Brauer's First Main Theorem

Burry quite correctly has pointed out that the theorem on p190 is not the full first main theorem: if b is a block of $N(D)$ with defect D then one needs that b^G also has defect D. This seems difficult.

Now Haweník has shown that every subgroup of the defect group of a block is the vertex of a module in the block. He uses Nagao's theorem plus the "non-zero-ness" of the Brauer homomorphism. Let's use his result and see if this is relevant to our difficulties.

Assume: $H = N(Q)$, b is a p-block of $N(Q)$ with defect Q and $B = b^G$ has defect group strictly larger than Q.

Lemma Under these hypotheses, there is a p-block b^* of $N(Q)$ with defect group strictly containing Q and $(b^*)^G = B$.

Proof. There is a p-subgroup R of H with $Q \subsetneq R$ and R contained in a defect group of B. Let U be an indecomposable module of vertex R and in B. Choose an indecomposable FH-module V with $V | U_H$ and V also having vertex R. Now $U | U \otimes B_{\delta G}$ so $V | U_H | U_H \otimes B_{\delta H}$. But $B_{\delta H}$ is a direct sum of blocks of H as δH-modules and modules whose vertices do not contain δQ; for we know about $B_{H\times H}$ and then it's easy to look at $B_{\delta H}$. $\therefore$ there must be a summand of $B_{\delta H}$ with vertex properly containing Q so there must be a block b^* of H of defect group properly containing Q and $b^* | B_{H\times H}$. Done.

This suggests a new plan to get at the main results module-theoretically.

Step 1. Definition + interpretation of defect groups.

Step 2. The correspondence $b \to b^G$.

Step 3. Nagao's theorem

Step 4. Hamernik's theorem (+ need existence of $b \ni b^G = B$)

Step 5. Local description of blocks (i.e. extended 1st main th.)

Step 6. First main theorem (use above lemma, what we already know + use step 5 to contradict defect gp of b^* not stabilizing the rest of b, i.e. 3rd main th. approach)

Let's do step 3 right now. We have G, B with idempotent E. Let $H = N(Q)$ be a p-local subgroup, b the sum of the the blocks corresponding to B, e the corr. idempotent.

Now $B_{H\times H} \simeq b \oplus c$ where the vertices of the indecomposable summands of c do not contain δQ. Now $be = b$ and $B = Be + B(1-e)$ is a direct sum of $H\times H$-modules, so every indec summand of the $H\times H$ module $B(1-e)$ has vertex not containing δQ. Now regarding $B(1-e)$ as an FH-module via $H \simeq \delta H$ and the δH action, it follows that every indec summand of the FH-module $B(1-e)$ has vertex not containing δQ. (This is an argument we've used before. The vertex of $B(1-e)$ is some δR and we have to look at how conjugates of this intersects δH.)

Hence, it suffices to prove the

__Lemma__ If V is an FG-module in B then $V_H(1-e) \,\big|\, V_H(1-e) \otimes B(1-e)$ where we are regarding $B(1-e)$ as an FH-module.

Let $V = U + W$ where $U = Ve$, $W = V(1-e)$. We know, from previous sections of these notes, that $V \,\big|\, V \otimes B$. We have maps of modules, $V \otimes B \to V$ and $V \to V \otimes B$ whose composite is 1_V. These maps $v \to v \otimes E$ and $v \otimes \beta \to v\beta$ for $v \in V$, $\beta \in B$. Now $B = Be + B(1-e)$ a direct sum as H-modules and projection on the second factor sends E to $E(1-e)$. $\therefore$ this is an FH-homomorphism of W to $W \otimes B(1-e)$ which sends each $w \in W$ to $w \otimes E(1-e)$. Also, the restriction of the maps $V \otimes B \to V$ to $W \otimes B(1-e)$ send $w \otimes \beta$ to $w\beta$, for $w \in W$, $\beta \in B(1-e)$. Now $\beta/e = 0$ as $\beta \in B(1-e)$ so $(w\beta)e = 0$ and $w\beta \in W$. The composite of these maps sends

$$\begin{aligned} W \to W \otimes E(1-e) &\to wE(1-e) \\ &= w(1-e) \\ &= w \end{aligned}$$

as $w \in V(1-e)$.

To do step 4 we need the steps in parentheses there, which we get from the following lemma of Burry:

Lemma (Burry). If U is an indecomposable FG-module with vertex R and trivial source and $Q \leq R$ then there is an indec. summand V of $U_{N(Q)}$ which has vertex containing $R \cap N(Q)$.

Pf (Burry) U_R has F as a summand so that $U_{R \cap N(Q)}$ does too. Hence, there is an indec summand V of $U_{N(Q)}$ such that $V_{R \cap N(Q)}$ has F as a summand, i.e. $V_{R \cap N(Q)}$ has a summand with vertex $R \cap N(Q)$.
$\therefore$ there is a vertex of V containing $R \cap N(Q)$.

Now Burry points out the consequence (already known with usual theory)

Prop If B is a block of G with defect group D and $Q \leq D$ then there is a block b of $N(Q)$ with $b^G = B$.

Pf B is an indec $G \times G$-module with vertex δD and trivial source. Hence, there is a summand of B_N, $N = N_{G \times G}(\delta Q)$ which has vertex containing δQ, even $N \cap \delta D$. Hence, there is a summand of $B_{H \times H}$, $H = N(Q)$, which has vertex containing δQ. This must be a block.

Now we turn to Harris's theorem and follow his argument.

<u>Theorem</u> (Harris) If B is a block with defect group D and $Q \le D$ then there is an indec module in B with vertex Q.

<u>Pf</u> Let b be a block of $N(Q)$ with $b^G = B$. By the Green correspondence, and Nagao's theorem, it suffices to produce an indecomposable $FN(Q)$ module in b which has vertex Q. Let S be a simple module in b and let U be its projective cover <u>as</u> $FN(Q)/Q$ module — which we can do as S is an $FN(Q)/Q$ module, Q being a p-group. This U works, as it is certainly relatively Q-projective while U_Q is a direct sum of F many times.

Wallace's radical problem

There is a question of Wallace about the dimension of the radical of group algebras (see Motose, Hokkaido Math J. 6 (1977) no 2, 255-259) which comes down to the following: assume F is an algebraically closed field of characteristic p, G is a finite group and P_F is a projective cover of the trivial FG-module F; if for any simple FG-module S, $P_F \otimes S$ is a projective cover of S does G have a normal Sylow p-subgroup?

Assume henceforth that G satisfies the hypotheses.

Lemma 1 Every simple FG-module S has dimension not divisible by p.

Proof We have $\mathrm{Hom}_{FG}(S \otimes P_F, S) \simeq F$ as $S \otimes P_F$ is the projective cover of S. Hence $\mathrm{Hom}_{FG}(P_F, S^* \otimes S) \simeq F$ so $S^* \otimes S$ has F as a composition factor with multiplicity one. But $S^* \otimes S \simeq \mathrm{Hom}_F(S, S)$ as FG-modules so we cannot have the elements of trace zero containing the "scalars." This proves the lemma.

Lemma 2 If H is a p-solvable group with a non-normal Sylow p-subgroup then H has normal subgroups L and K with $L \supseteq K$, L/K p-nilpotent and $O_p(L/K) = 1$.

Proof Let $L = O_{p,p',p}(H)$, $K = O_p(H)$.

<u>Proposition</u> (Wallace) If G is p-solvable then G has a normal Sylow p-subgroup.

<u>Pf</u> (New one) Apply Lemma 2 to G, assuming this proposition is false, and let L, K be the appropriate subgroups. Thus, H/K is such that there is a simple $F[H/K]$-module of dimension divisible by p – as there is a simple $F[O_{p'}(L/K)]$-module S_1 not stabilized by L.⊛ Hence, by Clifford's theorem, there is a simple FG-module of dimension divisible by $\dim_F S_1$, contradicting Lemma 1.

<u>Def.</u> A group H has the <u>multiplicative lift property</u> if $\varphi_1, \dots, \varphi_s$ are the irreducible Brauer characters mod p and there exist ordinary irreducible characters $\chi_1, \dots, \chi_s$ such that φ_i is the restriction of χ_i to the p'-elements and such that whenever $\varphi_i \varphi_j = \sum a_{ijk} \varphi_k$ then $\chi_i \chi_j = \sum a_{ijk} \chi_k$.

<u>Proposition</u>. A group H has the multiplicative lift property if, and only if, it has a normal Sylow p-subgroup.

<u>Pf</u> If P is a normal Sylow p-subgroup then this is clear, as P is in the kernel of each φ_i. For the converse, we can replace F by a splitting field – still denoted by F – such that F is the residue

⊛ By the Brauer lemma, if an elt fixes each character of $O_{p'}(L/K)$ then it fixes each class. Now use the relative coprime property.

class field of the complete discrete valuation ring R whose quotient field K is a splitting field of characteristic zero for H. Part of our assumption is that there is an RH-lattice L_i such that the FG-module $\bar{L}_i$ is irreducible with Brauer character φ_i while the KH-module $K \otimes L_i$ is irreducible with character χ_i.

Now the intersection of the kernels of the $\bar{L}_i$ is $O_p(H)$ so it suffices to show that $\bar{L}_i \otimes \bar{L}_j$ is semi-simple as then $F[H/O_p(H)]$ will be semi-simple and so $|H:O_p(H)|_p = 1$ as desired.

Suppose $\varphi_i \varphi_j = \sum a_{ijk} \varphi_k$, a_{ijk} non-negative integers. But then $\chi_i \chi_j = \sum a_{ijk} \chi_k$ so $L_i \otimes L_j$ has an R-form $\oplus a_{ijk} L_k$, since $(K \otimes L_i) \otimes (K \otimes L_j) = \oplus a_{ijk}(K \otimes L_k)$. Now L_{k_1} and L_{k_2} have no composition factors in common so every R-form of $\oplus a_{ijk} L_k$ is really a direct sum of R-forms of each of the $a_{ijk} L_k$. Hence, $\bar{L}_i \otimes \bar{L}_j$ is a direct sum of indecomposable modules each of which has all its composition factors isomorphic.

Hence, the projective modules for $H/O_p(H)$ have this property, in particular the projective cover of F so $\bar{H} = H/O_p(H)$ is p-nilpotent.

Suppose that H is not p-nilpotent; hence, there is an irreducible Brauer character ψ of $O_{p'}(\bar{H})$ not stable in $\bar{H}$. Let φ be a corr. irred Brauer character of $\bar{H}$ so that $\varphi | O_{p'}(\bar{H})$ is the sum of the conjugates of ψ. Thus $\varphi \bar{\varphi}$ involves the trivial character ι if χ, χ' are the corr ord irred characters then $\chi\chi'$ also involves the principal character. $\therefore \chi' = \bar{\chi}$.

Now $\chi | O_{p,p'}(H) \cap p'$-elements $= \psi$. $\therefore$ by Clifford theory, there is a maximal subgroup M of H, $|H:M| = p$, $H \rhd M$ with χ vanishing on $H - M$.

Let λ be a linear character of H/M, not the principal character. Then $\chi\lambda = \chi$ and $(\chi\chi', \lambda) = (\chi\lambda \cdot \chi', \chi_0) = (\chi\chi', \chi_0) = 1$, where χ_0 is the principal character. $\therefore$ by the multiplicative property, λ corresponds to one of the irreducible Brauer characters. But λ & χ_0 agree on all p'-elements of G, a contradiction.

Here's another proof suggested by Blauberman: Let N be the intersection of the kernels of the χ_i, $1 \leq i \leq s$. If $x \in N$ and x is a p'-element then $\chi_i(x) = \chi_i(1)$ so $\varphi_i(x) = \varphi_i(1)$ so x is in the kernel of all the φ_i's, by the usual argument. Hence $x = 1$ as the intersection of the kernels of all the φ_i's is $O_p(G)$.
Thus $N \subseteq O_p(G)$ so the number of ordinary irreducible characters of $G/O_p(G)$ is at most s, by the fact that $\chi_1, \ldots, \chi_s$ are the ordinary irreducible characters of G/N by Burnside's theorem. But s is the number of modular irreducible characters of $G/O_p(G)$ so $G/O_p(G)$ is a p'-group.

Filtrations and χ-modules

Humphreys states the following problem: "Given a PIM, I want to find a filtration of it such that each quotient is the reduction mod p of a complex irreducible representation." This certainly can be done; in fact much more can be said.

Let R be a usual valuation ring so that the field of quotients K and the residue class field F of R are splitting fields for G of characteristics 0 and p, respectively. If χ is an irreducible character of G in K then an FG-module U which is the reduction mod p of a representation affording χ is said to be a χ-module. That is, there is an RG-lattice M (RG-module, finitely generated and free as an R-module) with character χ such that $\bar{M} \simeq U$, where $\bar{M}$ is the reduction of M mod p. We also say M lifts U to R or M is a lift of U.

Proposition. Let V be an FG-module which lifts to the RG-lattice M. Let the character of M be $\chi_1 + \dots + \chi_n$ where each χ_i is an irreducible character of G in K. There exists a series of submodules of V,

$$V = V_0 \supset V_1 \supset V_2 \supset \dots \supset V_{n-1} \supset V_n = 0,$$

such that V_{i-1}/V_i is a χ_i-module, $1 \le i \le n$.

Proof. We proceed by induction on n. The case $n=1$ is trivial. Now M contains as submodules all the R-forms of $\chi_1 + \dots + \chi_n$ so there is a submodule M_n which is an R-form of χ_n. The set of all elements $m \in M$ with some $rm \in M_n$ is also an R-form of χ_n; call this N so N is a pure submodule of M and M/N is an RG lattice. Moreover, reducing mod p, using "tars," $\overline{M/N} \simeq \overline{M}/\overline{N}$ and M/N lifts $V/\overline{N}$. We are now done by induction.

This result appears to be useful in studying lifting. Let's look at an example: $G = A_5 \simeq SL(2,4)$, $p = 2$. The decomposition matrix:

	V_0	V_1	V_2	V_{12}
χ_1	1	0	0	0
χ_2	1	1	0	0
χ_3	1	0	1	0
χ_4	1	1	1	0
χ_5	0	0	0	1

Lifting V_{12} is no problem, as is V_0. Suppose we have an FG-module with composition factors such that it might lift to a χ_i, $2 \le i \le 4$. Does it? We'll use our result to answer this, together with the use of our knowledge of FG-modules.

Let P_0, P_1, P_2 be the projective covers of V_0, V_1, V_2 and use the usual diagrams. Let's concentrate on the reduction mod 2 of χ_4. One of these is a submodule of P_0 so it must be $\begin{array}{ccc} 2 & & 1 \\ & 0 & \end{array}$. From quotients get $\begin{array}{ccc} & 0 & \\ 1 & & 2 \end{array}$. Also same for P_0/V_0. If it were $\begin{array}{c} 1 \\ 0 \\ 1 \end{array}$ then, picture

$$\begin{array}{ccc} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \\ & 0 & \end{array}$$

there no submodule corresponding to χ_3. Hence, is $\begin{array}{c} 0 \\ 2 \end{array} + 1$ or $\begin{array}{c} 0 \\ 1 \end{array} + 2$. $\therefore$ get all of these: $\begin{array}{c} 0 \\ 2 \end{array} + 1$, $\begin{array}{c} 0 \\ 1 \end{array} + 2$, $\begin{array}{c} 2 \\ 0 \end{array} + 1$, $\begin{array}{c} 1 \\ 0 \end{array} + 2$. From P_1

will get $\begin{smallmatrix}1\\0\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\0\\1\end{smallmatrix}$ and so have all – by inspection – the FG-modules with composition factors V_0, V_1, V_2 (each once) lifting with the possible exception of $V_0 \oplus V_1 \oplus V_2$. Certainly, this can not be lifted by our methods.

What about reductions mod 2 of χ_2, χ_3. We just have seen for P_0 that we have

$$\begin{array}{ccc} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \\ & 0 & \end{array}$$

(or with 1 and 2 interchanged). There must be a reduction mod 2 of χ_2 on top of this, i.e. $V_0 \oplus V_1$ ∴ easy to get $V_0 \oplus V_1$, $\begin{smallmatrix}V_0\\V_1\end{smallmatrix}$, $\begin{smallmatrix}V_1\\V_0\end{smallmatrix}$, $V_0 \oplus V_2$, $\begin{smallmatrix}V_0\\V_2\end{smallmatrix}$, $\begin{smallmatrix}V_2\\V_0\end{smallmatrix}$ all lifting.

This is really just a generalization of the Thompson indecomposables. It works nicely to calculate some liftings for $SL(3, 2^n)$. Let $n' = N - n - \{1, \ldots, n\}$ in the usual notation so $C_{n'n'} = 2$. Hence $P_{n'}$ lifts to the sum of exactly two characters. It's now easy to describe some of their reductions modulo two

<u>Linear sources</u>

With the idea that modules with linear sources may be of interest, as they generalize permutation modules, we investigate them. Let K, R, F and G be as usual, p also the usual prime. If H is a subgroup of G then an RH lattice is linear if it is R as an R-module.

Our first result is quite easy:

<u>Proposition 1</u>. If G is a p-group and χ is an irreducible character of G in K then there is a one-to-one correspondence between isomorphism classes of R-forms of χ with a linear source and conjugacy classes of linear characters of subgroups of G which induce χ.

<u>Proof</u>. Let λ be a linear character of a subgroup H of G such that $\lambda^G = \chi$. Let U be the linear RH-module corresponding to H so $U \cong R$ as R-module and the element $h \in H$ operates on U by multiplication by $\lambda(h)$. Thus, U^G is an indecomposable module with linear source and has character χ.

In view of the Green correspondence, it remains only to see that these indecomposable modules all arise this way. Let V be such a module and let U be a linear source for it for the vertex H. But U^G is indecomposable so it is isomorphic with V. Hence, the linear character corresponding to U induces to χ.

And we are using the fact that the linear character determines U; for it's clear that this is how elements of H act.

We now look at groups which are not p-groups.

Theorem 1. Let P be a normal Sylow p-subgroup of G. If χ is an irreducible character of G of degree prime to p then χ has a unique R-form with vertex P and linear source. Moreover, every indecomposable RG-lattice with vertex P and linear source arises in this way.

If $m_p(G)$ is the usual McKay number and we now let $m_p'(G)$ be the number of indecomposable RG-lattices with vertex a Sylow p-subgroup of G and linear source then this theorem shows that $m_p(N(P)) = m_p'(N(P))$ if P is a Sylow p-subgroup of G. The Green correspondence gives $m_p'(N(P)) = m_p'(G)$ so we have the following consequence of the theorem:

Corollary 1. The McKay conjecture holds for the prime p for the group G if, and only if, $m_p(G) = m_p'(G)$.

Picture this:

$$\begin{array}{ccc} m_p(G) & \overset{?}{=} & m_p'(G) \\ \|? & & \| \\ m_p(N(P)) & = & m_p'(N(P)) \end{array}$$

Thus, one question mark can be erased if, and only if, the other one can be erased.

The main steps of the proof of the theorem is the following:

<u>Lemma 1</u>. If P is a normal Sylow p-subgroup of G, λ is a linear character of P, U_λ is an R-form of λ and $\lambda^G = \sum m_\chi \chi$ over characters χ of G then

$$U_\lambda^G \simeq \oplus\, m_\chi\, U_\chi$$

where U_χ is an R-form of χ.

Let's first see that this establishes the theorem. The characters χ with $m_\chi \neq 0$ will be characters of G/P' and hence of degree not divisible by p; hence, U_χ will have P as its vertex and so has U_λ as its source. Moreover, the character χ will not "appear" in μ^G, for a linear character μ of P, if μ is not conjugate to λ, by Clifford's theorem. Moreover, any RG-lattice with a linear source will be a summand of some U_λ^G so the theorem is established.

<u>Proof of the lemma</u>. Let K_λ be the kernel of λ, S_λ its stabilizer in $N(P)$ so S_λ/K_λ is the direct product of P_λ/K_λ and a p'-group M_λ/K_λ. Hence, $\lambda^{S_\lambda} = \sum_\psi \psi(1).\, \lambda \otimes \psi$, over the characters ψ of M_λ/K_λ. Also $U_\lambda^{S_\lambda} \simeq U_\lambda \otimes R[M_\lambda/K_\lambda]$ so $U_\lambda^{S_\lambda} \simeq \bigoplus_\psi U_\lambda \otimes U_\psi$ where U_ψ is the R-form of ψ. Induction to G from S_λ sends irreducibles to irreducibles by Clifford's theorem; the corresponding lattices are certainly indecomposable and the result is proved.

<u>Remarks</u>: Usual theory yields: U indec has vertex P, linear source $\Leftrightarrow$ U_P has rank one summand (i.e. U "<u>eigenmodule</u>")

$SL(2,2^n)$, an addendum

With the usual notation, we answer a question of G. Mason:

Proposition $P_\emptyset|_B \simeq FB$.

Proof #1. Let A be the cyclic subgroup of order 2^n+1 so $P_\emptyset \simeq (F_A)^G$ and thus by Mackey's theorem $P_\emptyset|_B$ is free.

Proof #2. First, we claim that $V_N|_B \simeq (F_H)^B$. Indeed, $V_0 \oplus V_N \simeq (F_B)^G$ so $F \oplus V_N|_B \simeq (V_0 \oplus V_N)_B \simeq ((F_B)^G)_B \simeq F \oplus (F_H)^B$, by Mackey's theorem.

Now $V_N \otimes V_N \simeq V_N \oplus P_0$ so

$$\begin{aligned} V_N|_B \oplus P_0|_B &\simeq (V_N \otimes V_N)_B \\ &\simeq V_N|_B \otimes V_N|_B \\ &\simeq (F_H)^B \otimes (F_H)^B \\ &\simeq \left(((F_H)^B)_H \otimes F_H\right)^B \\ &\simeq (F \oplus FH)^B \\ &\simeq (F_H)^B \oplus FB \end{aligned}$$

That is, in view of the preceding paragraph,

$$(F_H)^B \oplus (P_0)_B \simeq (F_H)^B \oplus FB,$$

so the result holds by Krull-Schmidt.

Decomposition numbers and $R(\theta)$

Suppose G is a group of Lie type and characteristic p. Let $R(\theta)$ be the Deligne-Lusztig character associated with the linear character θ of the torus T. If σ is the Steinberg character then $R(\theta)\sigma = \theta^G$. We can use this to study the decomposition modulo p of $R(\theta)$ provided we know the matrix A — see the paper on projectives and tensor products.

Let V be an irreducible FG-module and let d be the decomposition number corresponding to V and $R(\theta)$. Let $r(\theta)$ be the reduction mod p of $R(\theta)$, as it's a module. Then

$$d = \dim_F \operatorname{Hom}_{FG}(P_V, r(\theta))$$
$$= \dim_F \operatorname{Hom}_{FG}(X_V \otimes S, r(\theta))$$

where X_V is the formal module that comes from A, S is the Steinberg module. Hence

$$d = \dim_F \operatorname{Hom}_{FG}(X_V, S \otimes r(\theta))$$
$$= \dim_F \operatorname{Hom}_{FG}(X_V, \theta^G)$$
$$= \dim_F \operatorname{Hom}_{FT}(X_V/T, \theta)$$

which is just the corresponding multiplicity.

For $G = SL(2, 2^n)$ this gives Burkhardt's results quickly. This is because A is almost a permutation matrix. It's easy to carry out the details.

Irreducible maps

Let F be an algebraically closed field of characteristic two and $A = FA_5$. We shall exhibit an irreducible monomorphism $f: M \to N$ between two indecomposable A-modules such that $D\mathrm{Tr}\, f$ is not a monomorphism; in fact there will be no monomorphism from $D\mathrm{Tr}\, M$ to $D\mathrm{Tr}\, N$. This answers a question(*) of Auslander et al, in "Almost split sequences whose middle term has at most two indecomposable summands."

Let V_0, V_1, V_2 be the usual simple A-modules of dimensions one, two and two. Let $0 \to V_0 \to N \to V_2 \to 0$ be non-split so N is uniquely determined. It follows easily that $0 \to V_1 \to \Omega^2 N \to V_0 \to 0$ is non-split:

$$\begin{matrix} & & \begin{smallmatrix}1\\0\end{smallmatrix} \to \begin{smallmatrix}2\\0\\1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\0\end{smallmatrix} \\ \begin{smallmatrix}0\\1\end{smallmatrix} \to \begin{smallmatrix}2\\0\\1\end{smallmatrix} & & \end{matrix}$$

Also, similarly, $\Omega^2 V_0 \cong \begin{smallmatrix} & V_0 & \\ V_1 & & V_2\end{smallmatrix}$. Hence, there is no monomorphism of $\Omega^2 V_0$ to $\Omega^2 N$. Hence, it suffices to show that the embedding $f: V_0 \to N$ is irreducible.

However, the almost split sequence with last term N looks like

$$0 \to \Omega^2 N \to \ ? \ \to N \to 0$$

that is,

$$\begin{smallmatrix}0\\1\end{smallmatrix} \to \ ? \ \to \begin{smallmatrix}2\\0\end{smallmatrix}$$

so we want the middle term to be $V_0 \oplus \begin{smallmatrix}V_2\\V_0\\V_1\end{smallmatrix}$ as this will establish our claim.

If B is the middle term then we know its composition factors certainly. From the sequence, it also is clear that $\mathrm{soc}(B) \neq V_1$

(*) Would like to know if this happens in the category $\underline{C}$.

as we know the structure of the injective envelope of $\begin{smallmatrix}V_0\\V_1\end{smallmatrix} = \Omega^2(N)$.

Hence we have to find $B \subseteq P_1 \oplus P_0$.

Inside,

$$\begin{smallmatrix}1\\0\\2\\0\\1\end{smallmatrix} \oplus \begin{smallmatrix}&0&\\1&&2\\0&&0\\2&&1\\&0&\end{smallmatrix}$$

containing the socle is B. By Bongartz we get some cases:

$$\begin{smallmatrix}2\\0\\1\end{smallmatrix} \oplus 0 \;;\quad \begin{smallmatrix}0\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\0\end{smallmatrix} \;;\quad \begin{smallmatrix}&2\\0&\\1&0\end{smallmatrix} \;;\quad \begin{smallmatrix}&0&\\1&&2\\&&0\end{smallmatrix}\ (\leftarrow \text{out easily}) \quad \cdots$$

and others out easily as they don't fit. The third is just $\begin{smallmatrix}2\\0\\1\end{smallmatrix} \oplus 0$ again by mapping P_2 into it. The second is out as isomorphisms are not irreducible.

Another way: By duality, it suffices to show $\begin{smallmatrix}0\\2\end{smallmatrix} \to 0$ is irreducible. But for a simple module there is a criterion!

$$\begin{smallmatrix}1\\0\\2\\0\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\0\\1\\0\\2\end{smallmatrix} \to \begin{smallmatrix}&0&\\1&&2\\0&&0\\2&&1\\&0&\end{smallmatrix} \to 0$$

Take preimage of socle of P_0 in previous projective. Get a sequence

$$\begin{smallmatrix}&0&\\1&&2\end{smallmatrix} \to \begin{smallmatrix}0\\1\end{smallmatrix} \oplus \begin{smallmatrix}0\\2\end{smallmatrix} \to 0$$

and this – by the criterion – see a paper of Auslander ... – is the almost split sequence for V_0. $\therefore$ $\begin{smallmatrix}0\\2\end{smallmatrix} \to 0$ is irreducible, as desired.

Now we have $\begin{smallmatrix}0\\2\end{smallmatrix} \to 0$ is an irreducible epimorphism. Let's calculate the irred maps corr. to applying Ω^k and see whether we get epimorphisms or monomorphisms. We tabulate the results. (Circle the kernel or cokernel: guess this only)

e = epimorphism
m = monomorphism

k	$\Omega^k\binom{0}{2}$	$\Omega^k(0)$	
-2	1 0	2 1 0	m
-1	2 0 1	0 1 2 0 0 2 1	m
0	0 2	0	e
1	1 0 0 2 1 0	1 2 0 0 2 1 0	m
2	2 0 0 1 2 2 0	0 1 2	e
3	0 0 2 1 2 0	0 0 2 1 0	e
4	1 0 0 0 2 1 2 1 0 0	2 1 0 0 0 2 1 2 1 0 0	m
5	2 0 0 0 2 1 2 1 1 0 0	0 0 2 1 2 1 0	e
6	0 0 0 2 1 2 1 2 0 0	0 0 0 2 1 2 1 0 0	e

Now suppose S is a simple FG module for a gp G, field F. Let the projective cover of S be pictured by $\begin{smallmatrix}S\\M\\S\end{smallmatrix}$ (where M stands for "middle"). Let's guess some irreducible maps. First, let's calculate $\Omega^2\left(\begin{smallmatrix}S\\M\end{smallmatrix}\right)$:

$$0 \to \begin{smallmatrix}M\\S\end{smallmatrix} \to \begin{smallmatrix}S\\M\\S\end{smallmatrix} \to \begin{smallmatrix}S\\M\\S\end{smallmatrix} \to \begin{smallmatrix}S\\M\end{smallmatrix} \to 0$$

so $\begin{smallmatrix}M\\S\end{smallmatrix} = \Omega^2\left(\begin{smallmatrix}S\\M\end{smallmatrix}\right)$. Here's a guess for the almost split exact sequence:

$$\begin{smallmatrix}M\\S\end{smallmatrix} \to M \oplus \begin{smallmatrix}S\\M\\S\end{smallmatrix} \to \begin{smallmatrix}S\\M\end{smallmatrix}$$

so $M \to \begin{smallmatrix}S\\M\end{smallmatrix}$ is irred as is $\begin{smallmatrix}S\\M\\S\end{smallmatrix} \to \begin{smallmatrix}S\\M\end{smallmatrix}$. And of course, $\begin{smallmatrix}M\\S\end{smallmatrix} \to M$, $\begin{smallmatrix}M\\S\end{smallmatrix} \to \begin{smallmatrix}S\\M\\S\end{smallmatrix}$.

Now since $M \to \begin{smallmatrix}S\\M\end{smallmatrix}$ is irreducible so is $\Omega M \to \Omega\left(\begin{smallmatrix}S\\M\end{smallmatrix}\right)$. That is,

$$\Omega M \to S.$$

Here are the ones we've guessed so far:

$$\begin{tikzcd}
\begin{smallmatrix}M\\S\end{smallmatrix} \arrow[r] \arrow[dr] & \begin{smallmatrix}S\\M\\S\end{smallmatrix} \arrow[r] & \begin{smallmatrix}S\\M\end{smallmatrix} & \Omega M \arrow[r] & S. \\
& M \arrow[ur] & & &
\end{tikzcd}$$

Since irreducible maps between non-projective indecomposables are a stable equivalence invariant we're interested. Of course, monos & epis can correspond under such a relation. But if we could unravel that could we say what maps are epis & monos?

If we could then we could say a lot. For we can say in the stable category what epis & monos are (see paper by Auslander et al on the middle term of almost split exact sequences). We assume we're working over a group algebra. Let $\underline{\mathcal{C}}$ be the usual stable category. If U is a module, $U \in \mathcal{C}$ that is, write $\underline{U}$ when we consider $\underline{U} \in \underline{\mathcal{C}}$.

Prop $U \in \mathcal{C}$ is simple $\Rightarrow$ every non-zero map to $\underline{U} \neq 0$ is epi.

Pf. If U is simple this is trivial. If U is not simple want maps, φ to U such that φ not epi and $\underline{\varphi} \neq 0$.

Case 1 U not semisimple. Chose a maximal submodule M of U with $U \supsetneq M \supseteq \mathrm{soc}\, U$. $\therefore$ soc M = soc U. Let I be injective envelope of M so $M \subseteq I$. Let $\varphi: M \to U$. $\therefore \varphi$ not epi.

$\underline{\varphi} = 0 \Rightarrow \varphi$ extends to I. This is a mono as soc is mapped 1-1. $\therefore U$ is projective, $\underline{U} \neq 0$, RAA

Case 2 U semisimple. Express $U = V + X$ direct, V simple. $\varphi: V \to U$

Rest easy.

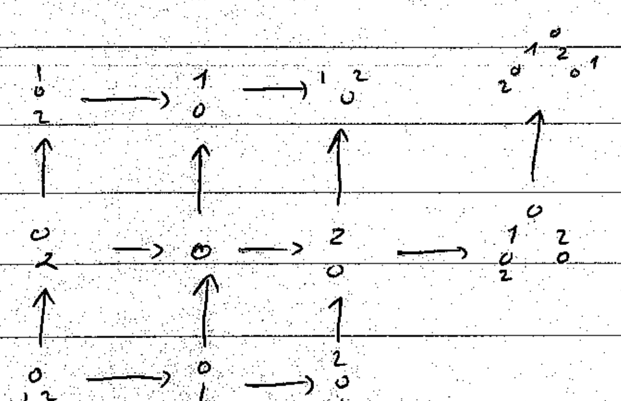
One last result:

Prop $U, V \in \mathcal{C}$, $\underline{\varphi}: \underline{U} \to \underline{V}$ is epi $\Leftrightarrow$ $\forall\, \underline{\psi}: \underline{V} \to \underline{S}$ S simple, $\underline{\psi} \neq 0$ have $\underline{\varphi}\,\underline{\psi} \neq 0$.

Pf. $\Rightarrow$: $\underline{\varphi}\,\underline{\psi} = 0 \Rightarrow \varphi\psi = 0$ as image is simple.

$\Leftarrow$: $\varphi: U \to V$ not epi $\therefore \exists\, \psi: V \to S \ni \varphi\psi \neq 0$ $\therefore \underline{\varphi}\,\underline{\psi} = 0$.

Let's return to the example on p 250: $0 \to \begin{smallmatrix}2\\0\end{smallmatrix}$. Let's show this is in $\underline{\mathcal{C}}$ – in Auslander et al notation. Let's calculate all irreducible maps, see what modules come in & keep going. We start and we seem to get the following – allowing some guesses:



There seem's to be a general pattern & we seem to be working with invertible modules (eg $\begin{smallmatrix}2\\0\end{smallmatrix} \otimes \begin{smallmatrix}0\\2\end{smallmatrix} \simeq 0 \oplus \begin{smallmatrix}2\\0\\1\\0\\2\end{smallmatrix}$.)

Here's the guess as to what's happening. There exist indecomposable modules V_{ij}, $i,j \in \mathbb{Z}\times\mathbb{Z}$, and maps $h_{ij}: V_{ij} \to V_{i+1,j}$, $u_{ij}: V_{ij} \to V_{i,j+1}$ such that the following conditions hold:

1) The h_{ij}, u_{ij} are irreducible;

2) $V_{ij} \xrightarrow{h_{ij}+u_{ij}} V_{i+1,j} \oplus V_{i,j+1} \xrightarrow{(u_{i+1,j},\, h_{i,j+1})} V_{i+1,j+1}$ is an almost split exact sequence;

3) $V_{00} \simeq V_0$;

4) Any indec module related to V_0 by sequences of irreducible maps is isomorphic to one of the V_{ij};

5) $\Omega^2 V_{ij} \simeq V_{i-1,j-1}$, $\Omega V_{ij} \simeq V_{i-2,j+1}$ (so $V_{ij} \simeq V_{i-3,j+3}$).

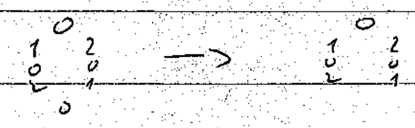
Picture:

$$\begin{tikzcd}
 & V_{01} \arrow[r] & V_{11} & \\
{} \arrow[r] & V_{00} \arrow[u] \arrow[r, "h_{00}"] & V_{10} \arrow[u] \arrow[r] & {} \\
 & {} \arrow[u] & &
\end{tikzcd}$$

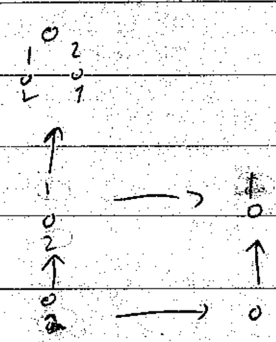
As on the last page, seem to be displaying the action of a group of invertible modules on V_0.

<u>However</u> <u>4)</u> is <u>false</u>. We're not in $\underline{C}$

By the above work on irreducible maps for a simple module S we get



is irreducible. Image is odd dimensional, so its Green correspondent is so this correspondent is invertible so the original image is also invertible. In the above picture, it comes in as follows:

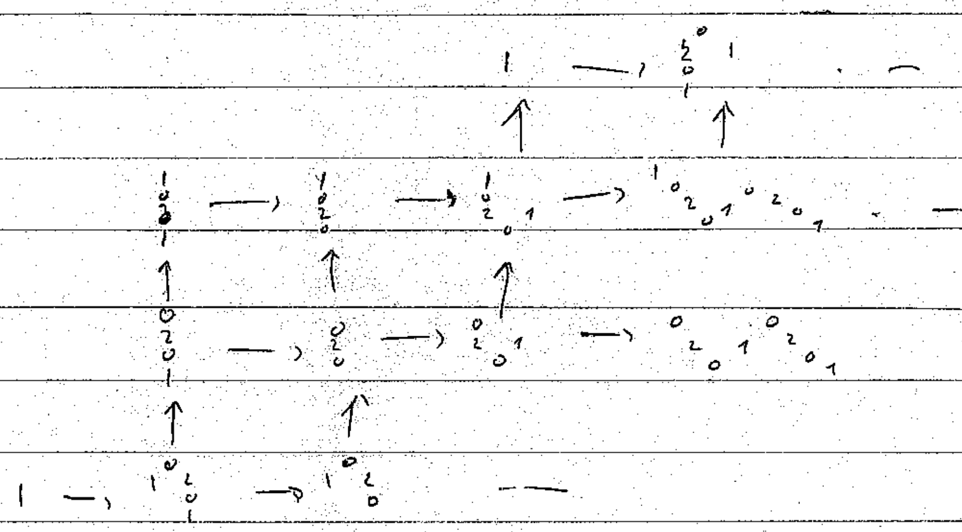


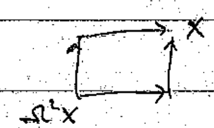
Seems likely that one of the equivalence classes, under irreducible maps, for FA_5 consists of the invertible modules plus the projective cover P_0 of V_0, i.e. the odd-dimensional indecomposable modules plus P_0.

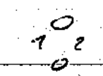
Seems there's an easy reason – for even dim indec are periodic by Green corr, and can't have irred maps between indec non projectives, one periodic, the other not.

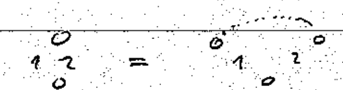
This should give an easy pf of some of the above

If we turn to the equivalence classes given by the other projectives we can make an educated guess. Seem to get the following picture:

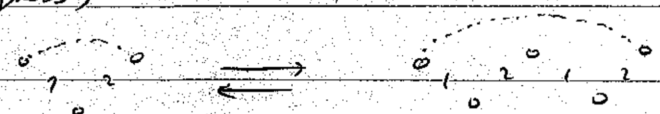


where:  so get a periodicity

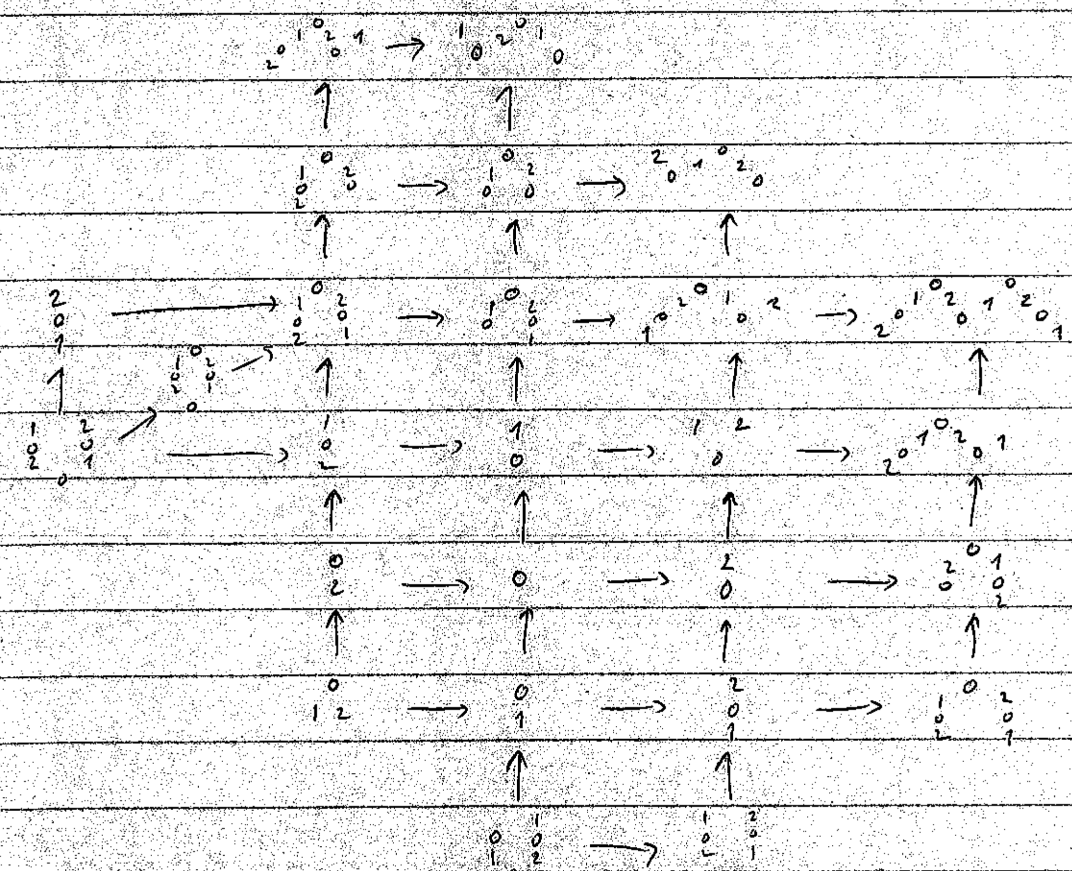
Here's a guess for a module in none of these classes:  .

That is  (dots = identify)

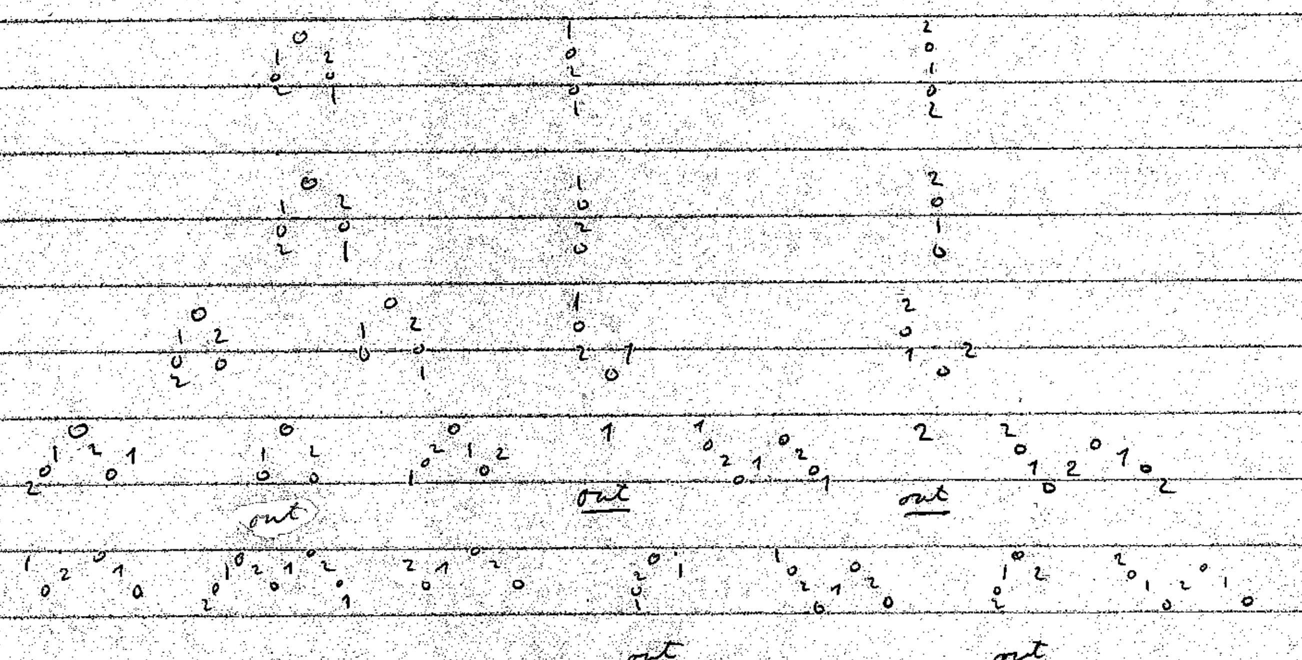
Some maps (a guess)



Let's look at the class containing V_0. Seems likely that we can correct the guess on page 256 by inserting P_0 at non-lattice points. Calculating, using rule 5) especially, we get the following

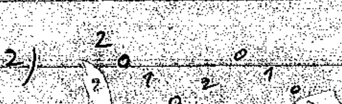
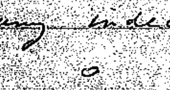


Now let's calculate the preprojective modules, using the algorithm described by Auslander: At each stage, take the modules you get from the preprojectives of a given level by irreducible maps, eliminate down to a generating set. The answer:



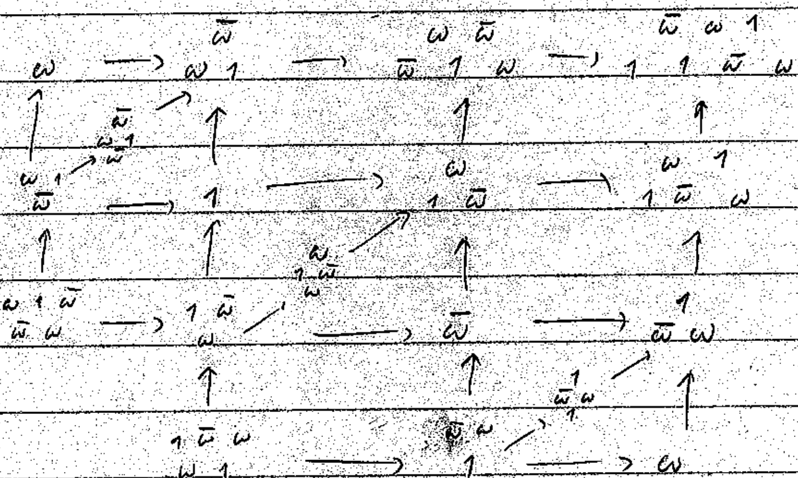
Tentative conclusions:

1) V_1 is a candidate but never recedes, i.e. not a preprojective (For V_1 doesn't come up again)

2)  is not a strong preprojective as there are infinitely many indecomposables mapping to the submodule  so mapping onto V_0 image of the preprojective

Let's compare with A_4. We get a similar situation, but with the projectives in a different position so no interesting examples arise.

Here's what the obvious calculations show.



Now let's return to A_5 and see if we can really establish the properties of the examples we've discussed. From general principles

$$\begin{smallmatrix}1\\0\\2\\0\\1\end{smallmatrix} \to \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$$

is irreducible – and the only such map from P_1, the proj. cover of V_1.
Now $\Omega^{-2}\left(\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}\right) = 1$, by direct calculation, so V_1 is certainly a "candidate". It remains only to show there are infinitely many indecomposables mapping onto V_1. Hence, E.T.S. $\mathrm{Ext}^n(V_0, V_1) \neq 0$ for arbitrarily many n, as V_0 is not periodic and V_1 is simple. But $\mathrm{Ext}^n(V_0, V_1) \cong \mathrm{Ext}^n(V_1, V_0)$ and we can easily calculate this from the periodicity of V_1.

Next, we want to prove that $\begin{smallmatrix}1&&\\0&&\\2&&1\\&0&\end{smallmatrix}$ is preprojective, but not strongly so.

a) $\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix} \to \begin{smallmatrix}1&&\\0&&\\2&&1\\&0&\end{smallmatrix} \to 1$ is an almost split exact sequence

Since $\Omega^2(1) = \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$ there is an almost split exact sequence with the given initial and terminal module. Since $\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$ is uniserial there must be a summand of the middle term in which $\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$ is injected. By composition length counting we get that the middle is indecomposable. It remains therefore to identify the middle term; it suffices to prove that $Ext^1(1, \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix})$ is one-dimensional. But

$$\Omega\left(\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}\right) = V_1$$

so

$$Ext^1(1, \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}) \cong Ext^2(1, 1)$$

which is as desired by the minimal resolution of V_1.

b) $\begin{smallmatrix}&&1\\&&0\\1&&2\\&0&\end{smallmatrix}$ is not strongly preprojective.

We shall use the exact sequence given in a). It suffices to show that if $k \equiv 1 \pmod{3}$ then $Hom(\Omega^k V_0, V_1) \neq 0$ and from the long exact sequence for the sequence of a) that

$$Hom(\Omega^k V_0, \begin{smallmatrix}&&1\\&&0\\1&&2\\&0&\end{smallmatrix}) \to Hom(\Omega^k V_0, 1)$$

is epic. But $Hom(\Omega^k V_0, V_1) \cong Ext^k(V_0, V_1) \cong Ext^k(V_1, V_0) \neq 0$. And to show the "onto" property, it's enough, again from the long exact sequence, that

$$Ext^1(\Omega^k V_0, \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}) = 0.$$

But

$$\mathcal{E}xt^1\left(\Omega^k V_0, \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}\right) \simeq \mathcal{E}xt^{k+1}\left(V_0, \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}\right)$$
$$\simeq \mathcal{E}xt^{k+1}\left(V_0, \Omega^{-1}(V_1)\right)$$
$$\simeq \mathcal{E}xt^{k+2}(V_0, V_1)$$
$$\simeq \mathcal{E}xt^{k+2}(V_1, V_0)$$
$$= 0.$$

This proves b)

To finish, we only need

c) $\begin{smallmatrix} & 1 \\ & 0 \\ 1 & 2 \\ 0 & \end{smallmatrix}$ is preprojective.

It suffices to show that if $\begin{smallmatrix} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \end{smallmatrix} \to U$ is irreducible with U indec then $\begin{smallmatrix} & 1 \\ & 0 \\ 1 & 2 \\ 0 & \end{smallmatrix}$ is not an image of such U. For, by gen'l principles, the preprojectives of level one (just after the projectives) are

$$\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix} \qquad \begin{smallmatrix}2\\0\\1\\0\end{smallmatrix} \qquad \begin{smallmatrix} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \end{smallmatrix}$$

and we know what follows $\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$ and $\begin{smallmatrix}2\\0\\1\\0\end{smallmatrix}$ by a). Now U is never projective as $\begin{smallmatrix} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \end{smallmatrix}$ is not the radical of an indec projective, nor a summand of such. Let's get U. Have

$$U^* \to \begin{smallmatrix} 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \\ & 0 & \end{smallmatrix}$$

irreducible so

$$\Omega^{-1}(U^*) \to V_0$$

is irreducible. But V_0 is simple and its minimal resolution shows that the almost split exact sequence with RHS V_0 ends as follows:

$$\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 2 \end{smallmatrix} \to 0$$

Hence

$$\Omega^{-1}(U^*) \simeq \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \text{ or } \begin{smallmatrix} 0 \\ 2 \end{smallmatrix}$$

so

$$U^* \simeq \begin{array}{ccc} 2 & & \\ 0 & & 0 \\ 1 & & 2 \\ & 0 & \end{array} \text{ or } \begin{array}{ccc} 1 & & \\ 0 & & 0 \\ 2 & & 1 \\ & 0 & \end{array}$$

so

$$U \simeq \begin{array}{ccc} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & \end{array} \text{ or } \begin{array}{ccc} & 0 & \\ 2 & & 1 \\ 0 & & \\ 1 & & \end{array}$$

and the proof of c) is complete.

Remark Auslander makes the following comment. To prove d) on p262 is immediate since the exact sequence in a) is almost split.

Remark V_1 is periodic so all the modules in the same class are too. For since projectives have middles & there enter the almost split exact sequences - known result - can get around in classes without using projectives. And need another known result: middle term of almost split exact seq. is periodic if ends are too.

$\therefore P/V_0 = \begin{array}{ccc} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \end{array}$ is not in this class. $\therefore$ more than one class containing projectives.

Let's look at $L_3(2) \cong L_2(7)$ in characteristic two. Use the following ideas and usual notation:

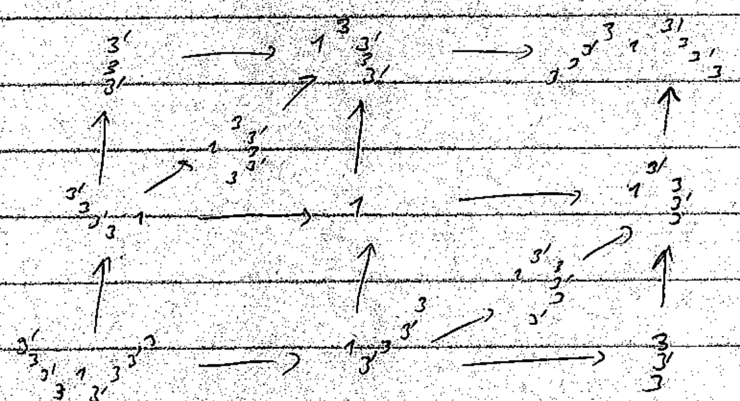
i) Can get almost split exact sequence for V_1 from resolutions;

ii) Know all projectives and mappings irreducibly;

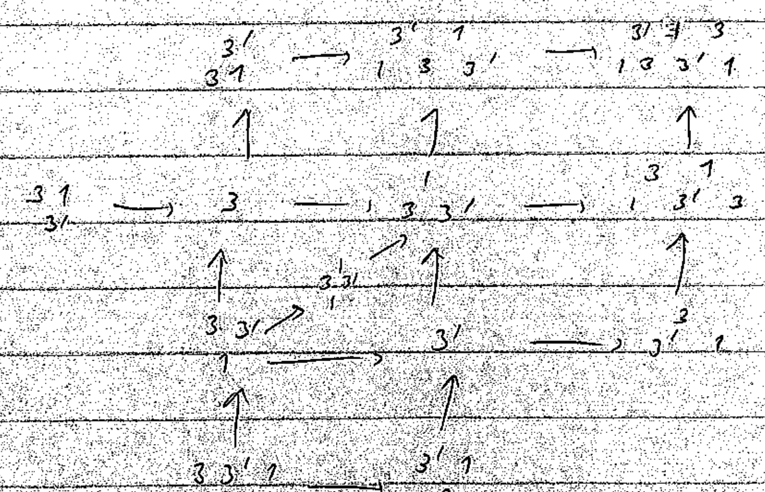
iii) Duality;

iv) Can apply Ω, e.g. $\Omega\left(\begin{smallmatrix} & 3' \\ 1 & 3 \\ & 3' \\ & 3 \end{smallmatrix}\right) = 3'$.

Here are the results:



(Duality $\stackrel{?}{=}$ diagonal flip)



(Duality $\stackrel{?}{=}$ 180 rotation)

Now let's produce an almost split exact sequence

First, $\Omega^{-2}\begin{pmatrix}3\\3'\end{pmatrix} = {}_{1}{}^{3'}{}_{3}{}_{3'}{}^{1}{}_{3}$ as is easy to check. Also

$\mathrm{Ext}^1({}_{1}{}^{3'}{}_{3}{}_{3'}{}^{1}{}_{3},\ \begin{smallmatrix}3\\3'\end{smallmatrix})$ is one-dimensional. Indeed, look at resol:

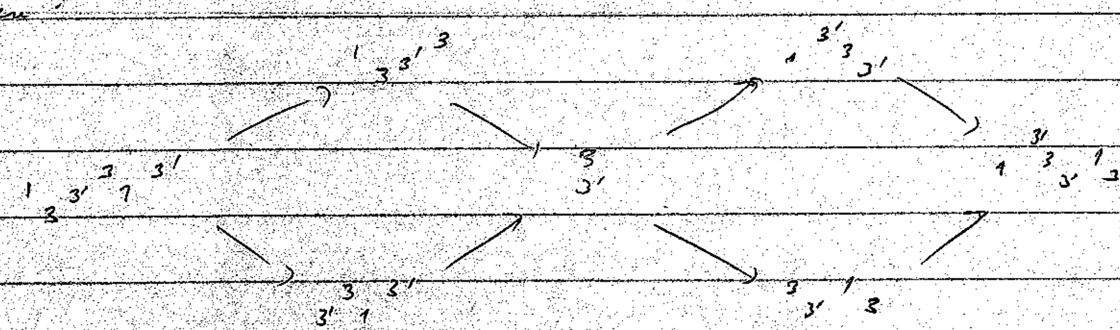
$$\begin{smallmatrix}&3'&\\1&&3\\&&3'\\&&3\\&&3'\end{smallmatrix} = \begin{smallmatrix}&1&\\3'&&3\\&1&\end{smallmatrix} \to {}_{1}{}^{3'}{}_{3}{}_{3'}{}^{1}{}_{3}$$

Easy to calculate. Now following is exact and non-split so is the almost split exact sequence:

$$\begin{smallmatrix}3\\3'\end{smallmatrix} \to {}_{1}{}^{3'}{}_{3}{}_{3'} \oplus {}^{3}{}_{3'}{}^{1}{}_{3} \to {}_{1}{}^{3'}{}_{3}{}_{3'}{}^{1}{}_{3}$$

Take duals, $\begin{pmatrix}3\\3'\end{pmatrix}^* = \begin{smallmatrix}3\\3'\end{smallmatrix}$ get same sort of sequence with maps epic instead of monic, a counterexample to Auslander et al query, provided we can show that $\begin{smallmatrix}3\\3'\end{smallmatrix}$ is not in the same irreducible mapping class as a projective.

Picture:



Recall that $\begin{smallmatrix}3 & & 3'\\ & 1 & \end{smallmatrix}$ and 3 generate freely a free abelian subgroup of the group of invertible modules. Here $U \otimes \begin{smallmatrix}3 & & 3'\\ & 1 & \end{smallmatrix} = \Omega(U)$ modulo projectives, as $\begin{smallmatrix}3 & & 3'\\ & 1 & \end{smallmatrix}$ is the radical of the projective cover of 1. Let's calculate $\Omega\left(\begin{smallmatrix}3\\3'\end{smallmatrix} \otimes 3\right)$. First, in the usual way

$$\begin{smallmatrix}3\\3'\end{smallmatrix} \otimes 3 = \begin{smallmatrix} & 3' & \\ 1 & & 3\\ & & 3'\end{smallmatrix} = \Omega^{-1}\left(\begin{smallmatrix}3\\3'\end{smallmatrix}\right)$$

so $\Omega\left(\begin{smallmatrix}3\\3'\end{smallmatrix} \otimes 3\right) = \begin{smallmatrix}3\\3'\end{smallmatrix}$. Let Δ be the operator corresponding to the invertible part of $\begin{smallmatrix}3' & & 3\\ & 1 & \end{smallmatrix} \otimes 3$ so $\Delta\left(\begin{smallmatrix}3\\3'\end{smallmatrix}\right) = \begin{smallmatrix}3\\3'\end{smallmatrix}$ and $\begin{smallmatrix}3\\3'\end{smallmatrix}$ is "Δ-periodic." Invertible part = radical of proj cover of 3.

<u>Question</u>: If U and V are indecomposable and non-projective modules, $f: U \to V$ is irreducible and one of U, V is Δ-periodic is also the other Δ-periodic?

Since an irreducible mapping class containing a projective is "connected" after leaving out the projectives – use the "middles" – if the answer is "yes" then $\begin{smallmatrix}3\\3'\end{smallmatrix}$ is not in such a class as 1 and 3 are not Δ-periodic by the freeness of the free abelian group mentioned above. (Rh. See Auslander–Reiten–Platzeck, Prop 1.3)

<u>Question</u>. If $A \to B \to C$ is an almost split exact sequence with A, C indecomposable non projective modules, is

$$A \otimes \begin{smallmatrix} & 3' & \\ & 3 & \\ 3' & & 1\\ & 3 & \end{smallmatrix} \to B \otimes \begin{smallmatrix} & 3' & \\ & 3 & \\ 3' & & 1\\ & 3 & \end{smallmatrix} \to C \otimes \begin{smallmatrix} & 3' & \\ & 3 & \\ 3' & & 1\\ & 3 & \end{smallmatrix},$$

after leaving out projectives at each end, almost split?

Answer of "yes" implies "yes" for above question. For say V

is Λ-periodic and let

$$A \to B \to V$$

be almost split so V is a summand of B. Now $A \simeq \Omega^2(V)$ so A is also Λ-periodic ∴ applying a suitable power of Λ we get the same sequence back with same middle term. Since U is a summand this yields – as Λ is 1-1 + onto on iso classes of indecomposable non-projective modules – that U is Λ-periodic.

But we can use a result of Auslander-Reiten (Rep th V, Prop 1.4) to answer the first question positively.(*) Have only to prove that our functor defines an equivalence modulo projectives. We're alright on objects, just have to deal with maps. That is, $\overline{\mathrm{Hom}}(U, V) \simeq \overline{\mathrm{Hom}}(\Lambda(U), \Lambda(V))$. But enough to show that when we use the dual invertible module, say Λ^*, we get identity on $\overline{\mathrm{Hom}}(U, V)$. But that is clear.

(*) Needs a little explanation of how to use the result (Prop 1.4) to answer the question. After applying Λ enough times will get V such, say $\Lambda^k V \simeq V$ (mod projectives), so must have $\Lambda^k U = U_1$, U_1 / middle term, so keep repeating powers of Λ + will be done.

Intersections and joins of subpairs

We use the notation of our paper with Broué, "Local methods in block theory." We have G, k, b.
We shall first show two conjectures are equivalent:

Conjecture 1 If (Q, b_Q) and (R, b_R) are b-subpairs contained in a b-subpair then there exists a unique block b_P of $C(P)$, $P = \langle Q, R \rangle$ such that
$$(Q, b_Q) \subset (P, b_P)$$
$$(R, b_R) \subset (P, b_P).$$

Remarks The uniqueness is the point. Existence follows from the assumption that $(Q, b_Q) \subset (S, b_S)$, $(R, b_R) \subset (S, b_S)$ for a suitable b-subpair (S, b_S). Note that uniqueness in S is already known.

Conjecture 2 If (S, b_S) and (T, b_T) are b-subpairs then there exists a b-subpair (U, b_U) contained in (S, b_S) and in (T, b_T) and containing any b-subpair contained in (S, b_S) and in (T, b_T).

Remark Should write $(S, b_S) \cap (T, b_T) = (U, b_U)$.
Thus $U \subset S \cap T$

Proposition Conjectures 1 and 2 are equivalent.

Pf 1) $\Rightarrow$ 2). Choose U of maximal order so that there exists a block b_U of $C(U)$ with $(U, b_U) \subset (S, b_S)$, $(U, b_U) \subset (T, b_T)$. Suppose that also $(V, b_V) \subset (S, b_S)$, $(V, b_V) \subset (T, b_T)$.

By conjecture 1, there exists a unique block b_W of $C(W)$, $W = \langle U, V \rangle$, with $(U, b_U) \subset (W, b_W)$, $(V, b_V) \subset (W, b_W)$. But $U, V \subset W \subset S$ and $U, V \subset W \subset T$ so there exist blocks b_W', b_W'' of $C(W)$ so that $(W, b_W') \subset (S, b_S)$ $(W, b_W'') \subset (T, b_T)$. But then $(U, b_U) \subset (W, b_W')$ as $(U, b_U) \subset (S, b_S)$ and also $(V, b_V) \subset (W, b_W')$ similarly. Thus, $b_W = b_W'$. Also $b_W = b_W''$ so we're done, by the maximality of U, as this now forces $V \subset U$

2) ⇒ 1) Suppose that the uniqueness fails and (Q, b_Q), (R, b_R) are contained in (P, b_P'), (P, b_P''). Let $(U, b_U) = (P, b_P') \cap (P, b_P'')$ so we must have $U \subsetneq P$. Hence, $(Q, b_Q) \subsetneq (U, b_U)$ or $(R, b_R) \subsetneq (U, b_U)$ by the fact that $\langle Q, R \rangle \subsetneq U$. This contradicts 2).

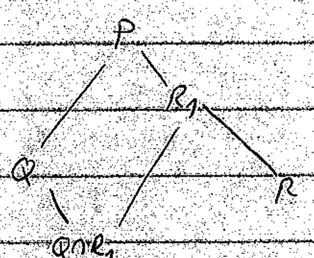
Now let's examine a special case of Conjecture 1).

<u>Conjecture 3</u> (Product Case) If (Q, b_Q) and (R, b_R) are b-subpairs contained in a b-subpair and $P = \langle Q, R \rangle = QR$ then there exists a unique block b_P of $C(P)$ such that $(Q, b_Q) \subset (P, b_P)$ and $(R, b_R) \subset (P, b_P)$.

<u>Lemma</u> In order to prove Conjecture 3) it suffices to deal with the case that $|P:Q| = |P:R| = p$.

That is, in particular, $(P, b_P) \triangleright (Q, b_Q)$ will be.

Pf. We prove conjecture 3), assuming the special case has been dealt with, and proceed by induction on $|P:Q|\cdot|P:R|$. We may assume that $|P:Q|>1$, $|P:R|>1$. Suppose that $|P:R|>p$; it suffices to show the conjecture now holds. Choose a subgroup R_1 with $P \supset R_1 \supset R$ and $|P:R_1| = p$ so $R_1 \supsetneq R$.



Suppose that (Q, b_Q), (R, b_R) are contained in (P, b'_P) and (P, b''_P). We want to show that $b'_P = b''_P$. We can arrange that

$$(R, b_R) \subset (R_1, b'_{R_1}) \subset (P, b'_P)\,;\ (R, b_R) \subset (R_1, b''_{R_1}) \subset (P, b''_P).$$

Let $(Q\cap R_1, b_{Q\cap R_1}) \subset (Q, b_Q)$ so $(Q\cap R), b_{Q\cap R}) \subset (P, b'_P)$ and (P, b''_P). Since $Q\cap R_1 \subset R_1$ we have $(Q\cap R, b_{Q\cap R}) \subset (R_1, b'_{R_1}) + (R_1, b''_{R_1})$. But $R_1 = (Q\cap R_1)R$ so $b'_{R_1} = b''_{R_1}$ by induction (applied to $R_1 = (Q\cap R_1)R$). Now we apply induction to $P = QR_1$ and we're done.

Let's list what we want to consider:

<u>Hypotheses</u> $P = QR$, $|P:Q| = |P:R| = p$

(Q, b_Q), (R, b_R) contained in a b-subpair

Here's the problem. There is a b-subpair (P, b_P) containing (Q, b_Q) and (R, b_R). Is b_P unique. Now $Br_P\, b_Q$ is a sum of primitive central idempotents, as is $Br_P\, b_R$. Is b_P the only one appearing in both sums? That is, do we have

$$Br_P\, b_Q \cdot Br_P\, b_R = b_P$$

ie.

$$Br_P(b_Q b_R) = b_P \; ?$$

SL(3, 4) in characteristic two.

Much of this is a very special case of work of Chastkofsky and of J. Archer (Oxford). But the structures for submodules we get are new and some of the arguments are new too. We use tensor products a lot rather than calculate the composition factors of a tensor product directly.

We use numbers for the simple modules in characteristic two, with * for duals and ‾ for algebraic conjugates. The blocks:

B_0 $\quad 1,\ 8,\ \bar{8},\ 3\otimes\bar{3},\ 3^*\otimes\bar{3}^*$

B_1 $\quad 3,\ \bar{3}^*,\ 3\otimes\bar{8},\ 3^*\otimes\bar{3},\ \bar{3}^*\otimes 8$

B_2 $\quad \bar{3},\ 3^*,\ \bar{3}\otimes 8,\ \bar{3}^*\otimes 3,\ 3^*\otimes\bar{8}$

B_3 $\quad 8\otimes\bar{8}$

<u>Lemma 1</u> $\quad 3\otimes 3 \cong \begin{matrix} 3^* \\ \bar{3} \\ 3^* \end{matrix}$, is uniserial.

By restriction to $SL(3,2) = L_3(2)$ know it is uniserial and by dimensions of simples has three composition factors $\therefore$ by blocks + duality,
* etc., result holds.

<u>Lemma 2</u> $\quad 3\otimes 3\otimes 3 \cong \begin{matrix} 1 \\ 3\otimes\bar{3} \\ 1 \end{matrix} \oplus 8 \oplus 8.$

From the first lemma, we get the composition factors: $\begin{matrix} 1\oplus 8 \\ 3\otimes\bar{3} \\ 1\oplus 8 \end{matrix}$.
From Lemma 1 easily get that 1 appears at top and bottom of $3\otimes3\otimes3$ each place just once: $(1, 3\otimes3\otimes3) = (3^*, 3\otimes 3) = 1$ (where $(\ ,\ ) = \dim \mathrm{Hom}(\ ,\ )$). Also

$$(3\otimes\bar{3},\ 3\otimes3\otimes3) = (3\otimes3^*\otimes\bar{3},\ 3\otimes3) = (\bar{3}\oplus(8\otimes\bar{3}),\ 3\otimes3)$$
$$= 0 \text{ by Lemma 1.}$$

Remains to deal with 8.

$$
\begin{aligned}
(8,\ 3\otimes 3\otimes 3) &= (1\oplus 8,\ 3\otimes 3\otimes 3) - 1\\
&= (3\otimes 3^*,\ 3\otimes 3\otimes 3) - 1\\
&= (3^*\otimes 3^*,\ 3^*\otimes(3\otimes 3)) - 1\\
&\geq (3,\ 3^*\otimes(3\otimes 3)) - 1 \qquad \text{as} \quad 3^*\otimes 3^* = \begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix}\\
&= (3\otimes 3,\ 3\otimes 3) - 1\\
&= 2-1\\
&= 1
\end{aligned}
$$

Have to do one better. But

$$(3^*\otimes 3^*,\ 3^*\otimes 3\otimes 3) = \left(\begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix},\ 3^*\otimes \begin{smallmatrix}3^*\\ 3\\ 3^*\end{smallmatrix}\right)$$

and at bottom of right hand term have $3^*\otimes 3^*$ so it's enough to show that another 3 drops to the bottom, i.e. $\left(3,\ 3^*\otimes \begin{smallmatrix}3^*\\ 3\\ 3^*\end{smallmatrix}\right)\geq 2$.

But

$$\left(3,\ 3^*\otimes \begin{smallmatrix}3^*\\ 3\\ 3^*\end{smallmatrix}\right) = (3\otimes 3,\ 3\otimes 3) = 2$$

so the lemma is proved.

<u>Lemma 3</u> $\bar{3}\otimes\bar{3}\otimes\bar{3} \cong \begin{smallmatrix}1\\ 3\otimes 3\\ 1\end{smallmatrix} \oplus \bar{8}\oplus\bar{8}$ with the first term <u>not</u> isomorphic to the first term of the previous result.

Indeed, the question is as to $(3\otimes 3\otimes 3,\ \bar{3}\otimes\bar{3}\otimes\bar{3})$? But

$$
\begin{aligned}
(3\otimes 3\otimes 3,\ \bar{3}\otimes\bar{3}\otimes\bar{3}) &= (3\otimes\bar{3}^*,\ 3^*\otimes 3^*\otimes\bar{3}\otimes\bar{3})\\
&= \left(3\otimes\bar{3}^*,\ \begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix}\otimes\begin{smallmatrix}\bar{3}^*\\ 3\\ \bar{3}^*\end{smallmatrix}\right)
\end{aligned}
$$

The right hand side has two submodules, namely

$$\begin{smallmatrix}3\\ \bar{3}^*\\ 3\end{smallmatrix}\otimes 3^*, \qquad 3\otimes\begin{smallmatrix}\bar{3}^*\\ 3\\ \bar{3}^*\end{smallmatrix}$$

which have all the composition factors isomorphic with $3 \otimes \bar{3}^*$.
$\therefore$ E.T.S each of these is uniserial with one $3 \otimes \bar{3}^*$ at the top, one at the bottom + nowhere else. Hence, only need

<u>Lemma 4</u> $\quad 3 \otimes \bar{3} \otimes \bar{3} \cong \begin{array}{c} 3 \otimes \bar{3}^* \\ 3^* \\ \bar{3} \\ 3^* \\ 3 \otimes \bar{3}^* \end{array}$ is uniserial.

We have the composition factors and a series of submodules as follows:
$$\begin{array}{c} 3 \otimes \bar{3}^* \\ \hline 3^* \\ \bar{3} \\ 3^* \\ \hline 3 \otimes \bar{3}^* \end{array}$$

But
$$\begin{aligned}(3^*, \, 3 \otimes \bar{3} \otimes \bar{3}) &= (3^* \otimes 3^*, \, \bar{3} \otimes \bar{3}) \\ &= (3^* \otimes \bar{3}^*, \, 3 \otimes \bar{3}) \\ &= 0 \quad \text{(as have different simples)}.\end{aligned}$$
This proves this lemma & the preceding one therefore holds too.

Now we turn to $3 \otimes 3 \otimes 3 \otimes 3$. We have to analyze $3 \otimes \begin{array}{c}1\\ 3\otimes\bar{3}\\ 1\end{array}$ and $3 \otimes 8$, where here the $\begin{array}{c}1\\ 3\otimes\bar{3}\\ 1\end{array}$ is the first one, as in Lemma 2.

<u>Lemma 5</u> $\quad 3 \otimes \begin{array}{c}1\\ 3\otimes\bar{3}\\ 1\end{array} \cong \begin{array}{c} 3 \\ 3^* \otimes \bar{3} \\ \bar{3}^* \\ 3 \\ \bar{3}^* \\ 3^* \otimes \bar{3} \\ 3 \end{array}$ is uniserial.

We have
$$3 \otimes \begin{array}{c}1\\ 3\otimes\bar{3}\\ 1\end{array} = \begin{array}{c} \hline 3 \\ \hline 3 \otimes 3 \otimes \bar{3} \\ \hline 3 \end{array}$$
$$\cong \begin{array}{c} \hline 3 \\ \hline 3^* \otimes \bar{3} \\ 3^* \\ \bar{3} \\ \bar{3}^* \\ 3^* \otimes \bar{3} \\ \hline 3 \end{array} \quad \text{by Lemma 4.}$$

But

$$\left(3,\ 3\otimes \begin{smallmatrix}1\\ 3\otimes\bar{3}\\ 1\end{smallmatrix}\right) = \left(1\oplus 8,\ \begin{smallmatrix}1\\ 3\otimes\bar{3}\\ 1\end{smallmatrix}\right) = 1$$

$$\left(3^*\otimes\bar{3},\ 3\otimes \begin{smallmatrix}1\\ 3\otimes\bar{3}\\ 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix}\otimes\bar{3},\ \begin{smallmatrix}1\\ 3\otimes\bar{3}\\ 1\end{smallmatrix}\right)$$

$$= 0$$

as no trivial composition factor.

The structure of $3\otimes 8$ is hard to get at. What do we know?

$$\begin{aligned} 3\otimes 8\oplus 3 &\cong 3\otimes(8\oplus 1)\\ &\cong 3\otimes 3\otimes 3^*\\ &\cong \begin{smallmatrix}3^*\\ 3\\ 3^*\end{smallmatrix}\otimes 3^*\\ &\cong \begin{array}{c}\begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix}\\ \hline 3\otimes 3^*\\ \hline \begin{smallmatrix}3\\ 3^*\\ 3\end{smallmatrix}\end{array}\end{aligned}$$

but further results elude us.

<u>Lemma 6</u> The permutation module on the 21 pts of $P_2(4)$ has the structure

$$1\oplus \begin{array}{ccc} & 3\otimes\bar{3} & \\ 1 & & 1\\ & 3^*\otimes\bar{3}^* & \end{array}.$$

Use characters, odd degree and fact that 1 is once at top + once at bottom.

(Geoff Mason has a geometric-combinatoric proof of this too).

Let P_i be the projective cover of i. We study tensor products with the Steinberg module $8 \otimes \bar{8}$. (These results known.)

<u>Lemma 7</u>
$$3 \otimes 8 \otimes \bar{8} = P_{3 \otimes \bar{8}},$$
$$\bar{3}^* \otimes 8 \otimes \bar{8} = P_{\bar{3}^* \otimes 8},$$
$$\bar{3} \otimes 8 \otimes \bar{8} = P_{\bar{3} \otimes 8},$$
$$3^* \otimes 8 \otimes \bar{8} = P_{3^* \otimes \bar{8}}.$$

Have only to work in the block B_1 and argue for $3 \otimes 8 \otimes \bar{8}$.

But
$$(3,\ 3 \otimes 8 \otimes \bar{8}) = (3 \otimes 3^*,\ 8 \otimes \bar{8}) = (1 \oplus 8,\ 8 \otimes \bar{8}) = 0,$$
$$(\bar{3}^*,\ 3 \otimes 8 \otimes \bar{8}) = (\bar{3}^* \otimes 3^*,\ 8 \otimes \bar{8}) = 0,$$
$$(\bar{3} \otimes 3^*,\ 3 \otimes 8 \otimes \bar{8}) = (\bar{3} \otimes 8,\ 3 \otimes 3 \otimes \bar{8})$$
$$= \left(\bar{3} \otimes 8,\ \begin{matrix} 3^* \\ 3 \\ 3^* \end{matrix} \otimes \bar{8}\right)$$
$$= \left(\bar{3} \otimes 8,\ \begin{matrix} 3^* \otimes \bar{8} \\ 3 \otimes \bar{8} \\ 3^* \otimes \bar{8} \end{matrix}\right).$$

But $\bar{3} \otimes \bar{8}$ is of dimension 24 so certainly has no comp factor iso to $\bar{3} \otimes 8$ so this is zero.
$$(3 \otimes \bar{8},\ 3 \otimes 8 \otimes \bar{8}) = (3 \otimes 3^* \otimes \bar{8},\ 8 \otimes \bar{8})$$
$$= ((1 \oplus 8) \otimes \bar{8},\ 8 \otimes \bar{8})$$
$$= (\bar{8} \oplus (8 \otimes \bar{8}),\ 8 \otimes \bar{8})$$
$$= 1.$$

Remains to deal with $\bar{3}^* \otimes 8$.

But

$$(\bar{3}^* \otimes 8,\ 3 \otimes 8 \otimes \bar{8}) = (\bar{3}^* \otimes 8, ((3 \otimes 8) \oplus 3) \otimes \bar{8}) \quad \text{by dims,}$$

$$= \left(\bar{3}^* \otimes 8, \left(\begin{smallmatrix} 3 \\ \bar{3}^* \\ 3 \end{smallmatrix} \Big/ \bar{3} \otimes 3^* \Big/ \begin{smallmatrix} 3 \\ \bar{3}^* \\ 3 \end{smallmatrix}\right) \otimes \bar{8}\right),$$

by 2 pages back,

$$\leq (\bar{3}^* \otimes 8,\ \bar{3} \otimes 3^* \otimes \bar{8}), \quad \text{by dims and composition factors,}$$

$$= (\bar{3}^* \otimes 3,\ \bar{3} \otimes 8 \otimes \bar{8})$$

$$= (3^* \otimes \bar{3},\ 3 \otimes \bar{8} \otimes 8)$$

$$= (3^* \otimes \bar{3},\ 3 \otimes 8 \otimes \bar{8})$$

$$= 0, \quad \text{from previous part of this proof,}$$

so lemma is proved.

<u>Lemma 8</u>. $3^* \otimes \bar{3} \otimes 8 \otimes \bar{8} \cong P_{3^* \otimes \bar{3}}$,

$\bar{3}^* \otimes 3 \otimes 8 \otimes \bar{8} \cong P_{\bar{3}^* \otimes 3}$.

$$(3^* \otimes \bar{3},\ 3^* \otimes \bar{3} \otimes 8 \otimes \bar{8}) = ((1 \oplus 8) \otimes (1 \oplus \bar{8}),\ 8 \otimes \bar{8}) = 1,$$

$$(3,\ 3^* \otimes \bar{3} \otimes 8 \otimes \bar{8}) = (3 \otimes 3,\ P_{\bar{3} \otimes 8}) = \left(\begin{smallmatrix} 3^* \\ \bar{3} \\ 3^* \end{smallmatrix},\ P_{\bar{3} \otimes 8}\right) = 0,$$

$$(\bar{3}^*,\ 3^* \otimes \bar{3} \otimes 8 \otimes \bar{8}) = (\bar{3}^* \otimes \bar{3}^*,\ P_{3^* \otimes \bar{8}}) = 0,$$

$$(\bar{3}^* \otimes 8,\ 3^* \otimes \bar{3} \otimes 8 \otimes \bar{8}) = (\bar{3}^* \otimes 8 \otimes \bar{8},\ 3^* \otimes \bar{3} \otimes 8)$$

$$= (P_{\bar{3}^* \otimes 8},\ 3^* \otimes \bar{3} \otimes 8)$$

$$= \text{mult of } \bar{3}^* \otimes 8 \text{ as comp factor of } 3^* \otimes \bar{3} \otimes 8$$

$$= 0$$

since $\bar{3} \otimes (3^* \otimes 8) \doteq \bar{3} \otimes (3 + \bar{3}^* + \bar{3} \otimes 3^* + 3 + \bar{3}^* + 3)$

and since

$$\bar{3} \otimes \bar{3} \otimes 3^*$$

doesn't have $\bar{3}^* \otimes 8$ as a factor

Finally, taking $-^*$ everywhere in the last argument gives

$$(3 \otimes \bar{8},\ \bar{3}^* \otimes \bar{3} \otimes 8 \otimes \bar{8}) = 0$$

For $\bar{3}^* \otimes 8$ in that argument becomes $3 \otimes \bar{8}$ & so on

<u>Lemma 9.</u> $3 \otimes \bar{8} \otimes 8 \otimes \bar{8} \cong P_3 \oplus P_{3\otimes\bar{8}} \oplus P_{3\otimes\bar{8}} \oplus P_{\bar{3}^*\otimes 8}$

$\bar{3}^* \otimes 8 \otimes 8 \otimes \bar{8} \cong P_{\bar{3}^*} \oplus P_{\bar{3}^*\otimes 8} \oplus P_{\bar{3}^*\otimes 8} \oplus P_{3\otimes\bar{8}}$

(and similarly for B_2)

This gives the following table for mult by Steinberg is part of the "A" matrix.

	P_3	$P_{\bar{3}^*}$	$P_{3^*\otimes\bar{3}}$	$P_{3\otimes\bar{8}}$	$P_{\bar{3}^*\otimes 8}$
3	0	0	0	1	0
3^*	0	0	0	0	1
$3^*\otimes\bar{3}$	0	0	1	0	0
$3\otimes\bar{8}$	1	0	0	2	1
$\bar{3}^*\otimes 8$	0	1	0	1	2

Now let's give the proof of the lemma.

$$\begin{aligned}(3,\ 3\otimes\bar{8}\otimes 8\otimes\bar{8}) &= (1\oplus 8,\ \bar{8}\otimes 8\otimes\bar{8}) \\ &= (\bar{8},\ 8\otimes\bar{8}) + (8\otimes\bar{8},\ 8\otimes\bar{8}) \\ &= 1\end{aligned}$$

$$\begin{aligned}(\bar{3}^*,\ 3\otimes\bar{8}\otimes 8\otimes\bar{8}) &= (\bar{3}^*\otimes 8\otimes\bar{8},\ 3\otimes\bar{8}) \\ &= (P_{\bar{3}^*\otimes 8},\ 3\otimes\bar{8}) \\ &= 0.\end{aligned}$$

$$(3\otimes\bar{8},\ 3\otimes\bar{8}\otimes 8\otimes\bar{8}) = (3\otimes 8\otimes\bar{8},\ 3\otimes\bar{8}\otimes\bar{8})$$

$$= (P_{3\otimes\bar{8}},\ 3\otimes\bar{3}\otimes\bar{3}^*\otimes\bar{3}\otimes\bar{3}^*)$$

$$-2(P_{3\otimes\bar{8}},\ 3\otimes\bar{8}) - (P_{3\otimes\bar{8}},\ 3)$$

$$= (P_{3\otimes\bar{8}},\ 3\otimes\bar{3}\otimes\bar{3}^*\otimes\bar{3}\otimes\bar{3}^*) - 2$$

$$= \left(P_{3\otimes\bar{8}},\ 3\otimes\begin{matrix}\bar{3}^*\\ 3\\ \bar{3}^*\end{matrix}\otimes\begin{matrix}\bar{3}\\ 3^*\\ \bar{3}\end{matrix}\right) - 2$$

$$= 4-2$$

$$= 2.$$

$$(\bar{3}^*\otimes 8,\ 3\otimes\bar{8}\otimes 8\otimes\bar{8}) = (\bar{3}^*\otimes 8\otimes\bar{8},\ 3\otimes 8\otimes\bar{8})$$

$$= (P_{\bar{3}^*\otimes 8},\ 3\otimes 3\otimes 3^*\otimes\bar{3}\otimes\bar{3}^*) - (P_{\bar{3}^*\otimes 8},\ 3\otimes 8)$$

$$-(P_{\bar{3}^*\otimes 8},\ 3\otimes\bar{8}) - (P_{\bar{3}^*\otimes 8},\ 3)$$

$$= (P_{\bar{3}^*\otimes 8},\ 3\otimes 3\otimes 3^*\otimes\bar{3}\otimes\bar{3}^*)$$

$$= 2(P_{\bar{3}^*\otimes 8},\ 3^*\otimes 3^*\otimes\bar{3}\otimes\bar{3}^*) + (P_{\bar{3}^*\otimes 8},\ \bar{3}\otimes 3^*\otimes\bar{3}\otimes\bar{3}^*)$$

$$= 4(P_{\bar{3}^*\otimes 8},\ 3\otimes\bar{3}\otimes\bar{3}^*) + 2(P_{\bar{3}^*\otimes 8},\ \bar{3}^*\otimes\bar{3}\otimes 3^*)$$

$$+ 2(P_{\bar{3}^*\otimes 8},\ \bar{3}^*\otimes 3^*\otimes\bar{3}^*) + (P_{\bar{3}^*\otimes 8},\ 3\otimes 3^*\otimes\bar{3}^*)$$

$$= 0+0+0+1$$

$$= 1.$$

$$(3^*\otimes\bar{3},\ 3\otimes\bar{8}\otimes 8\otimes\bar{8}) = (3^*\otimes 8\otimes\bar{8},\ \bar{3}^*\otimes 3\otimes\bar{8})$$

$$= (P_{3^*\otimes\bar{8}},\ \bar{3}^*\otimes 3\otimes\bar{8}) = \text{mult of } 3^*\otimes\bar{8} \text{ in } \bar{3}^*\otimes 3\otimes\bar{8}.$$

But this is zero, so lemma is proved, as

$$\bar{3}^*\otimes 3\otimes\bar{8} \doteq \bar{3}^*\otimes(\bar{3}^*\oplus(3^*\otimes\bar{3})\oplus 3\oplus 3\oplus\bar{3}^*).$$

We turn to B_0.

Lemma 10. $\delta \otimes \delta \otimes \bar{\delta} \cong P_{\bar{\delta}} \oplus (\delta \otimes \bar{\delta}) \oplus (\delta \otimes \bar{\delta})$

$\bar{\delta} \otimes \delta \otimes \bar{\delta} \cong P_{\delta} \oplus (\delta \otimes \bar{\delta}) \oplus (\delta \otimes \bar{\delta})$.

$(1, \delta \otimes \delta \otimes \bar{\delta}) = (\delta, \delta \otimes \bar{\delta}) = 0,$

$(\bar{\delta}, \delta \otimes \delta \otimes \bar{\delta}) = (\delta \otimes \bar{\delta}, \delta \otimes \bar{\delta}) = 1,$

$(3 \otimes \bar{3}, \delta \otimes \delta \otimes \bar{\delta}) = (\bar{3} \otimes \delta \otimes \bar{\delta}, 3^* \otimes \bar{\delta}) = (P_{\bar{3} \otimes \delta}, 3^* \otimes \bar{\delta}) = 0,$

$(3^* \otimes \bar{3}^*, \delta \otimes \delta \otimes \bar{\delta}) = (P_{3^* \otimes \bar{\delta}}, \bar{3} \otimes \delta) = 0,$

$(\delta \otimes \bar{\delta}, \bar{\delta} \otimes \delta \otimes \bar{\delta}) = (\bar{\delta}, \bar{\delta} \otimes \delta \otimes \bar{\delta}) \oplus (\delta \otimes \bar{\delta}, \bar{\delta} \otimes \delta \otimes \bar{\delta})$

$= ((1 \oplus \bar{\delta}) \otimes \bar{\delta}, \bar{\delta} \otimes \delta \otimes \bar{\delta})$

$= (3 \otimes \bar{\delta}, 3 \otimes \bar{\delta} \otimes \delta \otimes \bar{\delta})$

$= 2$, by Lemma 9.

$(\delta, \delta \otimes \delta \otimes \bar{\delta}) = (\delta \otimes \delta, \delta \otimes \bar{\delta}) = 0$ as $\delta \otimes \bar{\delta}$ is simple + dims are same.

Lemma 11 $3 \otimes \bar{3} \otimes \delta \otimes \bar{\delta} \cong P_{3 \otimes \bar{3}} \oplus (\delta \otimes \bar{\delta})$

$3^* \otimes \bar{3}^* \otimes \delta \otimes \bar{\delta} \cong P_{3^* \otimes \bar{3}^*} \oplus (\delta \otimes \bar{\delta})$

Hence, table for B_0 is as follows:

	P_1	P_δ	$P_{\bar{\delta}}$	$P_{3 \otimes \bar{3}}$	$P_{3^* \otimes \bar{3}^*}$	$\delta \otimes \bar{\delta}$
1	0	0	0	0	0	1
δ	0	0	1	0	0	2
$\bar{\delta}$	0	1	0	0	0	2
$3 \otimes \bar{3}$	0	0	0	1	0	1
$3^* \otimes \bar{3}^*$	0	0	0	0	1	1

Hence, "A" is calculated, as B_2 like B_1, except for $(\delta \otimes \bar{\delta}) \otimes (\delta \otimes \bar{\delta})$.

It remains to demonstrate Lemma 11.

$$(1,\ 3\otimes\bar{3}\otimes 8\otimes\bar{8}) = (3^*\otimes\bar{3}^*,\ 8\otimes\bar{8}) = 0,$$

$$(8,\ 3\otimes\bar{3}\otimes 8\otimes\bar{8}) = (3^*\otimes 8\otimes\bar{8},\ \bar{3}\otimes 8) = (P_{3^*\otimes\bar{8}},\ \bar{3}\otimes 8) = 0,$$

$$(\bar{8},\ 3\otimes\bar{3}\otimes 8\otimes\bar{8}) = \ \cdots\ = 0, \text{ similarly},$$

$$(3\otimes\bar{3},\ 3\otimes\bar{3}\otimes 8\otimes\bar{8}) = (P_{3\otimes\bar{8}},\ \bar{3}^*\otimes 3\otimes\bar{3}) = 1,$$

$$(3^*\otimes\bar{3},\ 3\otimes\bar{3}\otimes 8\otimes\bar{8}) = (3^*\otimes\bar{3}\otimes\bar{3}^*,\ P_{3\otimes\bar{8}}) = 0,$$

$$\begin{aligned}(8\otimes\bar{8},\ 3\otimes\bar{3}\otimes 8\otimes\bar{8}) &= (3^*\otimes\bar{8},\ (\bar{3}\otimes 8)\otimes(8\otimes\bar{8})) \\ &= (3^*\otimes\bar{8},\ P_{\bar{3}}\oplus P_{\bar{3}\otimes 8}\oplus P_{\bar{3}\otimes 8}\oplus P_{3^*\otimes\bar{8}}) \\ &= 1.\end{aligned}$$

Complexity of modules

§1 Let F be a field of characteristic p, G a finite group. Recall – and we review all this below – the complexity of an FG-module M, written $C_G(M)$ or $C(M)$.

Main Theorem. If M is an FG-module then $C_G(M)$ is the maximum of all $C_E(M_E)$ over all elementary abelian p-subgroups of G.

Rk: If M indec enough to use $E \leq$ vertex of U. As a corollary, by L16

Corollary 1 (Quillen) The Quillen dimension (at p) of G equals the p-rank of G.

$$\text{Here, we let } H(G) = \begin{cases} \bigoplus_{n \geq 0} H^n(G, F_p) & p = 2 \\ \bigoplus_{n \text{ even}} H^n(G, F_p) & p > 2 \end{cases} \quad \text{as in Quillen}$$

and the Quillen dimension is the Krull dimension of the finitely generated commutative algebra $H(G)$.

The corollary follows in this way: It's easy to see that $C(F) =$ p-rank of G and, if $q_p(G)$ is the Quillen dimension, to see that

$$q_p(G) \leq C(F_p).$$

An argument of Quillen shows that $q_p(G) \geq$ p-rank of G using Evens norm map.

Corollary 2 (Chouinard) An FG-module M is projective if, and only if, all the restrictions M_E, E an elementary abelian p-subgroup, are projective.

This follows as M is projective exactly when $C(M) = 0$.

<u>Corollary 3</u> An FG-module M is periodic if, and only if all the restrictions M_E, E an elementary abelian p-subgroup, are periodic.

This holds as periodicity of a module M is equivalent with $c(M) \leq 1$.

<u>Corollary 4</u> (Donovan) The maximum of the complexities of the modules in the block B of FG equals the p-rank of the defect group of B.

<u>§2</u> Let's remind ourself of the definition of complexity and develop some elementary properties that we may or may not need.

<u>Def 5</u> A sequence $a_0, a_1, \ldots$ of non-negative integers is almost PORC if there is a positive integer N and rat'l polys $f_0, f_1, \ldots, f_{N-1}$ such that

$$a_n = f_k(n)$$

provided that $n \equiv k$ (modulo N), with a finite number of exceptions.

<u>Lemma 6</u> If M is an FG-module and

$$\cdots \to P_1 \to P_0 \to M \to 0$$

is the minimal projective resolution of M then

$$\dim_F P_n$$

is almost PORC.

Proof Let $S_1, S_2, \ldots, S_t$ be the simple FG-modules so that $P_1, \ldots, P_t$ are their projective covers then

$$P_n \cong \bigoplus_i \left[\dim_F \mathrm{Ext}^n_{FG}(M, S_i)\right] P_i .$$

But

$$\mathrm{Ext}^n_{FG}(M, S_i) \cong \mathrm{Ext}^n_{FG}(F, M^* \otimes S_i)$$

so the sequence

$$\dim_F \mathrm{Ext}^n_{FG}(M, S_i)$$

is almost PORC by Vanhove – see Evans's paper on odd coho. Hence the result follows since a linear combination of almost PORC sequences is also almost PORC.

Def 7 If $a_0, a_1, \ldots$ is an almost PORC sequence then the growth $\gamma(a_n)$ of the sequence is one plus the maximum degree of polys.

Remember 0 is the unique poly. of degree -1.

Def 8 If M is an FG-module as in Lemma 6 then the complexity $c_G(M)$ is the growth of $\dim_F P_n$.

Lemma 9 If M is an FG-module then $c_G(M) = 0$ iff M is projective.

Proof $c_G(M) = 0$ iff $P_n = 0$ for all large n. But since projectives are injective and since $c_G(M) = 0$ implies that M has finite projective dimension it follows that M is projective.

The converse is clear as $0 \to M \to M \to 0$ is the desired resolution.

<u>Lemma 10</u> The FG-module M is periodic iff $C_G(M) \le 1$.

Pf For $C_G(M) \le 1$ iff M is "bounded" as in our Gelisoris Journal paper on periodicity. This is so iff M is periodic, by a result of that paper when F is algebraic and by a result of Eisenbud for general F.

<u>Lemma 11</u>. If E is an elementary abelian p-subgroup then

$$C_E(F) = m$$

where $|E| = p^m$.

Pf. FE has only the simple FG-module F and $\dim_F \mathrm{Ext}^n(F,F) =$ well known.

<u>Lemma 12</u>. If M is an FG-module then the following are equal:

1) $C_G(M)$;
2) The maximum $\gamma(\dim_F \mathrm{Ext}^n_{FG}(M,S))$ over simple FG-modules S;
3) The maximum $\gamma(\dim_F \mathrm{Ext}^n_{FG}(M,U))$ over all FG-modules U.

Pf (Sketch). 1) & 2) are equivalent by the pf of Lemma 6. The equivalence of 3) with these is a consequence of the next result (it's pf).

<u>Lemma 13</u>. If $0 \to U \to V \to W \to 0$ is an exact sequence of FG-modules then the complexity of any one of the three modules is at most the maximum complexity of the other two.

Pf Long exact sequence + estimates for growth.

<u>Lemma 14</u> If U and V are FG-modules then $C_G(U \otimes V) \le C_G(U)$.

Pf Tensor V with the projective resolution of U, term by term.

Lemma 15. If U is an FG-module then $c_G(U) \le c_G(F)$.
Pf Take $V = F$ in Lemma 14.

Lemma 16. If U and V are FG and FH modules, where H is a subgroups of G with V / U_H and U / V^G then

$$c_G(U) = c_H(V).$$

Pf As sketched in the Illinois J. paper.

Lemma 17. If U is an FG-module and H is a subgroup of G then

$$c_G(U) \ge c_H(U_H).$$

Pf The minimal resolution of U restricts to some projective resol. of U_H.

At this point we can also see how Donovan's theorem follows. Let D be a defect group of the block b of FG. Let b be the corresponding block of $N(D)$. Now each module in b has vertex contained in D so the complexities of the modules in b certainly are bounded by the p-rank of D.

Hence, it's enough to produce one module in b with complexity at least that big. Let S be a simple $FN(D)$-module in b so D is in the kernel of S and S_D is a multiple of F_D. But, by Lemma 17 $c_{N(D)}(S) \ge c_D(S_D) = c_D(F) =$ p-rank of D. Let U be the Green correspondent of S so U lies in b by the Nagao theorem. But $c_G(U) = c_{N(D)}(S)$ by Lemma 16 so $c_G(U) \ge$ p-rank of D and we're done.

§3. Let's prove the main theorem. Let P be a Sylow p-subgroup of G so $M \mid (M_P)^G$ as M is relatively P-projective. Thus, $c_G(M) = c_P(M_P)$ and M and M_P have the same restrictions to elementary abelian subgroups of P. Thus, we may assume that G is a p-group.

Proceeding by induction, we may assume that G is not itself elementary abelian. Hence, it suffices to show that $c_G(M)$ is at most the maximum of the $c_H(M_H)$ as H runs over the maximal subgroups of G, for $c_H(M_H) \leq c_G(M)$ by Lemma 17.

If H is a maximal subgroup of G there is a corresponding element of $H^1(G, F_p)$ and this has a Bockstein $\beta_H \in H^2(G, F_p)$ so we can take $\beta_H \in H^2(G, F)$. By a theorem of Serre, there exist maximal subgroups $H_1, \dots H_t$ such that $\beta_{H_1} \cdots \beta_{H_t} = 0$. Hence it suffices to study the codimension in $\operatorname{Ext}^{n+2}_{FG}(F, M)$ of $\beta_H \cdot \operatorname{Ext}^n_{FG}(F, M)$: for G has only the simple module F and $\operatorname{Ext}^*(F, M) \simeq \operatorname{Ext}^*(M^*, F)$. It is thus enough to show that there is a constant K such that the codimension of $\beta_H \operatorname{Ext}^n_{FG}(F, M)$ in $\operatorname{Ext}^{n+2}_{FG}(F, M)$ is at most $K\, n^{c_H(M_H)-1}$. This gives the codimension of $\beta_{H_1} \cdots \beta_{H_t} \operatorname{Ext}^n_{FG}(F, M) = 0$ in $\operatorname{Ext}^{n+2t}_{FG}(F, M)$ as roughly $tK\, n^{c_H(M_H)-1}$, which is all that is needed.

Let $F_0 \supseteq F_1 \supseteq \cdots$ be the filtration of $\operatorname{Ext}^*(F, M)$ for the spectral sequence given by $H \trianglelefteq G$.

<u>Lemma</u> (Evens) $\beta_H \cdot \operatorname{Ext}^n(F, M) = F_2(\operatorname{Ext}^{n+2}(F, M))$

A generalization of the Quillen - Venkov argument.

But $F_0(\mathrm{Ext}^n(F,M))/F_1(\mathrm{Ext}^n(F,M))$ is isomorphic with a subspace of $\mathrm{Ext}^0_{F[G/H]}(F, \mathrm{Ext}^n_{FH}(F, M_H))$ while $F_1(\mathrm{Ext}^n_{FG}(F,M))/F_2(\mathrm{Ext}^n_{FG}(F,M))$ is isomorphic with a section of $\mathrm{Ext}^1_{F[G/H]}(F, \mathrm{Ext}^{n-1}_{FH}(F, M_H))$. Now $\mathrm{Ext}^n_{FG}(F,M)$ and $\mathrm{Ext}^{n-1}_{FG}(F,M)$ have dimension, which as a function of n grow at the required rates. We have to see that applying $\mathrm{Ext}^0_{F[G/H]}(F,-)$ and $\mathrm{Ext}^1_{F[G/H]}(F,-)$ doesn't destroy this. But Ext^0 gives fixed pts so $\dim_F \mathrm{Ext}^0_{F[G/H]}(F,U) \le \dim_F U$. The resolution

$$\cdots \to J_1 \to F$$

with the usual Jordan blocks, has J_{p-1} as first kernel + this is a cyclic $F[G/H]$ module so $\dim_F \mathrm{Hom}_{F[G/H]}(J_1, U) \le \dim_F U$ and thus $\dim_F \mathrm{Ext}^1_{F[G/H]}(F,U) \le \dim_F U$. This proves the main theorem.

__§4__ Our last task is to prove Corollary 1. We have by Lemma 11 and the main theorem (now $F = F_p$)

$$c_p(F) = p\text{-rank of } G.$$

But, by Lemma 12,

$$c_p(F) \geq \gamma(\dim_F \mathrm{Ext}^n_{FG}(F,F))$$

so we need only show two things:

$$\gamma(\dim_F \mathrm{Ext}^n_{FG}(F,F)) = \text{Krull dim of } H(G)$$

$$\gamma(\dim_F \mathrm{Ext}^n_{FG}(F,F)) \geq p\text{-rank of } G.$$

We'll do these in order.

In the case $p=2$, $H(G)$ is a finite module over $H^{ev}(G)$ and $H(G) = H^{ev}(G)$ in case $p>2$, where H^{ev} is the even dim. terms. $\therefore q_p(G) = $ Krull dim of $H^{ev}(G)$. Quillen states – Spectrum etc. Annals v 94 p 556 – that this is the order of the pole at $t=1$ of

$$\sum_{n\geq 0}^{\infty} \left[\dim_F \mathrm{Ext}^n_{FG}(F,F)\right] t^n$$

Thus we only need the following result to achieve our first goal:

<u>Lemma</u> If $a_0, a_1, \ldots$ is almost PORC then

$$\sum a_n t^n$$

is rational with pole of order $\gamma(a_n)$ at $t=1$.

<u>Pf.</u> First observe, using Taylor series, that

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n.$$

Hence, our result holds for a "polynomial sequence" (obvious def) and also the residue at $t=1$ is <u>positive</u> as the leading coefficient of our poly is positive since we're dealing with sequences of non-negative integers.

In dealing with $\sum a_n t^n$ we can assume $a_0, a_1, \ldots$ is PORC and not almost PORC as this only changes things by a polynomial. To get different residue classes need terms $\frac{x^c}{(1-x^m)^k}$. The positivity of the residue means when we add up we get the pole to be the maximum of the poles of the terms.

For our last task, we need results from "A cohomological criterion for p-nilpotence," by Quillen, J. Pure + Applied algebra, v1 (1971), 361-72. Lemma 2.4 shows that if A is an elementary abelian p-subgroup then the image under restriction of $H^*(G,F)$ in $H^*(A,F)$ is f.g., containing a principal ideal of $H(A)^{N(A)}$, the invariant elements of $H(A)$ under $N(A)$. This shows that $H(A)^{N(A)}$ and the restriction image grow at the same rate so it suffices to show the same for $H(A)$ and $H(A)^{N(A)}$.
"E.T.S. $H(A)$ is a f.g. module over $H(A)^{N(A)}$. But $H(A)$ is a poly ring so E.T.S. $H(A)$ integral over $H(A)^{N(A)}$. But poly is $\prod_{n\in N(A)}(h - n(h))$ by symm fn theory.

In dealing with Poincaré series and the like we have skirted around some points. What we should do is get a reference (or a pf.) for the following presumably true statements.

<u>Hypotheses</u>. A is a finitely generated commutative algebra over a field k, graded by the non-negative integers. A_n is finite dimensional and let $a_n = \dim_k A_n$.

<u>Statements</u>

1) The sequence $a_0, a_1, a_2, \ldots$ is almost PORC.

2) $\sum_{n \geq 0} a_n t^n$ is rational.

3) The following are equal:

   a) $1 +$ maximum degree of the polys in 1);

   b) Order of the pole at $t=1$ of $\sum a_n t^n$

   c) Krull dimension of A.

Now 1) seems standard & we've given an argument that 1) $\Rightarrow$ 2). Also, we've seen that a) and b) under 3) are equal.

Cyclic defect

We restrict ourselves, though the results are more general, to the simplest and most basic case: k is algebraically closed of prime characteristic p, G is a finite group with $|G|_p = p$. We will develop another approach and carry it a certain distance. Let's also assume that a Sylow p-subgroup P of G is self-centralizing and that $e = |N(P) : P|$.

Prop 1 There are at least e non-projective simple kG-modules.

Pf For $0 \leq i < 2e$, choose a simple kG-module S_i such that $\mathrm{Ext}^i_{kG}(k, S_i) \neq 0$; it exists from the minimal resolution of k. Now $S_i \mid N(P) = T_i + Q_i$ where T_i is indec. & non-projective (as S_i is not projective) and Q_i is projective. Thus $\mathrm{Ext}^i_{kN(P)}(k, T_i) \neq 0$ as $\mathrm{Ext}^i_{kG}(k, S_i) \approx \mathrm{Ext}^i_{kG}(k, T_i)$ (by inspection for $i = 0$ where $S_0 = k$). Hence, no three of $S_0, S_1, \ldots, S_{2e-1}$ can be equal as there is, by inspection, no indec. $kN(P)$-module T with $\mathrm{Ext}^*_{kN(P)}(k, T)$ non-zero three times in the first $2e$ terms.

Th 2 There are exactly e non-projective simple kG-modules. The minimal projective resolution of k has each term indecomposable.

Pf The non-projective simple kG-modules are all in the principal block by our assumption. If S is such that $S_{N(P)} = T + Q$ as before and so $\mathrm{Ext}^i_{kG}(k, S) \neq 0$ is the case by inspection, for exactly two values of i in the usual range, and is one-dimensional as well $\therefore$ E.T.S. at most e such simples. But this is true by the second main theorem and the character values on $P^{\#}$. (Would like to avoid this use of characters!)

To avoid the use of characters we continue our development.

Prop 3 Let U and V be (indecomposable) kG-modules with simple "tops". If there is $j > 0$ with $\mathrm{Ext}^j_{kG}(k, U) \neq 0$ and $\mathrm{Ext}^j_{kG}(k, V) \neq 0$ then there is a homomorphism of U to V or one of V to U with one of the following holding: it is an epimorphism; it induces an isomorphism $\mathrm{Ext}^j(k, U) \cong \mathrm{Ext}^j(k, V)$.

Cor 4 There are at most e non-projective simple kG-modules.

(Another pf. For look at the case where U, V are simple!) Hence, have avoided characters.)

Before proving Prop 3, we need a preliminary result.

Lemma 5 If W_1 and W_2 are kG-modules with W a quotient module of $W_1 \oplus W_2$ and W having a simple top then W is a homomorphic image of W_1 or of W_2.

Pf W is the sum of the image of W_1 and the image of W_2 while the radical of W is the unique maximal submodule of W.

Pf (of Prop 3) Let $U_{N(P)} = X + P$, $V_{N(P)} = Y + Q$ direct sums with X, Y indecomposables, P, Q projective. Thus, $\mathrm{Ext}^j_{kN(P)}(k, X) \neq 0$, $\mathrm{Ext}^j_{kN(P)}(k, Y) \neq 0$ so, by inspection, we have that – after a relabelling if necessary – that Y is a homomorphic image of X. Hence, Y^G is a homomorphic image of X^G, i.e. $Y^G = V + S$ is a hom. image of $X^G = U + R$, sums direct, R, S projective.

Therefore, V is a homomorphic image of $U+R$. If V is a homomorphic image of U we are done; otherwise, by the lemma, we have that V is a homomorphic image of R.

Now, let's return to X, Y and the map of X onto Y. By inspection, it induces an isomorphism

$$\mathrm{Ext}^i_{kN(P)}(k, X) \simeq \mathrm{Ext}^i_{kN(P)}(k, Y),$$

and so the map of X^G onto Y^G induces an iso

$$\mathrm{Ext}^i_{kG}(k, X^G) \simeq \mathrm{Ext}^i_{kG}(k, Y^G)$$

and so

$$\mathrm{Ext}^i_{kG}(k, U) \simeq \mathrm{Ext}^i_{kG}(k, V)$$

is induced as claimed.

<u>Remark</u>: Seem to always get second conclusion of Prop 3.

We can use this result to get more information about the structure of the minimal projective resolution of k.

<u>Prop 6</u>. $\Omega^n(k)$ is uniserial, $n \geq 0$.

Pf. Let U and V be quotients of $\Omega^n(k)$; we shall show that one of these is a homomorphic image of the other. As a consequence of Cor 4 and previous argument we know that each term of the minimal projective resolution of k is indec. so each $\Omega^n(k)$ has a simple top and is thus indec. Applying Prop 3, using the natural maps of $\Omega^n(k)$ to U and to V to define non-zero cohomology, we get that, after relabelling if necessary, there is a hom $\mu: U \to V$ inducing an iso on cohomology:

$$\mathrm{Ext}^n_{kG}(k, U) \simeq \mathrm{Ext}^n_{kG}(k, V).$$

Let ρ, σ be natural maps $\Omega^n(U)$ to U, V respectively:

$$\begin{tikzcd} & \Omega^n(U) \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\ U & & V \end{tikzcd}$$

Now suppose that μ is not an epimorphism. Then $\mu(U) \subseteq \mathrm{rad}(V)$, as V also has a simple top. Now ρ, σ define non-zero elements of $\mathrm{Ext}^n_{kG}(k, U)$, $\mathrm{Ext}^n_{kG}(k, V)$ respectively; they give cocycles which are not coboundaries. Hence, by the above iso given by μ we have that $\rho\mu$ is also defining a non-zero element of $\mathrm{Ext}^n_{kG}(k, V)$. But $\mathrm{Ext}^n_{kG}(k, V) \cong \mathrm{Ext}^n_{kN(P)}(k, Y)$ is one-dimensional, so after changing things by scalar multiples we can assume that $\rho\mu - \sigma$ defines the zero element of $\mathrm{Ext}^n_{kG}(k, V)$. But $\rho\mu - \sigma$ is an epimorphism as $\Omega^n(U)\rho\mu \subseteq \mathrm{rad}\, V$ and σ is an epi. So this is a contradiction. Hence our claim holds and so the proposition now follows from the next result:

__Lemma 7.__ If W is a kG-module with a simple top and given any two quotients of W, one is always a hom. image of the other, then W is uniserial.

Pf. Let $L_0 = W \supseteq L_1 \supseteq L_2 \supseteq \cdots$ be the lower Loewy series of W. Choose n maximal such that W/L_n is uniserial. Hence, L_n/L_{n+1} is not simple. It must be homogeneous, by the hypothesis, for otherwise W/L_{n+1} has two non-isomorphic uniserial quotients of composition length $n+1$. Hence, there is a simple kG-module S and a quotient of L_n isomorphic with $S \oplus S$. This gives two quotients of W of comp. length

$n+1$ which are uniserial and have socles isomorphic with S. These are isomorphic and so the extensions of S by W/L_n they define are equivalent. Thus, in our extension of $S \oplus S$ by W/L_n we can find a "diagonal" S such that when we factor it out the resulting extension of S by W/L_n splits, a contradiction.