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Cartan matrices

The question is whether we can find a modular proof of Brauer's theorem that the determinant of the Cartan matrix is a power of p. So let G be a finite group, k an algebraically closed field of characteristic p.

First, we have, just as in our proof of Brauer's induction theorem,

Lemma 1. If M is a kG-module then there are p-local kG-modules L and L' and there are projective kG-modules P and P' such that

$$M \oplus L \oplus P \simeq L' \oplus P'.$$

Hence, if $|G|_p = p^n$, $n > 0$, we let $[U]$ be the class of the kG-module U in $G_0(kG)$, identify $K_0(kG)$ with a subgroup of $G_0(kG)$, it suffices to show that $p^n[U] \in K_0(kG)$ in order to prove Brauer's theorem. Hence, by Lemma 1, we may assume that $O_p(G) \neq 1$, as we may be proceeding by induction.

The next step will be to give a modular proof of the next result - which is proved using characters:

Lemma 2 If P is a projective kG-module then there are kG-modules U and V, each a direct sum of modules induced from p'-subgroups of G, such that $P \oplus U \simeq V$.

Proof. It suffices to show that if Φ is a generalized character of G vanishing on p-singular elements then Φ is an integral linear combination of characters induced from p'-subgroups, inasmuch as the character of a projective has this property.

We do this now by an argument of Brauer: Express the principal character 1_G of G via Brauer's Induction Theorem:

$$1_G = \sum a_i \lambda_i^G$$

where λ_i is a character of the elementary subgroups E_i of G. Hence,

$$\Phi = \sum a_i \left(\Phi|_{E_i} \cdot \lambda_i\right)^G$$

But $\Phi_{E_i} \lambda_i$ vanishes on the p-singular elements of E_i and E_i is certainly nilpotent so we can prove the result in E_i by inspection and hence have it in G.

Let's see how this can be used. Let $|O_p(G)| = p^m$, set $\overline{G} = G/O_p(G)$. By induction, $p^{n-m}[k] \in K_0(k\overline{G})$ where k is the trivial $k\overline{G}$-module. Hence, if we let H be a p'-subgroup of G we need only prove that the theorem holds in $O_p(G)H$. For then if V is a projective $kO_p(G)H$-module then $p^m[V] \in K_0(kO_p(G)H)$.

Hence, we are reduced - using Lemma 2 - to proving the result when $G = QH$, $Q = O_p(G)$, H is a p-complement. It's now also easy to see that we can assume that Q is a minimal normal subgroup of G.

This last case lies in with Knörr's work but again we need to find a modular proof a result proved using characters. Let φ be the Knörr character of H coming from Q; that is φ is the generalized character of H with $\varphi(h)$, $h \in H$, equal to $|Q|/|C_Q(h)|$. An easy calculation shows that φ^G is the character vanishing on p-singular elements and having value $|Q|$ on p-elements. It's projective because it comes from the p'-subgroup H. Hence $|Q|\,[k] \in K_0(G)$ and we're done.

Brauer's Induction Theorem (cont.)

We want to indicate a slightly different finish to our new proof of Brauer's theorem. At the end we have a minimal normal subgroup N of G, N an elementary abelian p-group, G a minimal counterexample.

Step 1. Every proper quotient group of G is elementary. (as before)

Step 2. G is not nilpotent (as before)

Step 3. Hence, N is the unique minimal normal subgroup of G.

Step 4. $C_G(N) \supsetneqq N$. Or else G splits over N, by Baer's theorem, and acts faithfully on N so we're done, as before.

Step 5. $C_G(N)$ is abelian. If not then $C_G(N)' \geq N$, by Step 3. $\therefore G/C_G(N)'$ is nilpotent as G/N is nilpotent. But $C_G(N)$ is nilpotent, as N is central in $C_G(N)$ and $C_G(N)/N$ ($\leq G/N$) is nilpotent. Hence, by a theorem of P. Hall, G is nilpotent, contradicting Step 2.

Step 6. Hence, $C_G(N)$ is a homocyclic p-group so $C_G(N)/\mho^1(C_G(N)) \cong N$ as G-modules so $N \subseteq Z(G)$ and thus G is nilpotent, again contradicting Step 2.

Characterizing certain groups of units

Let E be an elementary abelian p-group of order p^n and kE its group algebra over an algebraically closed field of characteristic p.

Theorem If F is a subgroup of the group of units of kE of augmentation 1 then the following are equivalent:

i) F is conjugate, by an automorphism of kE, with a subgroup of E;

ii) kE is a projective kF-module.

Remarks: Augmentation 1 means in $1+\mathrm{rad}(kG)$, that is, a p-subgroup of the units. Note that in ii) kF means the group algebra of F, not the k linear combinations of F inside kE.

Proof. Of course i) clearly implies ii) so we have to prove the converse. We assume ii) holds.

We begin by showing that if $u \in \mathrm{rad}(kE)$ then kE is free as a $k\langle 1+u\rangle$-module if, and only if $u \notin \mathrm{rad}^2(kE)$. If $u \notin \mathrm{rad}^2(kE)$ then $1+u$ is conjugate, via an automorphism of kE, with a generator of E, so half our claim is clear; this is easily seen to be a consequence of the isomorphism $kE \cong k[X_1,\dots,X_n]/(X_1^p,\dots,X_n^p)$. Conversely, say kE is free as a $k\langle 1+u\rangle$-module.

Let $1+u_1, \ldots, 1+u_n$ be a basis for the group E so the images of $u_1, \ldots, u_n$ in $\mathrm{rad}(kE)/\mathrm{rad}^2(kE)$ are a basis of that vector space. Consider the group $\langle 1+u_1, \ldots, 1+u_n, 1+u \rangle$. Consider a linear combination $a_1 u_1 + \cdots + a_n u_n + b u$, $a_i \in k$, $b \in k$. If some $a_i \neq 0$ and we assume that $u \in \mathrm{rad}^2(kE)$ then $1 + a_1 u_1 + \cdots + bu \in \mathrm{rad}(kE) - \mathrm{rad}^2(kE)$ and so kE is a free module for it. If all the $a_i = 0$ then this is still so, by our assumption on the action of u. (i.e. all Jordan blocks maximal size.) Hence, by Dade's lemma, giving a projectivity criterion, we have that kE is free as a $k\langle 1+u_1, \ldots, 1+u_n, 1+u\rangle$-module. This contradicts $u \in \mathrm{rad}^2(kE)$ as $u \in \mathrm{rad}^2(kE)$ now forces this group to be of order p^{n+1}.

Now let's finish the proof of the theorem. Let F be as in (ii). F is elementary abelian as kE is commutative and $u \in \mathrm{rad}(kE)$ implies $u^p = 0$. Let $F = \langle 1+y_1, \ldots, 1+y_m \rangle$ be a basis for F. Hence, all $c_1 y_1 + \cdots + c_m y_m + 1$, $c_i \in F$, act freely by assumption, if not all $c_i = 0$. $\therefore$ they are in $\mathrm{rad}(kE) - \mathrm{rad}^2(kE)$ and so $y_1, \ldots, y_m$ are linearly independent modulo $\mathrm{rad}^2(kE)$ and we're done.

A new proof of Kroll's lemma

The idea comes from a result Evens and I use in our work on Serre's question. Fix a group G, a field k of characteristic p and a normal subgroup H of G of index p. First, a preliminary result:

Lemma. If
$$0 \to V \to W \to U \to 0$$
is an exact sequence of kG-modules, $\alpha, \beta \in H^*(G,k)$ with α annihilating $H^*(G,U)$ and β annihilating $H^*(G,V)$ then $\alpha\beta$ annihilates $H^*(G,W)$.

Proof. As in work with Evens, use the long cohomology sequence.

Kroll's Lemma. If M is a kG-module, $\mathfrak{A}$ is the annihilator of $H^*(G,M)$ in $H^*(G,k)$ and $\mathfrak{B}$ is the annihilator of $H^*(H, M_H)$ in $H^*(H,k)$ then
$$\mathrm{Res}^G_H(\mathfrak{A}^p) \subseteq \mathfrak{B}.$$

Proof Let $\alpha_1, \dots, \alpha_p \in \mathfrak{A}$ so we are asserting that $\mathrm{res}^G_H(\alpha_1 \cdots \alpha_p) \in \mathfrak{B}$. The $H^*(G,k)$ module $H^*(H,M)$ - via restriction - is isomorphic with the $H^*(G,k)$-module $H^*(G, \mathrm{Ind}_H^G(M_H))$, by Shapiro's lemma (also due to Eckmann). Hence, by the lemma above, it suffices to show that $\mathrm{Ind}_H^G(M_H)$ has a series of submodules with exactly p factors and each isomorphic with M. However, this is exactly what happens:

$$\begin{aligned} \mathrm{Ind}_H^G(M_H) &\simeq \mathrm{Ind}_H^G(M_H \otimes k) \\ &\simeq M \otimes \mathrm{Ind}_H^G(k) \\ &\simeq M \otimes k[G/H] \end{aligned}$$

and $k[G/H]$ has a composition series of length p.

Kroll's Bockstein factorization

Let $H \triangleleft G$ be of index p and let

$$\gamma_1 : \quad 0 \to J_1 \to J_p \to J_{p-1} \to 0$$
$$\gamma_2 : \quad 0 \to J_{p-1} \to J_p \to J_1 \to 0$$

be exact sequences for $k[G/H] \approx k\mathbb{Z}_p$ regarded as kG-modules, where k is as usual a field of characteristic p. We prove that $\gamma_1 \gamma_2 = \beta$ where β is the Bockstein corresponding to G/H and the product is as cohomology classes using Yoneda products. It's enough to see this holds for $k\mathbb{Z}_p$ as then the result follows by applying the inflation map.

But it's quite clear that the composition is

$$0 \to J_1 \to J_p \to J_p \to J_1 \to 0$$

as usual as the "J_{p-1}" term drops out.

It remains only to see that the relevant connecting homomorphisms in Kroll's factorization are just Yoneda products with γ_1 and γ_2. This should be easy to check but we haven't yet done so.

A construction of a complex

Suppose $P \trianglelefteq G$, P a Sylow p-subgroup, G/P and P both abelian. Seems likely that we have a nice resolution of complex type. We do one case here which should generalize to part of the general argument. What is missing in this case is the construction of suitable uniserial modules.

We assume $p = 2$, $P \cong Z_2 \times Z_2 \times Z_2$, $G/P \cong Z_7$ with faithful action. Let k be an algebraically closed field of characteristic two, let $k \otimes P$ as kG module be isomorphic with $\lambda \oplus \lambda^2 \oplus \lambda^4$, with the obvious notation. Hence, by Gaschütz's theorem, or an easy cohomology calculation, we have uniserial modules

$$\begin{matrix} 1 \\ \lambda \end{matrix} \qquad \begin{matrix} 1 \\ \lambda^2 \end{matrix} \qquad \begin{matrix} 1 \\ \lambda^4 \end{matrix}$$

We form the three complexes – which are easily seen to make sense –

$$1 \leftarrow \begin{matrix} 1 \\ \lambda \end{matrix} \leftarrow \begin{matrix} \lambda \\ \lambda^2 \end{matrix} \leftarrow \begin{matrix} \lambda^2 \\ \lambda^3 \end{matrix} \leftarrow \begin{matrix} \lambda^3 \\ \lambda^4 \end{matrix} \leftarrow \cdots$$

$$1 \leftarrow \begin{matrix} 1 \\ \lambda^2 \end{matrix} \quad \begin{matrix} \lambda^2 \\ \lambda^4 \end{matrix} \leftarrow \begin{matrix} \lambda^4 \\ \lambda^6 \end{matrix} \leftarrow \cdots$$

$$1 \leftarrow \begin{matrix} 1 \\ \lambda^4 \end{matrix} \leftarrow \begin{matrix} \lambda^4 \\ \lambda \end{matrix} \leftarrow \begin{matrix} \lambda \\ \lambda^5 \end{matrix} \leftarrow \cdots$$

We take the tensor product and so get a resolution of 1; have to see that it is projective and minimal. The latter is O.K. as the minimal resolution restricts to the minimal resolution of $Z_2 \times Z_2 \times Z_2$ so we're O.K. by dimension counting.

All the terms of the tensor product complex are linear modules tensored with

$$\overset{1}{\lambda} \otimes \overset{1}{\lambda^2} \otimes \overset{1}{\lambda^4},$$

so we need only see this is projective. Enough to show that its socle is just 7. Have therefore to calculate

$$\mathrm{Hom}_{kG}\left(\lambda^i, \overset{1}{\lambda} \otimes \overset{1}{\lambda^2} \otimes \overset{1}{\lambda^4}\right).$$

This is

$$\mathrm{Hom}_{kG}\left(\lambda^i \otimes \left(\overset{1}{\lambda}\right)^*, \overset{1}{\lambda^2} \otimes \overset{1}{\lambda^4}\right).$$

Hence, we want to show, by usual arguments, that

$$\lambda^i \otimes \left(\overset{1}{\lambda}\right)^* = \begin{matrix}\lambda^{i+6}\\ \lambda^i\end{matrix}$$

$$\overset{1}{\lambda^2} \otimes \overset{1}{\lambda^4} = \begin{matrix}\lambda^2 & 7 & \lambda^4\\ & \lambda^6 & \end{matrix}$$

which does follow by the same "Hom manipulations" we're now using. Hence, done by inspection.

Small projective modules

Landrock and Michler have studied the case of the principal projective indecomposable P_I (cover of the trivial module I) having three composition factors. What about four?

Hypothesis. $\mathrm{rad}(P_I)/\mathrm{soc}(P_I) \cong X \oplus X$, X simple.

Would like to prove: $X \cong I$ so $p=2$, G is 2-nilpotent and its Sylow 2-subgroup is $Z_2 \times Z_2$. Assume: $X \not\cong I$. We will start to analyze this now.

Lemma 1 $X \cong X^*$

Pf. $P_I \cong P_I^*$.

Lemma 2 $\mathrm{rad}(P_X)/\mathrm{soc}(P_X) \cong I \oplus I \oplus Y$, for some module Y.

Rk: P_X is the projective cover of X.

Pf. The symmetry of the Cartan matrix implies that I occurs with multiplicity two as a composition factor of P_X. Moreover, $\dim \mathrm{Ext}^1(I, X) = \dim \mathrm{Ext}^1(X, I) = 2$ by the structure of P_I. Hence $\mathrm{rad}(P_X)/\mathrm{soc}(P_X)$ has a submodule and a quotient module isomorphic with $I \oplus I$.

Lemma 3 $p=2$.

Pf. The Cartan matrix has determinant a power of p and is

$$\begin{pmatrix} 2 & 2 & 0 & \cdots \\ 2 & ? & & \\ 0 & & & \\ \vdots & & & \end{pmatrix}$$

so it's even.

<u>Lemma 4</u>. $\dim X$ is odd.

Pf For $4 \mid \dim P_I$ by Lemma 3 so $1 + \dim X$ is even.

<u>Lemma 5</u> $\mathrm{Ext}(X, X) \neq 0$.

Pf We assume otherwise so $\mathrm{Hom}(Y, X) = 0$, Y as in Lemma 2. Now $\Omega^1(I) = \begin{smallmatrix} X & & X \\ & I & \end{smallmatrix}$. Let's look at $\Omega^2(I)$:

$$0 \to \Omega^2(I) \to \begin{smallmatrix} & X & \\ Y & I & I \\ & X & \end{smallmatrix} \oplus \begin{smallmatrix} & X & \\ I & I & Y \\ & X & \end{smallmatrix} \to \begin{smallmatrix} X & & X \\ & I & \end{smallmatrix} \to 0$$

Since I is not a composition factor of Y, we deduce that $\mathrm{Hom}(\Omega^2(I), I) \cong k \oplus k \oplus k$ so $\dim \mathrm{Ext}^2(I, I) = 3$. We already have that $\dim \mathrm{Ext}^1(I, I) = 0$. But there is no I or X at the top of Y so we get from the structure of P_I that $\mathrm{Hom}(\Omega^3(I), I) = 0$ so $\mathrm{Ext}^3(I, I) = 0$. This contradicts the relation between the $\mathbb{Z}$-cohomology of G and the k cohomology in view of the calculations of Ext^1 and Ext^2.

<u>Lemma 6</u> X is a submodule of $X \otimes X$.

Pf $\mathrm{Ext}^1(X, X) \cong \mathrm{Ext}^1(I, X^* \otimes X) = \mathrm{Ext}^1(I, X \otimes X)$, by Lemma 1. Hence, $\mathrm{Hom}(\begin{smallmatrix} X & X \\ I & \end{smallmatrix}, X \otimes X)$ modulo the maps that come from $\mathrm{Hom}(\begin{smallmatrix} & I & \\ X & & X \\ & I & \end{smallmatrix}, X \otimes X)$ is not zero, by the previous lemma. But

$$\mathrm{Hom}(I, X \otimes X) \cong \mathrm{Hom}(X^*, X) \cong \mathrm{Hom}(X, X)$$

is one-dimensional and $I \mid X \otimes X \cong X^* \otimes X$ as X is odd and $p = 2$. Hence, any homomorphism of $\Omega^1(I) \cong {}^{X}{}_{I}{}^{X}$ to $X \otimes X$ has I in its kernel. Thus, $\mathrm{Hom}(X, X \otimes X) \neq 0$ as $\mathrm{Hom}(\Omega^1(I), X \otimes X) \neq 0$ as $\mathrm{Ext}^1(X, X) \neq 0$.

Symmetric algebras and projective modules

Let k be a field, A a finite dimensional algebra with unit over k and we use only finitely generated modules.

Theorem. If A is symmetric and P is a projective A-module then $\mathrm{End}_A(P)$ is also symmetric.

We shall prove two lemmas; these two clearly give the result.

Lemma 1 If n is a positive integer then $M_n(A)$ is symmetric.

Lemma 2 If e is an idempotent element of A then $\mathrm{End}_A(Ae)$ is symmetric.

Proof (of Lemma 1). Let $X = (a_{ij})$, $Y = (b_{ij}) \in M_n(A)$ and define

$$(X, Y) = \sum_{i,j} (a_{ij}, b_{ji})$$

where $(\ ,\ )$ is also the inner product on A giving A symmetric. This form on $M_n(A)$ is clearly bilinear and symmetric. Let's see that it is associative. Let $Z = (c_{ij})$. Thus

$$(X, YZ) = \sum_{i,j} (a_{ij}, \sum_k b_{jk} c_{ki})$$

$$= \sum_{i,j,k} (a_{ij}, b_{jk} c_{ki})$$

while

$$(XY, Z) = \sum_{i,k} \left(\sum_j a_{ij} b_{jk} ,\ c_{ki} \right)$$

$$= \sum_{i,j,k} (a_{ij} b_{jk}, c_{ki})$$

so the form is associative.

Finally, we shall see that it is non-degenerate. Let a_{st} be a non-zero entry in X and choose $a \in A \ni (a_{st}, a) \neq 0$. Let W be the n by n matrix with only one non-zero entry, namely a, in the t, s position. Hence $(X, W) = (a_{st}, a) \neq 0$ as desired.

Proof (of Lemma 2). If $\varphi, \psi \in \mathrm{End}_A(Ae)$ then set

$$(\varphi, \psi) = (e\varphi, e\psi)$$

where the second form is the one for A. This is clearly bilinear and symmetric.

Let's prove that this form for $\mathrm{End}_A(Ae)$ is associative. Let $\zeta \in \mathrm{End}_A(Ae)$ as well. Choose elements $r, s, t \in A$ such that on Ae right multiplication by these elements coincides with φ, ψ, ζ, respectively. We can choose such elements as $A = Ae + A(1-e)$, and all endomorphisms of A are right multiplications. Hence

$$(\varphi\psi, \zeta) = (e(\varphi\psi), e\zeta)$$
$$= (ers, et)$$

But $er \in Ae$ so $ere = er$. Similarly $ese = es$ so

$$(\varphi\psi, \zeta) = (eres, et)$$
$$= (er, eset)$$

$$= (er, est)$$
$$= (\varphi, \psi\xi)$$

as required.

It remains to show that this form is non-degenerate. Suppose $(\varphi, \psi) = 0$ for all $\psi \in \mathrm{End}_A(Ae)$. Hence $(er, es) = 0$ for all $s \in A$ such that $Aes \subseteq Ae$. In particular, if $a \in A$ then $(er, e.ae) = 0$. That is, $(ere, ae) = 0$ or $(er, ae) = 0$, as $ere = er$ as we saw above. But thus $(er, Ae) = 0$; however, $(er, A(1-e)) = (A(1-e), er) = (A(1-e)e, r) = 0$ so $(er, A) = 0$ and $er = 0$ as required.

It may be possible to give a functorial proof using the criterion, told to us by Auslander, for A to be symmetric: the dualities D and $(, A)$ are isomorphic (the latter making sense inasmuch as A is certainly self-injective).

Using Auslander's functors try and get the following squares commutative and see this suffices: $(E = \mathrm{End}_A(P))$

$$\begin{array}{cccc} (P, \ ) : & \mathrm{mod}_A & \longrightarrow & \mathrm{mod}_{E^\circ} \\ & \downarrow D & & \downarrow D \\ (DP, \ ) : & \mathrm{mod}_{A^\circ} & \longrightarrow & \mathrm{mod}_E \end{array}$$

$$\begin{array}{ccccc} (P,\ ) & : & \mathrm{mod}_A & \longrightarrow & \mathrm{mod}_{E^\circ} \\ & & \Big\downarrow{\scriptstyle (\ ,A)} & & \Big\downarrow{\scriptstyle (\ ,A)} \\ ((P,A),\ ) & : & \mathrm{mod}_{A^\circ} & \longrightarrow & \mathrm{mod}_E \end{array}$$

We haven't tried to carry this through as the argument won't be any shorter.

We conclude with an immediate consequence, which may be well known:

<u>Corollary</u> If A and B are Morita equivalent algebras then if one is symmetric so is the other.

<u>Corollary</u> If k is a splitting field for A and A is the basic algebra of a block then $\mathrm{rad}\, A \supsetneq [A,A]$.

<u>Proof</u> A is symmetric, $A/\mathrm{rad}\, A$ is commutative and $[A,A]$ contains no ideal.

Algebras of type A_4

We are going to classify and study algebras A which have exactly three simple modules S_1, S_2, S_3 and projective covers with the following structures:

$$\begin{array}{ccc} S_1 & S_2 & S_3 \\ S_2\ S_3 & S_3\ S_1 & S_1\ S_2 \\ S_1 & S_2 & S_3 \end{array}$$

(and assume splitting field).

We're only interested in Morita equivalence classes, of course. Hence, we need only analyze the endomorphisms - acting on the right - of the direct sum of the three modules.

Let E_1, E_2, E_3 be the projections on the three projective modules so $E_1 + E_2 + E_3$ is "primitive decomposition" for the endomorphism algebra. Since all such expressions are conjugate anything we do with them is canonical.

Let T_{ij} be a non-trivial map of the i-th projective to the j-th regarded as an endomorphism of the sum, with T_{ii} having a simple image, $1 \le i, j \le 3$. Hence, these are well-defined up to scalar multiples.

What relations do they satisfy? Clearly, $E_i T_{jk}$ is $\delta_{ij} T_{jk}$ and the E_i's are orthogonal idempotents. Also $T_{ij} T_{kl}$ is zero unless $j = k$, $i \ne j$, $i = l$, in which case

$$T_{ij} T_{ji} = \lambda_{ij} T_{ii}$$

for some $0 \ne \lambda_{ij} \in k$.

Lemma $\lambda = \lambda_{12}\lambda_{21}^{-1}\lambda_{23}\lambda_{32}^{-1}\lambda_{31}\lambda_{13}^{-1}$ depends only on A.

<u>Proof</u>. We need only show that if we replace each T_{ij} by a non-zero scalar multiple then "λ" remains unchanged. Suppose we replace T_{ij} by $\mu_{ij} T_{ij}$, $\mu_{ij} \neq 0$. We get

$$(\mu_{12} T_{12})(\mu_{21} T_{21}) = \mu_{12}\mu_{21}\mu_{11}^{-1} \lambda_{12} (\mu_{11} T_1)$$
$$(\mu_{21} T_{21})(\mu_{12} T_{12}) = \mu_{21}\mu_{12}\mu_{22}^{-1} \lambda_{21} (\mu_{22} T_2)$$
$$\vdots$$

so λ goes to

$$\lambda \cdot (\mu_{12}\mu_{21}\mu_{21}^{-1}\mu_{12}^{-1}\mu_{11}^{-1}\mu_{22})(\quad)(\quad)$$
$$= \lambda \, \mu_{11}^{-1}\mu_{22}\mu_{22}^{-1}\mu_{33}\mu_{33}^{-1}\mu_{11} = \lambda.$$

<u>Theorem</u> Two algebras of A_4 type are Morita equivalent if, and only if, they have the same λ invariant.

<u>Proof</u>. It suffices to show the multiplication table above — and all possible change of bases — depends only on λ. It suffices to see that we can modify the T_{ij} as above so that $\lambda_{12} = \lambda_{21} = \lambda_{13} = \lambda_{31} = \lambda_{23} = 1$ so $\lambda_{32} = \lambda^{-1}$.

First, we can get $\lambda_{12} = 1$ by changing T_{12}. $\therefore T_{12}, T_{21}, T_{11}$ now fixed. Second, get $\lambda_{13} = 1$ by changing T_{13} so T_{13}, T_{31} also fixed. Third, get $\lambda_{21} = 1$ by modifying T_{22}. $\therefore T_{22}$ is also fixed. Fourth, get $\lambda_{31} = 1$ by changing T_{33} and now keeping T_{33} fixed. Fifth, get $\lambda_{23} = 1$ by changing T_{23}.

Theorem For each $0 \neq \lambda \in k$ there is an algebra of type A_4 and invariant λ.

Proof. The generators and relations we have give the desired algebra.

Theorem If A is an algebra of type A_4 with invariant λ then A is symmetric if, and only if, $\lambda = 1$.

Proof In view of the result of the last section, we need only show the basic algebra is symmetric if, and only if, $\lambda = 1$. (When the characteristic of k is two then kA_4 is symmetric so we need only half of this statement).

Now the annihilator of T_{ii} is of codimension one so $(E_i, T_{ii}) = \sigma_i \neq 0$, where we are assuming symmetry with inner product $(\ ,\ )$. Hence,

$$\lambda_{12}\sigma_1 = (E_1, \lambda_{12} T_{11}) = (E_1, T_{12}T_{21}) = (T_{12}, T_{21})$$
$$= (T_{21}, T_{12}) = (E_2, \lambda_{21} T_{22}) = \lambda_{21}\sigma_2 .$$
$$\therefore\ \lambda_{12}\lambda_{21}^{-1} = \sigma_1^{-1}\sigma_2 \text{ so } \lambda = \sigma_1^{-1}\sigma_2 \cdot \sigma_2^{-1}\sigma_3 \cdot \sigma_3^{-1}\sigma_1 = 1.$$

Conversely, assume $\lambda = 1$ so we can assume, by the proof above, that all $\lambda_{ij} = 1$. Let's calculate the form and then see that the result works. The above calculation shows $(E_i, T_{ii}) = (T_{ij}, T_{ji})$ all i, j so we can assume these are all 1 after multiplying by a scalar. We shall now see that all the other inner products vanish so the space is an orthogonal direct sum of six two dimensional spaces.

First, if $T_{ij}T_{kl} = 0$ then certainly $(T_{ij}, T_{kl}) = 0$ so three of these two dimensional spaces give an orthogonal direct sum. Now each T_{ii} annihilates all T_{ij} while if $j \neq i$ $(E_i, T_{ij}) = (T_{ij}, E_i) = 0$ as $T_{ij} E_i = 0$. Hence, the six-dimensional space we have constructed is orthogonal to the rest. The remaining calculations are as trivial.

To see this associative, define a linear functional on the algebra by

$$\varphi(\alpha_{11} T_{11} + \alpha_{22} T_{22} + \alpha_{33} T_{33} + \dots \quad) = \alpha_{11} + \alpha_{22} + \alpha_{33}.$$

It's easy to see that on basis elements x, y, $(x, y) = \varphi(xy)$. Hence, we're done.

We note, in closing, that we probably could have saved a little calculation by not introducing T_{ii} and just giving instead a relation between $T_{ij} T_{ji}$ and $T_{ji} T_{ij}$.

Commutators and Morita equivalence

If A_λ is the algebra of the last section for invariant λ then $\dim A_\lambda / [A_\lambda, A_\lambda] = \begin{cases} 3 & \lambda \neq 1 \\ 4 & \lambda = 1 \end{cases}$.
If we knew this was preserved under Morita equivalence then we would have another argument that only A_1 can come up for group algebras. Indeed let K, R, k be as usual in block theory and let B be a block algebra. Then $\dim_k B/[B,B] \geq k(B) = \dim_k Z(B)$. For if $\hat{B}$ is the corresponding block algebra over R then $\operatorname{rank}_R \hat{B}/[\hat{B},\hat{B}] = \dim_K K\otimes\hat{B}/[K\otimes\hat{B}, K\otimes\hat{B}]$ which is $k(B)$ and $\dim_k B/[B,B] \geq \operatorname{rank}_R \hat{B}/[\hat{B},\hat{B}]$.

We shall now establish what we desire:

Theorem. If A and B are Morita equivalent algebras then

$$\dim_k A/[A,A] = \dim_k B/[B,B].$$

For this, k is again just some field.

First, let φ be a linear functional on A with kernel containing $[A,A]$. This defines a symmetric bilinear and associative (but not necessarily non-degenerate) form on A in the usual way: $(a_1, a_2) = \varphi(a_1 a_2)$. And as usual we get a one-to-one correspondence between such functionals and such forms.

<u>Lemma 1</u> $\dim_k A/[A,A] = \dim_k M_n(A)/[M_n(A), M_n(A)]$.

<u>Proof</u>. Let φ be a linear functional for A as above. Define $\Phi : M_n(A) \to k$ as follows:

$$\Phi((a_{ij})) = \varphi(a_{11} + \cdots + a_{nn})$$

(i.e. $\Phi(X) = \varphi(\operatorname{tr} X)$). This is certainly linear and the map φ to Φ is linear and injective. Let's see Φ has the desired property for $M_n(A)$. Let $X = (a_{ij})$, $Y = (b_{ij})$. Thus $\Phi(XY) = \varphi(\sum_{i,j} a_{ij} b_{ji}) = \sum \varphi(a_{ij} b_{ji})$ while $\Phi(YX) = \sum \varphi(b_{ji} a_{ij})$ as required. Hence $\dim_k M_n(A)/[M_n(A), M_n(A)] \geq \dim_k A/[A,A]$.

It remains to demonstrate the reverse inequality. Let Φ be a linear functional of the right sort for $M_n(A)$ and define $\varphi : A \to k$ via $\varphi(a) = \Phi\left(\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}\right)$. So φ is of the right sort and the map sending Φ to φ is linear. It remains only to see that if $\varphi = 0$ then $\Phi = 0$. But every matrix is a linear combination of conjugates of matrices of the form $\begin{pmatrix} a & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & & & \end{pmatrix}$ so we're done as Φ is constant on conjugates.

The next result is clearly all we require:

<u>Lemma 2</u> If e is an idempotent of A such that every isomorphism class of indecomposable projective A-modules appears in a decomposition of Ae and $E = \operatorname{End}_A(Ae)$ then $\dim_k E/[E,E] \geq \dim_k A/[A,A]$.

Proof. Here we'll use the forms instead of the linear functionals. If $(\, , \,)$ is a form on A then we have defined, two sections ago, a form on E and this correspondence is linear. (We had more assumptions then but the arguments did establish this.)

It remains only to see that if the form is zero on E then the defining form on A is also zero. Let $\varphi \in E$ and suppose $e\varphi = er$. Now, by assumption, if $\psi \in E$ and $e\psi = es$ then $(er, es) = 0$ (the form on A). If $t \in A$ then

$$(er, t) = (e \cdot er \cdot e, t) = (t, e \cdot er \cdot e) = (te \cdot ere)$$
$$= (ere, te) = (er, ete) = 0$$

as $e \cdot te = (et)e \in Ae$. Thus, er is in the radical of the form for A.

Hence, eAe is in the radical of the form. Let P and Q be two indecomposable projective A-modules. Isomorphic copies appear in a decomposition of Ae. Hence, there is an element of eRe mapping the first copy to the second in any desired way, zero on the rest of a decomposition of Ae and certainly zero on $A(1-e)$. This element and all of its conjugates are in the radical of the form for A. But these conjugates span A so we are indeed finished.

We shall conclude by finishing up the work of the motivating discussion at the beginning of this section and put it together with the main result. We get

<u>Theorem</u> If A is Morita equivalent with a block algebra then

$$\dim_k A/[A,A] = \dim_k Z(A).$$

<u>Proof</u> It suffices to show that $[RG, RG]$ is an R-pure submodule of RG. But it is easily seen to be spanned by all $g-h$, where g and h are conjugate and so it's easy to give a basis for RG which demonstrates this result.

Remark: This is alright for all fields by using field extensions and the invariance of $\dim_k(A/[A,A])$ and $\dim_k Z(A)$ under such extensions.

We also want to discuss at least the prospects for a functorial-module-theoretic proof instead of the above argument. Let $\mathcal{F}$ be the vector space of symmetric bilinear associative forms on A.

<u>Lemma</u> $\mathrm{Hom}_{A\otimes A}(A, D(A)) \cong \mathcal{F}$.

Here we are regarding A and $D(A) = \mathrm{Hom}_k(A, k)$ as A-bimodules, i.e. $A \otimes A$ modules.

<u>Proof</u>. If $(\, , \,) \in \mathcal{F}$ define $\varphi : A \to D(A)$ via $\varphi(a) = (a, \;)$. The correspondence $(\, , \,)$ to φ is linear. It remains to see that φ is a module homomorphism and that the correspondence $(\, , \,)$ to φ is one-to-one and onto.

First, if $x, a, y \in A$ then $x \varphi_a y = \varphi_{xay}$ inasmuch as $(a, y r x) = (xay, r)$ whenever $r \in A$. Next, if $\varphi = 0$ then clearly $(\, , \,)$ is also zero. Finally, let $\varphi \in \mathrm{Hom}_{A \otimes A}(A, D(A))$. Define $(x, y) = \varphi_x(y)$. As soon as we show that $(\, , \,) \in \mathcal{F}$ then we will have established all the required facts. Clearly $(\, , \,)$ is bilinear. It is also commutative:

$$(y, x) = \varphi_y(x) = \varphi_{1 \cdot y}(x) = \varphi_1(yx)$$
$$(x, y) = \varphi_x(y) = \varphi_{x \cdot 1}(y) = \varphi_1(yx).$$

And it is associative:

$$(xy, z) = \varphi_{xy}(z) = \varphi_{x \cdot 1 \cdot y}(z) = \varphi_1(yzx),$$
$$(x, yz) = \varphi_x(yz) = \varphi_{x \cdot 1}(yz) = \varphi_1(yzx).$$

Here's another argument suggested by Auslander: $\mathrm{Hom}_{A \otimes A}(A, D(A))$ is determined by the images of 1, which will be "fixed-points" so we get $\mathrm{Hom}_{A \otimes A}(A, D(A))$ isomorphic with $D(A/[A, A])$ directly.

We get the known result directly from the above.

<u>Th</u>. A is symmetric if, and only if $A \simeq D(A)$ as bimodules

Now the rest of a module-theoretic proof might go as follows: $\mathrm{Hom}((\ ,A), D(\ )) \simeq D(A/[A,A])$ as k-spaces where the first term consists of the natural transformations between the functors $(\ ,A)$ and $D(\ )$ from mod_A to mod_{A°; hence, use projectives to define Morita equivalence and show a suitable diagram commutes.

The first statement is plausible, looking like some sort of Monade lemma:

$$\mathrm{Hom}((\ ,A), D(\ )) \simeq \mathrm{Hom}_{A\otimes A}(A, D(A)).$$

Lets make a few of the steps toward showing this.

We will show how to construct a natural transformation from $(\ ,A)$ to $D(\ )$ from a linear functional φ on A having $[A,A]$ in its kernel. If $U \in \mathrm{mod}_A$ define a map

$$\mathrm{Hom}_A(U,A) \to D(A)$$

by sending $f \in \mathrm{Hom}_A(U,A)$ to $\varphi \circ f \in \mathrm{Hom}_k(U,k) = D(A)$. This is a natural transformation: if $\psi \in \mathrm{Hom}_A(U,V)$, $V \in \mathrm{mod}_A$, need following to commute,

$$\begin{tikzcd}
\mathrm{Hom}_A(U,A) \arrow[r] & D(U) = \mathrm{Hom}_k(U,k) \\
\mathrm{Hom}_A(V,A) \arrow[r] \arrow[u, "(\ ,A)(\psi)"] & D(V) = \mathrm{Hom}_k(V,k) \arrow[u, "D(\psi)"']
\end{tikzcd}$$

But this is O.K.

$$\begin{tikzcd}
g \circ \psi \arrow[r] & \varphi \circ g \circ \psi \\
g \arrow[r] \arrow[u] & \varphi \circ g \arrow[u]
\end{tikzcd}$$

Next, need to know that all such natural transformations arise this way. Let φ be the image of $1 \in \mathrm{Hom}_A(A, A)$

$$1 \in \mathrm{Hom}_A(A,A) \to \varphi \in \mathrm{Hom}_k(A,k).$$

Need φ has certain properties and that it defines the natural transformation as above.

Brauer induction and the symmetric groups

We shall give another proof of Brauer's theorem on induced characters. We showed that the theorem holds for nilpotent groups as part of our previous argument. Hence, we need only show that every character is an integral linear combination of characters induced from nilpotent subgroups.

By Mackey's theorem, it suffices to prove the result holds for the symmetric groups Σ_n; we do this by induction on n, the case $n=1$ being trivial. For each partition $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots$ of n let π_λ be the permutation character corresponding to the Young subgroup $\Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots$. By induction, and since direct products of nilpotent groups are nilpotent, each π_λ, $\lambda \neq n$, is in the ideal of the character ring R_n of Σ_n of characters induced in the desired way. But the π_λ form a $\mathbb{Z}$-basis for R_n so the ideal we are after is already quite large.

There is a homomorphism of R_n onto $\mathbb{Z}$ given by evaluation on $x = (12 \cdots n)$. All π_λ, $\lambda \neq n$ are in the kernel so the kernel is the set of all integral linear combinations of the π_λ, $\lambda \neq n$. Hence, to prove the result it suffices to show there is χ in the ideal with $\chi(x) = 1$.

Let p be a prime divisor of n and let H be the permutation wreath product $F_{p(p-1)} \wr \Sigma_m$, $n = mp$, where $F_{p(p-1)}$ is the Frobenius group of order $p(p-1)$ on p letters. We assert that H contains a unique conjugacy class of n cycles.

Let's first see that this fact enables us to complete the proof. Let π be the permutation character of Σ_n corresponding to H. Our claim implies that $\pi(x) = 1$ for an n-cycle x of H. Indeed, $C(x) = \langle x \rangle \subseteq H$ and $H \supseteq N(\langle x \rangle)$ as all the generators of $\langle x \rangle$ are conjugate in H. Hence, if $g \in \Sigma_n$ and $g x g^{-1} \in H$ then $g \in H$. Moreover, π is in the ideal in question. For the principal character of H is an integral linear combination of characters induced from subgroups of the form E/B, B the "base group" of H and E/B elementary. Since we showed an independent proof of Brauer's theorem for solvable groups before, it now does follow that π is in the ideal. It remains now only to establish the stated property of H.

First, we require some notation. Let the partition of n used in constructing H be

$$\{1, 2, \ldots, p\} \cup \{p+1, \ldots, 2p\} \cup \cdots \cup \{\ldots mp\},$$

let $x_i = ((i-1)p+1, (i-1)p+2, \ldots, ip)$, F_i be the normalizer of $\langle x_i \rangle$ in the symmetric group on the i-th subset so $F = F_1 \cdots F_p$, a direct product, is the "base group" of H. Let Σ_m be embedded in H, so $H = F \cdot \Sigma_m$, by its action on $\{1, p+1, \ldots, (m-1)p+1\}$

"copied" on the "translates" of this set.

Let $x \in \Sigma_n$ be an n-cycle chosen so that $x^m = x_1 x_2 \cdots x_m = z$. Thus, $x \in C(z)$; but it's clear that $C(z) \subseteq H$ so $x \in H$ and H has an n-cycle.

Finally, let y be an n-cycle lying in H. Hence, y permutes the m sets whose partition was used above. Thus, $y^m = x_1^{a_1} \cdots x_m^{a_m}$, $0 < a_i < p$, $1 \le i \le m$. Hence, y^m is conjugate with z in F: choose $f \in F$ with $(y^f)^m = z$. Thus, choosing $g \in \Sigma_n$ so that $y^{fg} = x$ we have $(y^{fg})^m = z$ so fg conjugates $z = y^m$ to $z = x^m$ and $fg \in C(z) \subseteq H$ and the proof is complete.

Tame intersections in classical groups

First, let's deal with the general linear groups and then see how to modify our argument to handle the other groups. Let V be an n-dimensional vector space over the field with $q = p^f$ elements. Let $B \supseteq U$ be the Borel subgroup and its unipotent radical for a flag of V. Our result is stronger than our title suggests:

Proposition If Q is a subgroup of U and $Q = O_p(N(Q))$ then $Q = O_p(P)$ for a parabolic subgroup P containing B.

Proof. Let $0 = V_0 \subset V_1 \subset \cdots \subset V$ be the "fixed-point flag" of Q on V so V_1 is the fixed points of Q and V_2/V_1 is the fixed points of Q on V/V_1 and so forth. Let P be the parabolic corresponding to this flag so $P \supseteq B$ is still to be shown. Now $P \supseteq N(Q)$ as $N(Q)$ stabilizes the fixed-point flag of Q. Since $Q \subseteq O_p(P)$ — as $O_p(P)$ is the subgroup of P with trivial action on the quotients of the subspaces — we have $N(Q) \cap O_p(P) \subseteq O_p(N(Q)) = Q$ so $Q = O_p(P)$ as required.

Thus, the only subspaces of V left invariant by Q are the spaces of its fixed point flag and those subspaces between two spaces of the flag. But $Q \subseteq U$ so Q leaves invariant the flag of U. Hence, the fixed-point flag of Q is part of that of U so $P \supseteq B$ as desired. (This last step as the flag of U goes up one dimension at a time.)

Now suppose (,) is a form, one of the classical ones on V, and we let G be its automorphism group. We use flags of isotropic subspaces instead of flags of subspaces to imitate the above.

We have to show that there is a maximal isotropic subspace left invariant by Q. We let V_1 be the radical of the fixed-points of Q so $V_1 \neq 0$ as otherwise the fixed points would have an orthogonal complement which would also be Q invariant. Next let's look at $V_1^{\perp}$ so we get an induced form on $V_1^{\perp}/V_1$ so we have some non-zero radical for the "new" fixed points, and so on.

The rest seems to go as before.

Counting characters and modules

A study of various conjectures on numbers of characters and simple modules in blocks suggests the following as plausible at least as a provisional version of a correct result. With the usual notation:

Conjecture There is a one-to-one correspondence between simple modules and indecomposable modules with simple Green correspondents and this correspondence is compatible with the Brauer correspondence.

Let's see what this means. Let U be such an indecomposable module with Green correspondent V so V is a kL-module, $L = N(Q)$, Q a p-subgroup (perhaps $Q = 1$) of G. Thus V has vertex Q so L/Q has V as a projective simple module. In particular, Q is a Sylow intersection in L, Q is a tame intersection in G and $Q = O_p(L) = O_p(N_G(Q))$. We are claiming there is a simple kG-module S belonging to U, or, that is, to V. V is sort of the "weight" of S. Moreover, we are claiming that the block containing S corresponds to the block containing V under the Brauer map.

Hence, we are putting the simple kG-modules in one-to-one correspondence with certain of the kG-modules with trivial vertices, the "permutation components."

Let's explore some consequences of the conjecture as our first task. Fix a block B of G with defect group D and corresponding block b of $N(D)$.

<u>Consequence 1</u> $\ell(B) \geq \ell(b)$.

<u>Proof</u>. Let φ be the canonical character of $D\,C(D)$ so $\ell(b)$ is the number of Brauer characters of $N(D)$ over φ. Let T be the inertial group of φ so $|T : D\,C(D)|$ is a p'-number. Hence, by Clifford theory, $\ell(b)$ is the number of characters — ordinary or Brauer — of $T/D\,C(D)$. But these clearly correspond to <u>some</u> of the Green correspondents determining the simples of B, so we're done.

<u>Consequence 2</u> If B is TI then $\ell(B) = \ell(b)$.

<u>Proof</u>. It suffices to show there are no indecomposable modules to count other than those in the previous proof. For this, we need only show that if $Q \subsetneq D$ then there are no indecomposables to be counted with vertex Q. Hence, we need only demonstrate that $Q \subsetneq O_p(N(Q))$. We can assume $(Q, b_Q) \subset (D, b_D)$ with the usual subpair notation. Let $Q_1 = N_D(Q, b_Q)$ so $Q_1 \supsetneq Q$. Let $(Q, b_Q) \subset (Q_1, b_{Q_1})$ so if $x \in N(Q, b_Q)$ then $(Q, b_Q) \subset (Q_1^x, b_{Q_1}^x)$. As B is TI this forces

$Q_1 = Q_1^x$, $b_1 = b_1^x$. Hence, $Q_1 \lhd N(Q, b_Q)$ so $Q \subsetneqq O_p(N(Q, b_Q))$. But $N(Q, b_Q) \lhd N(Q)$ so $O_p(N(Q, b_Q)) \lhd N(Q)$ and we're done.

<u>Consequence 3</u> If B is nilpotent then $\ell(B) = 1$.

(a result of Broué and Puig.) We need a preliminary result:

<u>Lemma!</u> If N is a normal subgroup of index p in the group H and S is a projective simple kH-module then

$$S_N \simeq S_1 \oplus \dots \oplus S_p$$

a direct sum of p distinct projective simple kN-modules.

<u>Proof</u>. We have $S_N \simeq m(S_1 \oplus \dots \oplus S_t)$ with the obvious notation. Thus,

$$\begin{aligned} m^2 t &= \dim_k \mathrm{Hom}_{kN}(S_N, S_N) \\ &= \dim_k \mathrm{Hom}_{kH}(\mathrm{Ind}_N^H(S_N), S) \\ &= \dim_k \mathrm{Hom}_{kH}(k.H/N \otimes S, S) \\ &= \dim_k \mathrm{Hom}_{kH}(\overbrace{S \oplus \dots \oplus S}^{p}, S) \\ &= p \end{aligned}$$

so $m = 1$, $t = p$ as required.

<u>Proof (of Consequence 3)</u>. Let Q be a proper subgroup of D. By the previous arguments we need only show we get no contribution towards $\ell(B)$ from $N(Q)$. Let $(Q, b_Q) \subset (D, b_D)$ as before. We have the picture

$$\begin{tikzcd}[column sep=small]
 & N(Q) \arrow[d, dash] & \\
 & N(b_Q) \arrow[d, dash] & \\
 & QC(Q) \arrow[dl, dash] \arrow[dr, dash] & \\
Q & & C(Q) \quad b_Q
\end{tikzcd}$$

Before proceeding with the proof let's note something that follows from Q contributing to B:

<u>Lemma 2</u>. $\ell(b_Q) = 1$ and b_Q has defect Q.

<u>Proof</u>. The simple module with vertex Q implies that b_Q has a simple module which as $Q.C(Q)$-module has vertex Q. This implies the result: the blocks of $C(Q)$ and $C(Q)/Q \cap C(Q) = C(Q)/Z(Q)$ are the "same" as $Z(Q) \subseteq Z(C(Q))$; hence they are the "same" as the blocks of $Q\,C(Q)$.

Note: We haven't used the nilpotence assumption of Consequence 3.

Back to the proof. Let T be the simple $kQC(Q)$ module in b_Q. Since $|H : Q\,C(Q)|$ is a power of p, where $H = N(b_Q)$, T extends to H, as T is invariant in H as b_Q is and as T is the unique simple module in b_Q. Since $H/QC(Q)$ has only one simple module*, this extension S must be a projective module for H/Q <u>and</u> a constituent of the restriction to H of the simple

* it's a p-group by the nilpotence of B.

module which is assumed to exist in our hypothesis.
Let N be a normal subgroup of index p in H, $N \supseteq QC(Q)$.
Thus, Lemma 1, guarantees that S_N is no longer simple, contradicting the simplicity of T.

It's open at this point if the conjecture implies the conjectured results on abelian blocks and controlled blocks. As a guess, these are consequences.

There may also be indirect consequences by way of analogy. In particular, we are thinking of the McKay conjecture. Let's take a brief "detour" to discuss this possibility.

Part of the problem to be solved in dealing with our "weight" conjecture is the following. Let P be a Sylow p-subgroup of G; we need to associate to each simple $kN(P)$-module a simple kG-module, i.e. a simple kG-module to the Green correspondent of the simple $kN(P)$-module. In our reduction for the McKay conjecture we saw we led to correspond characters of p'-degree to certain R-forms of the irreducible representations of $N(P)/P'$. That is, to correspond certain characters to certain Green correspondents. It seems plausible that a similar idea is needed to settle both of these questions. It would be nice to be able to "prove" this by a theoretical connection.

As our next task, let's examine some cases when we can verify the truth of our conjecture. We begin with $GL(n, q)$, $q = p^f$, the prime p and we ignore the block part of the conjecture. We shall simply make a count of the simple Green correspondents and show that there are $(q-1)q^{n-1}$ which is the number of simple modules for $GL(n, q)$.

In view of the result of the previous section we need only count the number of simple modules, for each parabolic subgroup containing the Borel subgroup B, which are projective as modules for the parabolic modulo its unipotent radical (largest normal p-subgroup), and add up these numbers.

However, if the parabolic modulo its radical is a direct product of r $GL(*, k)$'s (including $GL(1,k)$'s) then the number of simples relevant is $(q-1)^r$. There are $\binom{n-1}{r-1}$ such parabolics so the sum we are after is

$$\sum_{r=1}^{n} \binom{n-1}{r-1} (q-1)^r = (q-1) \sum_{i=0}^{n-1} \binom{n-1}{i} (q-1)^i$$
$$= (q-1)\,(1 + (q-1))^{n-1}$$
$$= (q-1)\, q^{n-1}$$

just as desired.

It's also not hard to look at Carter and Lusztig's paper on these representations of $GL(n, q)$ (basically the principle series modulo p) and see that their parametrization of the simples can be given the way we want it so we have more than a count.

Next suppose that G is p-nilpotent, $G = PK$ with P a Sylow p-subgroup of G and $K = O_{p'}(G)$. First, suppose that φ is an irreducible character of K with stabilizer Q in P and ψ is the corresponding character of $C_K(Q)$. Hence, ψ is the unique character of $C_K(Q)$ appearing in $\varphi_{C_K(Q)}$ with multiplicity not divisible by p. We assert that Q is the stabilizer of ψ in $N_P(Q)$. Indeed, say $x \in N_P(Q)$ and $\psi^x = \psi$. Thus, φ and φ^x are both Q invariant and both correspond to ψ as above, contradicting properties of p-nilpotent groups.

Since Q is the stabilizer of ψ in $N_P(Q)$, ψ determines a simple $kN_G(Q) = k\,N_P(Q)\,C_K(Q)$ module which is projective modulo Q. The conjugacy class of φ determines a simple module of G uniquely up to isomorphism so we have a correspondence from simple kG-modules to the desired indecomposables. Properties of the above construction and of p-nilpotent groups clearly show this correspondence is one-to-one. It is necessary only to show that it is onto.

However, let Q be a subgroup of P so $N_G(Q) = N_P(Q)\,C_K(Q)$. A simple module of the desired sort corresponds to a character ψ of $C_K(Q)$ with stabilizer Q in $N_P(Q)$. Using the p-nilpotent group <u>QK</u> let φ be the character of K corresponding to ψ. It's enough to see that the stabilizer of φ in P is Q. But if it is larger, then it's larger in $N_P(Q)$ so Q is not the stabilizer of φ in $N_P(Q)$. Hence, we can choose $x \in N_P(Q)$, $\varphi^x = \varphi$

but $\varphi^x \neq \varphi$. Hence φ^x and φ occur in $\mathcal{C}_{C_K(Q)}$ with the same multiplicity, a contradiction.

We now turn to the symmetric groups. We will be interested in the principal block so we first record a useful result:

Lemma 3. If P is a p-subgroup of G then $N(P)$ has a "weight" for P and the principal block if, and only if, the following conditions hold:

a) $N(P)$ is p-constrained;

b) $O_{p',p}(N(P)) = P\,O_{p'}(N(P))$;

c) $N(P)/O_{p',p}(N(P))$ has a simple projective module.

If this holds, the simple projective modules of $N(P)/O_{p',p}(N(P))$ are the "weights" for P and the principal block.

Proof. First, suppose the conditions are satisfied. Thus, the simple modules for $N(P)/O_{p'}(N(P))$ are the ones in the principal block. The rest follows easily including the last assertion of the statement.

Conversely, suppose S is a weight as described. Thus, by Clifford's theorem, it follows that $P\,C(P)$ has a projective simple module in the principal block. Thus, $P\,C(P) = P\,O_{p'}(N(P))$ is a direct product. Since S is in the principal block, it has $O_{p'}(N(P))$ in its kernel so $N(P)/O_{p'}(N(P))$ has a simple projective with vertex $P\,O_{p'}(N(P))/O_{p'}(N(P))$. Thus, $P\,O_{p'}(N(P)) = O_{p',p}(N(P))$. Finally, any element that centralizes $P\,O_{p'}(N(P))/O_{p'}(N(P))$ centralizes P as $P\,O_{p'}(N(P)) \cong P \times O_{p'}(N(P))$.

We want to now verify the conjecture for the principal p-block of Σ_{p^2}; it will turn out that the heuristic calculation we made in the last volume is correct.

Let P be a non-identity subgroup satisfying the conditions of Lemma 3. We shall show that P is, up to conjugacy, the Sylow p-subgroup, the "base group" of $Z_p \wr Z_p$ or the regular subgroup which is elementary abelian. Let's first see that this suffices for the proof. The weights from the regular subgroup are $p-1$ in number, coming from $GL(2,p)$ (see last volume), the weights of a Sylow p-subgroup are $(p-1)^2$ in number and the base group gives the number of characters in blocks of defect zero in $Z_{p-1} \wr \Sigma_p$. That is, $k(Z_{p-1} \wr \Sigma_p) - p(p-1)$ so we want, as in the last volume, $k(Z_{p-1} \wr \Sigma_p) = \ell(B_0(\Sigma_{p^2}))$. (The term $p(p-1)$ subtracted corresponds to the principal p-block of $Z_{p-1} \wr \Sigma_p$ – the normalizer of the Sylow p-subgroup of that group is $Z_{p-1} \times F_{p(p-1)}$ (Frobenius group)). Hence, we only require the following for $n = p$:

<u>Lemma 4</u> $k(Z_{p-1} \wr \Sigma_n) = \ell(B_0(\Sigma_{np}))$.

<u>Proof</u>. Let $\lambda_1, \dots, \lambda_{p-1}$ be the distinct linear characters of Z_{p-1}. If λ is a linear character of the "base group" of $Z_{p-1} \wr \Sigma_n$ then its "coordinates" are given by various λ_i; say λ_i occurs b_i times. This determines λ up to conjugacy in $Z_{p-1} \wr \Sigma_n$. By Clifford theory, we now get that

$$k(Z_{p-1} \wr \Sigma_n) = \sum_{\substack{(b_1, \dots, b_{p-1}) \\ \Sigma b_i = n}} k(\Sigma_{b_1}) \cdots k(\Sigma_{b_{p-1}})$$

$$= \sum_{b_1 + \cdots + b_{p-1} = n} \pi(b_1) \cdots \pi(b_{p-1})$$

$$= \ell(B_0(\Sigma_{p^2}))$$

by general formulae (see Robinson, e.g.).

It remains to pin down P as stated. The case $p = 2$ is easily handled by inspection so we henceforth assume $p > 2$. Thus, if P has any fixed points, so it has at least p, then its centralizer is not right. Hence, either all the orbits of P are of size p or P is transitive.

First, assume there are p orbits, so P is inside an obvious base group. The subgroup of the normalizer of P which stabilizes these orbits normalizes the projection of P on each symmetric group on each orbit. Hence, $N(P)$ is contained in the normalizer of the base group so P is the base group or the centralizer of P is wrong – as it picks up the base group.

Last, suppose P is transitive. If it is of order p^2 then it is regular and there are two cases, each easily handled by inspection. If $|P| > p^2$ then P is not in a base group. $\therefore$ P has a unique maximal subgroup which is abelian and has non-identity elements with fixed points; call this Q so Q is the intersection of P and a base group. Thus, $N(P) \subseteq N(Q)$. But $N(Q)$ permutes the projections of Q on each of its p orbits. (There are p as P contains the center of a Sylow p-subgroup) Hence, $N(Q)$ normalizes the group generated by these projections, i.e. a base group. Thus $N(P)$ is in the normalizer of a base group so the base group must be in P, by the structure $N(P)$ must have, so P is a Sylow p-subgroup, as required.

We want to suggest a direction for a possible proof. It would be good if there were an abelian category, with simple objects corresponding to the weights, stable equivalent with the category of kG-modules. This is probably too much. But the idea is to construct, up to stable equivalence, the category of kG-modules from local information.

Let's see how to compute

$$\dim_k \overline{\mathrm{Hom}}_{kG}(k, U)$$

for an indecomposable kG-module U with Green correspondent V, a kL-module. We proceed by induction on the order of the vertex of U. Now

$$\mathrm{Ind}_L^G V \cong U \oplus E \oplus P$$

where P is projective, E is projective-free, so, by induction, $\dim_k \overline{\mathrm{Hom}}_{kG}(k, E)$ is known and, of course, $\dim_k \overline{\mathrm{Hom}}_{kG}(k, P) = 0$. But

$$\dim_k \overline{\mathrm{Hom}}_{kG}(k,U) + \dim_k \overline{\mathrm{Hom}}_{kG}(k,E) + \dim_k \overline{\mathrm{Hom}}_{kG}(k,P)$$
$$= \dim_k \overline{\mathrm{Hom}}_{kG}(k,\ U\oplus E\oplus P)$$
$$= \dim_k \overline{\mathrm{Hom}}_{kG}(k,\ \mathrm{Ind}_L^G V)$$

But $\mathrm{Hom}_{kG}(k, \mathrm{Ind}_L^G V) \cong \mathrm{Hom}_{kL}(k, V)$ and moreover, a map from k to $\mathrm{Ind}_L^G V$ is projective if and only if the corresponding map of k to V is projective. For if the kL map factors through a projective then the induced projective is a kG-projective, while if the kG-map factors through a projective then we're alright as the corresponding map is inside:

$$\begin{tikzcd} & \boxed{\text{proj}} \arrow[dr] & \\ k \arrow[ur] \arrow[rr] & & \mathrm{Ind}_L^G(V) = U + \cdots \end{tikzcd}$$

Hence,

$$\dim_k \overline{\mathrm{Hom}}_{kG}(k,U) + \dim_k \overline{\mathrm{Hom}}_{kG}(k,E)$$
$$= \dim_k \overline{\mathrm{Hom}}_{kL}(k,V).$$

Note that E is locally specified by the Burry - Carlson theorem! This is needed!

Another possible approach: generalize the Brauer map to non-central idempotents, presumably to those which are central modulo the radical. This looks like a good possibility.

Cyclic defect (cont.)

Again consider the case of a cyclic Sylow p-subgroups P of the groups G, $P = C_G(x)$ for each $1 \neq x \in P$, $|N(P):P| = e$. Let k be an algebraically closed field of characteristic p. We want to give another proof of the fact that G has at most e non-projective simple modules.

First, we detail something mentioned before.

Lemma. If V is a non-projective indecomposable $kN(P)$-module and $n \in \mathbb{N}$ then $\operatorname{Ext}^{2n+1}_{kN(P)}(k, V) \neq 0$ if, and only if λ^{n+2} is the top of V while $\operatorname{Ext}^{2n+2}_{kN(P)}(k, V) \neq 0$ if, and only if λ^{n+2} is the socle of V.

Here λ is the usual canonical one-dimensional $kN(P)$-module.

Proof (sketch) Inspect the minimal resolution of k and calculate.

Now to prove what we claimed above all we need is the following result:

Proposition. If S and T are simple kG-modules with $\operatorname{Ext}^n_{kG}(k, S) \neq 0$ and $\operatorname{Ext}^n_{kG}(k, T) \neq 0$ then $S \simeq T$.

Proof. The case $n = 0$ is trivial so assume $n > 0$. Let U and V be the Green correspondents of S and T, respectively, so

$$\mathrm{Ext}^n_{kN(P)}(k, U) \neq 0, \quad \mathrm{Ext}^n_{kN(P)}(k, V) \neq 0.$$

If n is odd then U and V have the same tops while if n is even then U and V have the same socle. In either case there is a non-zero map of one to the other which does not factor through projectives. Hence, the same holds for S and T, a contradiction.

Algebras of type A_5

We're interested in algebras that have projectives like the ones in the principal 2-block of A_5. So we work over an algebraically closed field k of characteristic 2 and we have an algebra A with three simple modules S_0, S_1 and S_2 with projective covers as follows:

$$
\begin{array}{ccc}
 & S_0 & \\
S_1 & & S_2 \\
S_0 & & S_0 \\
S_2 & & S_1 \\
 & S_0 &
\end{array}
\qquad\qquad
\begin{array}{c}
S_1 \\ S_0 \\ S_2 \\ S_0 \\ S_1
\end{array}
\qquad\qquad
\begin{array}{c}
S_2 \\ S_0 \\ S_1 \\ S_0 \\ S_2
\end{array}
$$

As for A_4, the endomorphism algebra of the direct sum of these three is what we have to pin down. It's eighteen dimensional, has a basis of three idempotents E_0, E_1, E_2 (obvious maps) and fifteen other maps:

$$T^l_{00},\ T^r_{00},\ T_{00},\ T^t_{01},\ T_{01},\ T^t_{02},\ T_{02}$$
$$T^l_{10},\ T^r_{10},\ T_{11},\ T_{12},\ T^l_{20},\ T^r_{20},\ T_{21},\ T_{22}$$

where T^*_{ij} maps the cover of S_i to the cover of S_j and where t = top, r = right, l = left (so the maps are the obvious ones.

Now we have to determine the multiplication table and see how canonical it is under allowable transformations. We list all non-zero products among the T's in the following way. An entry means the product is a non-zero multiple of the entry while a slash indicates a zero product.

	T_{00}^l	T_{00}^r	T_{00}	T_{01}^t	T_{01}	T_{02}^t	T_{02}	T_{10}^l	T_{10}^r	T_{11}	T_{12}	T_{20}^l	T_{20}^r	T_{21}	T_{22}
T_{00}^l		T_{00}				T_{02}									
T_{00}^r	T_{00}			T_{01}											
T_{00}															
T_{01}^t								T_{00}^l	T_{00}		T_{02}				
T_{01}								T_{00}							
T_{02}^t												T_{00}	T_{00}^r	T_{01}	
T_{02}													T_{00}		
T_{10}^l		T_{10}^r			T_{11}	T_{12}									
T_{10}^r				T_{11}											
T_{11}															
T_{12}													T_{10}^r	T_{11}	
T_{20}^l						T_{22}									
T_{20}^r	T_{20}^l				T_{21}		T_{22}								
T_{21}								T_{20}^l			T_{22}				
T_{22}															

This gives us twenty-four non-zero products. They come in two classes: the ones $T_{ij}^* T_{kl}^*$ such that $T_{kl}^* T_{ij}^*$ is also non-zero; the ones $T_{ij}^* T_{kl}^*$ such that $T_{kl}^* T_{ij}^* = 0$.

Let's tabulate all these in two sets of tables:

First, where the reverse is also not zero.

$$\begin{array}{c|c|c} & T_{00}^{l} & T_{00}^{r} \\ \hline T_{00}^{l} & & T_{00} \\ \hline T_{00}^{r} & T_{00} & \end{array} \qquad \begin{array}{c|c|c} & T_{10}^{l} & T_{10}^{r} \\ \hline T_{01}^{t} & & T_{00} \\ \hline T_{01} & T_{00} & \end{array}$$

$$\begin{array}{c|c|c} & T_{20}^{l} & T_{20}^{r} \\ \hline T_{02}^{t} & T_{00} & \\ \hline T_{02} & & T_{00} \end{array} \qquad \begin{array}{c|c|c} & T_{01}^{t} & T_{01} \\ \hline T_{10}^{l} & & T_{11} \\ \hline T_{10}^{r} & T_{11} & \end{array}$$

$$\begin{array}{c|c|c} & T_{02}^{t} & T_{02} \\ \hline T_{20}^{l} & T_{22} & \\ \hline T_{20}^{r} & & T_{22} \end{array} \qquad \begin{array}{c|c|c} & T_{12} & T_{21} \\ \hline T_{12} & & T_{11} \\ \hline T_{21} & T_{22} & \end{array}$$

The second set is as follows: $(K_i \in k)$

$$\begin{aligned}
T_{01}^{t}\, T_{10}^{l} &= K_1\, T_{00}^{l}, & T_{01}^{t}\, T_{12} &= K_2\, T_{02} \\
T_{02}^{t}\, T_{20}^{r} &= K_3\, T_{00}^{r}, & T_{02}^{t}\, T_{21} &= K_4\, T_{01} \\
T_{10}^{l}\, T_{00}^{r} &= K_5\, T_{10}^{r}, & T_{10}^{l}\, T_{02}^{t} &= K_6\, T_{12} \\
T_{20}^{r}\, T_{00}^{l} &= K_7\, T_{20}^{l}, & T_{20}^{r}\, T_{01}^{t} &= K_8\, T_{21} \\
T_{12}\, T_{20}^{r} &= K_9\, T_{10}^{r}, & T_{21}\, T_{10}^{l} &= K_{10}\, T_{20}^{l} \\
T_{00}^{l}\, T_{02}^{t} &= K_{11}\, T_{02}, & T_{00}^{r}\, T_{01}^{t} &= K_{12}\, T_{01}
\end{aligned}$$

Next, we want to use the fact that A is a symmetric algebra; it is because it is assumed and so the algebra of endomorphisms E we've just constructed is also, by an above section. Hence $[E,E] \subsetneq \operatorname{rad} E$ as $\operatorname{rad} E$ is an ideal and $[E,E] \subseteq \operatorname{rad} E$ as $E/\operatorname{rad} E \cong k \oplus k \oplus k$.

The second set, just above, shows that all the T's, save T_{00}, T_{11}, T_{22}, are in $[E, E]$. (It's also easy to use the E_i to see this another way.) Let's introduce some scalars for the first set above:

$$T_{00}^{l}\, T_{00}^{r} = \alpha_1\, T_{00}, \quad T_{00}^{r}\, T_{00}^{l} = \alpha_2\, T_{00}$$
$$T_{01}^{t}\, T_{10}^{r} = \beta_1\, T_{00}, \quad T_{10}^{r}\, T_{01}^{t} = \beta_2\, T_{11}$$
$$T_{01}\, T_{10}^{l} = \gamma_1\, T_{00}, \quad T_{10}^{l}\, T_{01} = \gamma_2\, T_{11}$$
$$T_{02}^{t}\, T_{20}^{l} = \delta_1\, T_{00}, \quad T_{20}^{l}\, T_{02}^{t} = \delta_2\, T_{22}$$
$$T_{02}\, T_{20}^{r} = \epsilon_1\, T_{00}, \quad T_{20}^{r}\, T_{02} = \epsilon_2\, T_{22}$$
$$T_{12}\, T_{21} = \eta_1\, T_{11}, \quad T_{21}\, T_{12} = \eta_2\, T_{22}$$

Let's list the linear combinations of T_{00}, T_{11}, T_{22} that lie in $[E, E]$. We tabulate them:

T_{00}	T_{11}	T_{22}
$\alpha_1 - \alpha_2$		
β_1	$-\beta_2$	
γ_1	$-\gamma_2$	
δ_1		$-\delta_2$
ϵ_1		$-\epsilon_2$
	η_1	$-\eta_2$.

Since all these scalars are non-zero we deduce that the symmetry of E yields

$$\alpha_1 - \alpha_2 = 0$$

$$\begin{vmatrix} \beta_1 & -\beta_2 & 0 \\ \delta_1 & 0 & -\delta_2 \\ 0 & \eta_1 & -\eta_2 \end{vmatrix} = 0$$

ie.
$$\beta_1 \eta_1 \delta_2 - \delta_1 \beta_2 \eta_2 = 0.$$

Let's next show that we can adjust the T's so that $\alpha_1 = \alpha_2 = \beta_1 = \cdots = \eta_1 = \eta_2 = 1$. This will leave only K_i, $1 \le i \le 12$, on page 51, to deal with. So keep our attention on the twelve equations on lines six through twelve on the previous page.

First, we can modify $T_{00}^{l}, T_{00}^{r}, T_{00}$ so $\alpha_1 = \alpha_2 = 1$. We now fix $T_{00}^{l}, T_{00}^{r}, T_{00}$. Next, adjust T_{01}^{t}, T_{11} so that $\beta_1 = \beta_2 = 1$ and now keep $T_{01}^{t}, T_{11}, T_{10}^{r}$ fixed.

Before going on we observe that we now have $\gamma_1 = \gamma_2$. Indeed from the six by eleven matrix on the previous page we have

$$\begin{vmatrix} \beta_1 & -\beta_2 & \\ \gamma_1 & -\gamma_2 & \\ \delta_1 & & -\delta_2 \end{vmatrix} = 0$$

where $\delta_2 \neq 0$, $\beta_1 = \beta_2 = 1$. This yields $\gamma_1 = \gamma_2$.

Hence, we can modify T_{10}^{l} so that $\gamma_1 = \gamma_2 = 1$ and we keep T_{10}^{l}, T_{01} fixed. Hence, eight of the T's are fixed and six of the scalars have been adjusted to one so far.

Next adjust T_{02}^{t}, T_{22} so that $\delta_1 = \delta_2 = 1$ and keep $T_{02}^{t}, T_{22}, T_{20}^{l}$ fixed. Next, as $\delta_1 = \delta_2$ we deduce that $\epsilon_1 = \epsilon_2$, just as in the previous paragraph, so now we can adjust T_{20}^{r} so $\epsilon_1 = \epsilon_2 = 1$ and keep T_{20}^{r} and T_{02} fixed.

Finally, the last equation on the last page now gives $\eta_1 = \eta_2$ so we can adjust T_{12} so $\eta_1 = \eta_2 = 1$.

Our relations are now the following plus the relations involving the K_i:

$$T^l_{00}\, T^r_{00} = T_{00}, \quad T^r_{00}\, T^l_{00} = T_{00}$$
$$T^t_{01}\, T^r_{10} = T_{00}, \quad T^r_{10}\, T^t_{01} = T_{11}$$
$$T_{01}\, T^l_{10} = T_{00}, \quad T^l_{10}\, T_{01} = T_{11}$$
$$T^t_{02}\, T^l_{20} = T_{00}, \quad T^l_{20}\, T^t_{02} = T_{22}$$
$$T_{02}\, T^r_{20} = T_{00}, \quad T^r_{20}\, T_{02} = T_{22}$$
$$T_{12}\, T_{21} = T_{11}, \quad T_{21}\, T_{12} = T_{22}.$$

Next, we shall explore the consequences for the K_i of the associative law.

$$(T^t_{01}\, T^l_{10})\, T^r_{00} = K_1\, T^l_{00}\, T^r_{00} = K_1\, T_{00}$$
$$T^t_{01}\,(T^l_{10}\, T^r_{00}) = K_5\, T^t_{01}\, T^r_{10} = K_5\, T_{00}$$

Hence $K_1 = K_5$. Similarly, replacing "1" by "2" and so forth, we have $K_3 = K_7$.

$$(T^t_{01}\, T_{12})\, T^r_{20} = K_2\, T_{02}\, T^r_{20} = K_2\, T_{00}$$
$$T^t_{01}\,(T_{12}\, T^r_{20}) = K_9\, T^t_{01}\, T^r_{10} = K_9\, T_{00}$$

so $K_2 = K_9$; similarly, $K_4 = K_{10}$.

$$(T^l_{10}\, T^r_{00})\, T^t_{01} = K_5\, T^r_{10}\, T^t_{01} = K_5\, T_{11}$$
$$T^l_{10}\,(T^r_{00}\, T^t_{01}) = K_{12}\, T^l_{10}\, T_{01} = K_{12}\, T_{11}$$

so $K_5 = K_{12}$ and similarly $K_7 = K_{11}$.

$$(T^l_{10}\, T^t_{02})\, T_{21} = K_6\, T_{12}\, T_{21} = K_6\, T_{11}$$
$$T^l_{10}\,(T^t_{02}\, T_{21}) = K_4\, T^l_{10}\, T_{01} = K_4\, T_{11}$$

so $K_4 = K_6$ and similarly $K_2 = K_8$. Hence, we now have

$$K_2 = K_8 = K_9;\ K_4 = K_6 = K_{10};\ K_1 = K_5 = K_{12};\ K_3 = K_7 = K_{11}.$$

Let's rewrite this:

$$T_{01}^t\, T_{10}^l = K_1\, T_{00}^l, \quad T_{01}^t\, T_{12} = K_2\, T_{02}$$
$$T_{02}^t\, T_{20}^r = K_3\, T_{00}^r, \quad T_{02}^t\, T_{21} = K_4\, T_{01}$$
$$T_{10}^l\, T_{00}^r = K_1\, T_{10}^r, \quad T_{10}^l\, T_{02}^t = K_4\, T_{12}$$
$$T_{20}^r\, T_{00}^l = K_3\, T_{20}^l, \quad T_{20}^r\, T_{01}^t = K_2\, T_{21}$$
$$T_{12}\, T_{20}^r = K_2\, T_{10}^r, \quad T_{21}\, T_{10}^l = K_4\, T_{20}^l$$
$$T_{00}^l\, T_{02}^t = K_3\, T_{02}, \quad T_{00}^r\, T_{01}^t = K_1\, T_{01}$$

Hence, we have these relations plus the ones at the top of the previous page.

Replace T_{02} by $K_2 T_{02}$ and T_{20}^r by $K_2^{-1} T_{02}$ so the first set of relations is unchanged & in the second we get $K_2 = 1$. Similarly, replace T_{01} by $K_4 T_{01}$ and T_{10}^l by $K_4^{-1} T_{10}^l$ and K_4 becomes 1 and all is alright. Next, replace T_{00}^l by $K_1 T_{00}^l$ and T_{00}^r by $K_1^{-1} T_{00}$; this lets $K_1 = 1$ and doesn't bother the relations on the last page as $K_1 K_1^{-1} = 1$. This leaves only K_3 possibly not equal to one.

But we use associativity once more!

$$(T_{02}^t\, T_{20}^r)\, T_{01}^t = K_3\, T_{00}^r\, T_{01}^t = K_3\, T_{01}$$
$$T_{02}^t\, (T_{20}^r\, T_{01}^t) = T_{02}^t\, T_{21} = T_{01}.$$

Hence, $K_3 = 1$ and we're done.

<u>Prop</u> All symmetric algebras of type A_5 lies in a single Morita equivalence class.

Projective modules for symmetric algebras

Our purpose is to give a module-theoretic proof of the standard fact about the socle of indecomposable projective modules. Let A be a symmetric algebra with defining form $(\ ,\ )$.

Prop. If P is an indecomposable projective A-module then $P/\mathrm{rad}\,P \cong \mathrm{soc}(P)$.

Pf. Suppose that this is false, $P/\mathrm{rad}\,P \cong S$ a simple A-module. Since A is self-injective, as it is symmetric, $\mathrm{soc}(P)$ is simple as is the socle of every indecomposable A-module. Express A as a direct sum of indecomposable projective modules

$$A \cong P_1 + \dots + P_r + Q_1 + \dots + Q_t$$

where $P_i \cong P$, $1 \le i \le r$, $Q_j \ncong P$, $1 \le j \le t$. As A is self-injective certain of the Q's have socle isomorphic with S, say $Q_1, \dots, Q_s$. Let $I = \mathrm{soc}(Q_1) + \dots + \mathrm{soc}(Q_s)$ so $\mathrm{soc}(A)/I$ has no composition factor isomorphic with S. Hence, if $\varphi \in \mathrm{End}_A(A)$ then $\varphi(I) \subseteq I$. Thus, I is an ideal as the elements of $\mathrm{End}_A(A)$ are the right multiplications.

Let $J = \{\varphi \mid \varphi \in \mathrm{End}_A(A),\ \varphi(A) \subseteq I\}$. Hence, J is a non-zero ideal of $\mathrm{End}_A(A)$. It's clearly non-zero as $\mathrm{soc}(Q_1) \cong S$ and since I is an ideal this easily shows that so is J.

We let π be the projection of A onto $P_1+\dots+P_r$ with kernel $Q_1+\dots+Q_t$. We claim that if $\varphi \in J$ then $\varphi\cdot\pi = \varphi$, $\pi\cdot\varphi = 0$. For $\varphi(A) \subseteq I$ and $\pi(I) \subseteq \pi(Q_1+\dots+Q_t) = 0$ while if $x \in Q_1+\dots+Q_t$ then $\varphi(x) = 0$, $\pi(x) = 0$ so $\varphi\cdot\pi(x) = 0$ and if $x \in P_1+\dots+P_r$ then $\varphi\cdot\pi(x) = \varphi(x)$. Consequently, $\varphi = \varphi\pi - \pi\varphi$.

Now $\mathrm{End}_A(A)$ is also a symmetric algebra: if ρ_a, ρ_b are two right multiplications in $\mathrm{End}_A(A)$ set $(\rho_a, \rho_b) = (b, a)$ and check all the required properties. Let $\varphi \in J$, $\varphi \neq 0$. $\therefore (\varphi, 1) = (\varphi\pi - \pi\varphi, 1) = (\varphi, \pi) - (\pi, \varphi) = 0$. Also, if $\alpha \in \mathrm{End}_A(A)$ then $\varphi\alpha \in J$ so $(\varphi\alpha, 1) = 0$, that is, $(\varphi, \alpha) = 0$ so $(\ ,\ )$ is degenerate, a contradiction.

An example of p-group moduli

We shall write down what is no doubt a standard example of moduli of isomorphism classes of p-groups. For each $\delta \in \mathbb{Z}/p\mathbb{Z}$ we shall define a p-group P_δ of order p^5, $p > 3$. P_δ / P_δ' is elementary of order p^2 and x_1 and x_2 are elements of P_δ such that $P_\delta' x_1$ and $P_\delta' x_2$ are a basis for P_δ / P_δ'. P_δ' is elementary of order p^3 with basis x_3, x_4, x_5 with $Z(P_\delta) = \langle x_4, x_5 \rangle$. The rest of the situation is determined by the following relations:

$$x_1^p = x_4, \quad x_2^p = x_5$$
$$[x_1, x_2] = x_3$$
$$[x_1, x_3] = x_4, \quad [x_2, x_3] = x_5^\delta, \quad p > 3.$$

Let's see directly that the groups P_δ are pairwise non-isomorphic. The only one with $[P_\delta', P_\delta]$ of order p is P_0, so we may now assume $\delta \neq 0$. Let H/P_δ' be a subgroup of order p in P_δ / P_δ'. Thus, $[H, P_\delta']$ is a subgroup of order p in $Z(P)$ as is $\mho^1(H)$. Suppose $H = \langle P_\delta', x_1^a x_2^b \rangle$, not both a and b divisible by p. Then $\mho^1(H) = \langle x_4^a x_5^b \rangle$ while $[H, P_\delta'] = \langle [x_1^a x_2^b, x_3] \rangle = \langle x_4^a x_5^{b\delta} \rangle$. Thus, if $\delta = 1$ these subgroups coincide for all H while if $\delta \neq 1$ these coincide if, and only if, H is either $\langle P_\delta', x_1 \rangle$ or $\langle P_\delta', x_2 \rangle$. Hence, we may now assume that $\delta \neq 1$ as well as $\delta \neq 0$.

The two subgroups $H_1 = \langle P'_\delta, x_1 \rangle$ and $H_2 = \langle P'_\delta, x_2 \rangle$ are not interchanged by any isomorphism. For if they were, using dots for "higher" terms,

$$x_1 \to x_2^a, \quad x_2 \to x_1^b$$

neither a nor b divisible by p. Hence

$$x_3 = [x_1, x_2] \to [x_2^a \cdots, x_1^b \cdots] = x_3^{-ab} \cdots$$
$$x_4 = x_1^p \to x_2^{ap} = x_5^a$$
$$x_5 = x_2^p \to x_1^{bp} = x_4^b$$
$$x_4 = [x_1, x_3] \to [x_2^a, x_3^{-ab}] = x_5^{-a^2 b \delta}$$
$$x_5^\delta = [x_2, x_3] \to [x_1^b, x_3^{-ab}] = x_4^{-ab^2}$$

Hence,

$$a = -a^2 b \delta$$
$$b\delta = -ab^2$$

so
$$\delta = (-ab)^{-1}$$
$$\delta = (-ab)$$

Hence, $\delta = -1$. Perhaps there is an automorphism in this case; no matter - if there is it distinguishes $\delta = -1$ and if not the remaining argument works.

Each H_i / P'_δ determines a subgroup of P'_δ in two ways. Let's compare how this is done. Take typical generators modulo P'_δ: x_1^a, x_2^b.

$$(x_1^a)^p = x_4^a \; ; \; (x_2^b)^p = x_5^b$$
$$[x_1^a, [x_1^a, x_2^b]] = [x_1^a, x_3^{ab}] = x_4^{a^2 b}$$
$$[x_2^b, [x_1^a, x_2^b]] = [x_2^b, x_3^{ab}] = x_5^{ab^2 \delta}$$

Now the ratios are $a^2b/a = ab$ and $ab^2\delta/b = ab\delta$ so their ratio is δ. $\therefore$ δ is an invariant.

We reached this by using Higman's representation theory approach. We start with the groups of order p^3, exponent p and class two. The "next layer" is spanned by

$$x_1^p,\ x_2^p,\quad [x_1, x_2, x_1],\ [x_1, x_2, x_2]$$

so we have, for $GL(2,p)$, the natural module and the tensor product of that and the determinant representation. Going to groups of order p^5 involves looking at two dimensional subspaces.

Let's just do the calculations:

$$\begin{array}{lll} x_1 \quad x_2 & & x_1^p = x_4, \quad x_2^p = x_5 \\ x_3 \quad x_4 \quad x_5 & & [x_1, x_2] = x_3. \end{array}$$

Say

$$[x_1, x_3] = x_4^\alpha x_5^\beta$$
$$[x_2, x_3] = x_4^\gamma x_5^\delta$$

Say

$$x_1 \to x_1^a x_2^b \cdots = x_1'$$
$$x_2 \to x_1^c x_2^d \cdots = x_2'$$

so

$$x_3 = [x_1, x_2] \to x_3^\Delta, \quad \Delta = ad - bc \neq 0.$$

What are the new $\alpha, \beta, \gamma, \delta$? Have to calculate $[x_1', x_3']$ in terms of x_4', x_5' and $[x_2', x_3']$ in terms of x_4', x_5'.

$$[x_1', x_3'] = [x_1^a x_2^b, x_3^\Delta] = (x_4^\alpha x_5^\beta)^{a\Delta}(x_4^\gamma x_5^\delta)^{b\Delta}$$
$$[x_2', x_3'] = [x_1^c x_2^d, x_3^\Delta] = (x_4^\alpha x_5^\beta)^{c\Delta}(x_4^\gamma x_5^\delta)^{d\Delta}$$

now

$$x_4 = (x_4')^{d\Delta^{-1}} (x_5')^{-b\Delta^{-1}}$$

$$x_5 = (x_4')^{-c\Delta^{-1}} (x_5')^{a\Delta^{-1}}$$

so

$$[x_1', x_3'] = (x_4')^{d\Delta^{-1}\alpha a\Delta} (x_5')^{-b\Delta^{-1}\alpha a\Delta}$$
$$(x_4')^{-c\Delta^{-1}\beta a\Delta} (x_5')^{a\Delta^{-1}\beta a\Delta}$$
$$(x_4')^{d\Delta^{-1}\gamma b\Delta} (x_5')^{-b\Delta^{-1}\gamma b\Delta}$$
$$(x_4')^{-c\Delta^{-1}\delta b\Delta} (x_5')^{a\Delta^{-1}\delta b\Delta}$$

$$= (x_4')^{d\alpha a - c\beta a + d\gamma b - c\delta b} (x_5')^{-b\alpha a + a\beta a - b\gamma b + a\delta b}$$

etc...

so
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as expected.

Consider $\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$ and an image also of the form $\begin{pmatrix} 1 & 0 \\ 0 & \delta' \end{pmatrix}$. Will show for the case $\delta \neq 0$, $\delta' \neq 0$ that $\delta = \delta'$ would follow.

Indeed, original matrix has trace $1+\delta$, det T. If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then image has trace $1+\delta' = (1+\delta)\det T$ and determinant $\delta' = \delta(\det T)^2$ forcing $\delta' = \delta \cdot \det T$, $\delta' = \delta \cdot (\det T)^2$ so $\det T = 1$ and so $\delta = \delta'$.

The canonical module

We wish to give a module-theoretic treatment of the last part of the Extended First Main Theorem. We fix D a p-subgroup of G such that $G = D\,C(D)$.

Proposition There is a one-to-one correspondence between blocks of G with defect group D and blocks of $D\,C(D)$ of defect 0.

(The simple module in these blocks is the canonical module; it corresponds to the canonical character.) Let k be usual field.

Lemma 1 Two simple kG-modules are in the same block of G if, and only if, they are in the same block of $\overline{G} = G/D$.

Proof. Let S and T be two distinct simple kG-modules and let

$$0 \to T \to U \to S \to 0$$

be exact; it suffices to show that D acts trivially on U. But $S_{C(D)}$ and $T_{C(D)}$ are simple and non-isomorphic, as $C(D)/Z(D) \cong \overline{G}$, so $\mathrm{End}_{C(D)}(U) = k$ or $k \oplus k$. But kD acts on U and induces endomorphisms, as D centralizes $C(D)$; but kD is a local algebra so rad (kD) annihilates U and so D acts trivially, as desired.

Hence, it makes sense to speak of a block b of G and the corresponding block $\bar{b}$ of $\bar{G}$ and this is a one-to-one correspondence. We wish to see that b has defect group D if and only if, $\bar{b}$ has defect group 1.

First, suppose b has defect group D and let S be a simple module in b. Hence, $S \mid (S_D)^G$. But since D acts trivially on S, we have $S \mid (S_1)^{\bar{G}}$ for S as a $k\bar{G}$-module, so S is projective and $\bar{b}$ is as required.

Next, let $\bar{b}$ have defect 0 and let S be the unique simple in $\bar{b}$, i.e. in b. Let P be the projective cover of S as a b-module. We are going to study b. Let $f = \dim_k S$. Now b is a right kD-module and so is $M_f(kD)$ by right multiplication by scalars. Hence the following makes sense:

<u>Lemma 2</u> There is an algebra isomorphism of b onto $M_f(kD)$ which is also a right kD-module isomorphism.

<u>Proof</u>. Now P_D is projective so $P_D \cong \overbrace{kD \oplus \cdots \oplus kD}^{m}$, where

$$\begin{aligned} m &= \dim_k \operatorname{Hom}_{kD}(m\,kD, k) \\ &= \dim_k \operatorname{Hom}_{kD}(P_D, k) \\ &= \dim_k \operatorname{Hom}_{kG}(P, k\bar{G}) \\ &= \dim_k S \\ &= f. \end{aligned}$$

Hence, $\dim_k P = f\,|D|$ as $\mathrm{End}_{kG}(P)$ is of dimension $|D|$ as the only composition factor of P is of dimension f.
Now $b \cong M_f(\mathrm{End}_{kG}(P)^{\circ})$ and $\dim_k b = f^2\,|D|$.

However, kG is a free kD-module, as it is projective, so as right kD modules, by dimension counting,

$$b \cong \overbrace{kD \oplus \cdots \oplus kD}^{f^2}.$$

Now b acting on b on the left gives endomorphisms of b as right kD-module, and so by dimension counting we get this is an isomorphism, which is exactly what we wanted.

To conclude, we need only the following:
Since $b \cong M_f(kD)$ we have a Morita equivalence sending a kD-module to a b module consisting of a column of length f over the kD-module:

$$V \to \begin{pmatrix} V \\ \vdots \\ V \end{pmatrix}.$$

<u>Lemma 3</u>. If V is a kD-module and U is the corresponding b-module then U / V^G.

<u>Remark</u>: Hence the proposition is proved, as every b-module is relatively D-projective.

<u>Proof</u> (of Lemma 3). Now $kG \otimes_{kD} V = V^G$ where we're using the fact that kG is a left kG-module

and a right kD-module, while V is a left kD-module.
Hence, as $kG = b \oplus \cdots$ we have

$$b\, V^G = b\, kG \otimes_{kD} V$$
$$= b \otimes_{kD} V$$

where we're considering b as a left k-module and right kD-module. But Lemma 2 applies, so as b-modules

$$b\, V^G = M_f(kD) \otimes_{kD} V$$
$$= M_f(V) \left(= \left\{ \begin{pmatrix} V & \cdots & V \\ \vdots & & \vdots \\ V & \cdots & V \end{pmatrix} \right\} \right)$$
$$\cong \overbrace{U \oplus \cdots \oplus U}^{f}$$

so we're done!

Carlson's conjecture consequences

Since this is now a theorem, due to Avrunin and Scott, we shall simply discuss some consequences.

Let M be a kG-module as usual, $\mathfrak{a}_M$ the annihilator in $H(G,k)$ of $\bigoplus_{S \text{ simple}} H^*(G, S\otimes M)$. We have

$$W(M) = V\left(H(G,k)/\mathfrak{a}_M \right)$$

the variety corresponding to the ring $H(G,k)/\mathfrak{a}_M$. Hence, $W(M) \subseteq W(k)$, the Quillen variety.

Quillen stratifies $W(k)$ into a union of $W(k)_E^+$, E running over representatives of all the conjugacy classes of elementary abelian p-subgroups of G. He proves, in particular, that for a point $x \in W(k)$ there is a unique smallest E giving rise to x by restriction; i.e. there is a point of $W(k_E)$ (a homomorphism of $H(E,k)$ to k) the Quillen variety of the kE-module k_E which goes to x by composition with res_E^G. We say E is a vertex for x.

Theorem If $x \in W(M)$ and x has vertex E then x is the composition of res_E^G and a point of $W(M_E)$.

Of course, $x \in W(k) \supseteq W(M)$ so it has a vertex. This result gives a stratification of $W(M)$ by intersecting with Quillen's stratification, each 'piece' a subvariety - up to isogeny - of a $W(k_E)^+$.

Now by my result with Evens on the varieties, there is an elementary abelian p-group E_0 such that x is the composition of $\mathrm{res}^G_{E_0}$ and an element of $W(M_E)$. By Quillen's result, we may assume that $E_0 \geq E$. We have the picture:

$$\begin{array}{ccc} H(E_0, k) & \qquad\qquad & H^*(E_0, M_{E_0}) \\ \downarrow \mathrm{res}^{E_0}_E & & \downarrow \mathrm{res}^{E_0}_E \\ H(E, k) & & H^*(E, M_E) \end{array}$$

We may assume, without any loss of generality, that $x \in W(M_{E_0})$. That is, x is a homomorphism of $H(E_0, k)$ to k, its kernel containing the annihilator $\mathfrak{A}$ of $H^*(E_0, M_{E_0})$, and that x is the composition of $\mathrm{res}^{E_0}_E$ and an element of $W(k_E)$. Let $\mathfrak{L}$ be the annihilator in $H(E_0, k)$ of the $H(E_0,k)$-module $H^*(E_0, \mathrm{Ind}^{E_0}_E(M_E))$, i.e. the annihilator of the $H(E_0, k)$ module $H^*(E, M_E)$ with action via the restriction map. Let K be the kernel of $\mathrm{res}^{E_0}_E$.

We are assuming that $\ker x \geq K$ as x arises by composition. We also have $\ker x \geq \mathfrak{A}$ by assumption. Hence, it suffices to show that $(\sqrt{K}, \sqrt{\mathfrak{A}}) \geq \mathfrak{L}$, i.e. that the variety of $\mathfrak{L}$ does contain the intersection of the variety of K and the variety of $\mathfrak{A}$. But $W(\mathfrak{L}) = W(\mathrm{Ind}^{E_0}_E(M_E))$ and

$$\mathrm{Ind}^{E_0}_E(M_E) \simeq k[E_0/E] \otimes M.$$

But Carlson has shown that his conjecture implies that

$$W(M_1 \otimes M_2) \geq W(M_1) \cap W(M_2)$$

for any two kE-modules M_1 and M_2. Since

$W(\mathcal{L}) = W(\mathrm{Ind}_E^{E_0}(M_E))$, $W(\Omega) = W(k[E_0/E])$ we're done.

One more consequence of Carlson's conjecture is the fact that his result on tensor products generalizes to arbitrary groups from elementary abelian groups.

<u>Theorem</u>. If M_1 and M_2 are kG-modules then
$$W(M_1 \otimes M_2) = W(M_1) \cap W(M_2).$$

<u>Proof</u>. One half is automatic so we take $x \in W(M_1) \cap W(M_2)$ and we have to demonstrate that $x \in W(M_1 \otimes M_2)$. Let E be the "vertex" of x so there is $y \in W((M_1)_E)$ and $z \in W((M_2)_E)$ such that $x = y \circ \mathrm{res}^G_E$, $x = z \circ \mathrm{res}^G_E$. Hence, it suffices to see that we can take $y = z$, as then $y = z \in W((M_1)_E) \cap W((M_2)_E) = W((M_1 \otimes M_2)_E)$ so that $x \in W(M_1 \otimes M_2)$ by a lemma from the afore-mentioned paper with Evens ($H \leq G$, M a kG-module $\Rightarrow \Omega_G(M) \subseteq \mathrm{res}_H^{-1}(\Omega_H(M_H))$).

Now $y, z \in W(k_E)^+_E$ – Quillen's notation – with the same image in $V(k)^+_E$. It's enough to see they're conjugate under $N(E)$, which is so by Quillen's paper, p 368 last line, J. Pure and Appl Alg. vol 1 (1971), 361–372.

Projective modules and normal subgroups

It is, in fact, easy to describe the relationship between the Cartan matrix of a group and of a quotient by a normal p-subgroup. Let us first proceed in more generality. Let N be a normal subgroup of the group G, k the usual algebraically closed field of prime characteristic p, $R_i = \mathrm{rad}^i(kN)$, $i \geq 0$. If M is a kG-module then set $M_i = R_i M$ so $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq 0$ and clearly, as N is normal, the M_i are kG-submodules. (For if $g \in G$ then $g\, \mathrm{rad}(kN)^i g^{-1} = \mathrm{rad}(kN)^i$.)

Proposition. If $i \geq 0$ then M_i/M_{i+1} is a homomorphic image of the tensor product

$$R_i/R_{i+1} \otimes M/M_1$$

of kG-modules, where G acts on kN by conjugation.

Proof. If $\alpha \in R_i$, $m \in M$ then $\alpha m \in R_i M = M_i$ so it's easy to see we have a vector space epimorphism

$$R_i/R_{i+1} \otimes_k M/M_1 \to M_i/M_{i+1}.$$

Moreover, if $\alpha \in R_i$, $m \in M$, $g \in G$ then $g\alpha g^{-1} \otimes gm$, the result of applying g to $\alpha \otimes m$, goes to $g\alpha g^{-1} g m = g \cdot \alpha m$, the result of applying g to the image of $\alpha \otimes m$.

Theorem. If N is a normal p-subgroup of G, S is a simple kG-module and P_S is its projective cover then P_S has a series of submodules whose successive quotients are $R_i/R_{i+1} \otimes \bar{P}_S$, where $\bar{P}_S$ is the projective cover of the simple $k\bar{G}$-module, $\bar{G} = G/N$.

Proof. The above result shows that P_S has a series with quotients being epimorphic images of the $R_i/R_{i+1} \otimes P_S/(\mathrm{rad}\, kN)P_S$. It suffices to see that $P_S/(\mathrm{rad}\, kN)P_S \cong \overline{P}_S$. For then we have $\dim_k P_S \leq (\dim_k kN)(\dim \overline{P}_S)$, while counting $\dim_k kG$, $\dim_k k\overline{G}$ in terms of decompositions into indecomposable projectives then forces $\dim_k P_S = (\dim_k kN)\dim_k \overline{P}_S$ and so all the epimorphisms are isomorphisms.

Now $\overline{P}_S$ is a homomorphic image of P_S, even of $P_S/(\mathrm{rad}\, kN)P_S$ as N is in the kernel of P_S and kN is local. On the other hand $P_S/(\mathrm{rad}\, kN)P_S$ is a $k\overline{G}$ module and has S as an image. Moreover, it is projective as a $k\overline{G}$ module – for say we have the usual set-up in the definition of projectivity:

$$\begin{tikzcd} & P_S \arrow[d] \arrow[dl, dotted] \\ U \arrow[r, two heads] & V \end{tikzcd}$$

If U and V are $k\overline{G}$ modules then the dotted map will also be a $k\overline{G}$ map. Hence $P_S/(\mathrm{rad}\, kN)P_S \cong \overline{P}_S$ as required.

Feit Cohomology

We're going to give a proof of Feit's theorem about his zero dimensional relative cohomology groups. This may already be in Green's work, using traces, but we will use maps factored through relatively projective modules.

<u>Def 1</u> If $\alpha : U \to V$ is a homomorphism of kG-modules then we say α is $\mathcal{H}$-projective, for a family of subgroups $\mathcal{H}$ of G, if α factors through a relatively $\mathcal{H}$-projective module

We need some preliminary results

<u>Lemma 2</u> Let H be a subgroup of the group G, X a kH module and M a kG-module. If $\alpha \in \mathrm{Hom}_{kG}(X^G, M)$ then the following diagram commutes:

$$\begin{tikzcd} & (M_H)^G \arrow[dr, "\rho"] & \\ X^G \arrow[ur, "(\alpha_X)^G"] \arrow[rr, "\alpha"'] & & M \end{tikzcd}$$

Here, $(\alpha_X)^G$ is the unique map of X^G to $(M_H)^G$ determined by $\alpha_X : X \to M$ and ρ is the natural collapsing map (sending $s \otimes m$ to sm).

<u>Proof</u> It suffices to see the composition agrees with α on X. But this is immediate:

$$\begin{tikzcd} & 1 \otimes \alpha(x) \arrow[dr] & \\ x \arrow[ur] \arrow[rr] & & \alpha(x) \end{tikzcd}$$

<u>Lemma 3</u> If we have maps

$$U \xrightarrow{\alpha} V \xrightarrow{\beta} W$$

of kG-modules, where α is $\mathcal{H}$-projective and β is $\mathcal{K}$ projective, for collections $\mathcal{H}$ and $\mathcal{K}$ of subgroups of G, then $\beta \cdot \alpha$ is $\{ H \cap sKs^{-1} \mid H \in \mathcal{H}, K \in \mathcal{K}, s \in G \}$ projective.

<u>Proof</u>. It's enough to assume X is a kH-module, $H \in \mathcal{H}$, Y is a kK-module, $K \in \mathcal{K}$, that we have a commutative diagram

$$\begin{tikzcd} & X^G \arrow[dr] & & Y^G \arrow[dr] & \\ U \arrow[ur] \arrow[rr, "\alpha"'] & & V \arrow[ur] \arrow[rr, "\beta"'] & & W \end{tikzcd}$$

as all the maps being considered are sums of such maps.

But by Lemma 2 we always have a commutative diagram

$$\begin{tikzcd} & ((Y^G)_H)^G \arrow[dr] & \\ X^G \arrow[ur] \arrow[rr] & & Y^G \end{tikzcd}$$

where the map X^G to Y^G is the composite of two maps in the previous diagram. However,

$$(Y^G)_H \simeq \bigoplus_s \left(s(Y)_{sKs^{-1} \cap H} \right)^H$$

so

$$((Y^G)_H)^G \simeq \bigoplus_s \left(s(Y)_{sKs^{-1} \cap H} \right)^G$$

as required.

Now let's have the usual set-up of the Green correspondence: G, $H \supseteq N(Q)$, $\mathfrak{X}$, $\mathfrak{Y}$, so Q is a p-subgroup of G.

<u>Def 4</u> If U_1 and U_2 are kG-modules and $\mathcal{H}$ is a family of subgroups of G then $\mathrm{Hom}_{kG}(U_1, U_2)_{\mathcal{H}}$ is the vector space of $\mathcal{H}$-projective maps of U_1 to U_2.

Now let U be an indecomposable kG-module with vertex $R \subseteq Q$, $R \not\in_G \mathfrak{X}$ and let V be the corresponding kH-module.

<u>Theorem 5</u> If M is any kG-module then

$$\mathrm{Hom}_{kG}(M,U)/\mathrm{Hom}_{kG}(M,U)_{\mathfrak{X}} \cong \mathrm{Hom}_{kH}(M_H,V)/\mathrm{Hom}_{kH}(M_H,V)_{\mathcal{Y}}$$

<u>Lemma 6</u> If N is any kH-module then

$$\mathrm{Hom}_{kH}(N,V)_{\mathcal{Y}} = \mathrm{Hom}_{kH}(N,V)_{\mathfrak{X}}$$

<u>Proof</u> Say we have

$$\begin{tikzcd} & Y \arrow[dr, "\beta"] & \\ N \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & V \end{tikzcd}$$

where Y is relatively $\mathcal{Y}$-projective. Hence, we have

$$N \xrightarrow{\gamma} V \xrightarrow{1} V$$

where γ is $\mathcal{Y}$-projective and 1_V is R-projective. Hence, by Lemma 3, γ is projective for the family of subgroups $(sQs^{-1} \cap H) \cap hRh^{-1}$, $s \in G - H$, $h \in H$. But these are contained in the subgroups $sQs^{-1} \cap Q$, as $R \subseteq Q$, so our assertion holds.

<u>Proof</u> (of Theorem 5). First, we have

$$\mathrm{Hom}_{kG}(M,U)/\mathrm{Hom}_{kG}(M,U)_{\mathfrak{X}}$$

$$\cong \mathrm{Hom}_{kG}(M,V^G)/\mathrm{Hom}_{kG}(M,V^G)_{\mathfrak{X}}$$

since $V^G \simeq U \oplus \cdots$ where the dots represent relatively $\mathcal{X}$-projective modules. But

$$\mathrm{Hom}_{kG}(M, V^G) \simeq \mathrm{Hom}_{kH}(M_H, V)$$

so it remains to show that under this isomorphism the $\mathcal{X}$-projective maps on the left-hand side correspond to the $\mathcal{Y}$-projective maps on the right-hand side.

But suppose we have

$$\begin{tikzcd} & Y \arrow[dr] & \\ M_H \arrow[ur] \arrow[rr] & & V \end{tikzcd}$$

where Y is relatively $\mathcal{Y}$-projective. Hence, the map $Y \to V$ is $\mathcal{Y}$-projective and so is $\mathcal{X}$-projective. $\therefore$ $Y^G \to V^G$ is also $\mathcal{X}$-projective so the diagram

$$\begin{tikzcd} & Y^G \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & V^G \end{tikzcd}$$

proves half of the desired result.

On the other hand, say we have

$$\begin{tikzcd} & X \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & V^G \end{tikzcd}$$

where X is relatively $\mathcal{X}$-projective so this yields

$$\begin{tikzcd} & X_H \arrow[dr] & \\ M_H \arrow[ur] \arrow[rr] & & (V^G)_H \end{tikzcd}$$

and X_H is relatively $\mathcal{Y}$-projective by one of the lemmas used in the proof of the Green correspondence (see 25 in our notes). Now $(V^G)_H = V \oplus s \otimes V \oplus \cdots$ is one decomposition of the kH-module $(V^G)_H$ so we have a projection of $(V^G)_H$

onto V which gives the maps of M_H to V corresponding to the maps of M to V^G. Hence, we have

$$\begin{tikzcd} & X_H \arrow[r] \arrow[dr] & (V^G)_H \arrow[d] \\ M_H \arrow[ur] \arrow[rr] & & V \end{tikzcd}$$

so the maps of M_H to V does factor through a relatively Y-projective module.

The theorem has as a consequence the case where M is also relatively Q-projective so we are dealing with a "section" – i.e. "subquotient" – of the category of kG-modules.

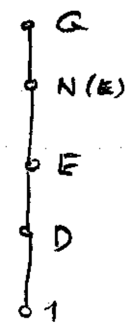
<u>The canonical module</u> (cont.)

We have made an error on page 64 in the proof of Lemma 2: there should be f^2 summands in the sum on line 5. Hence, to save the proposition, we shall prove the

<u>Lemma</u> If $\bar{b}$ has defect group 1 then b has defect group D.

Here the notation is as before. $G = D\,C(D)$, $\bar{G} = G/D$, b and $\bar{b}$ are the corresponding blocks of G and $\bar{G}$, each having the one simple module S so the $\bar{b}$-modules are precisely the b-modules with kernels containing D.

<u>Proof.</u> Let E be the defect group of b so $D \subseteq E$. Let β be the block of $N(E)$ which is the Brauer correspondent of b.



Let V be an indecomposable projective $kN(E)/E$-module which is in β. As $V \mid (V^G)_{N(E)}$ there is a kG module U which is indecomposable, has D in its kernel and satisfies $V \mid U_{N(E)}$. Hence, by Nagao's theorem, U lies in b, as V has vertex E. We can also look at $\bar{G}$ and we have $V \mid U_{\overline{N(E)}}$

since V as a $N(E)/D$-module has vertex E/D we can apply Nagao's Theorem again to deduce that the block of $\overline{N(E)}$ containing V is mapped by the Brauer map to the block containing U, that is, $\bar{b}$. Hence $\bar{E}$ is contained in a conjugate of the defect group of $\bar{b}$, i.e. $\bar{E} = 1$ and $E = D$.

We want to point out that the error we made sheds some interesting light on the case of a block b of a group G with defect group D and $b \simeq M_f(kD)$ (not the G above), namely, that the structure of b as a right kD-module need not be the original type we described in the Lemma 2 referred to above. For in that case, kD and b are Morita equivalent algebras and the map

$$V \to b \otimes_{kD} V = b \cdot V^G \simeq \overbrace{U \oplus \cdots \oplus U}^{f}$$

sends a kD-module V to the direct sum of f copies of the corresponding b-module. This doesn't happen always, e.g. $p = 3$, $G = SL(2,3)$, b the non-principal 3-block.

One final remark: Lemma 1 seems to be essentially as in a paper of Landrock on the Extended First Main Theorem.

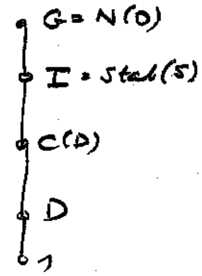
Abelian defect groups

Let B be a block of G with abelian defect groups. Our first task is to show that the "weights" of page 35 for B are only the obvious ones, namely, the simple modules for the Brauer correspondent of B. (Hence, if the weight conjecture holds then B and its Brauer correspondent will have the same number of simple modules and so, by the Second Main Theorem, the same number of irreducible characters.

Proposition. If E is a p-subgroup of G, S is a projective simple $kN(E)/E$-module, b is the block of $N(E)$ containing S and $b^G = B$ then E is a defect group of B.

Proof. Let S_1 be an indecomposable summand of $S_{C(E)}$ so S_1 is a projective simple $kC(E)/E$ module. If b_1 is the block of $C(E)$ containing S_1 then $b_1^{N(E)} = b$ so $b_1^G = B$. Now S_1 "is" a block of $C(E)/E$ of defect zero so, by the results redone in the previous section, we have that b_1 has defect group E. Hence, (E, S_1) is a Brauer pair, so, by Olsson's theorem, E contains the center of a defect group of B. These are abelian, by assumption, so the result is proved.

Now lets assume, in addition, that the defect group D of B is normal in G. We shall use the results of pages 69 and 70 to get at the (known) structure of the projective modules in B in certain cases.



Let S be the canonical module so the simple modules in B are exactly those whose restrictions to $C(D)$ involve S. These are easily determined by Clifford theory since $I/C(D)$ is a p'-group. Let S^e be an extension of S to I — the inertia group of S as usual. Let $T_1, \ldots, T_n$ be the simple $k\,I/C(D)$ modules so the distinct simple modules in B are

$$S_i = \left(S^e \otimes T_i\right)^G,$$

$1 \le i \le n$. Each S_i is also projective as a kG/D module, inasmuch as S is projective as a $kC(D)/D$ module so S^e is projective as a kI/D-module (as $I/C(D)$ is a p'-group and projectivity can be tested on p-groups). Hence, each S_i is its own projective cover as a kG/D module.

Let $R_0 \supseteq R_1 \supseteq \ldots$ be the radical series of kD considered as kG-modules under conjugation. If P_i is the projective cover of S_i as kG-module then we know that P_i has a filtration with the successive factors

$$R_j / R_{j+1} \otimes S_i .$$

That is,

$$R_j/R_{j+1} \otimes (S^e \otimes T_i)^G \simeq \left((R_j/R_{j+1})_I \otimes S^e \otimes T_i \right)^G.$$

Now suppose that D is cyclic of order p^n, $|I : C(D)| = e$ and λ is the one-dimensional module given by $D/\Phi(D)$. Hence, P_i has a series with p^n factors, these being

$$S_i, \quad S_i \otimes \lambda^G, \quad S_i \otimes \lambda^G \otimes \lambda^G, \ldots$$

By dimension counting and restriction to D we get easily that each P_i is uniserial as well.

Next, suppose that $p=2$ and $D \simeq Z_2 \times Z_2$. Consider the case that $e = |I : C(D)| = 3$ (the case $e=1$ being easier). Hence, T_1, T_2, T_3 are one-dimensional and we easily get the right answer. This is because kD has an easily calculated series and the Cartan matrix has such small entries.

Brauer's Induction Theorem revisited

A. Weil has suggested using embeddings in linear groups instead of symmetric groups to prove Brauer's theorem. We will do something in this direction.

Fixing a prime p, and using arguments from our first proof, we need only show that the principal character of G is an integral linear combination of "projective" characters and characters induced from subgroups of local subgroups. But $G \subseteq GL(n,p)$, for suitable n, and the Tits complex of $GL(n,p)$, regarded as a complex for G does the job.

Another approach: use the Quillen complex for G directly!

Inertial blocks

The set-up is as in section two of the Alperin - Broué paper so P is a normal subgroup of G b is a block of G (it's an idempotent now),

$$b = b_p + \ldots$$

is a sum of conjugate blocks of $C(P)$ (conjugate in G) and b_p is a block of $N(b_p)$, $b = Tr_{N(b_p)}^{G}(b_p)$.

Theorem If b has defect group P then

$$kGb \simeq M_{|G:N(b_p)|}(kN(b_p)b_p)$$

In particular, we have a Morita equivalence. (Here $M_n(A)$ is n by n matrices over A.) This is useful in passing between the two algebras in question.

Proof Let U be a module belonging to the algebra $kN(b_p)b_p$ (i.e. U is a $kN(b_p)$-module and $b_p U = U$). We shall show that the induced module U^G belongs to kGb and this correspondence is an equivalence of categories; this is enough to prove the result.

Let T be a transversal to $N(b_p)$ in G. Hence $U^G = \ldots \oplus (t \otimes U) \oplus \ldots$. Regarding this as a $kC(P)$-module, we have $t b_p t^{-1} \cdot t \otimes U = t \otimes U$, so $t \otimes U$ belongs to $kC(P)(t b_p t^{-1})$. Hence, any other conjugate of b_p annihilates $t \otimes U$ so we deduce that $b U^G = U^G$ and U^G belongs to kGb.

Also, if V is another such module then

$$\mathrm{Hom}_{kG}(U^G, V^G) \simeq \mathrm{Hom}_{kN(b_p)}((U^G)_{N(b_p)}, V)$$

$$\simeq \mathrm{Hom}_{kN(b_p)}(U \oplus \dots, V)$$

where now the dots represent modules in other blocks of $N(b_p)$ (the orbits of $N(b_p)$ on the other G-conjugates of b_p). Hence

$$\mathrm{Hom}_{kG}(U^G, V^G) \simeq \mathrm{Hom}_{kN(b_p)}(U, V)$$

Now we have to see this isomorphism is given by induction. But tracing through the isomorphisms we get that the map from U^G to V^G "contains" the original map of U to V so is the induced map.

Finally, we need only demonstrate that any module belonging to kGb is isomorphic with an induced module. Let W be a kGb-module so $W \mid (W_{PC(P)})^G$ as W is relatively P-projective since b has defect group P. Now $bW = W$ so

$$W = b_p W + \dots$$

over all the conjugates. As these summands are all conjugate in G we get that $W \mid (b_p W)^G$ so $W \mid ((b_p W)^{N(b_p)})^G$. But b_p acts as the identity on $(b_p W)^{N(b_p)}$ so we're done.

Remark: Easy to see $U \to U^G$ and $W \to b_p W$ are the correspondence and its inverse (the latter taken as $b_p \mathrm{Res}^G_{N(b_p)}(W)$).

Now we want to discuss some module-theoretic means of doing the above. Let β be a block of $PC(P)$ which is a "root" of b (so if β is an idempotent we can take $\beta = b_p$). Let $I = N(\beta)$ so $PC(P) \leq I \leq G$. Let β^I be the corresponding block of I.

The block β has a single simple module S, the canonical module. We also saw that I is the stabilizer of S in G. Since β has defect P we saw that β^I is the only block of I "over" β so a kI-module is in β^I if, and only if, its restriction to $PC(P)$ lies in β. (Remember the block β is self-conjugate in I.) Hence, if U is in β^I then $(U^G)_{PC(P)} \cong U_{PC(P)}' \oplus \cdots$ where the dots represent modules in other G-conjugates of β. This gives part of the above in a different way.

<u>Remark</u>: The first argument didn't need that P was the defect group of b, only that $PC(P)$ contained a defect group. In particular, if $C(P)$ does, as might happen analyzing cyclic blocks, then the result applies.

Multiplicities of Brauer trees

We're going to describe how to determine the multiplicity of a Brauer tree working with modules in characteristic p and not passing to characteristic zero. Let k be the usual algebraically closed field of characteristic p and let b be a cyclic block of G with defect group D. Let b_1 be the corresponding block of $N(\Omega_1(D))$. Let b have e simples, exceptional multiplicity m and let e_1, m_1 be these invariants for b_1. The standard methods give $e = e_1$. This comes about from a stable equivalence between b and b_1. This in turn implies that the Cartan matrices of b and b_1 have equal determinants (for the stable equivalence implies $G_0(b)/K_0(b) \cong G_0(b_1)/K_0(b_1)$). Hence, to see that $m = m_1$, it's enough to prove the following result:

<u>Proposition</u> If A is a Brauer tree algebra with e simple modules and exceptional multiplicity m then the determinant of the Cartan matrix of A is $em+1$.

This gives $m = m_1$. There are other possible approaches. Janusz's results on indecomposables gives a count which will do. The paper by Gabriel and Riedtmann also shows b is stably equivalent to a Nakayama algebra with same e and m and we can (as we do below) calculate the Cartan matrices of Nakayama algebras easily.

Now let's work on the proposition.

<u>Lemma 1</u> The result holds when the tree is a "star."

<u>Proof</u>. There are two cases depending on where the exceptional node is located. First consider the star

with Cartan matrix

$$\begin{pmatrix} m+1 & m & \cdots & m \\ m & m+1 & & \\ & & \ddots & \\ & & & m+1 \end{pmatrix}$$

which is $I + mJ$ where J is the e by e matrix (e is number of edges, i.e. number of simples) with all entries one. But

$$\det(I + mJ) = m^e \det\left(\tfrac{1}{m} I - (-J)\right) = m^e f(1/m)$$

where f is the characteristic polynomial of $-J$.

But $-J$ is of rank one and the vector $\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$ is an eigenvector for eigenvalue $-e$ so $f(x) = x^{e-1}(x+e)$.

Hence, the determinant is

$$m^e \left(\frac{1}{m}\right)^{e-1}\left(\frac{1}{m} + e\right) = m\left(\frac{1}{m} + e\right) = em+1,$$

as claimed.

Next suppose we have the tree

which has Cartan matrix

$$\begin{pmatrix} m+1 & 1 & \cdots & 1 \\ 1 & 2 & 1 \cdots & 1 \\ 1 & & \ddots & \\ \vdots & & & \\ 1 & & & 2 \end{pmatrix}$$

The determinant of this is, by the case just considered,

$$(m+1)\,(1.(e-1)+1) + x$$

where the first term is the first part of the first row expansion and x is the sum of the rest of the first row expansion of the determinant. But, by the case just considered for $m=1$, we know that

$$2\,(1.(e-1)+1) + x = 1.e+1$$

so

$$x = -e+1$$

and the determinant we're after is

$$(m+1)e - e + 1 = me+1$$

as claimed.

Next, we require a general lemma on determinants of matrices which have "overlapping" blocks.

Let M be an n by n matrix and let $1 < r < n$. Assume that $m_{ij} = 0$ if $i < r$ and $j > r$ or if $i > r$ and $j < r$. That is,

$$M = \left(\begin{array}{c||c} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \\ \hline\hline \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & \end{array}\right) \quad \text{row } r$$

$$\text{col } r$$

Let A_0 be the $r-1$ by $r-1$ matrix in the upper left corner, A the r by r located there. Let B_0 be the matrix in the last $n-r$ rows and columns and let B be the one of size $n-r+1$ by $n-r+1$ in that place.

<u>Lemma 2</u> $\det M = \det A \det B_0 + \det A_0 \det B$
$\qquad - (\det A_0)\, m_{rr}\, (\det B_0)$.

<u>Proof</u>. Consider a term

$$(-1)^\sigma m_{1\sigma(1)} \cdots m_{n\sigma(n)}$$

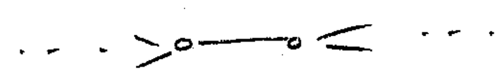
of $\det M$. If this is not zero then $\sigma(1), \ldots, \sigma(r-1) \leq r$. If, moreover, $\sigma(1), \ldots, \sigma(r-1) \leq r-1$ then this equals a term of $\det A_0 \det B$. On the other hand suppose $\sigma(k) = r$, $1 \leq k \leq r-1$. Then $\sigma(r) < r$; this follows by considering the columns. We must have a non-zero m_{ij} from each of the first $r-1$ columns and the r-th row is the last chance, as exactly $r-2$ of the first $r-1$ columns have been used. Thus, the term

$$(-1)^\sigma m_{1\sigma(1)} \cdots m_{r\sigma(r)} m_{r+1\,\sigma(r+1)} \cdots m_{n\sigma(n)}$$

corresponds to a term of $\det A \det B_0$ but one not involving m_{rr}. That is, it corresponds to a term of $\det A \det B_0 - (\det A_0)\, m_{rr} (\det B_0)$

We can now conclude:

Pf (of the proposition). We may assume the tree is not a star so that it follows easily that there is an edge which is not an "end." Hence, we can assume the tree looks like:



It has $e+f+1$ edges, e to the left of the middle edge shown, f to the right. Numbering the edges, first using the edges to the left, then the middle one and finally the ones on the right we see that the Cartan matrix is such that Lemma 2 applies and we can use induction on the submatrices.

By symmetry there are only two cases to consider: the exceptional vertex is to the left of the middle edge (strictly); the exceptional vertex is the left-hand vertex of the middle edge.

In the first case we have

$$\begin{aligned} \mathrm{Dec}\, C &= (m(e+1)+1)(f+1) + (me+1)(f+2) - (me+1)(2)(f+1) \\ &= (me+1)(f+1) + m(f+1) + (me+1)(f+1) + me+1 \\ &\quad - 2(me+1)(f+1) \end{aligned}$$

$$= m(e+f+1)+1$$

as required.

In the second case,

$$\begin{aligned} \det C &= (m(e+1)+1)(f+1) + (me+1)(m(f+1)+1) \\ &\quad - (me+1)(m+1)(f+1) \\ &= (me+1)(f+1) + m(f+1) + (me+1)m(f+1) + (me+1) \\ &\quad - (me+1)(m+1)(f+1) \\ &= mf + m + me + 1 \\ &= m(e+f+1)+1 \end{aligned}$$

so we're done.

<u>Remark</u>: If we consider a tree with e edges and each of the $e+1$ vertices having a multiplicity then the corresponding determinant should be

$$\sum_i m_1 \cdots \hat{m}_i \cdots m_{e+1}$$

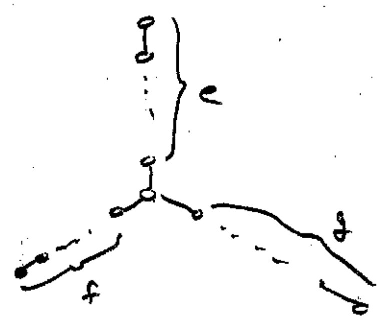
where the m_i are the multiplicities. The above argument should generalize easily to give this.

<u>Remark</u> Lemma 2 can be used in computing other determinants, e.g. the determinant of quadratic forms associated with reflection groups. Let's look at some cases. Here we have a graph and the matrix in question has rows and columns indexed by the <u>vertices</u> and there is an entry of -1 for joined vertices and there are 2's on the main diagonal. For example,

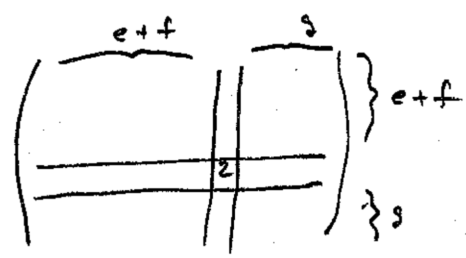
has matrix

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

which has determinant $n+1$ (by first row expansion and induction). Next consider



The matrix has form



and so has determinant

$$\begin{aligned}
&(e+f+2)(g+1) + (e+1)(f+1)(g+2) - 2(e+1)(f+1)(g+1) \\
&= (e+f+2)(g+1) + (e+1)(f+1) - (e+1)(f+1)(g+1) \\
&= (e+f+1)(g+1) + g+1 + (e+f+1) + ef \\
&\qquad - (e+f+1)(g+1) - ef(g+1) \\
&= g+1+e+f+1+ef - efg - ef \\
&= e+f+g+2-efg.
\end{aligned}$$

Let's carry on and derive further results. Let T be a tree and C its Cartan matrix. (So T is connected, undirected, etc. with e edges and $v = e+1$ vertices and no multiplicities.) For r with $1 \le r \le e$, we define an invariant $f_r(T)$ - and we also set $f_0(T) = 1$. Each subgraph with r edges of T is a forest - a union of trees - and we calculate the product of the number of vertices of the trees in the forest and add up over all forests with r edges and this is $f_r(T)$.

<u>Theorem</u> $$|C - \lambda I| = \sum_{i=0}^{e} (-1)^i f_{e-i} \lambda^i$$

This is immediate by the principal minors of C and our results above. Now let A be the "adjacency matrix of the line graph of T" so $A = C - 2I$. (Line graph is the graph whose "vertices" are the edges of T and so on.)

<u>Corollary</u> $$|A - \lambda I| = \sum_{i=0}^{e} \left(\sum_{j=0}^{e-i} (-1)^{i+j} \binom{i+j}{j} 2^j f_{e-i-j} \right) \lambda^i$$

<u>Proof</u>. We have $A - \lambda I = C - 2I - \lambda I = C - (\lambda+2) I$

so

$$|A - \lambda I| = \sum_{i=0}^{e} (-1)^i f_{e-i} (\lambda+2)^i$$

$$= \sum_{i=0}^{e} (-1)^i f_{e-i} \sum_{j=0}^{i} \binom{i}{j} 2^j \lambda^{i-j}$$

$$= \sum_{e \ge i \ge j \ge 0} (-1)^i f_{e-i} \binom{i}{j} 2^j \lambda^{i-j}$$

$$= \sum_{r=0}^{e} \Big(\sum_{\substack{e \geq i \geq j \geq 0 \\ i-j=r}} (-1)^{i} f_{e-i} \binom{i}{j} 2^{j} \Big) \lambda^{r}$$

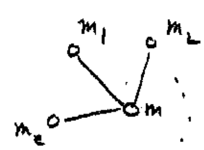
$$= \sum_{r=0}^{e} \Big(\sum_{j=0}^{e-r} (-1)^{j+r} f_{e-j-r} \binom{j+r}{j} 2^{j} \Big) \lambda^{r}$$

as required.

Our last task is to establish the formula for the Cartan matrix of a tree with multiplicities. So we have positive integers $m_1, \dots, m_v$ attached to the $v = e+1$ vertices of T and we define C_m as before, the Cartan matrix.

<u>Theorem</u> $|C_m| = \sum_{i=1}^{v} m_1 \cdots \hat{m}_i \cdots m_v$

<u>Proof</u> Let's begin with the star, labelling things a bit differently as follows:



so

$$C = \begin{pmatrix} m+m_1 & m & \cdots & m \\ m & m+m_2 & \cdots & m \\ \vdots & \vdots & \ddots & \\ m & m & & m+m_e \end{pmatrix}$$

$$= mJ + \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_e \end{pmatrix}, \quad \left(J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \right).$$

Hence

$$\det C = \det\left(mJ + \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_e \end{pmatrix}\right)$$

$$= \det\left(mJ\begin{pmatrix} m_1^{-1} & & \\ & \ddots & \\ & & m_e^{-1} \end{pmatrix} + I\right) \cdot m_1 m_2 \cdots m_e$$

$$= \det\left(1 \cdot I - \left(-mJ\begin{pmatrix} m_1^{-1} & & \\ & \ddots & \\ & & m_e^{-1} \end{pmatrix}\right)\right) \cdot m_1 \cdots m_e$$

$$= p(+1)\, m_1 \cdots m_e$$

where

$$p(\lambda) = \det(\lambda I - M)$$

and

$$M = -mJ\begin{pmatrix} m_1^{-1} & & \\ & \ddots & \\ & & m_e^{-1} \end{pmatrix}$$

But

$$p(\lambda) = \prod_{i=1}^{e} (\lambda - \lambda_i)$$

where

$$\lambda_1, \ldots, \lambda_e$$

are the eigenvalues of M. Since M has rank one (as J does) we can take $\lambda_1 = \cdots = \lambda_{e-1} = 0$. Also

$$M\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = -\begin{pmatrix} m & \cdots & m \\ m & \cdots & m \end{pmatrix}\begin{pmatrix} m_1^{-1} \\ \vdots \\ m_e^{-1} \end{pmatrix}$$

$$= -(m m_1^{-1} + \cdots + m m_e^{-1})\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

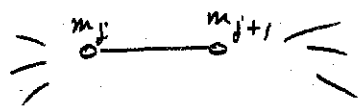
so $\lambda_e = -(m m_1^{-1} + \cdots + m m_e^{-1})$. Hence

$$p(1) = 1^{e-1}(1 + m m_1^{-1} \cdots + m m_e^{-1})$$

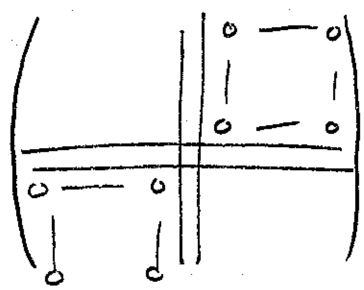
$$\det C = (1 + m m_1^{-1} + \cdots + m m_e^{-1})\, m_1 \cdots m_e$$

as needed!

We do the general case in the same way we did the special case of a Brauer tree. We pick an edge which is not an end and assume the vertices on it have multiplicities m_j and m_{j+1}, and that all m_i, $i \le j$, belong to the tree at one end of the edge and all m_i, $i \ge j+1$, belong to the tree at the other end. The picture



so the Cartan matrix is of the shape



where the "middle" entry is $m_j + m_{j+1}$.

If $1 \le s < t \le e$ denote

$$\Sigma_s^t = \sum_{i=s}^{t} m_s \cdots \hat{m}_i \cdots m_t$$

Hence, our determinant lemma yields that

$$\det C = \Sigma_1^{j+1} \cdot \Sigma_{j+1}^{v} + \Sigma_1^{j} \Sigma_j^{v} - \Sigma_1^{j}(m_j + m_{j+1}) \Sigma_{j+1}^{v}$$

by using induction. However,

$$\Sigma_1^{j+1} = \Sigma_1^{j} \cdot m_{j+1} + m_1 \cdots m_j$$

while

$$\Sigma_j^{v} = m_j \Sigma_{j+1}^{v} + m_{j+1} \cdots m_v$$

so

$$\det C = \Sigma_1^j\, m_{j+1}\, \Sigma_{j+1}^{\nu} + m_1 \cdots m_j\, \Sigma_{j+1}^{\nu}$$

$$+ \Sigma_1^j\, m_j\, \Sigma_{j+1}^{\nu} + \Sigma_1^j \cdot m_{j+1} \cdots m_{\nu}$$

$$- \Sigma_1^j\, m_j\, \Sigma_{j+1}^{\nu} - \Sigma_1^j\, m_{j+1}\, \Sigma_{j+1}^{\nu}$$

$$= m_1 \cdots m_j\, \Sigma_{j+1}^{\nu} + \Sigma_1^j\, m_{j+1} \cdots m_{\nu}$$

$$= \Sigma_1^{\nu}$$

as required.