RESEARCH NOTES

VOLUME VII

Research Notes

Volume VI

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Note: As before, and always, these notes need not be correct, complete or chronological!

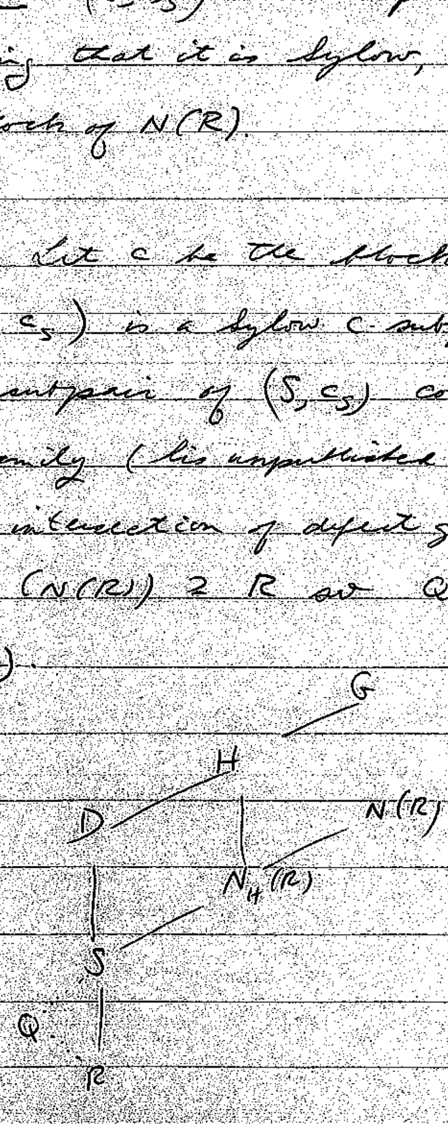
Control of fusion in blocks

We shall consider the following set-up. G is a finite group, b is a p-block, (D, b_D) is a Sylow b-subpair of G and H is a control subgroup for b, $H \supseteq D$.

Proposition. Let R be a subgroup of D with (R, b_R) extremal in (D, b_D). If $S = N_D(R)$ and (S, b_S) is the normalizer of (R, b_R) in (D, b_D) then $H \cap N(R)$ is a control subgroup for (S, b_S) in $N(R)$.

Remarks. (S, b_S) is a subpair of $N(R)$ as $C(S) \subseteq N(R)$. We are assuming that it is Sylow, that is for the appropriate block of $N(R)$.

Proof. Let c be the block of $N(R)$ in question so $(S, b_S) = (S, c_S)$ is a Sylow c-subpair of $N(R)$. Let (Q, c_Q) be a subpair of (S, c_S) coming from the Puig conjugation family (his unpublished work - at this point). Since Q is an intersection of defect groups for c, it follows that $Q \supseteq O_p(N(R)) \supseteq R$ so $QC(Q) \subseteq N(R)$ and $(Q, c_Q) = (Q, b_Q)$.



To prove the result, we take $x \in N_{N(R)}(b_Q) = N(R) \cap N(b_Q)$ and we must produce a factorization in terms of $C(Q)$ and $N_H(R) \cap N(b_Q)$. Let $x = ch$, where $c \in C(Q)$, $h \in H$, which exists by hypothesis. Since $Q \supset R$ we have $c \in C(R)$ so $h = c^{-1}x$ also normalizes Q and R. It remains only to see that h normalizes b_Q. But $c \in C(Q)$ certainly normalizes b_Q, x does by assumption, so h does and the proof is complete.

A similar result, with a similar proof, holds for extremal b-elements:

<u>Proposition</u> If $z \in D$ and (z, b_z) is an extremal b-element in (D, b_D) while $S = C_D(z)$ then $H \cap C(z)$ is a control subgroup for (S, b_S) in $C(z)$.

Now, let's <u>assume</u> in addition, that $H \supseteq C(D)$ so $b_D^H = \beta$ makes sense and $(D, b_D) = (D, \beta_D)$ etc. What we're after are the following guesses:

<u>Conjecture k</u>: The number $k(b)$ of ordinary irreducible characters in b equals the number $k(\beta)$ in β.

<u>Conjecture l</u>: The number $l(b)$ of modular irreducible characters in b equals the number $l(\beta)$ in β.

Especially interesting is the case that $H = N(b_D)$ as this covers the nilpotent block case, abelian D, TI blocks, b.

We're interested in the relation between these conjectures, possibly their equivalence. (Also in deeper questions about further structure more than just counting; for example, when are b and β stably equivalent?)

What about the l-conjecture implying the k-conjecture? Keeping our notation as above, let $\mathcal{X}$ be a subset of D such that the b-elements (x, b_x), $x \in \mathcal{X}$, are a full set of representatives of the b-subsections of G and such that each (x, b_x) is extremal in (D, b_D). One would like that the same statements then hold for all the (x, β_x) and that $l(b_x) = l(\beta_x)$, all $x \in \mathcal{X}$, so that the corollary of the second main theorem expressing $k(b)$ in terms of $l(b_x)$ and $k(\beta)$ in terms of the $l(\beta_x)$ can be applied. But it's not so easy.

For example, suppose $x \in \mathcal{X}$ and $S = C_D(x)$ so (S, b_S) is Sylow in $C_G(x)$. We have shown that $C_H(x)$ has the relevant control properties, but there's no reason to expect $C(S) \subseteq C_H(x)$, so no hope of applying the l-conjecture directly.

We're also interested in the "weak" k-conjecture and the "weak" l-conjecture — ie the k and l conjectures in the special case that $H = N(b_D)$ — and their relationship. Dade's work on Morita equivalence based on his endopermutation modules looks relevant. Perhaps also in the gen'l case above.

The question on the previous page in the second paragraph has a positive answer in the "weak" situation, that is, when $H = N(b_D)$. Indeed, by the local theorem on fusion of subsections, it suffices to prove the following:

<u>Lemma</u> Under the above hypotheses, when $H = N(b_D)$, and R is "extremal in D,"

$$N(b_R) = C(R)\, N_H(\beta_R).$$

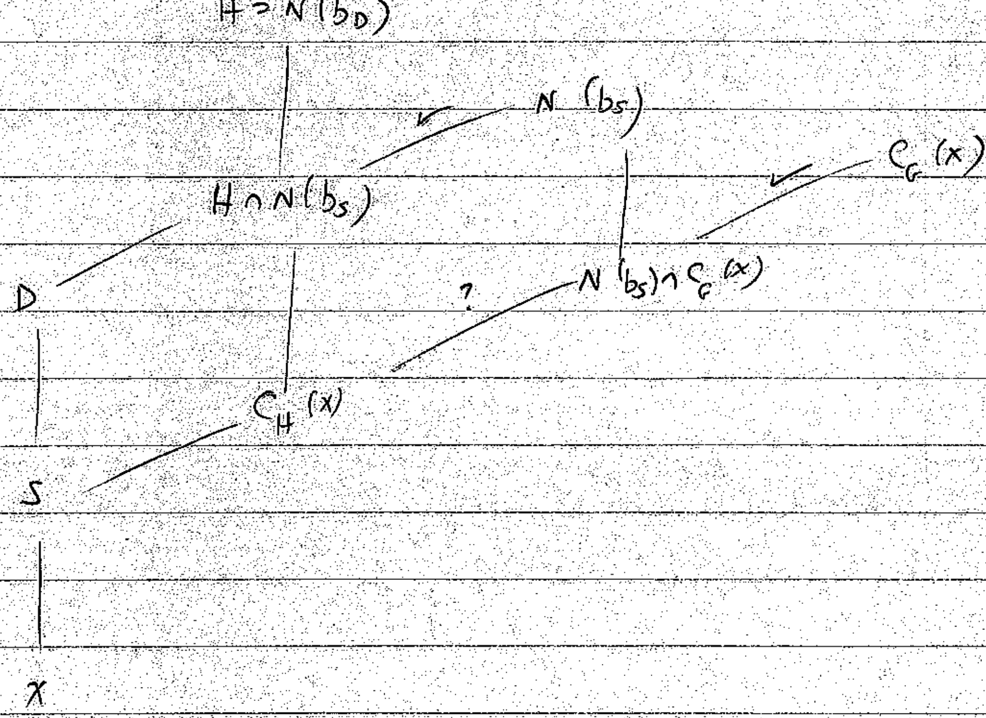
We mean no G-conjugate of R lying in D has $|N_D(\ )|$ larger than $|N_D(R)|$; that is (R, b_R), (R, β_R) are extremal.

<u>Proof.</u> $N(b_R) = C(R) \cdot H \cap N(b_R)$, as H is a control subgroup and such equations are equivalent to control. But $H \cap N(b_R) = N_H(\beta_R) = N(\beta_R)$ (as are working in H for β); indeed, since H normalizes $(D, b_D) = (D, \beta_D)$, to lie in these subgroups for $h \in H$ is equivalent to $h \in N(R)$, by the uniqueness properties of subpairs.

For the weak l-conjecture to imply the weak k-conjecture, this leaves the thorny question of

$$l(b_X) = l(\beta_X)\ ?$$

Let's just look at the relevant diagram in the very special case that $S = C_D(X)$ is normal in D.
It is as follows:



The two edges with checks ("✓") are places where the weak l-conjecture can be applied (as one can as $N(b_S) \cap C_G(x) \supseteq C_H(x)$ which already controls.) The question mark points out the problem. We seem to have reduced to a local situation: $N(b_S)$. This is why we hope Clifford theory helps.

McKay's diagonalization

McKay has made the following striking observation: If C is the Cartan matrix of an extended Dynkin diagram with single bonds then there is a character table with $X^{-1}CX$ diagonal. His results:

Diagram	Character table
A_n	cyclic
D_n	binary dihedral
E_6	binary tetrahedral ($\simeq SL(2,3)$)
E_7	binary octahedral
E_8	binary icosahedral ($\simeq SL(2,5)$).

He has also observed the reason. Let X be a character table of the finite group G. Let $\chi_1, \dots, \chi_n$ be the irreducible characters. Let χ be a character and let M be the non-negative integral matrix defined by

$$\chi_i \, \chi = \sum_j m_{ij} \, \chi_j$$

Then $X^{-1}MX$ is diagonal, that is, the columns of X are eigenvectors of M. This is an immediate consequence of the definition of M; the eigenvalue for the column corresponding to $g \in G$ is $\chi(g)$.

If M is the adjacency matrix of the extended Dynkin diagram then X also diagonalizes $2I - M$, the Cartan matrix.

This is what happens:

$$\begin{array}{ccccccc}
1 & 2 & 4 & 3 & 3 & 6 & 6 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \omega & \bar\omega & \omega & \bar\omega \\
1 & 1 & 1 & \bar\omega & \omega & \bar\omega & \omega \\
3 & 3 & -1 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & -1 & -1 & 1 & 1 \\
2 & -2 & 0 & -\omega & -\bar\omega & \omega & \bar\omega \\
2 & -2 & 0 & -\bar\omega & -\omega & \bar\omega & \omega
\end{array}$$

$\chi_5 \chi_1 = \chi_5$

$\chi_5 \chi_2 = \chi_6$

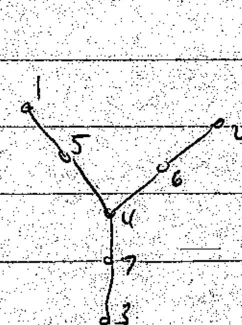
$\chi_5 \chi_3 = \chi_7$

$\chi_5 \chi_4 = \chi_5 + \chi_6 + \chi_7$

$\chi_5^2 = \chi_1 + \chi_4$

$\chi_5 \chi_6 = \chi_2 + \chi_4$

$\chi_5 \chi_7 = \chi_3 + \chi_4$



	1	2	4	3	6	4	8	8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	-1
χ_3	2	2	2	-1	-1	0	0	0
χ_4	3	3	-1	0	0	1	-1	-1
χ_5	3	3	-1	0	0	-1	1	1
χ_6	2	-2	0	-1	1	0	$\alpha+\alpha^7$	$\alpha^3+\alpha^5$
χ_7	2	-2	0	-1	1	0	$\alpha^3+\alpha^5$	$\alpha+\alpha^7$
χ_8	4	-4	0	1	-1	0	0	0

$\chi_1\chi_6 = \chi_6$

$\chi_2\chi_6 = \chi_7$

$\chi_3\chi_6 = \chi_8$

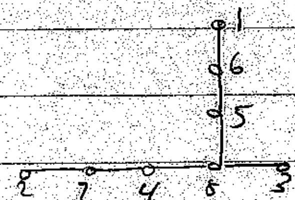
$\chi_4\chi_6 = \chi_7+\chi_8$

$\chi_5\chi_6 = \chi_6+\chi_8$

$\chi_6\chi_6 = \chi_1+\chi_5$

$\chi_7\chi_6 = \chi_2+\chi_4$

$\chi_8\chi_6 = \chi_3+\chi_4+\chi_5$



	1	2	4	3	6	5	5	10	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	0	0	$1+\lambda+\lambda^4$	$1+\lambda^2+\lambda^3$	$1+\lambda+\lambda^4$	$1+\lambda^2+\lambda^3$
χ_3	3	3	-1	0	0	$1+\lambda^2+\lambda^3$	$1+\lambda+\lambda^4$	$1+\lambda^2+\lambda^3$	$1+\lambda+\lambda^4$
χ_4	4	4	0	1	1	-1	-1	-1	-1
χ_5	5	5	1	-1	-1	0	0	0	0
χ_6	2	-2	0	-1	1	$\lambda+\lambda^4$	$\lambda^2+\lambda^3$	$-\lambda-\lambda^4$	$-\lambda^2-\lambda^3$
χ_7	2	-2	0	-1	1	$\lambda^2+\lambda^3$	$\lambda+\lambda^4$	$-\lambda^2-\lambda^3$	$-\lambda-\lambda^4$
χ_8	4	-4	0	1	-1	-1	-1	+1	+1
χ_9	6	-6	0	0	0	1	1	-1	-1

$\chi_6\chi_1 = \chi_6$

$\chi_6\chi_2 = \chi_9$

$\chi_6\chi_3 = \chi_6+\chi_8$

$\chi_6\chi_4 = \chi_1+\chi_9$

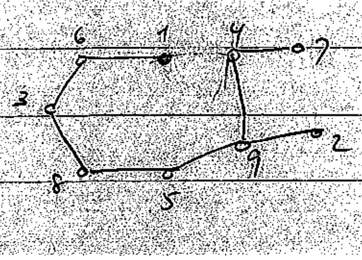
$\chi_6\chi_5 = \chi_8+\chi_9$

$\chi_6\chi_6 = \chi_1+\chi_3$

$\chi_6\chi_7 = \chi_4$

$\chi_6\chi_8 = \chi_3+\chi_5$

$\chi_6\chi_9 = \chi_2+\chi_4+\chi_5$



For the type A_n, a cyclic group of order $n+1$ has characters $\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^n$ and $(\lambda + \lambda^{-1}) \cdot \lambda^i = \lambda^{i-1} + \lambda^{i+1}$ giving the adjacency matrix of a cycle, as desired.

Our only contribution, is as follows, also mostly observed by Berger and McKay. We preserve our notation.

<u>Prop.</u> i) M is symmetric iff χ is real valued

ii) M has all diagonal entries 0 iff the "row sum" of χ is zero.

iii) M is connected iff χ is "faithful."

<u>Proof</u> i). $m_{ij} = (\chi_i \chi, \chi_j)$, $m_{ji} = (\chi_j \chi, \chi_i)$ so χ real valued implies M is symmetric. On the other hand, if M is symmetric and χ_1 is the principal character, then for any j,

$$(\chi, \chi_j) = (\chi_1 \chi, \chi_j) = m_{1j} = m_{j1} = (\chi_j \chi, \chi_1) = (\chi_j, \bar{\chi})$$
$$= (\bar{\chi}, \chi_j)$$

so $\chi = \bar{\chi}$

ii). The condition on row sums means that no constituent of χ occurs in the permutation representation on G by conjugation, whose character is $\Sigma \chi_i \bar{\chi}_i$. That is, iff $(\chi, \chi_i \bar{\chi}_i) = 0$ all i.

iii) We mean the associated representation is faithful. This is clear by a theorem of Burnside applied to decomposing the powers of χ.

<u>Remark</u>: Can produce other graphs by using $G \times H$ + $\chi \otimes \varphi$ where G, χ and H, φ work.

Permutation modules ($GL(n,q)$, Σ_n, $L_3(4)$)

We're interested in a special case: G a finite group, k (algebraically closed) field of characteristic p, P a Sylow p-subgroup of G, $M = (k_P)^G$, E is the endomorphism algebra of M. We want to get at the indecomposable summands of M, that is, the structure of E. This seems relevant to a number of problems:

1) Determination of simple kG-modules;

2) Trying to get stable equivalence or weaker conditions when one has control - Broué's theorem for a block of full defect;

3) The McKay problem;

Indeed, for 1), every simple kG-module is a homomorphic image of M. For 3), the ring we have already introduced is a "souped-up" version of E. There are a few things known that are relevant: The work on groups of Lie type, eg. Green on Sawada; Scott modules at least show some nice property.

Here are some tasks to start with: 1) Discuss a formula for the number of modular irreducibles in a block, and so get a formula for the number of ordinary irreducibles, using the Second Main Theorem; 2) Calculate E, M in some new case, for example for the symmetric groups; 3) Try to get stable equivalence in the cyclic and possibly nilpotent cases by using M and E; 4) Look at $L_3(4)$ again for $p=3$ this time at M and E rather than restriction, Here is the principle which we think governs $l(b)$, the number of modular irreducibles in the p-block b. Let D be the defect group, (D, b_D) a Sylow b-subpair. The more fusion there is the greater should be $l(b) - l(b_D)$, which should always be non-negative. We should have a formula for $l(b)$ in terms of the l's of various local subgroups.

For example, if $G = GL(n, q)$, q a power of p. To get at the number of non-projective modular irreducibles (lumping together all the p-blocks of full defect) we want to count modular irreducibles locally, adding up, but not counting twice.

Suppose $n = 3$ so B has $(q-1)^3$ irreducibles. Each of the two maximal parabolics has $q(q-1)^2$, i.e. $(q-1)^2$ more than B. Moreover,

$$(q-1)^3 + 2(q-1)^2 = (q-1+2)(q-1)^2$$
$$= (q-1)(q^2-1)$$

which is the number for G.

Next, let $n = 4$. We make a list of parabolics and the number of non-projective modular irreducibles:

G	$(q^3-1)(q-1)$
maximal parabolics	$q^2(q-1)^2$
minimal $(> B)$ parabolics	$q\ (q-1)^3$
B	$(q-1)^4$

We get these by looking at each case. Now let's figure the contributions from each of these locals to get at the total for G. We have for B

B	$(q-1)^4$

For the minimal parabolics we get

$$\text{minimal} \qquad q(q-1)^3 - (q-1)^4 = (q-1)^3.$$

For the maximal parabolics we must compute their number of unipotents and subtract the contributions from B and from the two minimal parabolics inside. We get

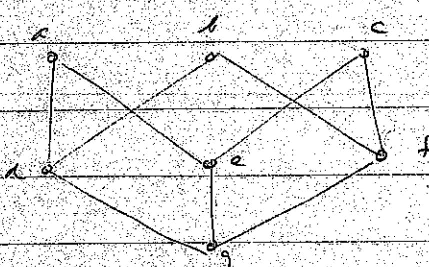
$$\begin{aligned} \text{maximal} \qquad & q^2(q-1)^2 - 2(q-1)^3 - (q-1)^4 \\ &= (q-1)^2\{q^2 - 2q + 2 - q^2 + 2q - 1\} \\ &= (q-1)^2. \end{aligned}$$

Hence, the total contributions are

$$\begin{aligned} &(q-1)^4 + 3(q-1)^3 + 3(q-1)^2 \\ &= \{(q-1+1)^3 - 1\}(q-1) \\ &= (q^3-1)(q-1), \end{aligned}$$

which is the number for G, as desired!

There seems to be a Möbius function at play. Let's draw a picture:



Parabolics & # of unipotents

$$a - (d-g) - (e-g) - g$$

$$g + (d-g) + (e-g) + (f-g) + (a - d - e + g) + (b - d - f + g) + (c - e - f + g)$$
$$= a + b + c - d - e - f + g$$

Let's see if this works for $GL(n,q)$. Just let's calculate an alternating sum of the modular unipotents. Should get zero.

$$(q^{n-1}-1)(q-1) - (n-1)q^{n-2}(q-1)^2 + \cdots \pm (q-1)^n$$

$$= -(q-1) + (q-1)\{q^{n-1} - (n-1)q^{n-2}(q-1) + \cdots \pm (q-1)^{n-1}\}$$

$$= -(q-1) + (q-1)\{(q-(q-1))^{n-1}\}$$

$$= -(q-1) + (q-1)$$

$$= 0,$$

as desired.

Now let's try and rationalize this. By Scott's results (unpublished) – and perhaps in other papers – the vertices of $(k_B)^G$ are the unipotent radicals (i.e. O_p) of the parabolics. Perhaps this is true for $(k_U)^G$, $U = P$ the upper triangular unipotent matrices. This may explain the terms, showing when a given simple appears in the summands of $(k_U)^G$.

Question: Under our general hypothesis, if P is abelian are there non-projective summands of $(k_P)^G$ which are in the principal block and have vertices properly contained in P?

For we'd like to only want $N(P)$ to get $l = l$ (i.e. local + global). Also this might indicate how to get near stable equivalence. If this is not the case we need further criteria to pick out the relevant terms of $(k_P)^G$.

Now let $G = \Sigma_{p^2}$ and let's guess the relevant local subgroups and the contributions to $l(b_0)$, b_0 the principal p-block of G.

We're guessing that in addition to the normalizer $N(P)$ we need only two more contributions, one from the normalizer of the base group, one from the normalizer of $Z_p \times Z_p$ (the one consisting of "long" elements). We won't count the normalizer of the product of k cyclic groups of order p (on kp letters) since this group has an abelian Sylow p-subgroup, the base group, so by ideas on the abelian case this is already counted.

First, let's look at $N(P)$. We can calculate this by doing so within the normalizer of the base group. We get:

$$\begin{array}{cc} & Z_{p-1} \\ Z_{p-1} \times & Z_p \\ & \\ Z_p \times \cdots \times Z_p & (p \text{ times}) \end{array}$$

so the contribution is $\underline{(p-1)^2}$.

From $Z_p \times Z_p \cdot GL(2,p)$ we get a contribution of $p(p-1) - (p-1)^2 = \underline{p-1}$ as there are $p(p-1)$ modular irreducibles here, and presumably the $(p-1)^2$ linear characters of $N(P)$ are counted in the normalizer of the Sylow p-subgroup of $Z_p \times Z_p \cdot GL(2,p)$.

Finally, consider the normalizer of the base group: $Z_p \times \cdots \times Z_p \cdot Z_{p-1} \wr \Sigma_p$ (the permutation wreath product). All the modular irreducibles are in the one block, by the structure, so the contribution here is $\underline{l(Z_{p-1} \wr \Sigma_p) - (p-1)^2}$.

Hence, we get the guess

$$l(b_0(\Sigma_{p^2})) = l(Z_{p-1} \wr \Sigma_p) + (p-1).$$

However, if $F_{p(p-1)}$ is the normalizer of the Sylow p-subgroup of Σ_p, a Frobenius group of order $p(p-1)$, then cyclic theory gives us

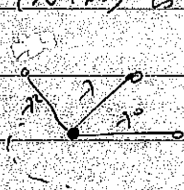
$$k(Z_{p-1} \wr \Sigma_p) - l(Z_{p-1} \wr \Sigma_p) = k(Z_{p-1} \times F_{p(p-1)}) - l(Z_{p-1} \times F_{p(p-1)})$$
$$= (p-1)\,p - (p-1)^2$$
$$= p-1$$

so our guess, in final form is

$$l(b_0(\Sigma_{p^2})) = k(Z_{p-1} \wr \Sigma_p).$$

This is correct when $p=2$ as $l(\Sigma_4) = 2$. When $p=3$ it's also O.K. For $l(\Sigma_9) = 16$ and there are two 3-blocks of defect 0, and two of defect 1 each having two modular irreducibles (coming from $\Sigma_3 \times \Sigma_6$, as Σ_6 has two characters of degree 9) so the left hand side is ten. But $k(Z_2 \wr \Sigma_3) = k(Z_2 \times \Sigma_4) = 10$.

Now let's turn our attention to E and M, as described at the start. Let suppose $|P| = p$ and $N(P)/P\,C(P)$ has order e. Let $\lambda^0, \lambda^1, \ldots \lambda^{e-1}$ be the modular irreducibles in b_0 for $N(P)$ where λ^1 is the natural representation, so the Brauer tree for $N(P)$ for $b_0(N(P))$ is



Thus $\Omega^{2i}(\lambda^0) = \lambda^i$. Let $k = S_0, S_1, \ldots S_{e-1}$ be the modular irreducibles of $b_0(G)$ – the edges of the tree. Each S_i is the simple top of two obvious uniserial modules – winding around a node. These modules are the Green correspondents of the $\Omega^i(\lambda^0)$. Hence, certain of them are the Green correspondents of the modular irreducibles λ^i

This is quite easy to describe in terms of the walk through the Brauer tree. Let $k = V_0, V_1, V_2, \dots, V_{2e-1} = k$ be the sequence of modular irreducibles corresponding to the edges traversed. Let U_i be the canonical uniserial corresponding to V_i and to the node attached to V_{i-1}. It's easy to see that

<u>Prop.</u> The Green correspondents of $\lambda^0, \lambda^1, \dots, \lambda^{e-1}$ are $U_0, U_2, U_4, \dots, U_{2e-2}$.

For example: $p=5$, $G = \Sigma_5$, tree is

$$\overset{1}{\circ}\underset{1}{\text{------}}\overset{4}{\circ}\underset{3_1}{\text{------}}\overset{6}{\circ}\underset{3_2}{\text{------}}\overset{4}{\circ}\underset{-1}{\text{------}}\overset{1}{\circ}$$

so Green correspondents are

$$1, \quad \begin{matrix}3_2\\3_1\end{matrix}, \quad -1, \quad \begin{matrix}3_1\\3_2\end{matrix}$$

and it's easy to see that

$$(k_p)^G = 1 \oplus (-1) \oplus \begin{matrix}3_1\\3_2\end{matrix} \oplus \begin{matrix}3_2\\3_1\end{matrix} \oplus 5_1 \oplus 5_2.$$

For example: $p=7$, $G = \Sigma_7$, tree is

$$\overset{1}{\circ}\underset{1}{\text{----}}\overset{6}{\circ}\underset{5}{\text{----}}\overset{15}{\circ}\underset{10}{\text{----}}\overset{20}{\circ}\underset{-10}{\text{----}}\overset{15}{\circ}\underset{-5}{\text{----}}\overset{6}{\circ}\underset{-1}{\text{----}}\overset{1}{\circ}$$

with Green correspondents

$$1, \quad \begin{matrix}10\\5\end{matrix}, \quad \begin{matrix}-5\\-10\end{matrix}, \quad -1, \quad \begin{matrix}-10\\-5\end{matrix}, \quad \begin{matrix}5\\10\end{matrix}$$

and it's easy to get from the character table that

$$(k_p)^G = 1 \oplus \begin{matrix}10\\5\end{matrix} \oplus \begin{matrix}-5\\-10\end{matrix} \oplus (-1) \oplus \begin{matrix}-10\\-5\end{matrix} \oplus \begin{matrix}5\\10\end{matrix} \oplus \begin{matrix} & 10 & \\ 5 & & -10 \\ & 10 & \end{matrix} \oplus \begin{matrix} & -10 & \\ 10 & & -5 \\ & -10 & \end{matrix} \oplus \left(\begin{matrix}\text{other}\\\text{blocks}\end{matrix}\right)$$

Another case: $p=3$, $G = \Sigma_3$. Tree: $\overset{1}{\circ}\underset{1}{\text{----}}\overset{2}{\circ}\underset{-1}{\text{----}}\overset{1}{\circ}$

$$(k_p)^G = 1 \oplus -1.$$

General pattern for Σ_p is clear. Aside from projectives and other blocks get $p-1$ modules in $(B_p)^{\Sigma_p}$ and the endomorphism algebra of their direct sum, a homomorphic image of E, has dimension $2 \cdot 1 + p-3 \cdot 2 = 2p-4$.

Now let's turn to $\Sigma_{2p} = G$. Let's start with $p = 3$. (Our work here agrees with James' tables)

3'-classes

		1^6	$1^4 2$	$1^2 4$	$1^2 2^2$	15	24	2^3
	6	1	1	1	1	1	1	1
	51	5	3	1	1	0	-1	-1
	42	9	3	-1	1	-1	1	3
	41^2	10	2	0	-2	0	0	-2
ordinary irreducibles	3^2	5	1	-1	1	0	-1	-3
	321	16	0	0	0	1	0	0
	2^3	5	-1	1	1	0	-1	3
	31^3	10	-2	0	-2	0	0	2
	$2^2 1^2$	9	-3	1	1	-1	1	-3
	21^4	5	-3	-1	1	0	-1	1
	1^6	1	-1	-1	1	1	1	-1
		1	1	1	1	1	1	1
		4	2	0	0	-1	-2	-2
modular irreducibles		6	0	0	-2	1	2	0
		4	-2	0	0	-1	-2	2
		1	-1	-1	1	1	1	-1
		9	3	-1	1	-1	1	3
		9	-3	1	1	-1	1	-3

D_0:

	1	4	6	4	1
1	1				
5	1	1			
10		1	1		
10			1	1	
5				1	1
1					1
5		1			1
16	1	1	1	1	1
5	1			1	

C_0:

	1	4	6	4	1
1	4	2	1	2	1
4	2	4	2	1	2
6	1	2	3	2	1
4	2	1	2	4	2
1	1	2	1	2	4

From Littlewood's tables, we can calculate the character

$$\chi(1_P^{\Sigma_6}) = \chi_6 + \chi_{51} + \chi_{42} + 2\chi_{41^2} + \chi_{3^2} + \chi_{2^3} + 2\chi_{31^3} + \chi_{2^21^2} + \chi_{21^4} + \chi_{1^6},$$

so using D we get composition factors (with obvious notation)

$$k_P^{\Sigma_6} \doteq 3S_1 \oplus 4S_4 \oplus 4S_6 \oplus 4S_{-4} \oplus 3S_{-1} \oplus S_9 \oplus S_{-9}$$

Since $k_P^{\Sigma_6}$ is stable under tensoring with S_{-1} we see that the only projective summands of $k_P^{\Sigma_6}$ are S_9, S_{-9} and possibly, with multiplicity at most one, the projective cover of S_6.

Let's determine how many times each simple occurs in the socle (+ at the top) of $k_P^{\Sigma_6}$. Have to determine $\mathrm{Hom}_P(k, S)$. But $P \leq A_6 \cong PSL(2,9)$ so can work with the known representations of this group.

For example,

$$\mathrm{Hom}_P(k, S_4) \simeq \mathrm{Hom}_P(S_2, S_2^{\varphi})$$

where these are modules for $SL(2,9)$, φ the Frobenius.

Thus, have to determine the matrices which intertwine the $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \alpha^3 \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & \alpha r + s \\ t & \alpha t + u \end{pmatrix}$$

$$\begin{pmatrix} 1 & \alpha^3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} r + \alpha^3 t & s + \alpha^3 u \\ t & u \end{pmatrix}$$

so $r = t = u = 0$. Hence S_4 and similarly S_{-4} occurs in the socle of $k_P^{\Sigma_6}$ once.

Now S_6 must be a sum of two three-dimensional modules on $A_6 \simeq PSL(2,9)$ so we also get easily that S_6 occurs twice in this socle. Hence, we have

Lemma $\mathrm{Soc}(k_P^{\Sigma_6}) \simeq S_1 \oplus S_{-1} \oplus S_4 \oplus S_{-4} \oplus S_6 \oplus S_6 \oplus S_9 \oplus S_{-9}$

To get more we have to look at the characters of $N(P)$ induced to Σ_6. We look at the classes of Σ_6 intersecting $N(P)$ and the characters of $N(P)/P$:

$\overbrace{\underbrace{\quad 1 \quad 4 \quad 4 \quad}_{P} \quad 6 \quad 12 \quad 9 \quad}^{\Sigma_3 \times \Sigma_3}$

#	1	4	4	6	12	9	6	12	18
	(1)	(123)	(123)(456)	(12)	(12)(456)	(12)(45)	(14)(25)(36)	(153426)	(1524)(36)
	1^6	$1^3 3$	3^2	$1^4 2$	$1\,2\,3$	$1^2 2^2$	2^3	6	$2\,4$
φ_1	1	1	1	1	1	1	1	1	1
φ_2	1	1	1	-1	-1	1	-1	-1	1
φ_3	1	1	1	1	1	1	-1	-1	-1
φ_4	1	1	1	-1	-1	1	1	1	-1
φ_5	2	2	-2	0	0	-2	0	0	0

(14)(25)(36)

(12) (45)

(123) × (456)

19

It's easy to calculate the restrictions from Σ_6:

	φ_1	φ_2	φ_3	φ_4	φ_5
6	1	0	0	0	0
51	0	0	1	0	0
42	1	-	0	-	-
41^2	-	-	0	-	1
3^2	-	-	1	-	-
321	-	-	-	-	-
2^3	-	-	0	1	-
31^3	-	-	-	-	1
$2^2 1^2$	-	1	-	-	-
21^4	-	-	0	1	-
1^6	-	1	-	-	-

(The dashes are zero filled in by counting $\dim \varphi_i^{\Sigma_6}$ or $k_p^{\Sigma_6}$ formula)

Hence, mod three,

$$\varphi_1^{\Sigma_6} \doteq S_1 \oplus S_9$$
$$\varphi_2^{\Sigma_6} \doteq S_{-1} \oplus S_9$$
$$\varphi_3^{\Sigma_6} \doteq S_1 \oplus S_{-4} \oplus S_{-4} \oplus S_{-1}$$
$$\varphi_4^{\Sigma_6} \doteq S_1 \oplus S_{-4} \oplus S_{-4} \oplus S_{-1}$$
$$\varphi_5^{\Sigma_6} \doteq S_4 \oplus S_{-4} \oplus S_6 \oplus S_6$$

Also $k_p^{\Sigma_6} \cong \varphi_1^{\Sigma_6} \oplus \varphi_2^{\Sigma_6} \oplus \varphi_3^{\Sigma_6} \oplus \varphi_4^{\Sigma_6} \oplus \varphi_5^{\Sigma_6} \oplus \varphi_5^{\Sigma_6}$

This information, together with the information on the node comes close to determining all the modules. In fact, using self-duality as well it all comes out easily:

$$\varphi_1^{\Sigma_6} \simeq S_1 \oplus S_9$$

$$\varphi_2^{\Sigma_6} \simeq S_1 \oplus S_9$$

$$\varphi_3^{\Sigma_6} \simeq \begin{tikzcd}[column sep=small, row sep=small] & S_4 \arrow[dl, dash] \arrow[dr, dash] & \\ S_1 \arrow[dr, dash] & & S_{-1} \arrow[dl, dash] \\ & S_4 & \end{tikzcd}$$

$$\varphi_4^{\Sigma_6} \simeq \begin{tikzcd}[column sep=small, row sep=small] & S_{-4} \arrow[dl, dash] \arrow[dr, dash] & \\ S_1 \arrow[dr, dash] & & S_{-1} \arrow[dl, dash] \\ & S_{-4} & \end{tikzcd}$$

$$\varphi_5^{\Sigma_6} \simeq \begin{tikzcd}[column sep=small, row sep=small] & S_6 \arrow[dl, dash] \arrow[dr, dash] & \\ S_4 \arrow[dr, dash] & & S_{-4} \arrow[dl, dash] \\ & S_6 & \end{tikzcd}$$

Hence, $E_0 \cong k \oplus k \oplus k[x]/(x^2) \oplus k[x]/(x^2) \oplus M_2(k[x]/(x^2))$.

(We can use this to make some guesses for the indecomposable projectives. For example

$$P_6 \overset{?}{\cong} \begin{tikzcd}[column sep=tiny, row sep=small] & & P_6 \arrow[dl, dash] \arrow[dr, dash] & & \\ & S_{-4} \arrow[dl, dash] \arrow[dr, dash] & & S_4 \arrow[dl, dash] \arrow[dr, dash] & \\ S_1 \arrow[dr, dash] & & S_6 \arrow[dl, dash] \arrow[dr, dash] & & S_{-1} \arrow[dl, dash] \\ & S_4 \arrow[dr, dash] & & S_{-4} \arrow[dl, dash] & \\ & & S_6 & & \end{tikzcd}$$

)

It would be a worthwhile project to generalize this to Σ_{2p}. Let's just make the guesses for the numbers of characters and guess the characters in the principal p-block of Σ_{2p}; should be then easy to verify.

Calculating from $N(P)$ we guess

$$\ell(b_0(\Sigma_{2p})) = 2(p-1) + \binom{p-1}{2} = p-1\,(2 + \tfrac{p-2}{2}) = \tfrac{1}{2}(p-1)(p+2)$$

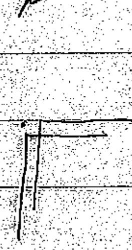
From $C(((12\cdots p)(p+1\cdots 2p)))$ we calculate the "ℓ-term" in counting $k(b_0(\Sigma_{2p}))$, get 2. From $C(((12\cdots p)))$ get $p-1$ so guess

$$k(b_0(\Sigma_{2p})) = \tfrac{1}{2}(p-1)(p+2) + p+1 = \frac{p^2+p-2+2p+2}{2} = \frac{p(p+3)}{2}$$

Now there are $2p$ "hooks" in the principal p-block — the fact that they're in b_0 is easy by removing hooks. That leaves

$$\frac{p(p+3)}{2} - 2p = \frac{p^2+3p-4p}{2} = \frac{p^2-p}{2} = \binom{p}{2}.$$

These are easy to get: (two p-hooks plus a node)



It's easy to see that the number of these is $(p-1)+(p-2)+\cdots+1$ as the first hook can be $2\,1^{p-2}$, $3\,1^{p-3}$, $\dots$, p and given that it's easy to determine the possibilities for the second.

These guesses are correct. For Olsson has verified my version of the McKay conjecture and the height zero - abelian defect conjecture for all Σ_n. Hence, our calculations are right. (Reference: J. Olsson, McKay numbers and heights of characters)

Let λ be the character (and also representation and module) for $N(P)/F_{p(p-1)} \times F_{p(p-1)}$ other than the principal character.

Conjecture. The Green correspondent of λ is
$$\begin{array}{ccc} & D_{2p-1,1} & \\ D_{2p} & & D_{p^2} \\ & D_{2p-1,1} & \end{array}$$

We seem to be close to this, perhaps even can prove it. The methods would also seem to generalize so it seems possible to get at the Green correspondents of the characters of $N(P)/P$ (i.e. simple $kN(P)$-modules), i.e. get at the main part of $(k_P)^{\Sigma_{2p}}$.

Let F_1, F_2 be the Frobenius groups, which are Sylow normalizers, on $\{1, \ldots, p\}$, $\{p+1, \ldots, 2p\}$, respectively. Let $F = F_1 \times F_2$ so $(k_F)^{N(P)} = k \oplus \lambda$. $\therefore$ we're interested in $(k_F)^{\Sigma_{2p}}$ so we start with $(k_F)^{\Sigma_p^{(1)} \times \Sigma_p^{(2)}}$, where $\Sigma_p^{(i)}$ is the symmetric group on $\{(i-1)p+1, \ldots, (i-1)p+p\}$. But
$$\begin{aligned}(k_F)^{\Sigma_p^{(1)} \times \Sigma_p^{(2)}} &= k_{F_1}^{\Sigma_p^{(1)}} \otimes k_{F_2}^{\Sigma_p^{(2)}} \\ &= (k \oplus \ldots) \otimes (k \oplus \ldots)\end{aligned}$$
where the dots denote projective terms. When we induce up to Σ_{2p} the terms, except for $k \otimes k = k$, will have vertices properly contained in P so the Green correspondents of k and λ come in $\left(k_{\Sigma_p^{(1)} \times \Sigma_p^{(2)}}\right)^{\Sigma_{2p}}$.

To get at this we induce the trivial character from $\Sigma_p^{(1)} \times \Sigma_p^{(2)}$ to Σ_{2p} and look at the part in the principal p-block. The induced character is — in notation of James' book — $[p][p]$. (As in Littlewood-Richardson)

But

$$[p][p] = [2p] + [2p-1,1] + \dots + [p,p]$$

from the example on page 52 of James' book (or use the Young rule or the Littlewood-Richardson rule — that will help in generalizing this argument to other simple $kN(P)$-modules).

The principal p-block part is, by inspection

$$[2p] + [2p-1,1] + [p,p].$$

The decomposition mod p for some of these follows from lines -6 to -2 on page 106 of James:

$$[2p-1,1] \doteq D_{2p-1,1} + D_{2p}$$

$$[p,p] \doteq D_{p,p} + D_{2p-1,1}$$

Since $[2p]$ is the trivial module we deduce: The composition factors of the Green correspondent of λ come from $D_{2p-1,1}$, D_{2p}, $D_{p,p}$, $D_{2p-1,1}$ and the remaining terms form a sum of modules with vertices properly contained in P.

However,

$$\dim D_{2p} = 1$$

$$\dim D_{2p-1,1} = 2p-2 \quad (= (2p-1)-1)$$

$$\dim D_{p,p} = \frac{(2p)!}{(p+1)!\,p!} - (2p-2).$$

Since $(1+p)(1-p) \equiv 1 \pmod{p^2}$ we have $\frac{(2p)!}{(p+1)!\,p!} \equiv \binom{2p}{p} \pmod{p^2}$

But $\binom{2p}{p} \equiv 2 \pmod{p^2}$; let P act on bisections of $\{1,\dots,2p\}$ so each el't of $P^{\#}$ fixes a unique bisection. Hence,

$$\dim D_{p,p} \equiv 4 \pmod{p^2}$$

The Green correspondent of λ has degree congruent to one modulo p, by its restriction to $N(P)$. Since we can assume $p > 3$ as we have already dealt with that case, a little case analysis gives that the Green correspondent has all four composition factors. (When $p=5$ could have $D_{2p+1,1}$, $D_{2p,1}$ and then D_{2p} and $D_{p,p}$ fit in summands of dimension divisible by p. But self-duality of λ eliminates this case.)

Now, in this correspondent, D_{2p} is not at the top or bottom, by reciprocity. Also, by self-duality, $D_{p,p}$ is not at top or bottom so get

<u>Prop</u> The Green correspondent of λ is as conjectured.

Now let's turn to $L_3(4)$ — see volume III, pages 41-43. We take $p=3$, $G=L_3(4)$ so $N(P) = Z_3 \times Z_3 . Q_8$. Since G has one class of involutions whose centralizers have order 64, it follows that the elements of order four in the Q_8 lie in the three classes of elements of order four in G. This, by using reciprocity, allows us to induce the characters of $N(P)/P$: (principal block part)

	1	20	35	35	35	64
1	1	1	1	1	1	1
i	0	1	2	0	0	1
j	0	1	0	2	0	1
k	0	1	0	0	2	1
2	0	0	0	0	0	2

Now using the decomposition matrix get the composition factors for inducing the simple $N(P)$ modules modulo three:

	1	19	$15^{(1)}$	$15^{(2)}$	$15^{(3)}$
1	5	5	2	2	2
i	3	4	3	1	1
j	3	4	1	3	1
k	3	4	1	1	3
2	0	2	2	2	2

Thus the principal block part of 1^G is the direct sum of the trivial module and the projective cover of the 19-dimensional module. It's also easy to rule out simple socles for all the Green correspondents.

Remark: Note that Cartan matrix on p42 of volume III is wrong — calculated incorrectly. Should be:

	1	19	15	15	15
1	5	4	1	1	1
19	4	5	2	2	2
15	1	2	2	1	1
15	1	2	1	2	1
15	1	2	1	1	2

This shows picture of $P(Q_5)$ on p 42 is wrong.

We can say some things more about restrictions. Let V be the 21-dimensional permutation module (pts of proj space). We can easily calculate the composition factors of its restriction to $N = N(Z_3 \times Z_3)$:

1	2	4	4	4
21	5	1	1	1

Set $V_N \doteq 4\times 1 + 3\times i + 3 \times j + 3\times k + 4 \times \text{"}2\text{"}$

We calculated the socle of V_N in volume III & get

$$V_N = \begin{matrix} 1 \\ 2 \\ ijk \\ 2 \\ 1 \end{matrix} \quad \oplus \quad \begin{matrix} 1\ i\ j\ k \\ \hline 2\ 2 \\ \hline 1\ i\ j\ k \end{matrix}$$

(and get $2\oplus 2$ in middle of Brauer correspondent as $Ext_N(2,2) \cong Ext_N(1, 2\otimes 2) = Ext_N(1, 1\oplus i\oplus j\oplus k) = 0$ by projective cover of 1, the first summand.)

Let's analyze the structure of i^G. From the restriction of $\begin{smallmatrix}1\\19\\1\end{smallmatrix}$ to N get

$$Hom_{kG}\left(i^G, \begin{smallmatrix}1\\19\\1\end{smallmatrix}\right) \cong Hom_{kN}\left(i, \begin{smallmatrix}1\\19\\1\end{smallmatrix}\Big|_N\right) \cong k$$

Since

$$Hom_{kG}(i^G, 1) = Hom_{kN}(i, 1) = 0$$

have

$$Hom_{kG}\left(i^G, \begin{smallmatrix}19\\1\end{smallmatrix}\right) \cong k.$$

But

$$\underbrace{Hom_{kG}(i^G, 1)}_{Hom_{kN}(i,1) = 0} \to \underbrace{Hom_{kG}\left(i^G, \begin{smallmatrix}19\\1\end{smallmatrix}\right)}_{\cong k} \to Hom_{kG}(i^G, 19) \to \underbrace{Ext^1_{kG}(i^G, 1)}_{Ext^1_{kN}(i,1) = 0}$$

so $Hom_{kG}(i^G, 19) \cong k$.

In fact, we can now establish the

<u>Lemma</u> The socle of i^G is $19 \oplus 15^{(1)}$ plus terms from other blocks.

<u>Pf.</u> If either $15^{(2)}$ or $15^{(3)}$ appeared in the socle, it would also be in the radical quotient and so would be a summand and thus the Green correspondent of i. But $15 \not\equiv 1$ (modulo 9).
Also $\mathrm{Hom}_{kG}(i^G, 1) = 0$. $\therefore$ remains only to see that $15^{(1)}$ occurs exactly once.

But does occur, or else principal block part would be in injective envelope of 19, impossible from the Cartan matrix and the composition factors of i^G. If it occurs twice or more then it's a summand — as it is a comp. factor with mult. three.

Now let's make some <u>guesses</u>.

<u>Conjectured</u> <u>Green</u> <u>correspondents</u>

$\underline{N}$	$\underline{G}$
1	1
$\begin{array}{ccc} & 1 & \\ i & j & k \\ 2 & & 2 \\ i & j & k \end{array}$	19
$\begin{array}{cc} & 2 \\ 1 & i \\ & 2 \end{array}$	$15^{(1)}$
i	$\begin{array}{ccc} & 19 & \\ 1 & 1 & 15^{(1)} \\ & 19 & \end{array}$
2	$\begin{array}{ccc} 15^{(1)} & 15^{(2)} & 15^{(3)} \\ 19 & & 19 \\ 15^{(1)} & 15^{(2)} & 15^{(3)} \end{array}$

Let's observe some consequences:

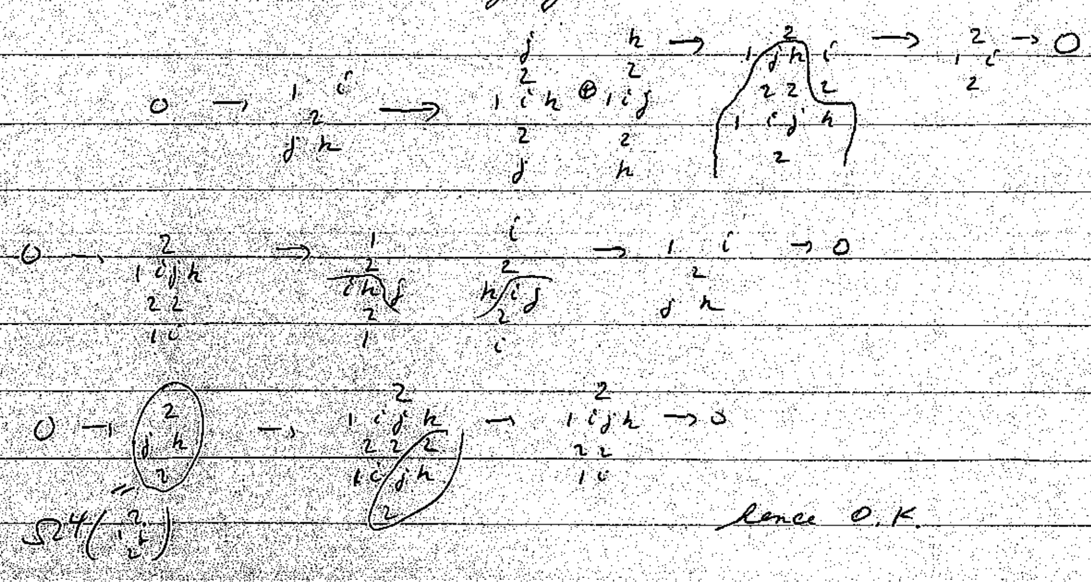
Remark: If the Green correspondents of the $15^{(4)}$ are as stated then $15^{(1)} \otimes 15^{(2)}$ is projective and so $\mathrm{Ext}^*(15^{(1)}, 15^{(2)}) = 0$.

Pf. Enough to show that $\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix} \otimes \begin{smallmatrix} & 2 & \\ 1 & & j \\ & 2 & \end{smallmatrix}$ is projective. But it has the same composition factors as $P_1 \oplus P_i \oplus P_j \oplus P_k$. Hence, need only show $\mathrm{soc}\left(\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix} \otimes \begin{smallmatrix} & 2 & \\ 1 & & j \\ & 2 & \end{smallmatrix}\right) = 1 \oplus i \oplus j \oplus k$.
But $\mathrm{Hom}\left(1, \begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix} \otimes \begin{smallmatrix} & 2 & \\ 1 & & j \\ & 2 & \end{smallmatrix}\right) = \mathrm{Hom}\left(\begin{smallmatrix} & 2 & \\ i & & 1 \\ & 2 & \end{smallmatrix}, \begin{smallmatrix} & 2 & \\ 1 & & j \\ & 2 & \end{smallmatrix}\right) = k$. Also $i \otimes \begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix} = \begin{smallmatrix} & 2 & \\ i & & 1 \\ & 2 & \end{smallmatrix}$, $i \otimes \begin{smallmatrix} & 2 & \\ 1 & & j \\ & 2 & \end{smallmatrix} = \begin{smallmatrix} & 2 & \\ i & & k \\ & 2 & \end{smallmatrix}$ etc. and can tensor again with i and get original module back. Thus, our claim is clear.

Note that the characters support this: $15^{(1)} \otimes 15^{(2)} \overset{3\text{-reg}}{=} \chi_5 + \chi_6 + 63 + 63$.

Conjecture: $15^{(4)}$ is periodic of period eight.

Argument: We show the same, at least heuristically, for $\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}$. Note that $P_1 \otimes 2 = P_2$. By counting and Loewy series is as is obvious, since $2 \otimes 2 = 1 + i + j + k$ and $P_1 \otimes 2 \otimes 2 = P_1 \oplus P_i \oplus P_j \oplus P_k$ has known series. We calculate roughly:



Hence O.K.

<u>Guess</u> $\mathrm{Ext}^1_{kG}(1, 19) = k \oplus k$

<u>Argument</u>: $\mathrm{Ext}^1_{kG}(1, 19) = \mathrm{Ext}^1_{kN}\left(1, \begin{smallmatrix} ijk \\ 22 \\ ijk \end{smallmatrix}\right)$

and

$$0 \to \mathrm{Ext}^1_{kN}\left(1, \begin{smallmatrix} 22 \\ ijk \end{smallmatrix}\right) \to \mathrm{Ext}^1\left(1, \begin{smallmatrix} ijk \\ 22 \\ ijk \end{smallmatrix}\right) \to \overset{=0}{\mathrm{Ext}^1(1, ijk)}$$

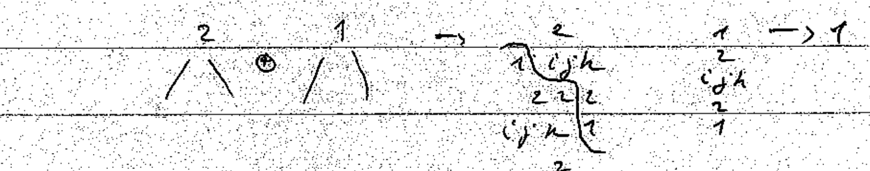
so it suffices to show that

$$\mathrm{Ext}^1\left(1, \begin{smallmatrix} 22 \\ ijk \end{smallmatrix}\right) = k \oplus k.$$

But

$$\underset{0}{\underset{\shortparallel}{\mathrm{Ext}^1(1, ijk)}} \to \mathrm{Ext}^1\left(1, \begin{smallmatrix} 22 \\ ijk \end{smallmatrix}\right) \to \underset{k \oplus k}{\underset{\shortparallel}{\mathrm{Ext}^1(1, 22)}} \to \mathrm{Ext}^2(1, ijk)$$

so it suffices to show that $\mathrm{Ext}^2(1, i \oplus j \oplus k) = 0$. But



so this seems likely. Depends on guess for the Green correspondent and the pictures.

<u>Conjecture</u>: Loewy series of projectives

1	19	$15^{(1)}$
$19 \quad 19$	$1, 1, 15^{(1)}, 15^{(2)}, 15^{(3)}$	19
$1 \quad 1 \quad 1 \quad 15^{(1)} \quad 15^{(2)} \quad 15^{(3)}$	$19 \quad 19 \quad 19$	$15^{(2)}, 15^{(3)} \quad 1$
$19 \quad 19$	$1, 1, 15^{(1)}, 15^{(2)}, 15^{(3)}$	19
1	19	$15^{(3)}$

<u>Lemma</u> $\mathrm{Ext}^1_G(1, 19) \simeq k \oplus k$

<u>Pf</u> By the argument on the previous page, it suffices to show that the Green correspondent of 19 has lower (&: upper, by duality) Loewy series $\frac{\frac{i\ j\ k}{2\ 2}}{i\ j\ k}$ and that $\mathrm{Ext}^2_N(1, i) = 0$. We prove these in turn.

<u>Lemma</u> The Green correspondent of 19 has Loewy series

$$\begin{array}{c} i\ j\ k \\ \hline 2\ 2 \\ \hline i\ j\ k \end{array}$$

<u>Pf</u> We already know these are the composition factors. The argument in volume III certainly shows that i, j, k are in the socle. By duality, it's enough to see that 2 is not and that $\mathrm{Ext}^1(2, 2) = 0$, that is $\mathrm{Ext}^1(1, 1+i+j+k) \simeq \mathrm{Ext}(2,2) = 0$, that is $\mathrm{Ext}^1(1, i) = 0$.

But if $\mathrm{Ext}^1(1, i) \neq 0$ then we would have an impossible two-dimensional module: it would be "triangular" in char 3 so $Z(S_2(N))$ would be in the kernel so S_3 would be too so it would be an indec. module for a four group.

Now if 2 is a submodule then it's easy to see that the Green correspondent must be

$$\begin{array}{c} 2\ i\ j\ k \\ \hline 2\ i\ j\ k \end{array}$$

But Ext^1 between the one-dimensional modules is zero - by the argument just given or using arguments like $\mathrm{Ext}^1(i, j) \simeq \mathrm{Ext}^1(1, i^* \otimes j) \simeq \mathrm{Ext}^1(1, k) = 0$. Since $\mathrm{Ext}^1(\ ,\ )$ is additive, we get that the Green correspondent is decomposable! For $\mathrm{Ext}(2,2) = 0$ also.

Lemma $Ext^2_N(1, i) = 0$.

Pf. From the composition factors of P_1 - the projective cover of 1 - and since $Ext^1(1, i) = 0$ get

$$P_1: \quad \begin{matrix} 1 \\ 2 \\ i\ j\ k \\ 2 \\ 1 \end{matrix}$$

Hence, the minimal resolution of 1 starts as follows:

$$P_2 \to P_1 \to 1$$

Hence $\Omega^2(1)$ has a simple socle, isomorphic with 1, and composition factors as follows: 2, 2, 1, i, j, k.

If $Ext^1(1, i) \neq 0$ then $Ext^1(1, j) \neq 0$, $Ext^1(1, k) \neq 0$ by automorphisms so i, j, k are at the top of $\Omega^2(1)$. Also $Ext^2_N(1, 1) \cong k$, by multiplicators, so as $\Omega^2(1)$ is indec get a contradiction as $Ext^1(2, 2) = 0$ means there is no place for second composition factor iso to 2.

Prop. The Green correspondent of i is as claimed $\left(\begin{matrix} 19 \\ 1\ 15^{(1)} \\ 19 \end{matrix}\right)$ if $P_{15^{(1)}}$ is a submodule.

Pf. The socle of the correspondent is known already to be $15^{(1)} \oplus 19$. ~~Hence~~, we claim that structure provided $P_{15^{(1)}}$ is a submodule of the correspondent. For then by comp factors, socles, and duality, the correspondent can only be as described or of the shape $\begin{matrix} 19 \\ 1 \\ 15^{(1)} \\ 1 \\ 19 \end{matrix}$. Hence, we need only show that if $\begin{matrix} 1 \\ 19 \end{matrix}$ is any non-split extension then $Hom_G(\begin{matrix} 1 \\ 19 \end{matrix}, i^G) \neq 0$, by the first lemma on page 30.

But

$$\mathrm{Hom}_G\left(\tfrac{1}{19}, i^G\right) = \mathrm{Hom}_N\left(\frac{\frac{1}{ijk}}{ij^2k}, i\right)$$

$$\neq 0$$

since $\mathrm{Ext}^1_N(1, i) = \mathrm{Ext}^1_N(1, j) = \mathrm{Ext}^1_N(1, k) = 0$. Q.E.D.

Now we must get the $P_{15^{(1)}}$. In any case we have that $X \subseteq P_{15^{(1)}} \oplus P_{19}$, where X is the correspondent. Let P, I be the projection and intersection of X with respect to $P_{15^{(1)}}$. We assume that X has no submodule isomorphic with $P_{15^{(1)}}$ so that

$$P_{15^{(1)}} \supsetneq P \supsetneq I \supsetneq 0,$$

the last inequality by composition factors as cannot have $X \subseteq P_{19}$ — or by rank of X. Thus rad $P_{15^{(1)}} = rP_{15^{(1)}} \supseteq P \supsetneq I \supseteq sP_{15^{(1)}} = \mathrm{soc}\, P_{15^{(1)}}$. The composition factors of $rP_{15^{(1)}} / sP_{15^{(1)}}$ are $1, 1, 19, 15^{(2)}, 15^{(3)}$. The quotient X / rX is $15^{(1)} \oplus 19$ so that P/I has "top" 19. Hard to finish this off!!

Note: The guess on page 14 is correct. See a letter from J. Olsson

<u>Prop.</u> The Green correspondent of $15^{(1)}$ is $1\,{}^{2}_{2}\,i$ as conjectured and $15^{(1)}\!\downarrow\! N \simeq 1\,{}^{2}_{2}\,i \oplus P_i$.

<u>Pf.</u> First, the composition factors are right. From the table on page 24 — plus using the one faithful character of N — get restriction of characters of G to N. Now use decomposition numbers to get desired results.

Next, we claim that the socle $s(15^{(1)}\!\downarrow\! N) \simeq i \oplus 2$. Indeed, the simple kN-module S is in the socle if, and only if, S maps to $15^{(1)}\!\downarrow\! N$, that is, exactly if S^G maps onto $15^{(1)}$. By what we know about 1^G, i^G, j^G, h^G this means that $s(15^{(1)}\!\downarrow\! N)$ is the direct sum of i and a multiple of 2. This multiple is not zero, by composition factors of $15^{(1)}\!\downarrow\! N$ versus P_i. It is not two or more, by the table on page 25 — or else $15^{(1)} \otimes 15^{(1)}$ splits off 2^G.

The Loewy series of P_2 is as described: $\begin{smallmatrix} & 2 & \\ h\ j & i & 1 \\ 2 & 2 & 2 \\ 1\ i & j & h \\ & 2 & \end{smallmatrix}$. For its easy to get that $P_2 \otimes P_2 = P_1 \oplus P_i \oplus P_j \oplus P_h$ and we know the series of these projectives and that $P_2 \simeq 2 \otimes P_1$.

So now we have $15^{(1)}\!\downarrow\! N \subseteq P_2 \oplus P_i$ and we use the Goursat type argument as on page 32. The top of the common P/S must be simple and must be 2; this if $15^{(1)}\!\downarrow\! N$ is not as described and thus is just indecomposable — the only possibility. The picture

$$15^{(1)}\!\downarrow\! N \subseteq \begin{matrix} & 2 & & \\ h & j & i & 1 \\ & 2\ 2\ 2 & & \\ 1 & i & j & h \\ & 2 & & \end{matrix} \oplus \begin{matrix} i \\ 2 \\ 1\ j\ h \\ 2 \\ i \end{matrix}$$

This 2 — at the top of P/S — is not the bottom 2 in P_2. Or else will have to get the two composition factors i simple and 1 from P_2 and so get 1 at top of $15^{(1)}\!\downarrow\! N$, contradicting the self-duality. Thus, the 2 at the top of the P/I in P_i is the top one and also, by the same argument the "1" in P_i is in the intersection $15^{(1)}/N \cap P_i$. Also, by counting number of times i occurs in $15^{(1)}/N$ and since it's at the top of $15^{(1)}/N$ we get that the i at the top of P_2 is in $15^{(1)}/N \cap P_2$. Mapping P_i to that i, image has socle 2 so can compute image. But so far in intersections the circled factors:

$$\begin{array}{cccc} & & 2 & \\ k & j & i & 1 \\ & 2 & 2 & 2 \\ 1 & i & j & k \\ & & 2 & \end{array} \qquad\qquad \begin{array}{ccc} & i & \\ & 2 & \\ 2 & j & k \\ & 2 & \\ & i & \end{array}$$

This gives eight factors so far. But the 2 at the top of P/I in P_i forces another j factor so get two of these, a contradiction.

The proof is now complete as have $P_i \subseteq 15^{(1)}/N$ and rest is trivial.

<u>Corollary</u> The product $15^{(1)} \otimes 15^{(2)}$ is projective and $Ext^*(15^{(1)}, 15^{(2)}) = 0$.

<u>Pf</u> As given before on basis of conjectured Green correspondents.

<u>Corollary</u> The Green correspondent of 2 is as conjectured, namely,

$$\begin{array}{ccc} 15^{(1)} & 15^{(2)} & 15^{(3)} \\ 19 & & 19 \\ 15^{(1)} & 15^{(2)} & 15^{(3)} \end{array}.$$

<u>Pf</u> We have $2 \subseteq 15^{(i)}/N$ so get 15's at top and bottom of Green correspondent. Also have composition

factors already. Also 19 is not at the top for if there is a map of 2^c into 19 then $2 \subseteq 19/N$. But the lemma on page 30 gives the Green correspondent of 19 and there aren't enough dimensions left to fit in P_2.
Hence, we're done unless we have the following:

$$\begin{array}{ccc} 15^{(1)} & 15^{(2)} & 15^{(3)} \\ & 19 & \\ & 19 & \\ 15^{(1)} & 15^{(2)} & 15^{(3)}. \end{array}$$

But then $P_{15^{(1)}}$ has at top: $\begin{array}{c} 15^{(1)} \\ 19 \\ 19 \end{array}$. By duality, also has $\begin{array}{c} 19 \\ 19 \end{array}$ at bottom of use $P_{15^{(1)}}$ / soc $P_{15^{(1)}}$. By composition factors get

$$P_{15^{(1)}} : \quad \begin{array}{c} 15^{(1)} \\ 19 \\ 19 \\ 15^{(1)} \end{array} \oplus X ,$$

where X has composition factors as follows: $1, 15^{(2)}, 15^{(3)}$.
This contradicts the $Ext(15^{(1)}, 15^{(*)}) = 0$.

Corollary $P_{15^{(1)}}$ has the conjectured structure: $\begin{array}{c} 15^{(1)} \\ 19 \\ 1 - 15^{(2)} - 15^{(3)} \\ 19 \\ 15^{(1)} \end{array}$

Pf. First we note that $Ext(1, 15^{(1)}) = 0$: Enough to see that $Ext^1(1, 1_2^{2c}) = 0$, but structure of P_1 gives this from composition factors easily. Hence, as $Ext(15^{(1)}, 15^{(*)}) = 0$ get

$$P_{15^{(1)}} : \quad \begin{array}{c} 15^{(1)} \\ 19 \\ X \\ 19 \\ 15^{(1)} \end{array}$$

where X has composition factors $1, 15^{(2)}, 15^{(3)}$. But this is then semisimple too, so done.

A similar argument gives:

<u>Corollary</u> P_1 has the conjectured structure: $\begin{matrix} & 1 & \\ 19 & & 19 \\ 1\ 1\ 1 & 15^{(1)}\ 15^{(2)} & 15^{(3)} \\ 19 & & 19 \\ & 1 & \end{matrix}$

<u>Pf</u> From what we know already get two 19's at top and two at bottom of the middle = rad P_1 / soc P_1. If these are unrelated then semisimplicity of rest is as above and done. Case left:

$$\begin{matrix} & 1 \\ & 19 \\ 19 \oplus & X \\ & 19 \\ & 1 \end{matrix}.$$

But then X is semisimple and $\mathrm{Ext}(19,1)$ is less than three dimensional, so done.

<u>Remaining conjectures</u>: Structure of P_{19};
Structure of Green correspondents of i, j, k;
Periodicity of $15^{(*)}$.

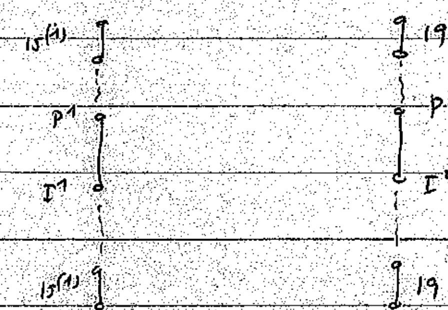
<u>Prop.</u> The Green correspondent of i is $\begin{matrix} & 19 & \\ 1 & 15^{(1)} & 1 \\ & 19 & \end{matrix}$ as conjectured.

<u>Pf</u> Already have $(i^G)_{B_0} \subseteq P_{15^{(1)}} \oplus P_{19}$; if a projective is in $(i^G)_{B_0}$ then it is $P_{15^{(1)}}$, by a composition factor count, so Green correspondent has composition factors $19, 1, 1, 15^{(1)}, 19$ with "top" and "bottom" both 19. Middle is semisimple, as in previous argument, so we're done in this case. ∴ can assume no projective in $(i^G)_{B_0}$ and shall argue to a contradiction.

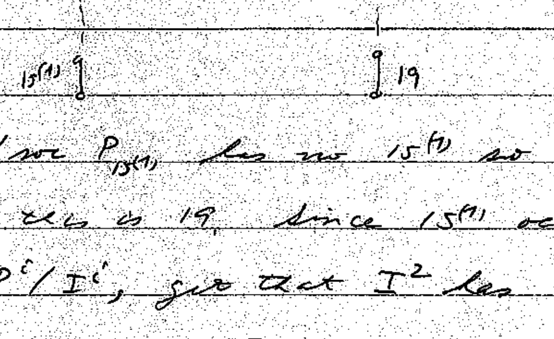
Hence, we can assume

$$\operatorname{soc}(P_{15^{(1)}} \oplus P_{19}) \subseteq (i^G)_{B_0} \subseteq \operatorname{rad}(P_{15^{(1)}} \oplus P_{19}).$$

Let P^1, I^1, P^2, I^2 be projections and intersections as in Goursat's theorem. Picture:



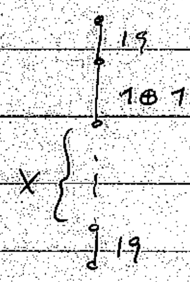
Now rad $P_{15^{(1)}}$ / soc $P_{15^{(1)}}$ has no $15^{(1)}$ so P^i/I^i has a simple top and this is 19. Since $15^{(1)}$ occurs thrice in $(i^G)_{B_0}$ and not in P^i/I^i, get that I^2 has $15^{(1)}$ twice, the full amount it occurs in P_{19}. From structure of $P_{15^{(1)}}$ know the $15^{(1)}$ occurs at the top and at the bottom of rad P_{19} / soc P_{19} and these are different $15^{(1)}$'s. Thus, I^2 has two images of $P_{15^{(1)}}$ both with socle 19. By inspection of P_{19} this is $\begin{smallmatrix}15^{(1)}\\19\end{smallmatrix}$ and $P_{15^{(1)}}$ / soc $P_{15^{(1)}}$. Picture:



This takes care of full $15^{(2)}, 15^{(3)}$ contribution to $(i^G)_{B_0}$ so $P^1/15^{(1)}$ does not involve either. Hence, by structure of $P_{15^{(1)}}$, $P^1/I^1 \cong 19$ and is 19 just above the socle. Hence P^1 does not involve the composition factor 1.

But 1 occurs thrice in $(i^{G})_{B_0}$ so does in P^2.

But have picture:



Here X has 1 as composition factor twice $\therefore$ $P^2 \nsubseteq \begin{smallmatrix} X \\ 19 \end{smallmatrix}$

so P^2 has a 1 at the top, a contradiction.

This leaves two things: P_{19} structure; $15^{(i)}$ periodicity.

Prop. P_{19} has the conjectured structure, namely,

$$\begin{array}{ccccc} & & 19 & & \\ 1 & 1 & 15^{(1)} & 15^{(2)} & 15^{(3)} \\ & 19 & 19 & 19 & \\ 1 & 1 & 15^{(1)} & 15^{(2)} & 15^{(3)} \\ & & 19 & & \end{array}$$

We proceed by a sequence of lemmas.

Lemma 1. There is an exact sequence

$$0 \to \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \to \begin{smallmatrix} i & j & k \\ 2 & & 2 \\ i & j & k \end{smallmatrix} \to \begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix} \to 0$$

where the middle term is the Green correspondent of 19.

Pf. There is such a submodule, an image of P_i. The quotient by this has i in its socle and also $2, j, k$ as other composition factors. If the 2 "slips past" the i

then so will j, k and the quotient will have i as a quotient, so there will be two i's at the top, a contradiction.
Similarly, if the j or the k is split over the 2 in the quotient we get a contradiction: The Green correspondent of 1_G, modulo its socle will not have a structure symmetrical in i, j, k; there are a number of cases.

<u>Lemma 2</u> $\mathrm{Ext}^1_{kG}(1_G, 1_G) \cong \mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right)$.

Pf. Using the previous lemma, we have the following exact sequence:

$$\mathrm{Hom}\left(\begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right) \to \mathrm{Hom}\left(\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right)$$

$$\to \mathrm{Ext}^1\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right)$$

But $\mathrm{Hom}\left(\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right)$ is two-dimensional and so the first map is clearly epic. As $\mathrm{Ext}^1_{kG}(1_G, 1_G) \cong \mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right)$, it suffices to prove that $\mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right) = 0$. But we have the exact sequence

$$0 \to \begin{smallmatrix} 1 \\ 2 \\ i \end{smallmatrix} \to \begin{smallmatrix} & i & \\ & 2 & \\ 1 & j & k \\ & 2 & \\ & i & \end{smallmatrix} = P_i \to \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \to 0$$

so we get the exact sequence

$$\underbrace{\mathrm{Hom}\left(\begin{smallmatrix} 1 \\ 2 \\ i \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right)}_{0} \to \mathrm{Ext}^1\left(\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}, \begin{smallmatrix} i & & j & & k \\ & 2 & & 2 & \\ i & & j & & k \end{smallmatrix} \right) \to \underbrace{\mathrm{Ext}^1\left(\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}, P_i \right)}_{0}$$

and the result is proved.

<u>Lemma 3</u> $1 + \dim \mathrm{Ext}^1_{kG}(19,19) = \dim \mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \right)$.

Pf. Using Lemma 1 again we have the exact sequence

$$0 \to \mathrm{Hom}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \right) \xrightarrow{\varphi} \mathrm{Hom}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} i & j & k \\ 2 & i & \\ i & j & k \end{smallmatrix} \right) \to \mathrm{Hom}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix} \right)$$

$$\to \mathrm{Ext}^1\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} i & j & k \\ 2 & i & \\ i & j & k \end{smallmatrix} \right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix} \right).$$

But we have the exact sequence

$$0 \to \begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix} \to \begin{smallmatrix} & i & \\ & 2 & \\ j & k & 1 \\ & 2 & \\ & i & \end{smallmatrix} = P_i \to \begin{smallmatrix} i \\ 2 \\ 1 \end{smallmatrix} \to 0$$

so the following is exact:

$$\underset{\overset{\parallel}{0}}{\mathrm{Hom}_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} i \\ 2 \\ 1 \end{smallmatrix} \right)} \to \mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix} \right) \to \underset{\overset{\parallel}{0}}{\mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, P_i \right)}$$

Hence, as $\dim \mathrm{Hom}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix} \right) = 1$, it suffices to show that φ is epic, in view of Lemma 2. That is, $\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}$ is not a submodule of the Green correspondent. But $\begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix}$ is a submodule so we would be getting a direct decomposition of the Green correspondent.

<u>Lemma 4</u> $\mathrm{Ext}^1_{kG}(19,19) = 0$.

Pf. It suffices to show, in view of the above, that $\dim \mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \right) = 1$. (That is, $\dim \mathrm{Ext}^1_{kN}\left(\Omega\left(\begin{smallmatrix} i \\ 2 \\ 1 \end{smallmatrix}\right), \Omega^{-1}\left(\begin{smallmatrix} 1 \\ 2 \\ i \end{smallmatrix}\right)\right) = 1$, so perhaps there is an Auslander–Reiten argument.) In fact, all we need is $\dim \mathrm{Ext}^1_{kN}\left(\begin{smallmatrix} j & & k \\ & 2 & \\ & i & \end{smallmatrix}, \begin{smallmatrix} & i & \\ & 2 & \\ j & & k \end{smallmatrix} \right) \leq 1$ as then $\dim \mathrm{Ext}^1_{kG}(19,19) \leq 0$.

But the exact sequence

$$0 \to \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix} \to \begin{smallmatrix} i \\ 2 \\ j\,h \end{smallmatrix} \to i \to 0$$

gives the exact sequence

$$\underset{\overset{\parallel}{0}}{\mathrm{Hom}\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, i\right)} \to \mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, \begin{smallmatrix} i \\ 2 \\ j\,h \end{smallmatrix}\right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, i\right)$$

We shall see that $\mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, i\right) = 0$ and then we're reduced to proving that $\dim \mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right) \leq 1$. Now

$$\mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, i\right) \simeq \mathrm{Ext}^1\left(i, \begin{smallmatrix} j\;i_h \\ 2 \\ j\,i\,h \end{smallmatrix}\right)$$

and the start of the minimal projective resolution of i is

$$\begin{smallmatrix} i \\ 2 \\ 1\,j\,h \\ 2 \\ i \end{smallmatrix} \qquad i$$

so it's clear that any cocycle is a coboundary in computing $\mathrm{Ext}^1\left(i, \begin{smallmatrix} i \\ 2 \\ j\,h \end{smallmatrix}\right)$. Now we must deal with $\mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right)$. But the exact sequence

$$0 \to \begin{smallmatrix} 2 \\ i \end{smallmatrix} \to \begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix} \to j \oplus h \to 0$$

gives the exact sequence

$$\underset{\overset{\parallel}{0}}{\mathrm{Hom}\left(\begin{smallmatrix} 2 \\ i \end{smallmatrix}, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right)} \to \mathrm{Ext}^1\left(j \oplus h, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} j\,h \\ 2 \\ i \end{smallmatrix}, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right) \to \mathrm{Ext}^1\left(\begin{smallmatrix} 2 \\ i \end{smallmatrix}, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right)$$

Also $\mathrm{Ext}^1\left(j \oplus h, \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}\right) = 0$ by cocycles and coboundaries:

$$\begin{smallmatrix} 1 \\ 2 \\ 1\,i\,h \\ 2 \\ j \end{smallmatrix} \longrightarrow j \qquad \begin{smallmatrix} 2 \\ j\,h \end{smallmatrix}$$

no cocycles !

Hence, we're reduced now to seeing that $\dim \operatorname{Ext}^1\left(\begin{smallmatrix}2\\ i\end{smallmatrix}, \begin{smallmatrix} & 2\\ j & k\end{smallmatrix}\right) \leq 1$.

We use cocycles and coboundaries:

$$\begin{array}{cccc} & & 2 & \\ 1 & j & k & i \\ & 2 & 2 & \\ 1 & j' & k & c \\ & & 2 & \end{array} \qquad \begin{array}{c} 2 \\ c \end{array}$$

The only possible cocycles are two dimensional; there is at least one coboundary so we're finally done.

<u>Pf of Prop</u>. From the Green correspondent of $1_{\hat{C}}$, i.e. $\begin{array}{ccc} & 19 & \\ 1 & 1 & 15^{(1)} \\ & 19 & \end{array}$ we have two 1's just under the 19 in P_{19} and two different 1's just above the socle. From $P_{15^{(1)}}$: $\begin{array}{ccc} & 15^1 & \\ & 19 & \\ 1 & 15^{(2)} & 15^{(3)} \\ & 19 & \\ & 15^1 & \end{array}$

get $15^{(2)}, 15^{(3)}$ just under 19 and different from the $15^{(2)}, 15^{(3)}$ on the 19 in the socle. Hence, have

$$\begin{array}{ccccc} & & 19 & & \\ 1 & 1 & 15^{(1)} & 15^{(2)} & 15^{(3)} \\ & & X & & \\ 1 & 1 & 15^{(1)} & 15^{(2)} & 15^{(3)} \\ & & 19 & & \end{array}$$

where the composition factors of X are $19, 19, 19$. $\therefore$ done by Lemma 4.

<u>Remaining conjecture</u>: Periodicity of $15^{(1)}, 15^{(2)}, 15^{(3)}$

<u>Prop.</u> The modules $15^{(1)}$, $15^{(2)}$, $15^{(3)}$ are periodic of period eight.

Again we proceed by a sequence of lemmas.

<u>Lemma 5</u> There is a unique module of the form $\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}$.

Pf Such exists; it's the Green correspondent of $15^{(1)}$. Looking at the top of P_2 below the 2, we see it's suffices to show that $Ext^1(\begin{smallmatrix} & 2 & \\ 1 & & i \end{smallmatrix}, 2)$ is one-dimensional. But we have the exact sequences as follows(*):

$$Ext^1(\begin{smallmatrix}2\\i\end{smallmatrix}, 2) \to Ext^1(\begin{smallmatrix} & 2 & \\ 1 & & i \end{smallmatrix}, 2) \to \underset{\substack{\parallel \\ 0}}{Ext^1(1, 2)}$$

$$Ext^1(2, 2) \to Ext^1(\begin{smallmatrix}2\\i\end{smallmatrix}, i) \to \underset{\text{one-dimensional.}}{Ext^1(i, 2)}$$

∴ the proof is complete.

<u>Remark</u>: By using tensor products get unique $\begin{smallmatrix} & 2 & \\ i & & j \\ & 2 & \end{smallmatrix}$ etc. Shall need this later.

<u>Lemma 6</u> The minimal projective resolution of $\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}$ begins

$$P_j \oplus P_k \to P_2 \to \begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix} \to 0.$$

<u>Pf</u> We just have to calculate the $Ext^1(\ ,\ \begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix})$ for simple kN-modules and apply duality. We just use cocycles and coboundaries and – in one case – Lemma 5. The pictures are as follows:

(*) Better: $Ext^1(\begin{smallmatrix}2\\i\end{smallmatrix}, 2) = Ext^1(\begin{smallmatrix}1ijk\\2\end{smallmatrix}, 1) = 0$ by P_1.

$$\begin{array}{c} 1 \\ 2 \\ i\ j\ k \\ 2 \\ 1 \end{array} \qquad \begin{array}{c} i \\ 2 \\ 1\ j\ k \\ 2 \\ i \end{array} \qquad \begin{array}{c} j \\ 2 \\ 1\ i\ k \\ 2 \\ j \end{array} \qquad \begin{array}{c} k \\ 2 \\ 1\ i\ j \\ 2 \\ k \end{array} \qquad \begin{array}{c} 2 \\ 1\ i\ j\ k \\ 2\ 2\ 2 \\ 1\ i\ j\ k \\ 2 \end{array}$$

cocycle dim:	1	1	1	1	1 (see below)
cobdry dim:	1	1	0	0	1 (use uniqueness)

Hence, to complete the proof only have to justify last column. To get cocycles and then using uniqueness of Lemma 5 need only that the tops j, k have two 2's under them. If not, contradict $\mathrm{Ext}^1(1\,{}^2_{}\,i, 2)$ being one-dimensional of the proof of Lemma 5.

<u>Lemma 7</u> $\Omega^2(1\,{}^2_2\,i) = \substack{1 \quad\ \ i \\ 2 \\ j \quad\ \ k}$.

<u>Pf.</u> From Lemma 6 we know that

$$\dim \Omega^2(1\,{}^2_2\,i) - 18 + 18 - 6 = 0$$

so $\Omega^2(1\,{}^2_2\,i)$ is indecomposable of dimension six with $\mathrm{soc} \cong j \oplus k$. A similar Euler characteristic calculation gives us that the composition factors of $\Omega^2(1\,{}^2_2\,i)$ are $1, i, j, k, 2$. We apply Goursat's theorem:

$$\Omega^2(1\,{}^2_2\,i) \subseteq \begin{array}{c} j \\ 2 \\ 1\ i\ k \\ 2 \\ j \end{array} \oplus \begin{array}{c} k \\ 2 \\ 1\ i\ j \\ 2 \\ k \end{array}$$

so the projection / intersection must have comp factors $2, 1, i$ as can't have the two 2's above the socles in the intersections. Rest is easy.

The Goursat argument also gives:

<u>Lemma 8</u> There is a unique module of the form $\substack{1 \quad\ \ i \\ 2 \\ j \quad\ \ k}$.

<u>Pf of Prop.</u> We'll use above results for the modules of the same shape to which the arguments apply. Here

$$\Omega^2\left(\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}\right) = \begin{smallmatrix} 1 & & i \\ & 2 & \\ j & & k \end{smallmatrix}$$

so

$$\Omega^2\left(\begin{smallmatrix} & 2 & \\ j & & k \\ & 2 & \end{smallmatrix}\right) = \begin{smallmatrix} j & & k \\ & 2 & \\ 1 & & i \end{smallmatrix}$$

and taking duals,

$$\Omega^{-2}\left(\begin{smallmatrix} & 2 & \\ j & & k \\ & 2 & \end{smallmatrix}\right) = \begin{smallmatrix} 1 & & i \\ & 2 & \\ j & & k \end{smallmatrix}$$

so

$$\begin{smallmatrix} & 2 & \\ j & & k \\ & 2 & \end{smallmatrix} = \Omega^2\left(\begin{smallmatrix} 1 & & i \\ & 2 & \\ j & & k \end{smallmatrix}\right) = \Omega^4\left(\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}\right)$$

Hence, similarly,

$$\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix} = \Omega^4\left(\begin{smallmatrix} & 2 & \\ j & & k \\ & 2 & \end{smallmatrix}\right) = \Omega^8\left(\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}\right)$$

so $\begin{smallmatrix} & 2 & \\ 1 & & i \\ & 2 & \end{smallmatrix}$ is periodic of period dividing eight. These calculations show its not periodic of period dividing four, so we're done

Let's tabulate our results. Use new notation

$15^{(1)}$	φ_1
$15^{(2)}$	φ_2
$15^{(3)}$	φ_3
i	f_1
j	f_2
k	f_3
19	ν
2	m
<u>old</u>	<u>new</u>

	G	N
Simple modules	$1, \varphi_1, \varphi_2, \varphi_3, \nu$	$1, f_1, f_2, f_3, n$
Correspondents	1	1
	φ_i	$\begin{matrix} n \\ 1 \;\; f_i \\ n \end{matrix}$
	$\begin{matrix} \nu \\ 1 \;\; 1 \;\; \varphi_i \\ \nu \end{matrix}$	f_i
	ν	$\begin{matrix} f_1 \;\; f_2 \;\; f_3 \\ n \;\; n \\ f_1 \;\; f_2 \;\; f_3 \end{matrix}$
	$\begin{matrix} \varphi_1 \;\; \varphi_2 \;\; \varphi_3 \\ \nu \;\; \nu \\ \varphi_1 \;\; \varphi_2 \;\; \varphi_3 \end{matrix}$	n
Projectives	$\begin{matrix} 1 \\ \nu \;\; \nu \\ 1 \;\; 1 \;\; 1 \;\; \varphi_1 \;\; \varphi_2 \;\; \varphi_3 \\ \nu \;\; \nu \\ 1 \end{matrix}$	$\begin{matrix} 1 \\ n \\ f_1 \; f_2 \; f_3 \\ n \\ 1 \end{matrix}$
	$\begin{matrix} \varphi_i \\ \nu \\ 1 \;\; \varphi_j \;\; \varphi_k \\ \nu \\ \varphi_i \end{matrix}$	$\begin{matrix} f_i \\ n \\ 1 \;\; f_j \;\; f_k \\ n \\ f_i \end{matrix}$
	$\begin{matrix} \nu \\ 1 \;\; 1 \;\; \varphi_1 \;\; \varphi_2 \;\; \varphi_3 \\ \nu \;\; \nu \;\; \nu \\ 1 \;\; 1 \;\; \varphi_1 \;\; \varphi_2 \;\; \varphi_3 \\ \nu \end{matrix}$	$\begin{matrix} n \\ 1 \;\; f_1 \;\; f_2 \;\; f_3 \\ n \;\; n \;\; n \\ 1 \;\; f_1 \;\; f_2 \;\; f_3 \\ n \end{matrix}$

Resolutions of extensions

This is clearly a most difficult question. We start with an easy case.

Prop If P is a normal Sylow p-subgroup of G, k is a field of characteristic p and U is a kG-module then the restriction to P of the minimal projective resolution of U is the minimal kP projective resolution of U_P.

Pf. It clearly suffices to show that if V is a kG-module then $\mathrm{rad}_{kG}(V) = \mathrm{rad}_{kP}(V_P)$, i.e. $\mathrm{rad}(kG)\,V = \mathrm{rad}(kP)\,V$. Hence, it suffices to see that $\mathrm{rad}(kG) \subseteq (\mathrm{rad}\, kP)\,kG$ as then $\mathrm{rad}\, kG \cdot V \subseteq \mathrm{rad}\, kP \cdot kG\,V = \mathrm{rad}\, kP \cdot V$. Hence, we want that $kG/(\mathrm{rad}\, kP)\,kG$ is semisimple as a right kG-module. This is semisimple as a kP-module so we only need establish the following statement: If W is a kG-module such that W_P is a multiple of the trivial module then W is semisimple. But this is obvious as W is a $k[G/P]$-module.

Remarks: 1) Probably there is a ring-theoretic proof describing the radical of kG in terms of $\mathrm{rad}(kP) = \Delta P$, the augmentation ideal of kP.

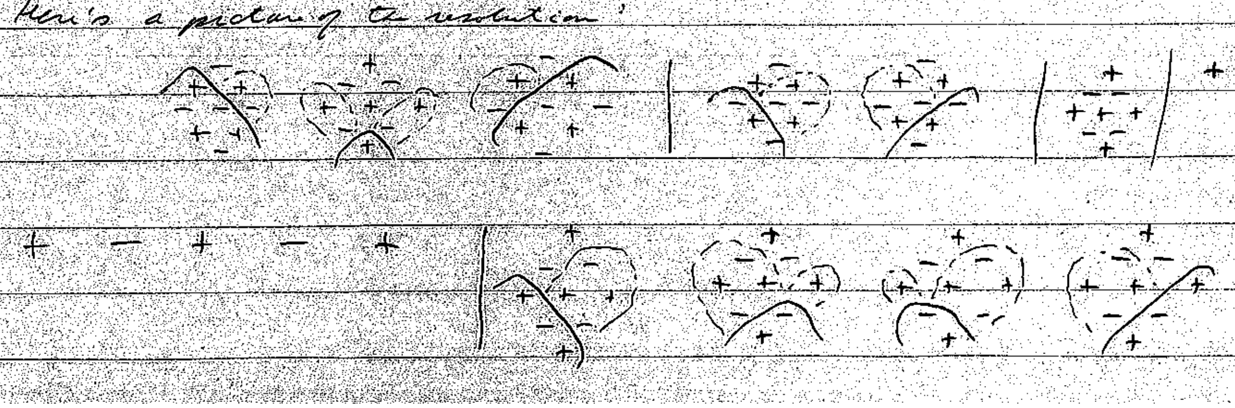
2) How far does this result generalize? What happens if $N \trianglelefteq G$, $p \nmid |G:N|$?

Our motivation is that it seems unlikely that all minimal resolutions have a nice "structure of a complex." For if U_p does then it's hard to see how U does for all of G. Let's concentrate on $N = Z_3 \times Z_3 \cdot Q_8$ of the last section, $p = 3$. We interested in $U = k$, the trivial module.

For $k \cdot Z_3 \times Z_3$ we know the minimal resolution of k, a tensor product giving a double complex. Let's look first now at $Z_3 \times Z_3 \cdot Z_2 \subset Z_3 \times Z_3 \cdot Q_8$. To identify the terms of the minimal resolution for $Z_3 \times Z_3 \cdot Z_2$ we need only look at the action of the Z_2 on the cohomology $H^*(Z_3 \times Z_3, k)$. We use + and − for eigenvalues +1, −1, respectively. The picture:

dim	$H^*(Z_3 \times Z_3, k)$
0	1^+
1	x^- $\quad y^-$
2	βx^- $\quad xy^+$ $\quad \beta y^-$
3	$\beta x . x^+$ $\quad \beta x . y^+$ $\quad \beta y . x^+$ $\quad \beta y . y^+$
4	$(\beta x)^{2+}$ $\quad \beta x . x . y^-$ $\quad \beta x . \beta y^+$ $\quad \beta y . xy^-$ $\quad (\beta y)^{2+}$
5	$(\beta x)^2 x^-$ $\quad (\beta x)^2 y^-$ $\quad \beta x . \beta y . x^-$ $\quad \beta x . \beta y . y^-$ $\quad (\beta y)^2 x^-$ $\quad (\beta y)^2 x^-$
6	

Here's a picture of the resolution:

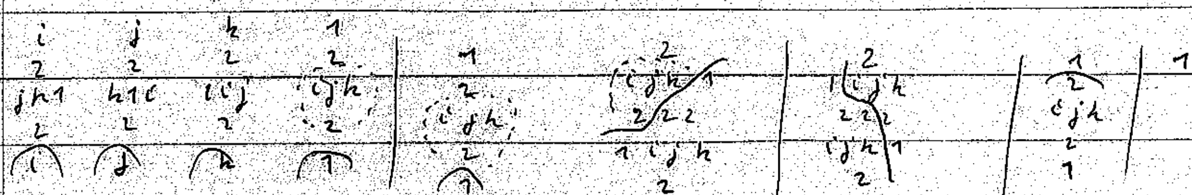


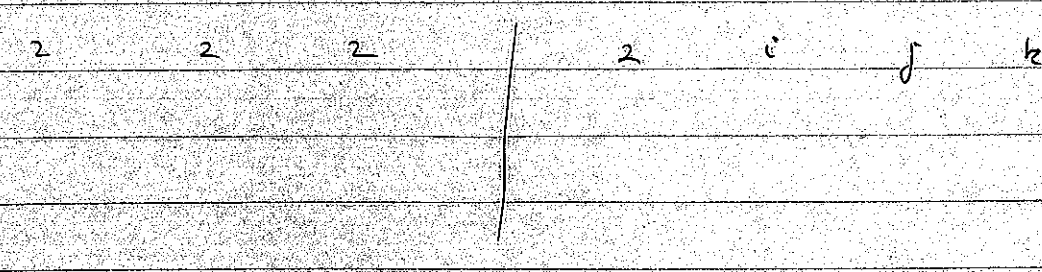
Here's a guess for the complex – note the 4×4 blocks:

$$
\begin{array}{cccc|cccc}
+ & & & & & & & \\
\hline
+ & - & & & & & & \\
- & + & + & & & & & \\
- & + & + & - & & & & \\
+ & - & - & + & & & & \\
\hline
+ & - & - & + & + & & & \\
- & + & + & - & - & + & & \\
- & + & + & - & - & + & + & \\
+ & - & - & + & + & - & - & + \\
\end{array}
$$

dim of + : 1, 0, 1, 4; 3, 0, 3, 8; 5 ---

But now it seems hard to go to $Z_2 \times Z_2$, Q_8 as all the "–" become covers of 2-dim simplex of N. An attempt:





A refinement of Scott's theorem

We're interested in Scott's work on permutation modules and permutation components – i.e. their indecomposable summands. We work over a field k of characteristic p, the results are way easily liftable by Scott's results. First, we recover the following trivial result, which we don't need, but supplies a context.

Prop. Let M be a kG-module, M indecomposable and P be a Sylow p-subgroup of G. The following are equivalent:

1) M is a permutation component;
2) M has trivial source;
3) M_P is a permutation module.

Now let's begin. Let X be a semigroup with G acting on X. Let P be a p-subgroup of G. We first see that a result from Alperin – Broué (the reading in §2) goes over.

Lemma $Br_P\left((kX)^G_P\right) = \left(k\, C_X(P)\right)^{N(P)}_P$.

Proof. If $a \in (kX)^P$ then

$$Br_P\, Tr_P^G(a) = Br_P\Big(\sum_s Tr^{N(P)}_{N(P)\cap sPs^{-1}}(s(a))\Big)$$
$$= Br_P\left(Tr_P^{N(P)}(a)\right)$$

inasmuch as when $N(P)sP \neq N(P)P$ then $N(P)\cap sPs^{-1} \not\geq_{N(P)} P$

so $\ker(Br_P) \cap (kX)^{N(P)} \geq (kX)^{N(P)}_{N(P)\cap sPs^{-1}}$.

Hence,

$$Br_P(Tr_P^G(a)) = Tr_P^{N(P)}(Br_P(a)),$$

as Br_P is a $N(P)$ map, so

$$Br_P(Tr_P^G(a)) \in (kC_X(P))_P^{N(P)}.$$

On the other hand, if $a \in (kC_X(P))_P^{N(P)}$, $a = Tr_P^{N(P)}(a')$, $a' \in kC_X(P)$ then, by the above calculation,

$$\begin{aligned} Br_P\, Tr_P^G(a') &= Tr_P^{N(P)}(Br_P(a')) \\ &= Tr_P^{N(P)}(a') \\ &= a \end{aligned}$$

so a is in the image, as required.

<u>Picture</u>:

$$\begin{array}{ccc} (kX)^P & \twoheadrightarrow & kC_X(P) \\ & & \cup\!\mid \\ \cup\!\mid & & kC_X(P)^{N(P)} \\ & & \cup\!\mid \\ (kX)_P^G & \twoheadrightarrow & (kC_X(P))_P^{N(P)} \end{array}$$

Next, say G acts on the set Ω. Let X consist of the linear transformations of $k\Omega$ which map one element of Ω to another and annihilate the rest (i.e. the e_{ij}). Thus X is a basis of $\mathrm{Hom}_k(k\Omega, k\Omega)$, X is a semigroup on which G acts. Also $(kX)^G = \mathrm{Hom}_{kG}(k\Omega, k\Omega)$.

A primitive idempotent of $(kX)^G$ corresponds to an indecomposable summand of $k\Omega$. The defect group of the idempotent is the vertex of the corresponding summand, by the trace form of the definition of the defect and the theorem of D. G. Higman.

Next, what is $kC_X(P)$? $C_X(P)$ consists of the elements of X corresponding to the subset Ω^P. That is, $kC_X(P) \simeq \mathrm{Hom}_k(\Omega^P, \Omega^P)$. Now let's look at $k\Omega^P$ under $N(P)$ action so

$$\mathrm{Hom}_{kN(P)}(\Omega^P, \Omega^P) \simeq (kC_X(P))^{N(P)}.$$

The primitive idempotents of $kC_X(P)^{N(P)}$ correspond to the indecomposable $kN(P)$ summands of $k\Omega^P$. The ones with vertex contained in P come from $kC_X(P)^{N(P)}_P$, again by the D. G. Higman result. Since we're dealing with Ω^P, the ones with vertex contained in P have vertex P.

Say e is a primitive idempotent of $(kX)^G_P$ with defect group P so $ek\Omega$ is an indecomposable module with vertex P. Then $Br_P(e) \neq 0$, $Br_P(e) \in (kC_X(P))^{N(P)}_P$ so $Br_P(e)\, k\Omega^P$ is an indecomposable $kN(P)$-module with vertex P. To see that this is the Green correspondent we need only see that

$$Br_P(e)\, k\Omega^P \mid (ek\Omega)_{N(P)}.$$

This is true by Scott's lemma 2 on page 112.

The lifting of idempotents in algebras now gives us what we want: an indecomposable is a summand of $k\Omega$ if, and only if its Green correspondent is a summand of the restriction of $k\Omega$ to the normalizer of its vertex. (This is Scott's "First Main Theorem.") Indeed, one help is clear.

Now let's see the other half. Let J be an indecomposable $kN(P)$-module with vertex P with $J \mid (k\Omega)_{N(P)}$. Let $J = f\,k\Omega$, $f \in (kC_X(P))_P^{N(P)}$. Note that f is also primitive in $kC_X(P)_P^{N(P)}$ as this is an ideal of $kC_X(P)^{N(P)}$: $f = f_1 + f_2$ then $f_1 = f_1 f$, $f_2 = f_2 f$. Applying the lemma on page 50, we get an idempotent $e \in (kX)_P^G$ with $Br_P(e) = f$; this is by the lifting of idempotents. We can assume e is also primitive since otherwise we can throw away a summand of e as the image under Br_P is primitive. By the results given above, we've established Scott's First Main Theorem.

But we get more:

__Theorem__. The multiplicities in Scott's theorem are equal.

(That is, the number of times the indecomposable divides $k\Omega$ is the number of times its Green correspondent does.)

This is because orthogonal families of idempotents lift in algebras: use $I \supset$ rad and I as in argument for lifting idempotents (see lectures Autumn '79). Say

$$f_1 + \cdots + f_k$$

are orthogonal idempotents, each primitive, in

$(kC_X(P))_P^{N(P)}$, all $f_i\, k\Omega$ isomorphic. Lift to
$e_1 + \dots + e_n$ ∴ can assume each e_i is primitive
in $(kX)_P^G$ and proceed as before.

Permutation components of defect zero type.

We're trying to generalize Scott's work on defect 0 and defect 1 in his paper on permutation modules. The following seems a start: (notation as above)

Theorem If V is a permutation component and $\mathrm{Hom}_{kG}(V,V) \cong k$ then the Green correspondent of V is simple.

Proof. Let Q be the vertex of V and U its Green correspondent so U is an indecomposable projective $kN(Q)/Q$-module. Assume U is not simple; it suffices to show that $\mathrm{Hom}_{kN}(U,U) / \mathrm{Hom}_{kN}(U,U)_{\mathfrak{X}}$ ($N = N(Q)$, usual $\mathfrak{X}$) is of dimension at least two.

Now $1 \in \mathrm{Hom}_{kN}(U,U)$ is not in the ideal $\mathrm{Hom}_{kN}(U,U)_{\mathfrak{X}}$ as U is not relatively $\mathfrak{X}$-projective.
$\therefore$ the ideal is proper and so is contained in $J(\mathrm{Hom}_{kN}(U,U))$.
$\therefore$ it's enough to show that the map of U onto its socle (with kernel $J(U)$) is not in $\mathrm{Hom}_{kN}(U,U)_{\mathfrak{X}}$ as this map is certainly not an isomorphism and so is in $J(\mathrm{Hom}_{kN}(U,U))$.

But suppose W is a relatively $\mathfrak{X}$-projective kN-module and there is a composition of maps

$$U \xrightarrow{\alpha} W \xrightarrow{\beta} U$$

giving the "top to bottom" map of W. Then $U\alpha \subseteq W^Q$ — the Q fixed pts — as U is a kN/Q-module and $W/\ker\beta$ is fixed by Q for the same reason. Hence it suffices to prove the following:

<u>Lemma</u>. If W is a relatively $\mathfrak{X}$-projective kN-module then $W \cdot \mathrm{Rad}\, kQ \supseteq W^Q$.

For $W \cdot \mathrm{Rad}\, kQ$ is the smallest kQ-submodule of W with trivial quotient as $kQ/\mathrm{Rad}\, kQ \cong k$.

<u>Proof</u>. It suffices to show this holds for $\mathrm{Ind}_R^N W_0$ where $R \subsetneq Q$ and W_0 is a kR-module. Hence, it's enough to deal with $(\mathrm{Ind}_R^N W_0)_Q$ so all we have to show is that it's true for a kQ-module induced from a proper subgroup of Q. But if V is a kQ-module and $V \neq k$ and V is indecomposable then $V(\mathrm{Rad}\, kQ) \supseteq V^Q$ by indecomposability!

Now where? Let k, R, K be usual rings. Let U / kX, X a G-set, $\hat{U}$ the lift of U to an RG-module with $\hat{U} / RX$. We're interested in the case that U behaves like a simple in a block of defect 0 — i.e. no maps either way with non-isomorphic summands of kX and $\mathrm{Hom}_{kG}(U,U) = k$. Does the vertex of U have to be maximal in some sense? Is $K \otimes \hat{U}$ simple with some set of maximal p-parts?

In general, can we determine the blocks of $\mathrm{End}_{kG}(kX)$ locally? Are there blocks of finite representation type generalizing cyclic blocks?

some methods appear plausible. We try and get at maps between summands of kX module projectives by an inductive procedure. More generally, say

$\mathrm{Ind}\, U_1 = U \oplus \ldots$

$\mathrm{Ind}\, V_1 = V \oplus \ldots$

so by induction on vertices have all maps between "lower" modules, in error term, so can subtract from $\mathrm{Hom}_{kG}(\mathrm{Ind}\, U_1, \mathrm{Ind}\, V_1)$ and get $\mathrm{Hom}_{kG}(U, V)$. Perhaps.

Another idea: Enough, as $\mathrm{Hom}_k(U, V) \simeq U^* \otimes V$ as kG-modules, to calculate for permutation components $\mathrm{Hom}_{kG}(k, U \otimes V)$, i.e. to find the Scott module summands. Do this by subtraction as above.

Complexity and a question of Serre

In our paper, Evens and I asked about a module version of the geometric consequences of Quillen's dimension theorem. Serre has written with a good suggestion: "Question. Is it true that $\mathrm{Supp}_G M = \bigcup_E \mathrm{Supp}_E M_E$?" Here, modulo some standard results about Supp, probably in Atiyah-Macdonald says Kaplansky, this is about the radical modulo the annihilator of $\bigoplus_S H^*(G, M\otimes S)$ in $H^{ev}(G,k)$, where the sum is over all simple kG-modules S. Hence, we seem to want the following: If $x \in H^*(G,k)$ then a power of x annihilates $\bigoplus_S H^*(G, M\otimes S)$ if, and only if, for every elementary abelian p-subgroup E of G, some power of $x_E = \mathrm{Res}^G_E(x)$ annihilates $H^*(E, M_E)$.

The Quillen-Venkov approach looks promising, with the aim of getting the "β's" into the picture. Evens thinks he can handle the case of G a p-group. Hence, we seem to need only the following result:

Proposition. Let $x\in H^*(G,k)$ and P be a Sylow p-subgroup of G. The following are equivalent:

(1) a power of x annihilates $\bigoplus_S H^*(G, M\otimes S)$;

(2) a power of x_P annihilates $H^*(P, M_P)$.

Proof. Let S be a simple kG-module and choose a kP-module V such that $S \mid \mathrm{Ind}_P^G(V)$. We want to show (2) implies (1) so assume (2) holds. It suffices to show that a power of x annihilates

$H^*(G, M \otimes \mathrm{Ind}_P^G(V)) \to H^*(G, \ )$ is additive.
But $H^*(G, M \otimes \mathrm{Ind}_P^G(V)) \simeq H^*(G, \mathrm{Ind}_P^G(M_P \otimes V))$ as the modules are isomorphic. Moreover, we have the $H^*(G, k)$ isomorphism of modules, by Shapiro's lemma,

$$H^*(G, \mathrm{Ind}_P^G(M_P \otimes V)) \simeq H^*(P, M_P \otimes V).$$

Hence, it's enough to see that we want a power of x_P annihilating $H^*(P, M_P \otimes V)$. But $M_P \otimes V$ has a filtration with each quotient isomorphic with V so we're O.K. by the long exact sequence. That is, if

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is an exact sequence of kP-modules and $y \in H^*(P, k)$ annihilates $H^*(P, V_i)$, $i = 1, 3$ then as the long exact sequence is a module homomorphism, we get y^2 annihilating $H^*(P, V_2)$.

So we seem to have the result!

Identification of Specht modules

There are many definitions of the Specht modules which we wish to compare. Let S^λ be the Specht module as defined by James, $\mathfrak{S}_\lambda$ as by Specht (Math. Z. v39) and V_λ as by Carter and Lusztig.

Theorem $V_\lambda \simeq \mathfrak{S}_\lambda \simeq S^\lambda$.

Remark: Towber also defines $YS^\alpha \subseteq \Lambda^\alpha$ and asserts that $YS^\alpha \simeq S^{\alpha^*}$. (He also claims that is $V_\lambda(E) = E_\lambda$, E_λ the Carter-Lusztig functor applied to E, and that $V_\lambda(E) = (\Lambda^{\lambda^*}(E^*))^*$.)

We shall first show that $V_\lambda \simeq \mathfrak{S}_\lambda$ and we begin by establishing an identity. Let $x_1, \dots, x_s$ be non-commuting variables, $y_1, \dots, y_s$ commuting ones. Let W be the subspace in the algebra generated by the x's (i.e. isomorphic with the tensor algebra) of total degree s and degree 1 in each variable; let U be the vector space of polynomials in the y's of total degree $\frac{s(s-1)}{2}$ consisting of terms in which the individual degrees are $0, 1, 2, \dots, s-1$. Let Σ_s act on W by place permutations and on U by variable permutations.

Lemma There is a unique module isomorphism of W onto U which sends each $x_{i_1} \cdots x_{i_s}$ to $y_{i_1}^{s-1} y_{i_2}^{s-2} \cdots y_{i_{s-1}}^{1} y_{i_s}^{0}$.
This map sends $\sum_{\sigma \in \Sigma_s} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(s)}$ to $\prod_{i<j} (y_i - y_j)$.

Proof (of the lemma). The first part is clear.
Let $L = \sum_{\sigma\in\Sigma_s} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(s)}$ and let R be its image so we have to prove that R is the discriminant. If we fix k, l $1 \le k < l \le s$, then the terms in L come in pairs differing only by a transposition of x_k and x_l, these terms differing in sign. Hence, $L(y_1, \dots, y_s)$ vanishes if we put $y_k = y_l$. $\therefore$ by unique factorization the generic discriminant divides L; by degrees, the multiple is a scalar. We can get this to be one inductively as $\sum_{\sigma(1)=1} (-1)^\sigma x_1 x_{\sigma(2)} \cdots x_{\sigma(s)}$ goes to $y_1^{s-1} \prod_{1<i<j\le s} (y_i - y_j)$.

Remark: Note that we just have $U \cong W$, two free modules.

Now let's show that $V_\lambda \cong \Theta_\lambda$. Let λ be a partition of n, $\lambda = (\lambda_1, \lambda_2, \dots)$, $n = \sum \lambda_i$. Let $\mathfrak{X}_\lambda$ be the space of x's in which x_1 occurs μ_1 times, x_2 μ_2 times and so on, where $\mu = \lambda^*$ is the dual partition. Let $\mathcal{Y}_\lambda$ be the space of monomials in the y's in which μ_1 y's occur to the zero-th power, μ_2 to the first power and so. These spaces have dimension $\frac{n!}{\mu_1! \cdots \mu_{\lambda_1}!}$ and the $\mathfrak{X}_\lambda$ and $\mathcal{Y}_\lambda$ are isomorphic

permutation modules for Σ_n for the same Young subgroup.
Note that (in Carter-Lusztig notation) $\Phi_\lambda \in \mathfrak{X}_\lambda$
as $\Phi_\lambda = \Phi^\mu = \left(\sum (-1)^{\sigma_1} x_{\sigma_1(1)} \cdots x_{\sigma_1(\mu_1)} \right) \left(\sum (-1)^{\sigma_2} x_{\sigma_2(1)} \cdots x_{\sigma_2(\mu_2)} \right) \cdots$
We have a map

$$x_{i_1} \cdots x_{i_n} \to \cdots y_j^m \cdots$$

when m is such that x_{m+1} occurs in the j-th place, and this clearly gives the isomorphism $\mathfrak{X}_\lambda = \mathcal{Y}_\lambda$.

Hence it suffices to see that Φ_λ the generator for V_λ goes to the generator for $\mathfrak{S}_\lambda$. But

$$\Phi_\lambda = \Phi^\mu = \left(\sum(-1)^{\sigma_1} x_{\sigma_1(1)} \cdots x_{\sigma_1(\mu_1)}\right)\left(\sum(-1)^{\sigma_2} x_{\sigma_2(1)} \cdots x_{\sigma_2(\mu_2)}\right)\cdots$$

so this is clear from the lemma as the usual generator for $\mathfrak{S}_\lambda$ is the product of discriminants.

Next, let S^λ be James' Specht module. Let M^λ be the space spanned by tabloids. Claim $M^\lambda \cong \mathfrak{X}_\lambda$. Map a tabloid to a monomial with x_1 in the i-th place if i is in the first row, x_2 in the i-th place if i is in the second row and so on. Let t be the tabloid $\begin{array}{|l} 1 \\ 2 \\ \downarrow \end{array}$ so $t \to (x_1 x_2 \cdots)(x_1 \cdots)$

and so the signed column operator sends the polytabloid to the generator of V_λ. This proves the theorem.

Duality for Specht modules

Towers asked about a duality between rows and columns with regard to maps between Specht modules. We shall answer this.

Let Σ_n be a symmetric group, S^λ a Specht module for Σ_n working over the field k of characteristic p. The duality sends S^λ to S^{λ^*}, where λ^* is the dual partition.(*)

Theorem $\mathrm{Hom}_{k\Sigma_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\Sigma_n}(S^{\mu^*}, S^{\lambda^*})$

(Probably very well-known).

Pf. Now $S^{1^n} \otimes (S^{1^n})^* \cong S^n \cong k$ so

$$\mathrm{Hom}_{k\Sigma_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\Sigma_n}(S^\lambda \otimes S^{1^n}, S^\mu \otimes S^{1^n})$$
$$\cong \mathrm{Hom}_{k\Sigma_n}((S^{\lambda^*})^*, (S^{\mu^*})^*)$$

by a result in James' book, page 33, so

$$\mathrm{Hom}_{k\Sigma_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{k\Sigma_n}(S^{\mu^*}, S^{\lambda^*})$$

as required.

Proposition. S^λ and S^{λ^*} have dual submodule lattices. In particular if λ is p-regular, then S^{λ^*} has a simple socle.

Pf. S^λ and $S^\lambda \otimes S^{1^n}$ have isomorphic submodule lattices, by definition of S^{1^n}. If λ is p-regular then $S^\lambda / J(S^\lambda)$ is simple, by James' work.

(*) Actually get a duality, tracing all maps through.

Let's apply this for $p=3$, $n=6$ and to the principal block. The decomposition matrix is

dim		1	4	1	6	4
		6	51	3^2	41^2	321
1	6	1				
5	51	1	1			
5	3^2		1	1		
10	41^2		1		1	
16	321	1	1	1	1	1
5	2^3	1				1
10	31^3				1	1
5	21^4			1		1
1	1^6			1		

Have immediately,

$$S^6 = D^6, \quad S^{51} = \begin{matrix} D^{51} \\ D^6 \end{matrix}, \quad S^{3^2} = \begin{matrix} D^{3^2} \\ D^{51} \end{matrix}, \quad S^{41^2} = \begin{matrix} D^{41^2} \\ D^{51} \end{matrix}, \quad S^{1^6} = D^{3^2}$$

And, by the proposition, as $(2^3)^* = 3^2$, $(31^3)^* = 41^2$, $(21^4)^* = 51$, get the corresponding Specht modules are uniserial. Also, by a result of James, in §24, $D^6 \subseteq S^{2^3}$ so

$$S^{2^3} = \begin{matrix} D^{321} \\ D^6 \end{matrix}$$

Let's go after S^{31^3}. Here

$$(S^{31^3})^* \cong S^{41^2} \otimes S^{1^6} = \begin{matrix} D^{41^2} \otimes D^{3^2} \\ D^{51} \otimes D^{3^2} \end{matrix}$$

which is uniserial with a six-dimensional top. Hence, as all the simples are self-dual, get

$$S^{31^3} = \begin{matrix} D^{321} \\ D^{41^2} \end{matrix}$$

and also that $D^{51} \otimes D^{3^2} \cong D^{321}$.

Next, let's get S^{21^4}. Now $S^{21^4} \otimes S^{1^6} = (S^{51})^*$ so $\mathrm{Hom}_{k\Sigma_6}(S^{21^4} \otimes S^{1^6}, D^6) \neq 0$ so $\mathrm{Hom}_{k\Sigma_6}(S^{21^4}, D^{3^2}) \neq 0$ so

$$S^{21^4} = \begin{matrix} D^{3^2} \\ D^{321} \end{matrix}$$

Last, let's look at S^{321}. Since $[321]$ is regular for $p=3$ and self-dual, we get that S^{321} has a simple socle of dimension four so

$$S^{321} = \begin{matrix} D^{321} \\ \\ \\ D^{51} \end{matrix} \Big\} \text{ comp factors } D^6, D^{3^2}, D^{41^2}.$$

<u>Claim</u>:

$$S^{321} = \begin{matrix} & D^{321} & \\ D^6 & D^{3^2} & D^{41^2} \\ & D^{51} & \end{matrix}$$

Now $[321] \uparrow [3^2]$ in Carter-Lusztig(*) moving boxes notation so $\mathrm{Hom}(S^{3^2}, S^{321}) \neq 0$ so $\begin{matrix} D^{3^2} \\ D^{51} \end{matrix} \subseteq S^{321}$. Also $[321] \uparrow [41^2]$ so $\mathrm{Hom}(S^{41^2}, S^{321}) \neq 0$ and $\begin{matrix} D^{41^2} \\ D^{51} \end{matrix} \subseteq S^{321}$.

It remains to get "a" $\begin{matrix} D^6 \\ D^{51} \end{matrix} \subseteq S^{321}$.

(*) Also seems O.K. by results in James' work.

But we do this completely indirectly. As $[321] = [321]^*$ we have that the submodule lattice of S^{321} is anti-isomorphic with itself. $\therefore$ this is true for $J(S^{321}) / \mathrm{Soc}(S^{321})$. Also dimensions match up so there are only two possibilities:

$$\begin{matrix} D^6 \\ D^{3^2} \end{matrix} \oplus D^{41^2}, \quad D^6 \oplus D^{3^2} \oplus D^{41^2}$$

But $\mathrm{Ext}(D^6, D^{3^2}) = 0$ as Σ_6 / A_6 would be the group acting on such an extension and its order two is prime to three.
$\therefore$ our claim holds.

Next, let's relate this to James results on duals of Specht modules. He defines, if λ is column p-regular, $D_\lambda = (S^\lambda)^* / J((S^\lambda)^*) =$ "top" of $(S^\lambda)^*$. Hence,

$$\begin{aligned} D_\lambda &\simeq (\mathrm{Soc}\, S^\lambda)^* \\ &\simeq \mathrm{Soc}\, S^\lambda \qquad \text{(as all simples are self-dual)} \\ &\simeq S^{1^n} \otimes \mathrm{Soc}(S^\lambda \otimes S^{1^n}) \\ &\simeq S^{1^n} \otimes \mathrm{Soc}((S^{\lambda^*})^*) \\ &\simeq S^{1^n} \otimes \mathrm{Top}\,(S^{\lambda^*}) \\ &\simeq S^{1^n} \otimes D^{\lambda^*} \qquad \text{(as } \lambda^* \text{ is p-regular)} \end{aligned}$$

We've proved:

<u>Prop</u> $D_\lambda \simeq S^{1^n} \otimes D^{\lambda^*}$.

<u>Query</u> (equivalent to a question of James!): $D_\lambda = D^\mu$ some p-regular μ. What is it? I.e. what is $S^{1^n} \otimes D^\alpha$?

Corollary If λ is column p-regular then the multiplicity of D_λ as a composition factor of $(S^\mu)^*$ equals the multiplicity of D^{λ^*} as a composition factor of S^{μ^*}.

Proof. The first multiplicity is the multiplicity of $D_\lambda \otimes S^{1^n}$ in $(S^\mu)^* \otimes S^{1^n}$, or with the obvious notation,

$$\begin{aligned}((S^\mu)^* : D_\lambda) &= ((S^\mu)^* \otimes S^{1^n} : D_\lambda \otimes S^{1^n}) \\ &= ((S^\mu \otimes S^{1^n})^* : D^{\lambda^*}) \\ &= (S^{\mu^*} : D^{\lambda^*})\end{aligned}$$

as desired.

There is one case where we can answer our query:

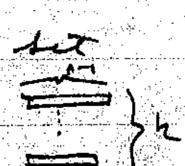
Prop $D_{1^n} \simeq D^{(h+1)^r h^{p-1-r}}$ when $n = h(p-1)+r$, $0 \le r < p-1$.

Pf. Now $D_{1^n} \simeq S^{1^n}$. There is a unique p-regular partition μ with $\mathrm{Hom}(S^\mu, S^{1^n}) \neq 0$; then $D_{1^n} \simeq D^\mu$. But

$$\begin{aligned}\mathrm{Hom}(S^\mu, S^{1^n}) &\simeq \mathrm{Hom}(S^\mu \otimes S^{1^n}, S^n) \\ &\simeq \mathrm{Hom}((S^{\mu^*})^*, S^n) \\ &\simeq \mathrm{Hom}(S^n, S^{\mu^*}).\end{aligned}$$

We can use James' criterion for S^n as a submodule. Let

$$\mu^* = (p-1, \dots, p-1, r)$$

μ^*: 

and so

$$\mu = (\overbrace{h+1, \dots, h+1}^{r}, \overbrace{h, \dots, h}^{p-1-r})$$

as desired.

Hooks and homomorphisms

The theory of blocks of weight one in Σ_n (i.e. defect Z_p) gives such complete information that it allows us to get some maps between Specht modules and hence a chance to get explicit formulae for maps between Weyl modules.

Let B be such a p-block, with $\lambda_1, \lambda_2, \ldots, \lambda_p$ the p partitions corresponding to B. With lexicographic ordering on the λ's assumed the decomposition matrix is known:

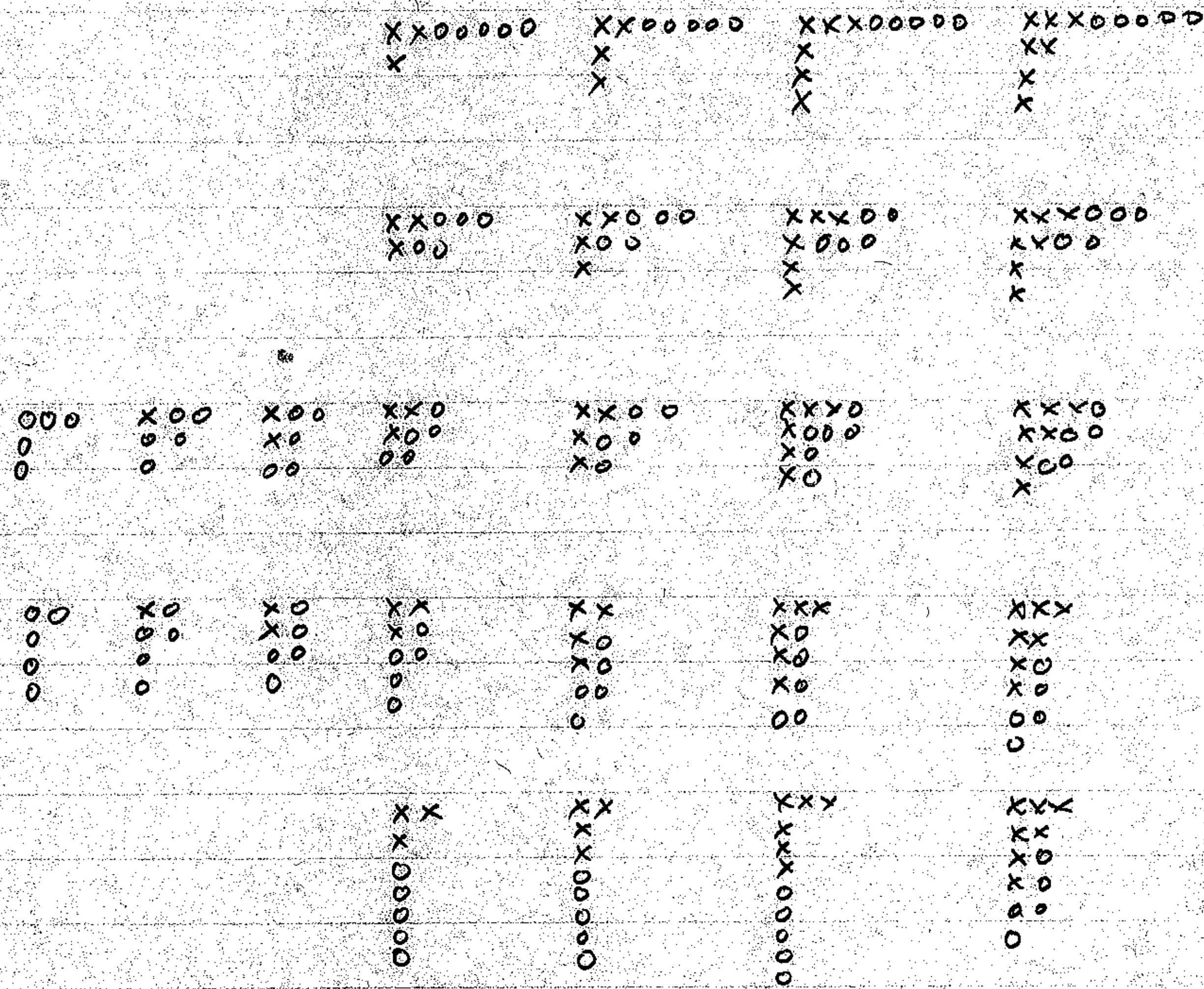
$$
\begin{array}{c|cccccc|}
S^{\lambda_1} & 1 & & & & & \\
S^{\lambda_2} & 1 & 1 & & & & \\
S^{\lambda_3} & 0 & 1 & 1 & & & \\
\vdots & & & & & & \\
S^{\lambda_{p-1}} & & - & - & - & 1 & 1 \\
S^{\lambda_p} & & & & & & 1
\end{array}
$$

The $\lambda_1, \ldots \lambda_p$ each have one removable p-hook of leg length $0, 1, \ldots, p-1$ respectively as $\lambda_1, \ldots, \lambda_{p-1}$ are certainly p-regular as a p-core certainly is. Hence, we get an exact sequence:

$$0 \to S^{\lambda_1} \to S^{\lambda_2} \to \cdots \to S^{\lambda_{p-1}} \to S^{\lambda_p} \to 0.$$

It may be that looking at the proofs of the structure of B will be useful in constructing maps along the lines of Towber's methods.

Let's give an heuristic argument: Σ_{12}, $p = 5$.



Map guess for the Λ's of Tartan:

$$y_1\, y_2\, y_8\, y_9$$
$$y_4\, y_5\, y_{10}\, y_3$$
$$y_6\, y_7\, y_{12}$$
$$y_{11}$$
$$\uparrow$$
$$y_1\, y_2\, y_3$$
$$y_4\, y_5$$
$$y_6\, y_7$$
$$y_8\, y_9$$
$$y_{10}\, y_{11}$$
$$y_{12}$$

Blocks of weight two

The question: to what extent does the way two p-hooks are removed from a Young diagram to give a p-core determine the corresponding decomposition numbers. We give some calculations, using the tables from James' book.

$p=2$, $B_1(\Sigma_4)$.

	4	$3\,1$
4	1	
$3\,1$	1	1
2^2		1
$2\,1^2$	1	1
1^4	1	

Removed hooks

$p=2$ $B_1(\Sigma_5)$

	5	$3\,2$
5	1	
$3\,2$	1	1
$3\,1^2$	2	1
$2^2\,1$	1	1
1^5	1	

$p=2$ $\quad B_2(\Sigma_7)$

	61	43
61	1	
43	1	1
41^3	2	1
$2^3 1$	1	1
21^5	1	

$p=2$ $\quad B_2(\Sigma_{10})$

	721	541
721	1	
541	1	1
521^3	2	1
$32^3 1$	1	1
321^5	1	

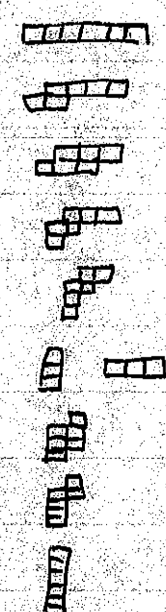
The evidence suggests the decomposition depends only on the removed hooks, the shapes of the connected components. Let's turn to $p=3$, this is perhaps indicative of $p>2$ where the defect group is $\mathbb{Z}_p \times \mathbb{Z}_2$, as opposed to $\mathbb{Z}_2 \wr \mathbb{Z}_2 \cong D_8$.

$p=3 \quad B_1(\Sigma_6)$

	6	51	3^2	41^2	321
6	1				
51	1	1			
3^2		1	1		
41^2		1		1	
321	1	1	1	1	1
2^3	1				1
31^3				1	1
21^4			1		1
1^6			1		

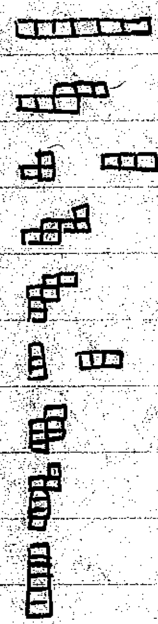
$p=3 \quad B_1(\Sigma_7)$

	7	52	43	421	321^2
7	1				
52	1	1			
43		1	1		
421	1	1	1	1	
321^2	1		1	1	1
41^3				1	
2^31	1				1
2^21^3			1		1
1^7			1		



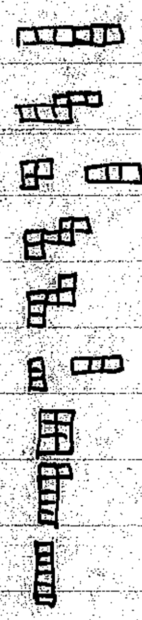
$p = 3$ $B_2(\Sigma_{10})$

	91	64	62^2	$4^2 2$	$3^2 2 1^2$
91	1				
64		1			
62^2	1	1	1		
$4^2 2$		1	1	1	
$3^2 2 1^2$	1		1	1	1
61^4			1		
$32^3 1$	1				1
$32^2 1^3$				1	1
31^7				1	



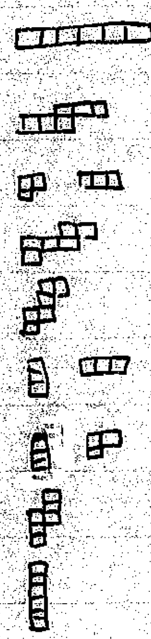
$p = 3$ $B_1(\Sigma_8)$

	8	53	521	431	$3^2 1^2$
8	1				
53		1			
521	1	1	1		
431		1	1	1	
$3^2 1^2$	1		1	1	1
51^3			1		
2^4	1				1
$2^2 1^4$				1	1
21^6				1	



$p = 3$, $B_3(\Sigma_{11})$

	91^2	641	632	542	$3^2 2^2 1$
91^2	1				
641		1			
632	1	1	1		
542		1	1	1	
$3^2 2^2 1$	1		1		1
61^5			1		
$3^2 2 1^3$			1	1	1
$32^3 1^2$					1
31^8				1	



It seems that the slope does indeed determine the decomposition numbers. For example,

1 1

1 1

1

⋮

Let's go back to $p=2$ and let's demonstrate that the pattern shown is general. We will go from Σ_4 to Σ_7 by an argument which clearly generalizes as the 2-cores are the partitions $[s\ s-1\ \cdots\ 2\ 1]$.

We calculate $\operatorname{Ind}_{Y}^{\Sigma_7}(S^{[21]} \otimes S^{\lambda})$ projected on the block with core $[21]$ when λ is a partition of 4 and Y is the appropriate Young subgroup. We use the Littlewood–Richardson rule (see James' book).

λ:	61	43	41^3	$2^3 1$	21^5
4	1	0	0	0	0
31	0	1	1	0	0
2^2	0	1	0	1	0
21^2	0	0	1	1	0
1^4	0	0	0	0	1

The Littlewood-Richardson pictures:

$$\begin{array}{llllll} \times & \times & 1 & 1 & 1 & 1 \\ \times \end{array} \qquad \begin{array}{llll} \times & \times & 1 & 1 \\ \times & 1 & 2 \end{array} \qquad \begin{array}{llll} \times & \times & 1 & 1 \\ \times \\ 1 \\ 2 \end{array} \qquad \begin{array}{llll} \times & \times & 1 & 1 \\ \times & 2 & 2 \end{array}$$

$$\begin{array}{ll} \times & \times \\ \times & 1 \\ 1 & 2 \\ 2 \end{array} \qquad \begin{array}{llll} \times & \times & 1 & 1 \\ \times \\ 2 \\ 3 \end{array} \qquad \begin{array}{ll} \times & \times \\ \times & 1 \\ 1 & 2 \\ 3 \end{array} \qquad \begin{array}{ll} \times & \times \\ \times \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

From the decomposition matrix of Σ_4 we get two characters of projectives:

$$\chi^{4} + \chi^{31} + \chi^{21^2} + \chi^{1^4}$$
$$\chi^{31} + \chi^{2^2} + \chi^{21^2}.$$

Applying the same process we get characters of projectives, tabulated as follows:

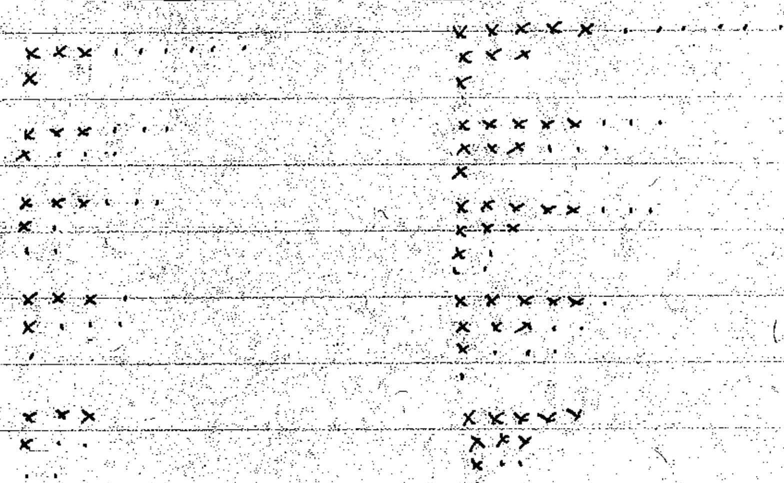
61	43	41^3	$2^3 1$	21^5
1	1	2	1	1
0	2	2	2	0

These must be linear combinations with integral coefficients of the indecomposable projective characters. Also we must use half the second row as that half gives a character which also vanishes on 2-singular elements. The two rows must have non-negative coefficients in their linear combinations as they are projective, not just generalized characters vanishing on 2-singular elements. Together these conditions give the projectives as desired.

61	43	41^3	$2^3 1$	21^5
1	1	2	1	1
0	1	1	1	0

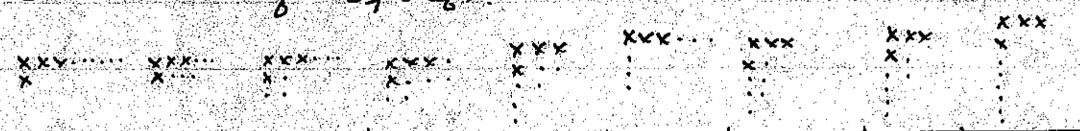
Let's return to $p = 3$. Let's display two blocks which have the same pattern of hook removal:

$B_2(\Sigma_{10})$ $\qquad\qquad$ $B(\Sigma_{15})$



We should have the same decomposition matrices and so forth. Let's try to get these numbers for $B_2(\Sigma_{10})$ by a method which will clearly generalize. We use the Littlewood-Richardson rule as for $p = 2$, using the core and the principal 3-block of Σ_6.

We tabulate the projection on $B_2(\Sigma_{10})$ of $\mathrm{Ind}_{\Sigma_4\times\Sigma_6}^{\Sigma_{10}}(\chi^{[31]}\otimes -)$ where the blank stands for the characters of $B_1(\Sigma_6)$.



111111	1								
11111 2		1	1						
111 222		1		1					
1111 2 3			1	1		1			
111 22 3			1	1	1				
11 22 33							1		
111 2 3 4					1	1			
11 2 3 4 5								1	
1 2 3 4 5 6									1

Now we take the columns of the decomposition matrix of $B_1(\Sigma_6)$ to get indecomposable projectives for $B_1(\Sigma_6)$ and we apply the above process to get projectives in $B_2(\Sigma_{10})$:

	91	64	62^2	$4^2 2$	$3^2 21^2$	61^3	$32^3 1$	$32^2 1^2$	31^6
Q_1	1	1	2	1	1	0	1	0	0
Q_2	0	2	2	3	1	1	0	0	0
Q_3	0	1	1	2	1	0	0	1	1
Q_4	0	0	2	2	2	2	0	0	0
Q_5	0	0	1	1	2	1	1	1	0

The indecomposables from James' tables:

P_1	1	0	1	0	1	0	0	0	0
P_2	0	1	1	1	0	0	0	0	0
P_3	0	0	1	1	1	1	0	0	0
P_4	0	0	0	1	1	0	0	1	1
P_5	0	0	0	0	1	0	1	1	0

Hence, hope to deduce – so it generalizes eg to $B(\Sigma_{15})$ – that

$$Q_1 = P_1 + P_2$$
$$Q_2 = 2P_2 + P_3$$
$$Q_3 = P_2 + P_4$$
$$Q_4 = 2P_3$$
$$Q_5 = P_3 + P_5$$

Cores of Young diagrams

The preceding examples suggest that the "shape" of the cores determine the nature of the decomposition matrix. In fact, the above suggest a way of at least bounding the number of Morita equivalence classes of blocks in symmetric groups with a given type of defect group, i.e. a given weight. Since everything is defined over the field k_p of p elements and since the number of characters and modular characters in a block depends only on the weight, this means bounding the decomposition numbers by a function of the weight. Hence, it's enough to exhibit some projections, as linear combinations of characters in the block, such that the coefficients are bounded and every indecomposable projective appears when these projections are expressed in terms of the indecomposables.

Let's give an explicit conjecture which would do all this. Let B be a p-block of Σ_n of weight w with core H_K, K a partition of k. Hence, $n = k + w.p$.

Conjecture. The projections on B of the characters

$$\mathrm{Ind}_{\Sigma_k \times \Sigma_{wp}}^{\Sigma_n} (\chi^K \otimes \chi),$$

as χ runs over the principal p-block $B_1(\Sigma_{wp})$, are linearly independent.

If this is true, then linear combinations of characters in B are projective, of number equal to the number of indec. projectives, and when expressed in terms of the characters of B the coefficients are bounded by the Littlewood-Richardson rule.

To get an idea of what's happening, let's try and describe the cores. The 2-cores are easily described: they are the partitions $[r, r-1, \ldots, 2, 1]$.

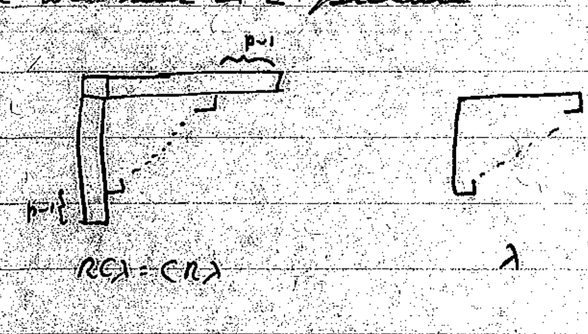
Let's describe a construction for p-cores; the idea is that eliminating the first row or column of a p-core leaves a p-core so we want to study the reverse process. We fix a prime p. If λ is a partition then $C\lambda$ is the partition which has one more column than λ, a new first column $p-1$ boxes longer than the first column of λ. Similarly $R\lambda$ has a new first row, $R\lambda = [\lambda_1+p-1, \lambda_1, \lambda_2, \ldots]$.

<u>Lemma</u> If λ is a p-core then so is $C\lambda$ and $R\lambda$.

<u>Proof</u>. We prove it for $C\lambda$; the argument for $R\lambda$ is similar - or use $R\lambda = (C\lambda^*)^*$. We need only compute the hook lengths for the first column of $C\lambda$. The last $p-1$ are $p-1, p-2, \ldots, 2, 1$. The first ones in the first column are p bigger than the hook lengths in the first column of λ, as is easily seen. Hence none of these is divisible by p.

<u>Lemma</u> For any partition λ, $RC\lambda = CR\lambda$.

<u>Pf</u> All we need is a picture:



We say that λ is a p-hardcore if λ is a p-core and $\lambda \neq R\lambda'$, $\lambda \neq C\lambda'$ for any partition λ'. Note that, of course, if λ is a p-core, $\lambda = R\lambda'$ then λ' is a p-core, as the hook lengths for λ' are among those of λ.

<u>Proposition</u> If $p > 2$ then every p-core has a unique expression

$$R^i C^j \eta,$$

where $i, j \geq 0$, η is a p-hardcore, and each such expression is a p-core.

<u>Proof</u> These expressions are p-cores, by the first lemma. Clearly, every p-core has such expression so what has to be done is prove the uniqueness. This is easily done by induction: $R^i C^j \eta = R^{i'} C^{j'} \eta'$ implies that i is positive if, and only if i' is and the same for j, j' as $RC = CR$. Just peel off an R or a C. The reason for this is as follows: There is an R present iff $\lambda_1 - \lambda_2 = p-1$.

<u>Theorem</u> The 3-hardcores are $[\phi]$ and $[1]$.

<u>Proof</u> Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_n > 0$ be a partition. Suppose $\mu = \lambda^*$ and $\mu_1 - \mu_2 \neq p-1$; i.e. $\lambda \neq C\lambda'$ for some λ'. We must show that λ is not a p-core, that $\lambda = \phi$ or $\lambda = [1]$, or that $\lambda = R\lambda'$ for some λ'. If $\mu_1 - \mu_2 > p-1$ then clearly λ is not a p-core so we may assume $\mu_1 - \mu_2 < p-1$. (Here $p=3$ but we go on for general p for a while.)

We need one general fact: <u>Assume</u> λ is a p-core.

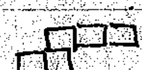
<u>Lemma</u> If $\lambda_i - \lambda_{i+1} = p-1$ then $\lambda = R^i[\lambda_{i+1}, \dots, \lambda_n]$.

<u>Proof</u> It's enough to see that if $i > 1$ then $\lambda_{i-1} - \lambda_i = p-1$ also. A picture:

λ_{i-1}
λ_i
λ_{i+1}

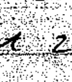
We will get a p-hook if $\lambda_i - \lambda_{i+1} \neq p-1$, as is easily seen. Just start with the end of the $i-1$st row and work backwards and down on the rim.

Now let's return to the theorem and we now take $p=3$. Hence, $\mu_1 - \mu_2$ is 0 or 1. First, say $\mu_1 = \mu_2$. If $\mu_3 > 0$ we're done, so the rim must look as follows:



or else we get a 3-hook. Actually, we could apply the lemma directly to the last row, but we don't have to; we get $\lambda = R^i \emptyset$.

Say $\mu_1 - \mu_2 = 1$. We get the rim at the end of the 1st 2 columns



so the third column must also be μ_2 so $\mu_4 < \mu_3$ or we get a 3-core so the lemma implies $\lambda = R^i[1]$ and we're done.

Now we take $p = 4$; the combinatorics doesn't need p a prime to define cores and so on. Let's look at this case to see what's going on.

Let's give some 4-hookcores. There is $\emptyset$ and there are the 2-hookcores $T(i) = [i, i-1, \dots, 1]$, $i \geq 1$. Let, for $i \geq 1$,

$$H_2(i) = [\,i, i, i-2, i-2, \dots \quad]$$
$$H_1(i) = [\,i, i-2, i-2, i-4, i-4, \dots\,]$$

(all the way down to 1's or 2's)

<u>Theorem</u> The following are the distinct 4-hookcores:

1) $\emptyset$;
2) $T(i)$, $i \geq 1$;
3) $H_1(i)$, $i > 1$;
4) $H_2(i)$, $i \geq 1$.

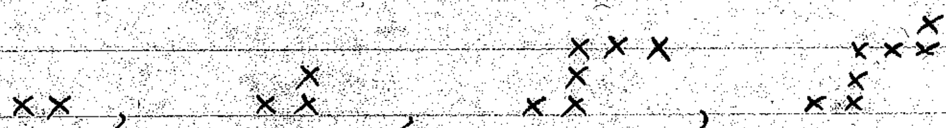
<u>Pf</u> These are clearly distinct and the first two types are clearly 4-hookcores. It remains to show that the H's are 4-cores and that any 4-hookcore is one of these.

First, if we remove the first row or first column from one of the H's we get another H. Hence we need only determine the hook length of the box in the upper left hand corner. It is as follows:

$H_1(i)$	i even	$2(i-1)$
	i odd	$2i-1$
$H_2(i)$	i even	$2i-1$
	i odd	$2i$.

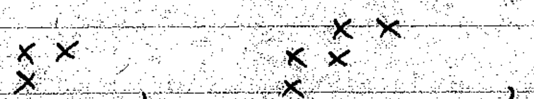
Now we come to the substantial part of the proof. We build up the rim until we see the pattern. The method is to see what $\lambda_i - \lambda_{i+1}$ can be, not allowing this to be greater than or equal to three, by the lemma on page 83 and by the fact we have a core.

Case 1 : $\mu_2 = \mu_1$



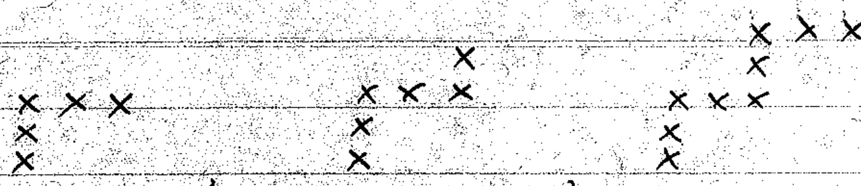
etc..

Case 2 : $\mu_2 = \mu_1 - 1$.



... etc...

Case 3 : $\mu_2 = \mu_1 - 2$



etc,..

and the result is proved.

Note that the diagrams are being built up, row by row from the bottom and happily there is only one case that doesn't imply a four hook or that we hit the "R-operation".

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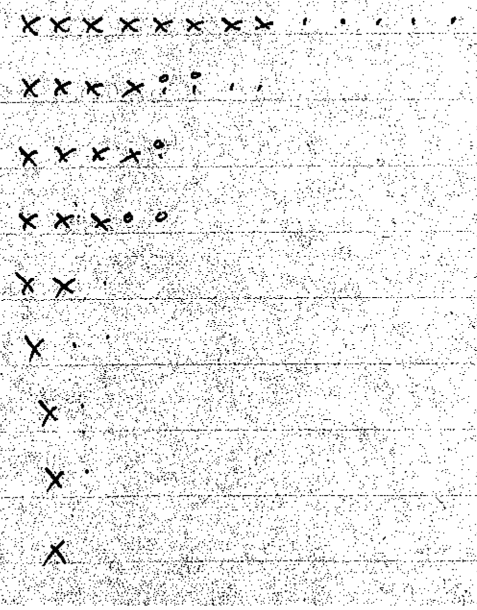
It's clear now what's happening. Just have to consider the first column hook lengths and the criterion for cores. What is going on is that we're "losing" residue classes as we pass from the shortest first column hook to the longest, i.e. from the last row to the first. As long as we keep hook lengths from a fixed set of classes we will keep repeating the "pattern".

This also makes clear why one can add exactly p p-hooks to a p-core. They correspond to the hook 1^p and the losses of the $p-1$ non-zero residue classes. (This is hinted in "On the p-quotients and star diagrams of the symmetric groups," by H. Farahat, Proc. Camb. Phil. Soc. 49, (1953) p 158.)

Let's pursue this to get that the p-hooks are of p leg lengths and occur in lexicographic order. First, an example: $p=5$, first column hook lengths are 1, 2, 3, 4, 6, 7, 9, 11, 12, 17. (Dots and circles the hooks.)



The heads of the hooks are in the last row before we "lose" a residue class.

Now let's sketch the proof. Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_s > 0$ and $h_i = \lambda_i + s - i$. Let $0 < r < p$ and suppose h_j is the last occurrence of a first column hook length in the class r — counting from $h_s, h_{s-1}, \ldots$ (If it never occurs then modify the ensuing argument.)
Let $h_{j-k} < h_j + p$, $h_{j-k-1} > h_j + p$. Run a hook from the $(j-k, \lambda_{j-k}+1)$ position to the $(j, \lambda_j + 1)$ position along the rim; this has length $h_{j-k} - h_j + 1$. Now add more nodes to the head of this one to the $(j-k)$-th row. We have to see this is alright. The condition we need is that

$$p - (h_{j-k} - h_j + 1) \leq \lambda_{j-k-1} - \lambda_{j-k}.$$

(The condition on leg lengths follows from the number of residue classes lost increasing and the lexicographic condition is clear from the construction.) That is,

$$\begin{aligned} p - h_{j-k} + h_j - 1 &\leq h_{j-k-1} - s + (j-k-1) \\ &\quad - h_{j-k} + s - (j-k) \\ p - h_{j+k} + h_j - 1 &\leq h_{j-k-1} - h_{j-k} - 1 \\ p + h_j &\leq h_{j-k-1} \end{aligned}$$

which holds.

Green correspondence and Brauer induction.

We're going to present a new proof of Brauer's induction theorem (that every character is an integral linear combination of characters induced from elementary subgroups — we ignore the refinement to linear characters). We shall use the Green correspondence, so that Brauer's theorem will be a "global" result following from the "local" Green correspondence.

Let K be a number field which is a splitting field for the finite group G. Let O be the completion of the integers of K with respect to a prime divisor of the rational prime p, $p \mid |G|$. We say that an OG-module (always assumed to be a lattice) is p-local if it is the direct sum of modules induced from p-local subgroups.

<u>Proposition 1</u>. If M is an OG-module then there are p-local OG-modules L and L' and projective OG-modules P and P' such that

$$M \oplus L \oplus P \cong L' \oplus P'.$$

<u>Proof</u>. We may assume M is indecomposable and proceed by induction on the order of the vertex of M. If M is projective the result is clear, so we need only establish the induction step. Let Q be a vertex of M and let U be the $ON(Q)$-module which is the Green correspondent of M. Hence,

$$\operatorname{Ind}_{N(Q)}^{G} U = M \oplus M_1',$$

where every indecomposable summand of M_1 has vertex smaller than Q. Hence, by induction, with the obvious notation,

$$M_1 \oplus L_1 \oplus P_1 \simeq L_1' \oplus P_1'.$$

Hence

$$\begin{aligned} \operatorname{Ind}_{N(Q)}^{G}(U) \oplus L_1 \oplus P_1 &\simeq M \oplus M_1 \oplus L_1 \oplus P_1 \\ &\simeq M \oplus L_1' \oplus P_1' \end{aligned}$$

and we're done.

Corollary 2 If χ is a character of G then there are gen'd characters Λ_p, Φ_p of G such that

$$\chi = \Lambda_p + \Phi_p$$

where Λ_p is an integral linear combination of characters induced from p-local subgroups and Φ_p vanishes on p-singular elements.

Proof. Let M be an $\mathcal{O}G$-module affording χ.

Corollary 3 If G is a non-identity group then every character of G is an integral linear combination of characters induced from local subgroups.

Proof. It suffices to prove this for the principal character 1_G as the integral linear combinations described form an ideal in the character ring. But

$$1_G = \prod_{p \mid |G|} (\Lambda_p + \Phi_p)$$

where each

$$1_G = \Lambda_p + \Phi_p$$

is as in Corollary 2. Each term in the expanded product is of the right sort clearly save $\prod_{p \mid |G|} \Phi_p$. But this term vanishes on <u>each</u> non-identity element of G so is a multiple of the regular representation and so is induced from <u>any</u> subgroup of G.

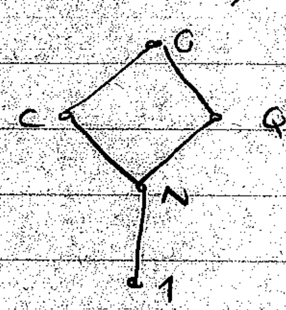
Now we prove Brauer's induction theorem by induction on $|G|$, letting G be a minimal counterexample. Hence, the conclusion of Brauer's theorem fails for 1_G for if it holds for 1_G then it holds for any character.

First, there is p such that $O_p(G) \neq 1$. Otherwise every local subgroup is proper so Corollary 3 and the minimality of G contradict the fact that the theorem fails for G.

Second, G has a unique minimal normal subgroup N. Otherwise, G is isomorphic with a subgroup of the direct product of groups of smaller order, so the minimality of G and Mackey's theorem yield a contradiction. For subgroups of elementary groups are elementary.

Third, G/N is elementary. Otherwise $1_{G/N}$ is the required linear combination using elementary subgroups of G/N and the truth of the theorem for the pre-image in G of each elementary subgroup of G/N yields a contradiction.

Hence, we have the picture:



where C/N is a cyclic q'-group, Q/N is a q-group, G/N is the direct product of C/N and Q/N, q being a prime.

Next, we assume that $N \subseteq O_p(G)$. Certainly $N \subseteq O_p(G)$ as $O_p(G) \neq 1$. The structure of G/N guarantees that $p \nmid |G : O_p(G)|$. Hence by Maschke's theorem, $O_p(G)$ is not elementary abelian. $\therefore N \subseteq \Phi(O_p(G))$ as $1 \neq \Phi_p(O_p(G)) \trianglelefteq G$. As G/N is nilpotent we deduce that $N \subseteq Z(G)$. As[⊛] $N \subseteq \Phi(O_p(G))$ this implies that G is elementary, a contradiction.

Hence, G is a semi-direct product, $G = KN$, and no non-trivial linear character of N is stabilized by G. Thus, $\mathrm{Ind}_K^G(1_K) = 1_G + \ldots$ where the dots correspond to characters of G which do not involve 1_N in their restriction to N; for $\mathrm{Ind}_K^G(1_K)$ restricted to N is ρ_N, the regular character. Thus, we are done, by Clifford's theorem as all the characters represented by the dots are induced from proper subgroups.

⊛ Or: G is nilpotent and we contradict the uniqueness of N.

Row removal for the symmetric groups

James suggests there may be a nice result about decomposition numbers for the symmetric groups with respect to removing the first columns of partitions. Perhaps there is a similar result for rows.

Conjecture Let $\lambda, \mu \vdash r$ with $\lambda_1 = \mu_1 = s$ and $\lambda', \mu' \vdash r-s$ the partitions obtained from λ and μ, resp, by removing the first row. If μ is p-regular then so is μ' and

$$(S^\lambda : D^\mu) = (S^{\lambda'} : D^{\mu'}).$$

Here $(S^\lambda : D^\mu)$ is the multiplicity that the "top" D^μ of S^μ occurs in S^λ. It's clear that μ' is p-regular

Sketch of how to try and prove this: First, would like James' results on decomposition of the Specht modules and Weyl modules to hold in a greater range – i.e. when they make sense. Then, using V's for Weyl modules, F's for their tops, for $GL(s)$ as λ^*, μ^* have s parts,

$$(S^\lambda : D^\mu) = (V_{\mu^*} : F_{\lambda^*})$$
$$(S^{\lambda'} : D^{\mu'}) = (V_{(\mu')^*} : F_{(\lambda')^*})$$

Hence, we need a first column removal result for Weyl modules. Let's assume all these Weyl modules around are isomorphic

Now λ^*: $\lambda_1^* \geq \dots \geq \lambda_s^*$. The corresponding weight is $a_1\omega_1 + \dots + a_s\omega_s$ with $a_i = \lambda_i^* - \lambda_{i+1}^*$ so $\lambda_1^* = a_1 + \dots + a_s$, $\lambda_2^* = a_2 + \dots + a_s$. Hence, want to lower a_s by one. This is just - should be - tensoring with $(\det)^{-1}$. That is, $V_\omega \otimes \det = V_{\omega + \omega_s}$ so submodule structures are the same and rest is easy.

Growth and Krull dimension

Evens has asked for the justification for the statement, in section three of our paper, relating growth and Krull dimension. Let's do it in more generality (the case $A = H^*(G, k)$, A below, is the one of our paper). Let A be a finitely generated $\mathbb{Z}$-graded commutative algebra over the field k with $A_n = 0$, $n < 0$, $A_1 = k.1$. Hence, each A_i is finite-dimensional and A has a growth $\gamma(A)$. Let $K(A)$ be the Krull dimension of A.

Lemma $K(A) = 1 + \gamma(A)$.

By our hypotheses, there is a modulus N and rat'l polynomials $f_0, f_1, \ldots, f_{N-1}$ such that, with finitely many exceptions, $\dim A_n = f_r(n)$ if $n \equiv r$ (modulo N). Hence, the maximum of the degrees of the f_i is $\gamma(A)$. It follows that if ΣA is the graded vector space with $(\Sigma A)_n = \bigoplus_{i=0}^{n} A_i$ ($(\Sigma A)_n = 0$ if $n<0$) then $\gamma(\Sigma A) = \gamma(A) + 1$ so we want to see that $\gamma(\Sigma A) = K(A)$.

Let $K(A) = d$ so, by Bourbaki, A is a finitely generated module over a polynomial subalgebra $k[x_1, \ldots, x_d]$. Let $y_1, \ldots, y_e$ be the module generators.

First, we show that $\gamma(\Sigma A) \geq d$. Indeed, suppose $\gamma(\Sigma A) \leq d-1$. Let M be a positive integer such that the i-th component of all the x_j's are zero if $i > M$. We may also assume, without loss of generality, that the zero-th component of each x_j is zero. Hence, with the obvious notation,

$$x_j = x_{j1} + \dots + x_{jM}.$$

Hence, all the monomials

$$x_1^{a_1} \cdots x_d^{a_d}$$

with $M(a_1 + \dots + a_d) \leq n$ are linearly independent and lie in $(\Sigma A)_n$. There are too many and our first claim holds.

Second, we must see that $\gamma(\Sigma A) \leq d$. It suffices to calculate a spanning set for A, take the projections of these on each $A_0 \oplus \dots \oplus A_n$ and count the number of the spanning elements which have a non-zero projection and have this turn out right. For the spanning set we take all the monomials

$$x_1^{a_1} \cdots x_d^{a_d} y_j,$$

$1 \leq j \leq c$. Since the zero components of the x_i are zero for the projection to be non-zero we must have $a_1 + \dots + a_d \leq n$. This proves the result.

Let's conclude by seeing why the result on the modulus N and polynomials $f_0, \dots, f_{N-1}$ holds. In fact, it follows from the theorem of Hilbert and Serre quoted on page 232 of volume II of Zariski-Samuel.

We let $z_1, \dots, z_s$ be a set of generators for A, each of positive degree and homogeneous. Let N be the least common multiple of the degrees of the z_j and let w_i be the power of z_i which has degree N. Hence, A is

a finitely generated module over $k[w_1, \ldots, w_s]$. Let
$$A^{(i)} = \bigoplus_{j=0}^{\infty} A_{i+jN}$$
$i = 0, \ldots, N-1$; hence $A^{(i)}$ is a graded module for $k[w_1, \ldots, w_s]$ with A_{i+jN} the elements of degree j. Apply the Hilbert-Serre result to each $A^{(i)}$ and we're done.