Research Notes
Volume 8

Research Notes

Volume VIII

As usual, these notes may not be correct, complete or chronological. They were collected "off and on."

Contents

Broué's Stable Equivalence

We fix a weakly p-embedded subgroup H of a group G. Thus, H has order divisible by p and whenever $Q \neq 1$ is a p-subgroup of H then

$$N_G(Q) = N_H(Q)O_{p'}(N_G(Q)).$$

In particular, of course, H contains a Sylow p-subgroup of H. The main result is the following:

<u>Theorem</u>. The algebras $B_0(G)$ and $B_0(H)$ are stably equivalent.

We begin by establishing a connection between modules for $B_0(G)$ and $B_0(H)$. Suppose that V is an indecomposable and non-projective $B_0(H)$-module. Let $f_H(V)$ be a Green correspondent of V so there is a vertex Q for V such that $f_H(V)$ is an indecomposable $kN_H(Q)$-module with vertex Q. In particular, $f_H(V)$ lies in $B_0(N_H(Q))$ by Nagao's theorem. Since this implies that $O_{p'}(N_H(Q))$ lies in the kernel of V, there is a "lift" of $f_H(V)$ to a $kN_G(Q)$-module; denote this by $\tilde{f}_H(V)$. But modules in $B_0(N_H(Q))$ are all lying in a single block of $N_G(Q)$ so it must be $B_0(N_G(Q))$, by looking at the trivial module) thus, $\tilde{f}_H(V)$ lies in $B_0(N_G(Q))$. Moreover, $\tilde{f}_H(V)$ also has Q as a vertex: this is easy to see since induction and multiplication by the central idempotent $1/|O_{p'}(N_G(Q))| \sum_{x \in O_{p'}(N_G(Q))} x$ are commuting operations. Hence, there is a

Green correspondent $g_G(\bar{f}_H(V))$, an indecomposable kG-module with vertex Q and this also lies in $B_0(G)$ by Nagao's theorem. Conversely, if U is an indecomposable $B_0(G)$-module with vertex Q then it has a Green correspondent $f_G(U)$, which is a $kN_G(Q)$-module, this is the lift of the $kN_H(Q)$-module $\bar{f}_G(U)$ and this has a Green correspondent $g_H(\bar{f}_G(U))$. These maps, on isomorphism classes, are inverses of each other so we have this result:

<u>Lemma 1</u> We have defined a one-to-one correspondence between non-projective indecomposable $B_0(H)$-modules and non-projective indecomposable $B_0(G)$-modules.

To get the stable equivalence, we need information about maps; this comes by relating the above correspondence with induction. We shall establish two results:

<u>Proposition 2</u>. If V is an indecomposable and non-projective $B_0(H)$-module then V^G is isomorphic with the direct sum of the corresponding $B_0(G)$-module and a module which is the direct sum of projective modules and modules lying in non-principal blocks of G.

<u>Proposition 3</u>. If U is an indecomposable and non-projective $B_0(G)$-module then U_H is isomorphic with the direct sum of the corresponding $B_0(H)$-module and a module which is the direct sum of projective modules

and modules lying in non-principal blocks of H.

The Theorem is a consequence of these two propositions in a standard fashion. Suppose that U_1, U_2 are appropriate kG-modules, V_1, V_2 the corresponding kH-modules. Thus,

$$\begin{aligned}\overline{\mathrm{Hom}}_{kG}(U_1, U_2) &\simeq \overline{\mathrm{Hom}}_{kG}(V_1^G, U_2)\\ &\simeq \overline{\mathrm{Hom}}_{kH}(V_1, (U_2)_H)\\ &\simeq \overline{\mathrm{Hom}}_{kH}(V_1, V_2).\end{aligned}$$

<u>Proof</u> (of Proposition 3). Let V be the module for kH corresponding with U. By Burry-Carlson-Puig, to show that $V \mid U_H$, it suffices to prove that $f_H(V) \mid U_{N_H(Q)}$, where Q is the vertex of V and U. But $\mathfrak{f}_H(V) \mid U_{N_G(Q)}$, by assumption, so $f_H(V) \mid U_{N_H(Q)}$ as $f_H(V) \mid \mathfrak{f}_H(V)_{N_H(Q)}$. Moreover, the multiplicity of V as a summand of U_H must be one. Otherwise, by the Green correspondence, there is an indecomposable summand Y of $U_{N_G(Q)}$, with vertex containing no conjugate of Q in $N_G(Q)$, that is, not containing Q, such that $f_H(V) \mid Y_{N_H(Q)}$. But then $f_H(V)$ can't have a vertex containing Q.

Next, suppose that W is a non-projective indecomposable $B_0(H)$-module with $W \mid U_H$. Let R be the vertex of $f_H(W)$ and, therefore, of W. Hence, there is an indecomposable summand X of $U_{N_G(R)}$ such that $f_H(W) \mid X_{N_H(R)}$. Hence, X has vertex containing R so $X \in B_0(N_G(Q))$ by Nagao's theorem and the Third Main Theorem. Hence, $X \simeq \tilde{f}_H(W)$. Thus, by Burry-Carlson-Puig, U is the Green correspondent of X and we must have $R = Q$ and $V \simeq W$.

<u>Proof</u> (of Proposition 2). Let U be the correspondent of V, Q the vertex as usual. Now, by the Green correspondence, $f_H(V)^H$ is the direct sum of V and modules with vertices smaller than Q. Hence, U / V^G if, and only if, $U / f_H(V)^G$; moreover, the multiplicity as a summand will be the same. But $f_H(V)^{N_G(Q)}$ is the direct sum of $\tilde{f}_H(V)$ and modules with smaller vertices than Q or with vertex Q and in non-principal blocks. Thus, by the Green correspondence, the Third Main Theorem and Nagao's Theorem, we have that U / V^G with multiplicity one. Note that not only is this multiplicity correct, but the argument shows that no other indecomposable $B_0(G)$-module with vertex Q is a direct summand of V^G.

Finally, suppose that W is a non-projective indecomposable $B_0(G)$-module with vertex R (in H) and W / V^G; it suffices to prove that R and Q are conjugate in H in order to complete the proof of the Proposition. We have that $(V^G)_{N_G(R)}$ has an indecomposable summand with vertex R lying in $B_0(N_G(R))$ (i.e. $f_G(W)$). But

$$(V^G)_{N_G(R)} \simeq \bigoplus_{s \in N_G(R) \backslash G / H} \left(s(V)_{sHs^{-1} \cap N_G(R)} \right)^{N_G(R)}.$$

Hence, there is s with $f_G(W) \mid \left(s(V)_{sHs^{-1} \cap N_G(R)} \right)^{N_G(R)}$.

Since R is normal in $N_G(R)$ and $f_G(W)$ has vertex R, we must have $R \subseteq sHs^{-1}$, that is, $s^{-1}Rs \subseteq H$. But H is weakly p-embedded, so there is $h \in H$ with $h^{-1}s^{-1}Rsh = R$. Thus, we may replace s by sh, without any loss, and we now have $s \in N_G(R)$. Thus, $sHs^{-1} \cap N_G(R) = s(H \cap N_G(R))s^{-1}$ $= sN_H(R)s^{-1}$, so after a conjugation by s, we now have

$$f_G(W) \mid (V_{N_H(R)})^{N_G(R)}.$$

But $f_G(W)$ has vertex R, so applying Nagao's theorem to $V_{N_H(R)}$ we deduce that there is an indecomposable summand X with vertex containing R, X lying in $B_0(N_H(R))$ and $f_G(W) \mid X^{N_G(R)}$. However, we claim that $X^{N_G(R)}$ is the direct sum of $\tilde{X}$, the lift, and modules lying in non-principal blocks; this suffices for the proof as then $\tilde{X} = f_G(W)$, $X = \bar{f}_G(W)$ so X is a Green correspondent of V, by Burry–Carlson–Puig, so $Q = R$. Indeed, $(k_{O_{p'}(N_H(R))})^{O_{p'}(N_G(R))}$ is the direct sum of k and non-trivial modules. Since $O_{p'}(N_H(R)) = H \cap O_{p'}(N_G(R))$ and $X \mid X^{N_G(R)}$, a little dimension counting gives the claim.

Stable equivalence and p-groups

In studying group representations one often has the following situation. We have A and B algebras over a field k of characteristic p. There are functors f and g from the category of A-modules to the category of B-modules and the other way, respectively, which preserve exact sequences, $Ext^1(\ ,\ )$ and which induce a stable equivalence in the following way: If U is an indecomposable non-projective A-module then $f(U)$ is the direct sum of such a B-module and a projective B-module, g has a similar property, f and g are inverses of each other on the non-projective indecomposables (disregarding the projective summands) and so on. We shall examine this situation but note that much that we do only use part of these now fixed hypotheses.

<u>Theorem</u>. If P is a non-identity p-group and $B = kP$ then A is the direct sum of a semisimple algebra and a non-semisimple local algebra. Moreover, the indecomposable and non-projective kP-module corresponding with the non-projective simple A-module is an endo-trivial module.

First, let's prove a lemma and do so in more generality than we actually require.

<u>Lemma</u> The A-module U is projective if, and only if, $\overline{Hom}_A(U, S) = 0$ for all simple A-modules S.

Here, as usual, $\overline{\mathrm{Hom}}_A(U,S)$ is the quotient of $\mathrm{Hom}_A(U,S)$ by the projective maps of U to S, that is, the homomorphisms which factor through a projective.

<u>Proof.</u> Let P_U be a projective cover of U and let φ be a homomorphism of P_U onto $U/\mathrm{rad}\,U$ (so the kernel is $\mathrm{rad}(P_U)$). Thus, P_U is also a projective cover of $U/\mathrm{rad}\,U$ and φ exhibits. Since our hypothesis implies that every homomorphism of U to a semisimple module is projective there is a homomorphism ρ of U to P_U such that $\varphi \circ \rho = \eta$ where η is the natural map of U onto $U/\mathrm{rad}\,U$. But ρ must also be onto or

$$\begin{tikzcd} P_U \arrow[r, "\varphi"] & U/\mathrm{rad}\,U \\ U \arrow[u, "\rho"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

else $\mathrm{rad}(P_U) + \rho(U) \subsetneqq P_U$ and $\varphi \circ \rho$ is not onto, a contradiction. Thus $P_U \mid U$ so $P_U \cong U$ as U is a homomorphic image of U. Thus, U is projective; the converse is obvious so the lemma is proved.

<u>Proof</u> (of the theorem). Since $P \neq 1$, not all kP-modules are projective so the same holds for A. Since then A is the direct sum of a semisimple algebra and a non-zero algebra with no semisimple summand and the second summand has all the properties of A it follows that we may assume, without any loss of generality, that A has no semisimple summand, that is, no simple A-module

is projective. We wish to prove that A is local.

Let $S_1, \ldots, S_n$ be the simple A-modules and let $U_1, \ldots, U_n$ be the corresponding indecomposable and non-projective kP-modules. Hence, as $\overline{\mathrm{Hom}}_A(S_i, S_j) = 0$ if $i \neq j$ (as $\mathrm{Hom}_A(S_i, S_j) = 0$ then), it follows from our hypotheses that $\overline{\mathrm{Hom}}_{kP}(U_i, U_j) = 0$ so $\overline{\mathrm{Hom}}_{kP}(U_i \otimes U_j^*, k) = 0$. Thus the lemma, taking kP as the algebra it applies to, gives us that $U_i \otimes U_j^*$ is projective so $\mathrm{Ext}^1_{kP}(U_i \otimes U_j^*, k) = 0$, $\mathrm{Ext}^1_{kP}(U_i, U_j) = 0$, $\mathrm{Ext}^1_A(S_i, S_j) = 0$.

Thus, $A = A_1 + \cdots + A_n$ is the direct sum of local algebras A_i, $1 \leq i \leq n$, where S_i is the A_i-module which is simple. Let U be the non-projective indecomposable A-module corresponding to the trivial kP-module k so, with a renumbering if necessary, U is an A_1-module. Since the functor $\mathcal{S}$ preserves exact sequences, we deduce that every A-module is the direct sum of an A_1-module and a projective module. Thus, each A_i-module is projective if $i > 1$; thus we deduce that $n = 1$.

(Here is another argument avoiding the use of the preservation of exact sequences at this point. If V is a non-projective indecomposable kP-module then $\overline{\mathrm{Hom}}_{kP}(V, k) \neq 0$, by the lemma, so $\overline{\mathrm{Hom}}_A(W, U) \neq 0$ when U is the indecomposable and non-projective A-module corresponding with V. Thus, $\mathrm{Hom}_A(W, U) \neq 0$ so W must lie in the block A_1.)

Finally, let V be the indecomposable kP-module corresponding with the simple A-module S ($= S_1$). The preservation of exact sequences and the fact that projective kP-modules certainly have dimensions divisible by p imply that $p \nmid \dim V$. Thus,

$$V \otimes V^* \simeq k \oplus X$$

for a kP-module X. We must show that X is projective. But $\overline{\mathrm{Hom}}_{kP}(V, V) \simeq \overline{\mathrm{Hom}}_A(S, S) \simeq \mathrm{Hom}_A(S, S) \simeq k$ so $\overline{\mathrm{Hom}}_{kP}(V \otimes V^*, k) \simeq k$, that is $\overline{\mathrm{Hom}}_{kP}(X, k) = 0$ as $\overline{\mathrm{Hom}}_{kP}(k, k) \simeq k$. The lemma now gives the desired conclusion.

As an application, note that Frobenius' theorem on normal p-complements is a consequence using Broué's (or mine) stable equivalence for the principal block and the observation that the conclusion of Frobenius' theorem only requires the principal block to be a local algebra.

<u>Remark</u>: By using "$\otimes V^*$" we can get a new stable equivalence in which S and k correspond. When can we prove that this is a Morita equivalence?

Fusion and the principal block

We are going to examine the relation between control of fusion and our "projective weight" conjecture in the case of the principal block. We denote the principal p-block of G by $B_0(G)$, as usual. Let us also introduce some vocabulary in connection with our conjecture (or question). If Q is a p-subgroup of G, $L = N(Q)$ and kL has a simple module with vertex Q (i.e. is projective as a module for L/Q) lying in a block of L corresponding with the block B of G then we say that L is a B-parabolic subgroup of G. Note that L determines Q since $Q = O_p(L)$ follows from the existence of the required simple module. (Note: Perhaps $G = N_G(1)$ should also be parabolic ?)

Proposition 1 If the normalizer of a Sylow p-subgroup of G controls strong fusion in G then the only $B_0(G)$-parabolic subgroups of G are the normalizers of the Sylow p-subgroups.

Proof. Let $L = N(Q)$ be a $B_0(G)$-parabolic; we wish to deduce that Q is a Sylow p-subgroup of G. We know, as we have shown before in discussing the weight problem, that Q is a Sylow p-subgroup of $O_{p',p}(L)$ and L is p-constrained. Let R be a Sylow p-subgroup of L containing Q and P a Sylow p-subgroup of G containing R; hence $R = N_P(Q)$ and it suffices to prove that $R = Q$ as then $P = Q$.

Our hypothesis on strong control means, in particular, that $L \subseteq C(Q)\,N(P)$ so $L \subseteq C(Q).\,L \cap N(P) \subseteq C(Q)\,N_L(R)$

since any element that normalizes Q and P must normalize $R = N_P(Q)$. But L is p-constrained so $C(Q) \subseteq Q\, O_{p'}(L)$ and so $L \subseteq O_{p'}(L) N_L(R)$. Thus, R is a p-subgroup of $O_{p',p}(L)$ so $|R| \leq |Q|$ and $Q = R$; the result is proved.

Let $l_0(H)$ denote the number of simple kH-modules lying in $B_0(H)$ for any group H. Hence, we have a consequence:

Corollary 2. If the projective weight conjecture holds for $B_0(G)$ and the normalizer of a Sylow p-subgroup P of G controls strong fusion then

$$l_0(G) = l_0(N_G(P)).$$

Proof. This is immediate from the Proposition.

We can almost get a converse as well. Recall the strongly p-embedded conjecture: If the group H has a strongly p-embedded subgroup then it has a block of defect zero.

Proposition 3. If P is a Sylow p-subgroup of G and $l_0(G) = l_0(N_G(P))$ then $N(P)$ controls strong fusion in G, provided the projective weight conjecture holds for $B_0(G)$ and the strongly p-embedded conjecture holds for sections of G.

Proof. Assuming these two conjectured results hold as stated, we must show that if $N(P)$ does not control strong fusion then $N(P)$ and its conjugates are not the only $B_0(G)$-parabolic subgroups. However, we now know, by Puig's thesis, that there is an essential subgroup; there is a non-identity p-subgroup Q, not a Sylow p-subgroup, such that Q is a Sylow p-subgroup of $QC(Q)$ and $N(Q)/QC(Q)$ has a strongly p-embedded subgroup. Thus $QC(Q) = Q\,O_{p'}(L)$, where $L = N(Q)$, so L is p-constrained. But $L/QC(Q)$ has a projective simple module and this must now be lying in $B_0(L)$ so L is a $B_0(G)$-parabolic subgroup of G; the proof is complete.

Now define, again as usual, $k_0(H)$ to be the number of irreducible characters in $B_0(H)$. We shall establish the

Theorem 4. If P is a Sylow p-subgroup of G and $N(P)$ controls strong fusion in P then

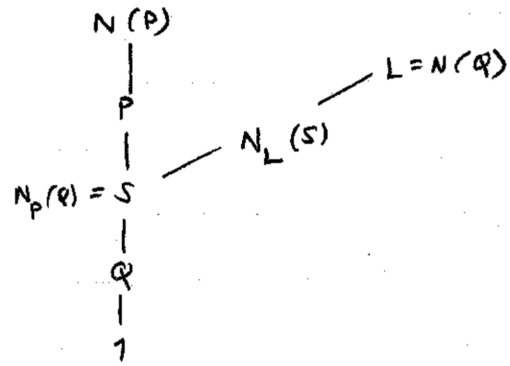
$$k_0(G) = k_0(N(P))$$

provided the projective weight conjecture holds.

We shall fix these hypotheses (though generally we don't need the weight conjecture until later) and proceed in a number of steps.

Lemma 5. In each p-local subgroup, the normalizer of a Sylow p-subgroup controls strong fusion.

Proof. Let Q be a non-identity p-group in G and set $L = N(Q)$; without any loss of generality we shall assume that $Q \subseteq P$ and $N_P(Q) = S$ is a Sylow p-subgroup of L. We have the following picture:



Let T be a subgroup of S; we want to show that $N_L(T) \subseteq C_L(T)\, N_L(S)$. First, suppose that $Q \subseteq T$. If $l \in N_L(T)$ then express $l = cn$, $c \in C(T)$, $n \in N(P)$, as our fixed hypothesis allows us to do. $\therefore n = c^{-1}l \in C(T)L \subseteq C(Q)L \subseteq L$ so n normalizes Q as well as P. Hence, n also normalizes S, so $l = cn \in C(T)\, N_L(S)$. For an arbitrary subgroup T we now have

$$\begin{aligned} N_L(T) &\subseteq N_L(TQ) \\ &\subseteq C_L(TQ)\, N_L(S), \quad \text{as } TQ \supseteq Q, \\ &\subseteq C_L(T)\, N_L(S) \end{aligned}$$

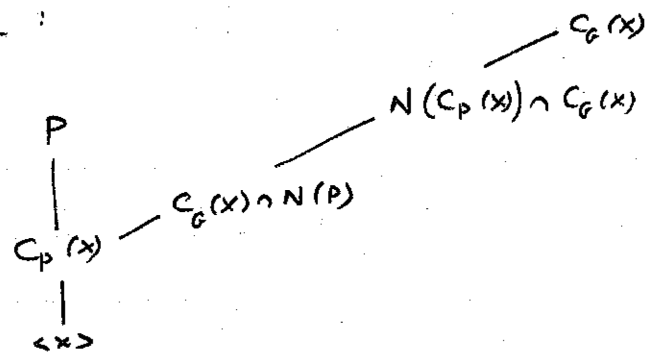
and the result is proved.

Remarks. 1. Puig says this result is in his thesis and he shows $L \cap N(P)$ controls. (In fact, our argument does this too.

2. The same applies to the group $C(X)$, X an

element of P. The proof is virtually the same.

The idea for proving Theorem 4 is to use the Second main Theorem, in connection with Corollary 2. Say $x \in P$ and $C_P(x)$ is a Sylow p-subgroup of $C_G(x)$. We have the following picture:



Now $N(C_P(x)) \cap C_G(x)$ controls strong fusion in $C_P(x)$ by Lemma 5, so by Corollary 2, $l_0(C_G(x)) = l_0(N(C_P(x)) \cap C_G(x))$. But we must go after $l_0(C_G(x) \cap N(P))$!

Remark (Puig) The first remark above can be applied to make things easier!

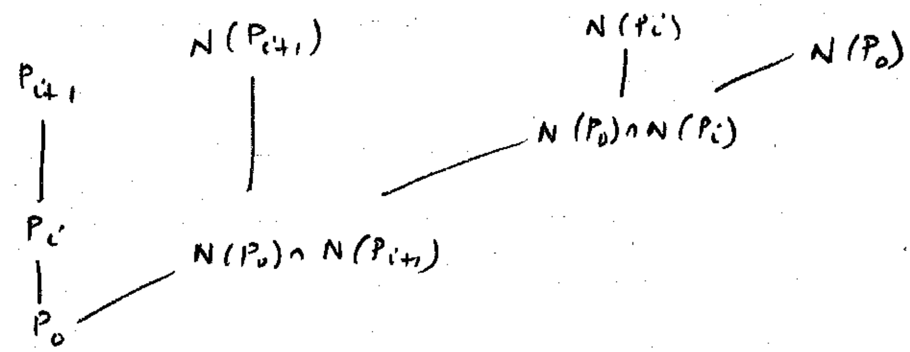
Lemma 6. If Q is an extremal subgroup of P then
$$l_0(N(Q) \cap N(N_P(Q))) = l_0(N(Q) \cap N(P)).$$

Proof. We are assuming that $N_P(Q)$ is a Sylow p-subgroup of $N(Q)$. By replacing Q by a conjugate, if necessary, we may assume, without any loss of generality, that when we set $P_0 = Q$, $P_1 = N_P(P_0)$, …, $P_{n+1} = N_P(P_n)$, … that each P_{n+1} is a Sylow p-subgroup of $N(P_n)$. We shall prove a result now that is stronger than the stated one, namely,

$$\ell_0(N(P_0) \cap N(P_1) = \ell_0(N(P_0) \cap N(P_i)))$$

for all $i \geq 1$. (This is stronger, as for some large n, $P_n = P$.)

We have that $N(P_0) \cap N(P_{i+1}) \leq N(P_0) \cap N(P_i)$ since if $g \in G$ normalizes P_0 and P_{i+1} then it normalizes $P_1 = N_P(P_0) = N_{P_{i+1}}(P_0)$, $P_2 = N_P(P_0) = N_{P_{i+1}}(P_0)$, $\ldots$. Hence, we have the following picture:



Let us begin by examining $N(P_i)$. It has P_{i+1} as a Sylow p-subgroup so $P_{i+1} \cap C(P_i)$ is a Sylow p-subgroup of $C(P_i)$. But

$$P_{i+1} \cap C(P_i) \leq C(P_0) \leq P_1 \leq P_i$$

so

$$P_{i+1} \cap C(P_i) \leq P_i \cap C(P_i) = Z(P_i).$$

Hence, $C(P_i)/Z(P_i)$ is a p'-group: $P_i\, C(P_i) = P_i K$, a direct product of P_i and a p'-group K. By Lemma 5, we have

$$N(P_i) = C(P_i) . N(P_i) \cap N(P_{i+1})$$

so

$$N(P_i) = K . N(P_i) \cap N(P_{i+1}),$$

that is, $N(P_i)/K$ has a normal Sylow p-subgroup and so $N(P_i)$ is p-solvable of p-length one.

We can now complete the proof. Since $K \subseteq N(P_0) \cap N(P)_i$, and we already have seen that $N(P_i) = N(P_i) \cap N(P_{i+1}) \cdot K$, we have that

$$N(P_0) \cap N(P_i) = N(P_0) \cap N(P_{i+1}) \cdot K.$$

Therefore, $\ell_0(N(P_0) \cap N(P_i)) = \ell_0(N(P_0) \cap N(P_{i+1}))$ so an obvious induction now completes the proof.

Remarks 1. The hypothesis on strong fusion means that Q is always extremal and that there is no need to change Q at the beginning of the proof.

2. The same holds for $C(x)$, $x \in P$. Just use $P_0 = \{x\}$, $P_1 = C_p(x)$, $P_2 = N_p(P_1), \ldots$ and mimic the above argument.

Now Lemma 5 and our fixed hypothesis imply that $\ell_0(N(Q)) = \ell_0(N(Q) \cap N(N_p(Q)))$. Hence, the last result now implies the next, immediately.

<u>Lemma 7</u>. If Q is a subgroup of P then $\ell_0(N(Q)) = \ell_0(N(Q) \cap N(P))$.

The same holds for $C(x)$ as before. The Second Main Theorem now completes the proof of the theorem.

Broué's Morita Equivalence.

Broué has proved the following result (undoubtedly over rings instead of our usual field):

Theorem. Let f be a primitive central idempotent in kN where N is a normal subgroup of G. Let H be the stabilizer of f in G and set $e = Tr_H^G f$. It follows that e is a central idempotent of kG and

$$kGe \simeq M_{|G:H|}(kHf).$$

Of course, f is still central in kH so kHf is an algebra. Note that e is clearly central in G by the trace definition. Moreover, if $g \in G$, $g \notin H$ then $gfg^{-1} \neq f$ so gfg^{-1} is another primitive central idempotent of kN and gfg^{-1} and f are orthogonal; hence, it follows that e is an idempotent.

Now, from the theory of Morita equivalences, it suffices to prove that Ind_H^G gives an equivalence between the category of kHf-modules and kGe-modules. Let V be a kHf-module so

$$Ind_H^G V = \bigoplus_{t \in G/H} t \otimes V.$$

Now

$$\begin{aligned} tft^{-1} \cdot t \otimes v &= tf \cdot 1 \otimes v \\ &= t \cdot 1 \otimes v \\ &= t \otimes v \end{aligned}$$

while if $s \notin tH$ then

$$tft^{-1} \cdot s \otimes v = s(s^{-1}tft^{-1}s) 1 \otimes v$$

$$= s(s^{-1}t)\,f(s^{-1}t)^{-1}\cdot 1\otimes v$$
$$= 0$$

as conjugates of f, distinct from f, will annihilate $1\otimes v$. We deduce that $e = Tr_H^G f$ induces the identity on $Ind_H^G V$ so that module is a kGe-module.

Every kGe-module is of this sort too. For if U is such a module then $eU = e$ so $U = \sum_{t\in G/H} tft^{-1}U$ is direct, from which it is easy to see that $U \cong Ind_H^G(fU)$. Hence, it remains to deal with maps. If U_1, U_2 are kGe-modules then

$$\begin{aligned}
Hom_{kG}(U_1, U_2) &\cong Hom_{kG}(U_1, Ind_H^G fU_2)\\
&\cong Hom_{kH}((U_1)_H, fU_2)\\
&\cong Hom_{kH}\Big(\bigoplus_{t\in G/H} tft^{-1}U_1 \;, fU_2\Big)\\
&\cong Hom_{kH}(fU_1, fU_2)
\end{aligned}$$

just as required.

Control of fusion and blocks

Our interest here is to what extent our results for the principal block can be extended to arbitrary blocks. Our main result is as follows:

<u>Proposition 1.</u> If b is a p-block of G, (D, b_D) is a Sylow b-subpair and $N(D, b_D)$ controls strong fusion in (D, b_D) then
$$\ell(b_D^{N(D,b_D)}) = \ell(b)$$
provided the projective weight conjecture holds.

Our first task is to relate the b-parabolic subgroups to b-local subgroups, that is, to normalizers of subpairs (Q, b_Q) where $Q \neq 1$ and (Q, b_Q) is a b-subpair. Suppose that $Q \neq 1$ is a p-subgroup such that $N(Q)$ is a b-parabolic. Let b_Q be a block of $Q\,C(Q)$ covered by β where β is the block of $N(Q)$ containing the simple module V with vertex Q such that $\beta^G = b$. Thus $b_Q^{N(Q)} = \beta$ and, since $Q\,C(Q) \trianglelefteq N(Q)$ and $Q\,C(Q)$ contains the centralizer in $N(Q)$ of the defect groups of any of its blocks, b_Q and its conjugates in $N(Q)$ are the only blocks with this property. The blocks $b_Q^{N(Q, b_Q)}$ and $b_Q^{N(Q)} = \beta$ are Morita equivalent by induction, by Broué's Morita equivalence (see the preceding section of these notes). Hence, $b_Q^{N(Q, b_Q)}$ has a simple module with vertex Q and, up to conjugacy and isomorphism, this b-local corresponds one-to-one with the p-local $N(Q)$. (Remember the statement above about b_Q and its conjugates. The converse is true too. If we start with b_Q and $b_Q^{N(Q,b_Q)}$ has the right sort of simple module then the Morita equivalence shows that $N(Q)$ is a b-parabolic. (It's important to note the easy remark that Broué's Morita equivalence preserves vertices.)

Hence, we can define a parabolic b-local as $N(Q, b_Q)$ as above, where we must keep the block of $N(Q, b_Q)$ in mind. The reason is that $Q = O_p(N(Q, b_Q))$ by the existence of the simple module with vertex Q. Up to conjugacy these are in one-to-one correspondence with the b-parabolic subgroups and the simple modules in question similarly match.

Therefore, in order to prove the Proposition, we need only demonstrate the following:

<u>Lemma 2</u>. Under the above hypotheses $N(D, b_D)$ is, up to conjugacy, the only b-local parabolic.

Thus, we are in a position to mimic the approach we used when we studied the principal block, but it is more difficult. Assume that $L = N(Q, b_Q)$ is a parabolic b-local and $(Q, b_Q) \subsetneqq (D, b_D)$. Let (R, b_R) be the normalizer subpair of (Q, b_Q) in (D, b_D) so $Q \subsetneqq R$ and, by our hypothesis on control, (R, b_R) is a Sylow b_Q^L-subpair of L.

We next observe that $R\,C(Q)$ is a normal subgroup of L. Indeed, $L \subseteq C(Q)\,N(D, b_D)$ so $L \subseteq C(Q)(N(D,b_D) \cap N(Q,b_Q))$. But the second factor, the intersection, is contained in $N(R, b_R)$ so $L \subseteq C(Q)\,N(R)$, that is, the image of R in $L/Q\,C(Q)$ is normal.

Let $\beta = b_Q^{RC(Q)} = b_R^{RC(Q)}$ (these are equal as $(Q, b_Q) \triangleleft (R, b_R)$) so β has defect group containing R. Therefore, β and its conjugates are the blocks of $RC(Q)$ covered by b_Q^L (as $\beta^L = b_Q^L$ covers β). Let S be a simple kL-module lying in b_Q^L with vertex Q; it exists by assumption. Let T be a simple $kRC(Q)$-module lying in β with $T \mid S_{RC(Q)}$; this exists by Clifford's theorem and basic facts on covering. Now T has vertex Q: T is projective as a module for $RC(Q)/Q$ as S is projective as a module for L/Q; T_Q is a direct sum of trivial modules as S_Q is such a sum. Thus, T is its own projective cover as a module for $RC(Q)/Q$. Let us determine the composition factors of the projective cover of T as a module for $RC(Q)$; we use the Alperin-Collins-Sibley theorem. As a module for $RC(Q)$ under conjugation, kQ has only trivial composition factors because $RC(Q)$ induces on Q only automorphisms of order a power of p. Hence, T is the only composition factor of its projective cover as $kRC(Q)$-module. Hence, T is the unique simple module in β so Q is the defect group of β as it is the vertex of T. This contradicts $R \supsetneq Q$, so the lemma is proved and so is the proposition.

We shall conclude by observing that a "converse," analogous to what occurs for the principal block, does not hold. Let G be the semi-direct product of an elementary abelian group E of order p^3 by $SL(2,p) = H$ with irreducible action. Thus, kG has simple modules of dimensions $1, 2, \ldots, p$. By our construction, every defect group must contain E so each block is of full defect. Suppose that $p \neq 2$; hence the First Main Theorem implies (from $Z(H) \cong \mathbb{Z}_2$) that G has exactly two p-blocks, the principal p-block and one other, $B_1(G)$. Now $B_0(G)$ has the simples of dimensions $1, 3, \ldots, p-2$ while $B_1(G)$ has the ones of dimension $2, 4, \ldots, p-1$ and the p-dimensional simple lies in one of these two blocks. Let P be a Sylow p-subgroup of G and let $B_0(N(P))$ and $B_1(N(P))$ be the Brauer correspondents of $B_0(G)$ and $B_1(G)$, respectively. Thus

$$\ell(B_0(N(P))) = \ell(B_1(N(P))) = \frac{p-1}{2}.$$

If the p-dimensional simple lies in $B_1(G)$ then $\ell(B_0(G)) = \frac{p-1}{2}$ so, by the "converse" for the principal block, we have a contradiction to the fact that $N(P)$ does not control strong fusion. Hence, $\ell(B_1(G)) = \frac{p-1}{2}$, $\ell(B_0(G)) = \frac{p+1}{2}$ so $\ell(B_1(G)) = \ell(B_1(N(P)))$ but it's easy to see that there is no strong control for B_1 which would occur if there were a converse.

Applications of "local" modules

In our proof of the Brauer Induction Theorem we introduced "local" modules, ones induced up from local subgroups. In particular, we deduced a result from the Green correspondence, expressing all modules as "virtual" direct sums of local modules and projectives.

Let us begin, by using these ideas to obtain a new proof of the Brauer lift. Let R, k be as usual. If U is a kG-module then we wish to produce RG-lattices M and N such that

$$U \oplus \bar{M} \sim \bar{N}$$

where the tilde denotes the relation of having the same composition factors and the bars denote reduction modulo p. Now there are local modules L and L′ and projective modules Q and Q′ such that

$$U \oplus L \oplus Q \simeq L' \oplus Q'.$$

Now Q and Q′ lift to RG-lattices. If $O_p(G) = 1$ then, by induction, the Brauer lift works for L and L′ so we are done. If $O_p(G) \neq 1$ then we note that it suffices to prove the result when U is simple. But then $O_p(G)$ is in the kernel of U so U is a $k[G/O_p(G)]$ module and we can apply induction to U for $G/O_p(G)$. This completes the proof.

Next, we wish to introduce a refinement of the lemma on "local" modules, and this is a canonical complex, which we dub the Green complex for every kG-module. We define it for indecomposable

kG-modules and use direct sums for arbitrary modules. We now proceed by induction on the order of the vertex of the indecomposable kG-module U. If it is 1 then we define the complex, by setting $C_0 = U$, $C_n = 0$, $n > 0$ and define an augmentation which is the identity map. Suppose that U has vertex $Q \neq 1$ and Green correspondent V. We now set $C_0 = V^G$ and express $C_0 = U \oplus X$ and use the Green complex of X, already defined as each indecomposable summand of X has vertex smaller than Q, to give the complex for U. If

$$\cdots \to D_1 \to D_0 \to X$$

is the complex for X then

$$\cdots \to C_2 = D_1 \to C_1 = D_0 \to C_0 = V^G \to U$$

is the complex for U. Each term of the complex is the direct sum of a local module and a projective module and each map splits so the lemma is a consequence of the construction. The complex is also clearly canonical.

We can relate this process to a graph the nodes of which are the Green correspondents for indecomposable kG-modules (up to conjugacy and isomorphism of course). If V_1 and V_2 are such modules then there is a directed edge from V_1 to V_2 if $U_2 \mid V_1^G$ where U_2 is the kG-module corresponding to V_2 (and where we assume V_1 and V_2 are distinct (up to conjugacy and isomorphism).

A number of questions suggest themselves.

1) Does this complex have an intrinsic characterization?

2) Is it related to other known complexes?
3) What about a calculation for the trivial module for $G = GL(n, q)$, q a power of p.
4) Can we use Mackey's theorem and Burry-Carlson-Puig to describe the nodes at distance d from a given one.
5) Can we use this complex to efficiently calculate $\overline{Hom}_{kG}(U_1, U_2)$ for kG-modules U_1, U_2 (where we mean homomorphisms modulo projective maps as usual).

Cohomology algebra radicals

Let k be a field of characteristic p and G a finite group. We wish to "explain" the dichotomy, $p=2$, $p\neq 2$, that arises in complexity theory. The answer is entirely trivial.

Theorem. With the above notation, let J be the Jacobson radical of $H^*(G,k)$.

1) J is a homogenous nil ideal.
2) $H^*(G,k)/J$ is a commutative affine ring.
3) $H^*(G,k)/J \cong H^{ev}(G,k)/J(H^{ev}(G,k))$ if $p\neq 2$.

Proof. Suppose that $x_1,\dots,x_n$ are homogenous nilpotent elements of $H^*(G,k)$. Then each term of a high power of $x_1+\dots+x_n$ involves some x_i to a high power and so is zero, by the graded commutativity. Moreover, if a is homogenous then so is ax_i; hence, the set of elements N of $H^*(G,k)$ whose homogenous components are nilpotent is a nil ideal so $N\subseteq J$. If we establish that $H^*(G,k)/N$ is a commutative affine ring then $N=J$ and 2) holds.

But if $p\neq 2$ and x is homogenous of odd degree then $x^2=0$ as $x^2=-x^2$ by graded commutativity. Thus, $H^*(G,k)=N+H^{ev}(G,k)$ so $H^*(G,k)/N$ is certainly commutative. We now assert that every nilpotent element of $H^*(G,k)$ lies in N (so thus, $H^*(G,k)/N$ is affine by the Hilbert Nullstellensatz). This will now establish the first two assertions. Let

$$x_1 + \cdots + x_n = x$$

be in $H^*(G, k)$, nilpotent, each x_i homogeneous and with

$$\deg x_1 < \cdots < \deg x_n.$$

Assume $x \notin N$ and minimize $\deg x_n$ among all such elements. But the "leading term" of any power of x is a power of x_n so x_n must be nilpotent. Thus $x_n \in N$ so

$$x \equiv x_1 + \cdots + x_{n-1} \pmod{N}$$

and so some power of $x_1 + \cdots + x_{n-1}$ lies in N (as some power of x is zero and lies in N consequently); but this power is then nilpotent, as N is nil, so by minimality of $\deg x_n$, $x_1 + \cdots + x_{n-1} \in N$ so $x \in N$, a contradiction.

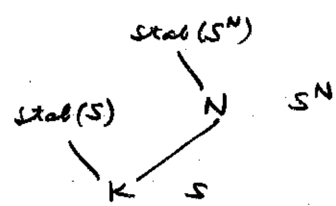
The third statement is immediate now.

Projective simple modules and normal subgroups

The result was suggested by remarks of D. Robinson and he seems to have proved it also, but by different methods.

<u>Theorem</u>. Let $1 \triangleleft K \triangleleft N \triangleleft G$ be normal subgroups of G with N/K a p-group. There is a one-to-one correspondence between the G-orbits of projective simple kN-modules with p' inertial index and the G-orbits of projective simple kK-modules with p' inertial index.

<u>Proof</u>. We begin by defining a map of the "suitable" (i.e. projective, simple and of p' inertial index) modules for K to the "suitable" modules for N. Let S be a "suitable" kK-module and, in particular, the stabilizer of S in N is just K, inasmuch as N/K is a p-group and S has p' inertial index. Thus, S has exactly $|N:K|$ conjugates in N so $(S^N)_K$ is the direct sum of $|N:K|$ distinct conjugates of S in N. Hence, by Clifford's theorem, S^N is simple, as a simple submodule of S^N would have the same summands as S^N on restriction to K. Moreover, since S is projective so also is S^N. We assert that S^N has p' inertial index. We have the picture:



Now certainly $\mathrm{Stab}(S) \subseteq \mathrm{Stab}(S^N)$ and $\mathrm{Stab}(S) \cap N = K$. We claim that $\mathrm{Stab}(S^N) = \mathrm{Stab}(S)N$; the First Isomorphism Theorem then gives $\mathrm{Stab}(S)/K \cong \mathrm{Stab}(S^N)/N$ and our claim on S^N is valid. Suppose $g \in \mathrm{Stab}(S^N)$ so $g \in \mathrm{Stab}((S^N)_K)$. Hence, there is $y \in N$ such that $gy^{-1} \in \mathrm{Stab}(S)$, as $g \otimes S \cong y \otimes S$ for a suitable y, that is, $g \in \mathrm{Stab}(S)N$ as required.

We have therefore defined a map from the "suitable" kK-modules to the "suitable" kN-modules. It is also clearly one-to-one on orbits; that is, if S_1 is another "suitable" kK-module and S^N and S_1^N are G-conjugate then so are S and S_1. Indeed, S and S_1 are summands of $(S^N)_K$ and $(S_1^N)_K$, respectively, so S^N and S_1^N being G-conjugate means the components of their restrictions are too.

Hence, it remains only to show that if T is a "suitable" kN-module then $T \cong S^N$ for a "suitable" kK-module to be chosen. We simply let S be a simple summand of T_K. Since N/K is a p-group and k has characteristic p it follows that S has an extension S^e to $\mathrm{Stab}_N(S)$. Thus, by Clifford theory, $T \cong (S^e)^N$. $\therefore\ S^e \mid T_{\mathrm{Stab}_N(S)}$ so S^e is projective as T is projective. Hence,

$$S^e \mid \left((S^e)_K\right)^{\mathrm{Stab}_N(K)} \cong S^{\mathrm{Stab}_N(S)}$$

but the latter is indecomposable by Green's Indecomposability Criterion so $S^e \cong S^{\mathrm{Stab}_N(S)}$. Hence, by dimension counting, $\mathrm{Stab}_N(S) = K$. Therefore, S is a projective simple kK-module and its inertial index <u>in</u> N is not divisible by p. Moreover $S^N \cong T$ as $S = S^e$.

Hence, it suffices to prove that S has a p'inertial index in G. But $\mathrm{Stab}_G(S) \leq \mathrm{Stab}_G(T)$, as $T = S^N$, so $|\mathrm{Stab}_G(S)N : N|$ is not divisible by p. Moreover this is equal to $|\mathrm{Stab}_G(S) : \mathrm{Stab}_G(S) \cap N| = |\mathrm{Stab}_G(S) : K|$, as we have seen, and this is the inertial index of S in G. The result is proved.

Blocks of quotient groups.

G. Robinson has proved the following result: there is a one-to-one correspondence between blocks of a group G and blocks of $G/\Phi(P)$, where P is a normal p-subgroup of G and $\Phi(P)$ is its Frattini subgroup.

We wish to point out that this is clear from a module-theoretic point of view. Indeed, if S, T are kG-modules and simple, so they are also modules for $G/\Phi(P)$, then it suffices to show that if $\mathrm{Ext}^1_{kG}(S,T) \neq 0$ then $\mathrm{Ext}^1_{kG/\Phi(P)}(S,T) \neq 0$. However, if we have an exact sequence of kG-modules,

$$0 \to S \to U \to T \to 0$$

then P is trivial on T and on U so induces an elementary abelian group of automorphisms on U, that is, $\Phi(P)$ is trivial on U so this sequence is also a sequence of $kG/\Phi(P)$-modules. Whether the sequence splits or not is independent of $\Phi(P)$!

2-permutation modules for GL(3, 2)

We shall examine the structure of the p-permutation modules for $p=2$ and $G = GL(3,2)$, as defined by Broué and Puig, i.e. summands of kG permutation modules, k algebraically closed of characteristic p. This means obtaining the structure of $(k_P)^G$ where $P = 1, Z_2, Z_2\times Z_2^{(1)}, Z_2\times Z_2^{(2)}, Z_4, D_8$. Of course, for $P=1$, we already know the components:

$$P=1: \quad \begin{matrix} & 1 & \\ 3 & & 3^* \\ & 1 & \end{matrix} \qquad \begin{matrix} & 3 \\ & 3^* \\ 1 & 3 \\ & 3^* \\ & 3 \end{matrix} \qquad \begin{matrix} & 3^* \\ & 3 \\ 1 & 3^* \\ & 3 \\ & 3^* \end{matrix} \qquad 8$$

Let $Z_2\times Z_2$ be the four subgroup such that k is a submodule for the restriction of V_3 to its normalizer and let $Z_2\times Z_2^*$ and V_3^* be the other such pair. Now let Σ_4 and Σ_4^* be these normalizers. Hence,

$$\mathrm{Hom}_{kG}(k_{\Sigma_4}^G, V_3) \cong \mathrm{Hom}_{k\Sigma_4}(k, (V_3)_{\Sigma_4}) = k$$
$$\mathrm{Hom}_{kG}(V_3^*, k_{\Sigma_4}^G) \cong \mathrm{Hom}_{kG}(k_{\Sigma_4}^G, V_3) = k$$

Since $k_{\Sigma_4}^G$ has a 2-dimensional endomorphism algebra, by double transitivity, we get

$$k_{\Sigma_4}^G \cong 1 \oplus \begin{matrix} 3 \\ 3^* \end{matrix}$$

Similarly,

$$k_{\Sigma_4^*}^G \cong 1 \oplus \begin{matrix} 3^* \\ 3 \end{matrix}$$

But

$$k \mid k_{\Sigma_4}^{D_8}, \qquad k \mid k_{\Sigma_4^*}^{D_8}$$

and $V_8 \mid k_{D_8}^G$ as V_8 is projective and $\mathrm{Hom}_{kD_8}(k, V_8) \neq 0$ so we deduce, by dimension counting that the components for D_8 are as follows:

$$P = D_8 \qquad 1 \qquad \begin{smallmatrix}3\\3^*\end{smallmatrix} \qquad \begin{smallmatrix}3^*\\3\end{smallmatrix} \qquad 8$$

Our next task is to deal with the four subgroups and by duality it's enough to do one. The answers are as follows:

$$P = Z_2 \times Z_2 : \quad \begin{array}{ccccccc} & 3 & & & & & 1 \\ 1 & & 3^* & & & & \\ & & & 3 & & & \\ & & & & 3^* & & \end{array} \qquad \begin{smallmatrix}3^*\\3\end{smallmatrix} \qquad 8$$

$$P = Z_2 \times Z_2^* : \quad \begin{array}{ccccccc} & 3^* & & & & & \\ 1 & & 3 & 3^* & & & 1 \\ & & & & 3 & & \end{array} \qquad \begin{smallmatrix}3\\3^*\end{smallmatrix} \qquad 8$$

Now $k_{Z_2 \times Z_2}^{\Sigma_4} = \begin{smallmatrix}k\\k\end{smallmatrix} \oplus W \oplus W$, where W is a 2-dimensional simple module. Moreover, $k_{D_8}^{\Sigma_4} = k \oplus W$ so since we have $k_{\Sigma_4}^G$, $k_{D_8}^G$ we deduce that $W^G \cong \begin{smallmatrix}3^*\\3\end{smallmatrix} \oplus 8$.
Hence, we have the picture

$$k_{Z_2 \times Z_2}^G \cong \frac{1 \oplus \begin{smallmatrix}3\\3^*\end{smallmatrix}}{1 \oplus \begin{smallmatrix}3\\3^*\end{smallmatrix}} \oplus \begin{smallmatrix}3^*\\3\end{smallmatrix} \oplus \begin{smallmatrix}3^*\\3\end{smallmatrix} \oplus 8 \oplus 8$$

We can use reciprocity as usual to calculate socles and radical quotients and deduce that the first bit looks as follows:

$$\begin{array}{c} 1 \quad 3 \\ \hline \vdots \\ \hline 1 \quad 3^* \end{array}$$

(where socle is contained in radical as V_1 cannot be a summand as its vertex is D_8). But there is a middle which is an extension of $\begin{smallmatrix}3\\3^*\end{smallmatrix}$ by $\begin{smallmatrix}3\\3^*\end{smallmatrix}$; if this is a direct sum then the 1 at the bottom is under $\begin{smallmatrix}3\\3^*\end{smallmatrix} \oplus \begin{smallmatrix}3\\3^*\end{smallmatrix}$ so both 3^*'s "slip by" and there are two 3^*'s in the socle. Hence the extension is not split so it must be in fact $\begin{matrix}3\\3^*\\3\\3^*\end{matrix}$ by inspection of the projective cover of V_3. Now V_1 is not a summand so we get a maximal submodule of the form

$$\begin{array}{ccccc} & 3 & & & \\ / & & \diagdown & & \\ 1 & & & 3^* & \\ & & & \diagdown & \\ & & & & 3 \\ & & & & \diagdown \\ & & & & \quad 3^* \end{array}$$

and since the module is self-dual we get the desired structure.

Next, let's turn to Z_4 and we first guess the answer to be

$$P = Z_4: \quad \begin{array}{ccccccccccccc} & 3 & & & & & 1 & & & & & 3^* & \\ 1 & & 3^* & & & & & & & & 3 & & \boxed{} \\ & & & 3 & & & & & & 3^* & & & \\ & & & & 3^* & & & & 3 & & & & \end{array} \qquad 8$$

where the box is the "1" at the very left, so the module is a "band."

Again we can easily compute socles and radical quotients and the middle we're after is an extension

$$\frac{\begin{matrix}3\\3^*\end{matrix} \oplus \begin{matrix}3^*\\3\end{matrix}}{\begin{matrix}3\\3^*\end{matrix} \oplus \begin{matrix}3^*\\3\end{matrix}}$$

as the whole first summand is

$$\frac{1 \oplus \begin{matrix}3\\3^*\end{matrix} \oplus \begin{matrix}3^*\\3\end{matrix}}{1 + \begin{matrix}3^*\\3\end{matrix} \oplus \begin{matrix}3\\3^*\end{matrix}}$$

using $k_{Z_4}^{D_8}$. (We also know from the detailed Green correspondence, computing intersections, that this summand is indecomposable as there aren't "enough dimensions" for any projectives when one remembers that 3 and 3^* will play equal roles. Now we can analyze the extensions of the middle and keep track of the consequences for the socle and it's easy to get that it is

$$\begin{matrix}3\\3^*\\3\\3^*\end{matrix} \quad \oplus \quad \begin{matrix}3^*\\3\\3^*\\3\end{matrix}$$

By symmetry, get one summand splits over the V_1 in the socle if and only if the other does. Using duality - i.e. self-duality - get that our guess is correct.

Lastly, we turn to $P = Z_2$ and the guess is

$$P = Z_2 : \quad \begin{array}{ccccccccc} & 3^* & & & & & & 3 & \\ 1 & & 3 & & 1 & & 3^* & & \square \\ & & & 3^* & & 3 & & & \end{array} ; \quad 1 \begin{matrix}3^*\\3\\3^*\\3\\3^*\end{matrix} ; \quad 1 \begin{matrix}3\\3^*\\3\\3^*\\3\end{matrix} ; \quad 8$$

Of course V_8 occurs 4 times in $(k_{Z_2})^G$ as kG is an extension of $(k_{Z_2})^G$ by $(k_{Z_2})^G$. Let's see that P_3, the projective cover of V_3 occurs.

$$\operatorname{Hom}_{kG}(P_3, k_{Z_2}^G) \simeq \operatorname{Hom}_{kZ_2}((P_3)_{Z_2}, k)$$
$$= \operatorname{Hom}_{kZ_2}(J_2 \oplus \dots \oplus J_2, k) \simeq \overbrace{k \oplus \dots \oplus k}^{8}$$
$$\operatorname{Hom}_{kG}(P_3/V_3, k_{Z_2}^G) \simeq \operatorname{Hom}_{kZ_2}(\overbrace{J_2 \oplus \dots \oplus J_2}^{6} \oplus J_1, k) \simeq \overbrace{k \oplus \dots \oplus k}^{7}$$

Also P_1, the cover of V_1 does not occur as the two relevant restrictions are $J_2 \oplus J_2 \oplus J_2 \oplus J_2$ and $J_2 \oplus J_2 \oplus J_2 \oplus J_1$. Hence, there is (by the Green correspondence in detailed form) just one non-projective summand and it has the right composition factors.

Again it's easy to compute the socle and the radical quotient. The radical modulo socle then has two composition factors, V_3 and V_{3^*}, so using the outer automorphism which switches these two, it must be their sum so the radical and socle series of our last module are as follows:

$$\begin{array}{c} 1 \oplus 3 \oplus 3^* \\ \hline 3 \oplus 3^* \\ \hline 1 \oplus 3 \oplus 3^* \end{array}$$

Then using socle information must have for the bottom two layers, modulo V_1, that one has

$$\begin{array}{c} 3 \\ 3^* \end{array} \oplus \begin{array}{c} 3^* \\ 3 \end{array}$$

so by duality have

$$\begin{array}{cc} \multicolumn{2}{c}{1} \\ \hline 3^* & 3 \\ 3 & 3^* \\ 3^* & 3 \\ \hline \multicolumn{2}{c}{1} \end{array}$$

and now can proceed just as in the last case and the guess is established.

. Blocks of p-permutation modules

The motivation is a generalization of block theory, from the point of view that it is a study of the module structure of kG, k of characteristic p. The hoped for generalization would be to $k[G/H]$, the permutation module on the cosets of the subgroup H.

Let's have the usual set-up of three rings: $\mathcal{O} \to k$, $K \supseteq \mathcal{O}$. Let Ω be a G-set and $\mathcal{O}\Omega$ and $k\Omega$ the corresponding $\mathcal{O}G$ and kG-modules. We assume that k is large enough so that the indecomposable summands of $k\Omega$ are absolutely indecomposable, and that K is large enough when this is required.

Our point of view on blocks is that the right thing to consider the blocks of $\mathrm{End}(\mathcal{O}\Omega)$, or what is the same, the blocks of $\mathrm{End}(k\Omega)$. These are the "same" because of the correspondence between summands of $\mathcal{O}\Omega$ and $k\Omega$ which "respects" maps, in view of Fitting's theorems on the structure of endomorphism rings. We are going to study now just the generalization of blocks of defect 0, which seems the proper place to start, and we shall do this in several ways.

First, suppose that U is an indecomposable summand of $k\Omega$ and $\mathrm{End}(U) = k$ (so if $\hat{U}$ is the corresponding summand of $\mathcal{O}\Omega$ then $\mathrm{End}(\hat{U}) = \mathcal{O}$) so that U is a generalization of a projective simple module, the summands of kG with the same property. Some time ago we did prove the following: the Green correspondent of U is simple; thus, if Q is the vertex of U then its Green correspondent is a projective simple $k[N(Q)/Q]$-module.

Next, we wish to consider simple summands of $\mathrm{End}(k\Omega)$. Since such a summand has itself a single simple module and because k is large enough, Fitting tells us the following: a simple summand of $\mathrm{End}(k\Omega)$ is a matrix algebra $M_r(k)$ and corresponds with an indecomposable summand U of $k\Omega$ such that $\mathrm{End}(U) = k$, U is a summand of $k\Omega$ with multiplicity r and if

$$k\Omega = \overbrace{U \oplus \cdots \oplus U}^{r} \oplus V$$

then $\mathrm{Hom}_{kG}(U, V) = \mathrm{Hom}_{kG}(V, U) = 0$. This means we have

$$\mathcal{O}\Omega = \overbrace{\hat{U} \oplus \cdots \oplus \hat{U}}^{r} \oplus \hat{V}$$

where $\mathrm{Hom}_{\mathcal{O}G}(\hat{U}, \hat{V}) = \mathrm{Hom}_{\mathcal{O}G}(\hat{V}, \hat{U}) = 0$ and $M_r(\mathcal{O})$ is a summand of $\mathrm{End}(\mathcal{O}\Omega)$. And of course, each matrix algebra over $\mathcal{O}$ so arises, again by Fitting. Hence, we have the

<u>Proposition 1</u>. The following sets are in one-to-one correspondence:

1) Matrix algebras over k which are blocks of $\mathrm{End}(k\Omega)$;

2) Matrix algebras over $\mathcal{O}$ which are blocks of $\mathrm{End}(\mathcal{O}\Omega)$;

3) Isomorphism classes of indecomposable summands U of $k\Omega$ such that $\dim_k \mathrm{Hom}(U, k\Omega) = \dim_k \mathrm{Hom}(k\Omega, U) = r$ where r is the multiplicity of U as a summand of $k\Omega$;

4) Isomorphism classes of indecomposable summands $\hat{U}$ of $\mathcal{O}\Omega$ such that $\mathrm{rank}_{\mathcal{O}}\, \mathrm{Hom}(\hat{U}, \mathcal{O}\Omega) = \mathrm{rank}_{\mathcal{O}}\, \mathrm{Hom}(\mathcal{O}\Omega, \hat{U}) = r$ where r is the multiplicity of $\hat{U}$ as a summand of $\mathcal{O}\Omega$.

Before going on we wish to point out that sometimes these dimensions (ranks) in 3) (4)) coincide even though not necessarily with r.

<u>Proposition 2</u>. If U is a p-permutation module and $\Omega = G/Q$ for a p-subgroup Q then
$$\mathrm{Hom}_{kG}(U, k\Omega) \simeq \mathrm{Hom}_{kG}(k\Omega, U).$$

<u>Proof</u>. By reciprocity we have
$$\mathrm{Hom}_{kG}(U, k\Omega) \cong \mathrm{Hom}_{kQ}(U_Q, k).$$
But U_Q is also a p-permutation module and since Q is a p-group it follows that U is a permutation module, the direct sum of trivial modules induced up to Q from subgroups of Q, and is self-dual. Thus,
$$\begin{aligned}\mathrm{Hom}_{kQ}(U_Q, k) &\simeq \mathrm{Hom}_{kQ}(U_Q^*, k)\\ &\simeq \mathrm{Hom}_{kQ}(k, U_Q)\\ &\simeq \mathrm{Hom}_{kG}(k\Omega, U)\end{aligned}$$
as required.

These summands of $\mathcal{O}\Omega$ can also be discussed in terms of characters. Let $\hat{U}$ be such a module, χ its character (that is, of $K \otimes \hat{U}$ the KG-module). Thus χ is irreducible as $\mathrm{Hom}(K \otimes \hat{U}, K \otimes \hat{U}) = K$ as maps are "preserved" over any ring. Moreover, the multiplicity of χ in the permutation character π of Ω is r, the multiplicity for $\hat{U}$ as summand, because of $Hom_{OG}(\hat{U}, \hat{V}) = 0$ giving $Hom_{KG}(K\otimes\hat{U}, K\otimes\hat{V}) = 0$.

Let's now put this in terms of idempotents. We assume now that $\Omega = G/H$ for a subgroup H of G and we identify $End\, O\Omega$ with what we shall write as $O[H\backslash G/H]$, the O-algebra with basis elements the double coset sums in OG divided by $|H|$ so $O[H\backslash G/H] \subseteq KG$. This gives the identification with the centralizer ring of matrices of 0's and 1's a la Scott. We have that $O[H\backslash G/H]$ is an order in $K[H\backslash G/H]$. Now Curtis-Fossum Math Z. (107) 1968, shows how $K[H\backslash G/H]$ embeds in KG, matrix algebras inside matrix algebras corresponding to the irreducible characters appearing in $K\Omega$. The unit element of $O[H\backslash G/H]$ is $e_H = \frac{1}{|H|}\sum_{h\in H} h$. Hence, the blocks of $O[H\backslash G/H]$ are sums of central idempotents for KG times e_H, such expressions that lie in $O[H\backslash G/H]$. Such expressions do lie in $K[H\backslash G/H]$, either by the Curtis-Fossum picture or the equation $e_\varphi e_H = e_H e_\varphi e_H$, for any irreducible character φ.

Hence, for a module $\hat{U}$ as above, we have that $e_\chi e_H \in O[H\backslash G/H]$. But the argument used less, just that there is a block of $O[H\backslash G/H]$ such that all the corresponding summands of $O\Omega$ have characters a multiple of χ. For that gives a block of $O[H\backslash G/H]$ inside the matrix summand of KG corresponding to χ. And the converse is clear.

Let's note that in Scott's "defect 0" case we have this condition:

<u>Proposition 3</u>. If $|G : H \cap H^x|_p$ divides $\chi(1)$ for all $x \in G$ then $e_\chi e_H \in \mathcal{O}[H\backslash G/H]$.

<u>Proof</u>. As e_χ is central and e_H is an idempotent we have

$$e_\chi e_H = e_H e_\chi e_H$$
$$= \chi(1)/|H|^2|G| \cdot \sum_{\substack{h,k\in H \\ g\in G}} \overline{\chi(g)}\, hgk$$

Hence, the coefficient of $x \in G$ is $\chi(1)/|H|^2|G| \sum_{\substack{k,h\in H \\ g\in G \\ hgk = x}} \overline{\chi(g)}$.

Thus, we have to let g range over HxH and for each g compute the number of times there are pairs (h,k), $h,k \in H$, with $hgk = x$. This number is $|H \cap H^x|$. (For if $u \in H \cap H^x$, $u = v^x$, $v \in H$ then

$$(v^{-1}h)g(ku) = v^{-1}xu = xv^{-x}u = x,$$

while if $h'gk' = x$ then

$$x = h'gk' = (h'h^{-1})hgk(k^{-1}k') = (h'h^{-1})x(k^{-1}k').\ )$$

Hence, the coefficient of x is

$$\chi(1)\,|H \cap H^x| / |H|^2|G| \sum_{g\in HxH} \overline{\chi(g)}$$

$$= \chi(1)\,|H\cap H^x| / |G|\,|H| \;\; \chi\Big(\frac{1}{|H|}\sum_{g\in HxH} g\Big)$$

But $e_H e_\chi e_H \in K[H\backslash G/H]$ so we want to look at the coefficient of the sum of all elements in HxH <u>divided</u> by $|H|$ and it is

$$\chi(1)\,|H\cap H^x| / |G| \;\; \chi\Big(\frac{1}{|H|}\sum_{g\in HxH} g\Big)$$

We want this to be in $\mathcal{O}$. But, by assumption, the first factor is indeed, while the second is a character value for $\mathcal{O}[H\backslash G/H]$. This is also in $\mathcal{O}$ (and is a character value as stated, by Curtis-Fossum and Tamaschke) by an argument in Tamaschke's paper in the first volume of the Journal of Algebra (see the part with the reference to D. Higman). Indeed, the idea is clear: $\mathcal{O}[H\backslash G/H]$ is isomorphic with Scott's matrix centralizer ring, all the bases elements going to 0,1 matrices so their characteristic roots are algebraic integers. Q.E.D.

Remark: We also can calculate $e_H e_\chi$ directly and compare the results! We have

$$e_H e_\chi = \frac{1}{|H|} \frac{\chi(1)}{|G|} \sum_{\substack{h\in H \\ g\in G}} \overline{\chi(g)}\, gh$$

$\therefore$ the coefficient of x is $\chi(1)/|G||H| \sum_{g\in x H} \overline{\chi(g)}$.
In particular, this expression can only depend on the double coset of x even though it seems at first to depend only on the coset!

Remark: The condition $e_\chi e_H \in \mathcal{O}[H\backslash G/H]$ can occur without being in the situation of Proposition 1. Look at our analysis of the 2-permutation modules of $G = GL(3,2)$. There is a block for G/D_8 with two modules $\hat{U}_1$, $\hat{U}_2$ both of dimension six with the same character.

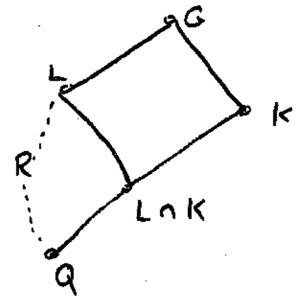
Returning to Scott's result. Our result, Proposition 3, shows that under this situation, there are isomorphism classes of indecomposable summands $\hat{U}_1, \ldots, \hat{U}_s$ of $O\Omega$ with the character Ψ_j of $\hat{U}_i$ being $a_j \chi$ and with χ not involved in any other summands of $O\Omega$. Now suppose that we knew that each $\hat{U}_i$ had a vertex a Sylow p-subgroup of some "minimal" $H \cap H^x$, a la Scott. Then Scott's argument (6.5) (Transactions A.M.S. 175 (1973), pp 116-7) shows that $\frac{1}{a_i} \Psi_{f(e_i)} = \frac{1}{a_j} \Psi_{f(e_j)}$, in his notation. But $\Psi_{f(e_i)}$, $\Psi_{f(e_j)}$ are principal indecomposable characters and if they are linearly dependent then they are equal so $\hat{U}_i$ and $\hat{U}_j$ have the same Green correspondent and are equal. Hence, there is just one $\hat{U}_i$, call it $\hat{U}$, with character $a\chi$. We still want $a=1$.

The p-rationality that Scott wants is no problem as the character of any summand of $O\Omega$ is p-rational. This can be seen by a standard type argument but Dave Benson gives a better one: it's enough to do for cyclic groups – which is dead easy. (The tedious argument means keeping track of Galois action and seeing that the reductions of certain conjugates, from O to k, coincide (over k) so must coincide over O.

We have considered two generalizations of blocks of defect zero. The first might be called p-permutation modules which are of "one-sided defect 0 type" while the second could be called of "two-sided defect 0 type" but they should be called summands of this sort because it depends on the particular Ω (e.g. see $GL(3,4)$ again). We wish to investigate the independence from Ω. We shall here stick to k (though it's easy enough to work with $\mathcal{O}$).

Definition 1. The p-permutation module U is two-sided defect zero type (or shortly, 0-type) if it is indecomposable and whenever $U \mid k\Omega$ for any transitive G-set Ω then U is a two-sided defect zero type summand.

Suppose $\Omega = G/H$ and $U \mid k\Omega$. Then $U \mid k[G/P]$, where P is a Sylow p-subgroup of H so the condition is that $Hom_{kG}(U, k[G/P])$ (which equals in dimension to $Hom_{kG}(k[G/P], U)$, by Proposition 2) has dimension the multiplicity of U as a summand of $k[G/P]$. And of course the condition for G/P implies the one for G/H as $k[G/H] \mid k[G/P]$ as $k_H \mid k[H/P]$. Hence, the definition really refers to transitive sets of the form G/P, P a p-subgroup of G.



Let $V = U_L$ so V is a projective simple $k[L/Q]$-module and is the Green correspondent of U (e.g. by Burry-Carlson-Puig). Now

$$\mathrm{Hom}_{kG}(U, k[G/R]) = \mathrm{Hom}_{kR}(U_R, k)$$
$$= \mathrm{Hom}_{kR}(V_R, k)$$

and this is of dimension

$$\dim V / |R:Q|$$

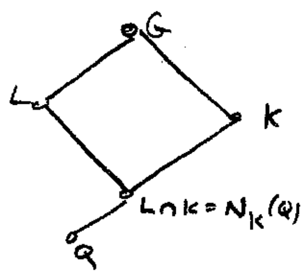
since V_R is a projective $k[R/Q]$-module. Thus, by Proposition 2 – or by the same arguments with left and right variables interchanged – it suffices in order to show that 1) now holds, to see that U is a summand of $k[G/R]$ at least $\dim V / |R:Q|$ times. But

$$k[L/R] \cong V \oplus \dots \oplus V \oplus \cdots$$

where the multiplicity of V is $\dim_k \mathrm{Hom}_{kL}(k[L/R], V)$ $= \dim_k \mathrm{Hom}_{kR}(k, V_R) = \dim V / |R:Q|$ as V_R is a projective $k[R/Q]$-module. This is because V is projective modulo Q so the "hom" calculation gives the multiplicity as summand. But if we induce $k[L/R]$ to G we get $k[G/R]$ and each V induced up gives U as a summand so we get the required number of summands and 1) holds now as a consequence of 4).

But $U_L = V \oplus \cdots$ so $U_Q = V_Q \oplus \cdots$ so the equality $\dim V = \dim_k \mathrm{Hom}_{kG}(k, V_Q \oplus \cdots)$ gives us that $U_Q = V_Q$, $U_L = V$ and, in particular, that Q is contained in the kernel of U.

Let K be the kernel of U so $K \cap L$ is the kernel of $U_L = V$ and so Q is a Sylow p-subgroup of $K \cap L$ inasmuch as V is projective for L/Q. Hence, as L is the normalizer of Q, we have that Q is a Sylow p-subgroup of K, or else the normalizer of Q in a Sylow subgroup of K containing it would properly contain Q. By the Frattini argument, we have that $G = LK$, giving us the picture



Now $U_L = V$ and U_K is trivial so U is just the "lift" of V to G/K and so we have established 4).

Next, let's assume that 4) holds. Let K be the kernel of U, Q a Sylow p-subgroup of K so Q is the vertex of U. Since $|K : Q|_p = 1$, we have that $L = N(Q)$ contains a Sylow p-subgroup of G. Hence, if R is a p-subgroup of G containing Q then we may as well, for the purpose of establishing 1), assume that $Q \subseteq R \subseteq L$. Hence, the picture that we now have is the following:

Our main result is the

Theorem. The p-permutation module U is of 0-type if, and only if, it is a projective simple module for a quotient group.

In fact, let's more precisely prove that the following statements are equivalent:

1) U is a 2-sided type 0 summand of $k[G/R]$ whenever R is a p-subgroup of G containing the vertex of U (and assume U indec.);

2) U is as in 1) but just for $k[G/Q]$, Q a vertex of U;

3) U is a 2-sided type 0 module;

4) U is a projective simple module for a quotient of G.

To prove this let's first assume that 2) holds and let V be the Green correspondent of U so that V is a projective simple $k[N(Q)/Q]$-module, as we recalled earlier, as $\mathrm{End}(U) \simeq k$. Thus, as kL-modules, $L = N(Q)$

$$k[L/Q] \simeq \underbrace{V \oplus \cdots \oplus V}_{\dim V} \oplus \cdots$$

so by the Green correspondence we have

$$k[G/Q] \simeq \underbrace{U \oplus \cdots \oplus U}_{\dim V} \oplus \underbrace{\cdots\cdots}_{\text{no } V \text{ summand}}$$

and

$$\dim V = \dim_k \mathrm{Hom}_{kG}(k[G/Q], \ ) = \dim_k \mathrm{Hom}_{kQ}(k, \ _Q).$$

Now it is clear that 1) ⇒ 2) as 2) is a special case of 1) so we have proven the equivalence of 1), 2) and 4). Now 1) implies 3) since, as we remarked before, the module $k[G/P]$ is a summand of $k[G/H]$, where P is a Sylow p-subgroup of H. Finally, 3) implies 2) as U is certainly a summand of $k[G/Q]$ where Q is a vertex of U. Hence, the theorem is proved.

Now we ask: what do we need to prove to show that a module is of type 0? Suppose U is a 2-sided type 0 summand of $k[G/Q]$, Q a p-subgroup; does it have the property for other p-subgroups? We don't want to ask this for arbitrary subgroups in view of the $GL(3,2)$ examples.

<u>Lemma</u> Let $R \geq Q$ be p-subgroups of G. If U is a 2-sided defect 0 type summand of $k[G/Q]$ then it is also for $k[G/R]$.

Rk: This gives an implication in the proof of the four statements 1)–4) being equivalent.

<u>Proof</u>. Our hypothesis gives immediately that any non-zero homomorphism of $k[G/Q]$ to U is a split epimorphism: if $\varphi : k[G/Q] \to U$, $\varphi \neq 0$ then there is $\psi : U \to k[G/Q]$ such that $\varphi \circ \psi = 1_U$. Similarly, and dually, if there is a non-zero map of U to $k[G/Q]$ then

there is a suitable map the other way, that is, we are dealing with a split monomorphism. The first property passes to quotients of $k[G/Q]$, the second to submodules of $k[G/Q]$, so they both hold for $k[G/R]$. Hence, express

$$k[G/R] = U \oplus \cdots \oplus U \oplus X$$

where $U \nmid X$; there are such U's as $\mathrm{Hom}_{kG}(k[G/R], U) \cong \mathrm{Hom}_{kR}(k, U_R) \neq 0$. Now if $\mathrm{Hom}_{kG}(X, U) \neq 0$ then there is a map of $k[G/R]$ to U which is the obvious extension with all the U's in the kernel. Hence, there is a split epimorphism of X to U, a contradiction. Similarly, if $\mathrm{Hom}_{kG}(U, X) \neq 0$ then also U / X. The lemma is proved.

Hence, if P is a Sylow p-subgroup of G then the 2-sided defect 0 type summands of $k[G/P]$ are the ones for all p-subgroups of G. They correspond one-to-one with the matrix summands of the algebra $k[P\backslash G/P] = k \otimes_{\mathcal{O}} \mathcal{O}[P\backslash G/P]$. Question: Are all these modules of type 0? That is, if they occur for P do they also occur for their vertex, that is, can we do the opposite of the lemma and go down to the vertex instead of going up?

Keep P fixed, let U be one of these modules for P and let Q be the vertex of U. We want to relate U and $k[G/Q]$. Let V be the Green correspondent of U, so V is a kL-module, $L = N(Q)$. The number of summands isomorphic to U in $k[G/Q]$ is equal to $\dim V$, by the Green correspondence and the fact that V is a summand of $k[L/Q]$ with multiplicity $\dim V$ (remember, V is simple as well as projective). On the other hand $\mathrm{Hom}_{kG}(k[G/Q], U) \cong \mathrm{Hom}_{kQ}(k, U_Q)$ has dimension equal to the number of components of U_Q. But $U_L = V \oplus \cdots$ so $\dim V$ is the number of trivial summands of U_Q. Hence, does U_Q have non-trivial summands?

On the other hand, we do know something about U and P. Let m be the multiplicity of U as a summand of $k[G/P]$, so, by hypothesis,

$$\begin{aligned} m &= \dim_k \mathrm{Hom}_{kG}(k[G/P], U) \\ &= \dim_k \mathrm{Hom}_{kP}(k, U_P) \\ &= \#\text{ components of } U_P . \end{aligned}$$

We can also calculate m another way. By Burry-Carlson-Puig, m is equal to the number of summands isomorphic with V in a decomposition of $k[G/P]_{N(Q)}$. But

$$k[G/P]_{N(Q)} \simeq \bigoplus_{P\backslash G/N(Q)} k[N(Q)/\, P^x \cap N(Q)]$$

Now V has vertex Q so, as $Q \triangleleft L = N(Q)$, we will get no summands isomorphic with V if $Q \nleq P^x \cap N(Q)$, that is if $Q \nleq P^x$. Hence, we're interested in the number of times V is a summand of

$$\bigoplus_{\substack{x \in P\backslash G/L \\ Q \leq P^x}} k[L/\, P^x \cap L]$$

But if $Q \leq P^x$ then the number of times V is a summand in $k[L/P^x \cap L]$ is $\dim_k \mathrm{Hom}_{kL}(k[L/P^x \cap L], V)$ $= \dim_k \mathrm{Hom}_{k[P^x \cap L]}(k, V_{P^x \cap L}) = \dim_k V / |P^x \cap L : Q|$, as V is projective for $P^x \cap L/Q$. Hence, we deduce that

$$m = \sum_{\substack{x\in P\backslash G/L \\ P^x \supseteq Q}} \frac{\dim_k V}{|P^x\cap L : Q|} = \Bigg(\sum_{\substack{x\in P\backslash G/L \\ P^x \supseteq Q}} \frac{1}{|P^x\cap L : Q|}\Bigg)\dim_k V$$

Hence, we have the condition

$$\#\text{ components of } U_P = \Bigg(\sum_{\substack{x\in P\backslash G/L \\ P^x \supseteq Q}} \frac{1}{|P^x\cap L : Q|}\Bigg)\dim_k V$$

as a consequence of our hypothesis. How do we use it?

We could have proved the lemma much more easily. We forgot to include and use the following result:

<u>Proposition</u> 4. If U is a 2-sided defect 0 type summand for $k[G/Q]$, where Q is a p-subgroup of G, then U is simple.

<u>Proof</u>. If not then, as U is indecomposable, we have $\operatorname{rad} U \supseteq \operatorname{soc} U$. Let S be a simple epimorphic image of U so $\operatorname{End}(U) \cong k$ yields $\operatorname{Hom}_{kG}(S, U) = 0$. But $\operatorname{Hom}_{kG}(S, k[G/Q]) = \operatorname{Hom}_Q(S_Q, k) \neq 0$ so there is an indecomposable summand V of $k[G/Q]$ with $\operatorname{Hom}_{kG}(S, V) \neq 0$. Thus, $V \not\cong U$ and $\operatorname{Hom}_{kG}(U, V) \neq 0$, a contradiction.

The lemma is now easier because the condition on U is $U \not\subseteq \operatorname{rad} k[G/Q]$.

Permutation modules for Lie type groups

We wish to give a proof of a well-known result but in a more general context. So let G and k be as usual and let P be a Sylow p-subgroup of G. Instead of assuming that G is a Chevalley group we only assume the following:

1) If S is any simple kG-module then $\dim(\mathrm{soc}(S_P)) = 1$;

2) Our "projective weight" conjecture holds for G.

(That is, in 4, there is a one-to-one correspondence between simple kG-modules and "projective Green correspondents" which are also simple. We just need the numbers match.)

<u>Theorem.</u> Under the above hypotheses,

$$(k_P)^G \simeq \bigoplus_S U_S$$

where the sum runs over the (isomorphism classes of) simple kG-modules S and $U_S / \mathrm{rad}(U_S) \simeq S$.

<u>Proof.</u> First, we have

$$\dim_k \mathrm{Hom}_{kG}(k_P^G, S) = \dim_k \mathrm{Hom}_{kP}(k, S_P) = 1$$

by assumption. Hence, we need only prove that the number of indecomposable summands of $(k_P)^G$ is at least as large as the number of simples, that is, the number of projective weights, by 2).

However, let Q be a p-subgroup of G, V a simple $k[N(Q)]$-module which is projective as a module for $N(Q)/Q$. Let U be its Green correspondent. We need only demonstrate that $U \mid (k_p)^G$. By Burry-Carlson-Puig, we need only show that $V \mid ((k_p)^G)_{N(Q)}$. By Mackey's theorem, we need only find a Sylow p-subgroup R of G with $V \mid (k_{R \cap N(Q)})^{N(Q)}$. However, let Q_1 be a Sylow p-subgroup of Q and let R be a Sylow p-subgroup of G containing Q_1 so $R \cap N(Q) = Q_1$. We have

$$\mathrm{Hom}_{kN(Q)}((k_{Q_1})^{N(Q)}, V) \cong \mathrm{Hom}_{kQ_1}(k, V_{Q_1}) \neq 0$$

as $V_{Q_1} \neq 0$. Thus, V is a homomorphic image of $(k_{Q_1})^{N(Q)}$, as it is simple. But $(k_{Q_1})^{N(Q)}$ is really a $kN(Q)/Q$ module, as $Q \trianglelefteq N(Q)$ and $Q \subseteq Q_1$. Since V is projective as a $k[N(Q)/Q]$-module, we must have $V \mid (k_{Q_1})^{N(Q)}$ just as required.

The "tightness" of the argument means we can deduce a few things about G (it is presumably more general than a Lie type group, or is it?). We get, with the same notation;

i) If R_1 is any Sylow p-subgroup of G containing Q then $R_1 \cap N(Q)$ is a Sylow p-subgroup of $N(Q)$;

ii) $\dim \mathrm{Hom}_{k[N(Q)]}((k_{Q_1})^{N(Q)}, V) = 1$.

The second is true because the theorem implies that $U \mid (k_p)^G$ with multiplicity one. The first is true, or we can get $V \mid ((k_p)^G)_{N(Q)}$ with multiplicity greater than 1, as any R_1 with $R_1 \supseteq Q$ and R_1 not containing a Sylow of $N(Q)$, corresponds to another summand of $(k_p)^G{}_{N(Q)}$, and $R_1 \cap N(Q)$ can be used in place of Q_1.

Correspondences and normal subgroups

Harris and Knorr have proved a result (which may well already be in Dade's papers) connecting the Brauer correspondence with Clifford theory. We begin by giving a module-theoretic treatment.

We fix notation: let b be a block of the normal subgroup N of G with D a defect group of b. We now describe the complete set-up before stating the result. Let L be a subgroup of G containing $N_G(D)$ and set $K = L \cap N$ so $K \supseteq N_N(D)$. Let β be the block of K corresponding with b under the Brauer correspondence.

Theorem 1. There is a one-to-one correspondence between the blocks of L covering β and the blocks of G covering b.

It also follows from the proof that the correspondence preserves defect groups: this is because, as we shall see, L contains the normalizer in G of the defect group of any block of L covering β and because the one-to-one correspondence is given by the Brauer map.

Proof. Now β also has defect group D so if $\mathcal{B}$ is a block of L covering β then $\mathcal{B}$ has a defect group $\mathcal{D}$ with $\mathcal{D} \cap K = D$. Therefore, $N_G(\mathcal{D}) \subseteq N_G(\mathcal{D} \cap N) = N_G(\mathcal{D} \cap K) = N_G(D) \subseteq L$ so $\mathcal{B}^G$ is defined and is the block corresponding to $\mathcal{B}$ under the First Main Theorem applied to L and G. Hence, we have a one-to-one maps from blocks of L covering β to blocks of G. Moreover, $\mathcal{B}^G$ does cover b. Indeed, as "trimodules," $\beta \mid \mathcal{B}_{K\times K}$, $\mathcal{B} \mid (\mathcal{B}^G)_{L\times L}$ so $\beta \mid ((\mathcal{B}^G)_{N\times N})_{K\times K}$ and, by the Burry-Carlson-Puig result, $b \mid (\mathcal{B}^G)_{N\times N}$, that is, $\mathcal{B}^G$ covers b. Finally, suppose that B is a block of G covering b; it remains only to show that there is a block $\mathcal{B}$ of L covering β such that $\mathcal{B}^G = B$. We have $\beta \mid b_{K\times K}$ and $b \mid B_{N\times N}$ so $\beta \mid B_{K\times K}$, that is, $\beta \mid (B_{L\times L})_{K\times K}$. But $B_{L\times L}$ is the direct sum of blocks of L and indecomposable $k[L\times L]$-modules with vertices <u>not</u> containing δD, by the standard lemma. Since β has vertex δD, as a $k[K\times K]$-module, we deduce that there is a block $\mathcal{B}$ of L with $\mathcal{B} \mid B_{L\times L}$ and $\beta \mid \mathcal{B}_{K\times K}$. Hence, $\mathcal{B}$ covers β so, in particular, $\mathcal{B}^G$ is defined. But $\mathcal{B} \mid B_{L\times L}$ so $B = \mathcal{B}^G$ and the theorem is proved.

Our next result is an analogous result for modules. Suppose that U is an indecomposable kG-module, V is an indecomposable kN-module and $V \mid U_N$. Then, there is a vertex of U which contains a vertex Q of V so Q is contained in the intersection of N and a vertex of U. If this containment is an equality then we shall say that U <u>covers</u> V. Now let L be a subgroup of G containing $N_G(Q)$; set $K = L \cap N$ so $K \supseteq N_N(Q)$. In particular, there is an indecomposable kK-module W which is the Green correspondent of V. We can now state the theorem we are after.

Theorem 2. There is a one-to-one correspondence between indecomposable kL-modules covering W and indecomposable kG-modules covering V.

It is also true that the correspondence preserves vertices as it is given, as we shall see, by the Green correspondence.

Proof. Suppose that X is an indecomposable kL-module covering W. Let R be a vertex of X with $R \cap K = Q$. Thus, $R \cap N = Q$ so $N_G(R) \subseteq N_G(R \cap N) = N_G(Q) \subseteq L$. Hence, let U be the Green correspondent of X so $W \mid U_K = (U_N)_K$ as $W \mid X_K$ and $X \mid U_L$. Thus, by the Burry-Carlson-Puig result, $V \mid U_N$. Since U also has vertex R we have that U covers V. Hence, we have a one-to-one map from indecomposable kL-modules covering W to indecomposable kG-modules covering V.

Finally, suppose that U is an indecomposable kG-module covering V. As above, U has a Green correspondent X which is an indecomposable kL-module, since L contains the normalizer of a vertex of U. Now $L \subseteq LN \subseteq G$ so let Y be the Green correspondent of U in LN. It follows easily that X and Y are Green correspondents (consider the restriction of Y to LN using Green's theorem). We have the following subgroups and modules to consider:

$$\begin{tikzcd}[row sep=small]
 & & G \arrow[d, dash] & U \\
 & & LN & Y \\
X & L \arrow[ur, dash] \arrow[d, dash] & N \arrow[u, dash] & V \\
W & K \arrow[ur, dash] & &
\end{tikzcd}$$

To conclude the proof we need only show that $W \mid X$ as then X must cover W since X has the same vertex as U while W has the same vertex as V.

First, we claim that $V \mid Y_N$. Since $U \mid Y^G$ and $V \mid U_N$ it suffices to prove that $V \nmid (g \otimes Y)_N$ whenever $g \in G$, $g \notin LN$. (For $Y^G = \bigoplus_{g \in G/LN} (g \otimes Y)$.) Suppose R is a vertex of Y with $R \cap N = Q$ so therefore $g \otimes Y$ is a $k[gLNg^{-1}]$-module with vertex gRg^{-1}. Therefore, each indecomposable summand of $(g \otimes Y)_N$ has a vertex contained in $N \cap gRg^{-1} = g(N \cap R)g^{-1} = gQg^{-1}$. But V has vertex Q so if $V \mid (g \otimes Y)_N$ then there is $x \in N$ with $x^{-1}Qx = gQg^{-1}$, that is, $xg \in N_G(Q) \subseteq L$ so $g \in NL = LN$, a contradiction.

Now $Y \mid X^{LN}$, as X and Y are Green correspondents, so $V \mid (X^{LN})_N$. However, $(X^{LN})_N \cong (X_K)^N$ by Mackey's theorem, so $V \mid (X_K)^N$. Hence there is an indecomposable summand W_1 of X_K such that $V \mid W_1^N$. But X has vertex R and $R \cap K = Q$ so W_1 has a vertex which is conjugate in L with a subgroup of Q. But V has vertex Q and any vertex of X must contain a vertex of V, as $V \mid W_1^N$. Hence, there is $l \in L$ such that Q^l is a vertex of W_1 and of V so $Q^l = Q^n$ for some $n \in N$. Hence, $ln^{-1} \in N(Q) \subseteq L$ so $n^{-1} \in L$. Thus, $Q = Q^{ln^{-1}}$ is a vertex of $n^{-1} \otimes W_1$ (this is a kK-module as $n^{-1} \in L$) and $(n^{-1} \otimes W_1)^N$ also has V ($\cong n^{-1} \otimes V$) as a summand. Thus, $n^{-1} \otimes W_1$ is the Green correspondent of V so $n^{-1} \otimes W_1 \cong W$. But $n \in L$ so $n^{-1} \otimes W_1 \mid X_K$ since $W_1 \mid X_K$ and we are done.

Blocks and parabolics

As motivation, let us return to our work on fusion and blocks. We dealt with the following situation: (Q, b_Q) is a subpair and $L = N(Q, b_Q)$ has a simple module S lying in $b_Q^L = b$ with vertex Q.

<u>Proposition 1</u>. Under the above hypothesis the following assertions hold:

1) (Q, b_Q) is a Brauer subpair;

2) $O_p(L/QC(Q)) = 1$;

3) If $QC(Q) \subseteq N \triangleleft L$ and b_Q^N is nilpotent, then $N = QC(Q)$.

<u>Proof</u>. Let T be a simple summand of $S_{QC(Q)}$; since $b_Q^L = b$ and b_Q is stable in L, it follows that b_Q is the only block of $QC(Q)$ covered by b, so T lies in b_Q. But T has vertex Q: since $T | S_{QC(Q)}$ it has vertex contained in Q but T_Q is a direct sum of trivial modules as Q is in the kernel of S (since $Q \subseteq O_p(L)$). Thus, T as a $kQC(Q)/Q$-module lies in a block of defect zero. Therefore, by the one-to-one correspondence between blocks of $QC(Q)$ and $QC(Q)/Q$, the block of $QC(Q)$ containing T, that is b_Q, has defect group Q. Thus 1) is proved.

Next, let $K/QC(Q)$ be a non-identity normal p-subgroup of $L/QC(Q)$. Now $(b_Q^K)^L = b_Q^L = b$ so b covers b_Q^K and no other block of K since b_Q is stable in L. Let V be a simple summand of S_K so

V lies in b_Q^K and, moreover, V has vertex Q (just as in the previous paragraph. Just as in the section on fusion and blocks above, the Alperin-Collins-Sibley theorem implies that V is the only simple module in b_Q^K (K induces only p-automorphisms on Q) and so b_Q^K has defect group equal to the vertex of V. But applying the Clifford theory of blocks (see the Alperin-Burry paper, for example), we obtain, since b_Q is stable in K and $K/QC(Q)$ is a p-group, that the defect group of b_Q^K covers (in the group-theoretic sense) the quotient $K/QC(Q)$, which is a contradiction. Hence, 2) is also established.

Finally, suppose that $QC(Q) \subseteq N \triangleleft L$ as in 3). Thus, b_Q^N has a unique simple module so its vertex is a defect group of b_Q^N; we can produce this just as in the previous paragraph. We will have a contradiction, just as in the previous paragraph, unless $N/QC(Q)$ is a p'-group. But then (Q, b_Q) is a b_Q^N-subpair so the nilpotence implies only p-automorphisms being induced on Q by N so $N \subseteq QC(Q)$, again a contradiction.

This suggests some general ideas.

<u>Lemma 2.</u> If (Q, b_Q) is a Brauer subpair and $QC(Q) \subseteq K \triangleleft L = N(Q, b_Q)$ then b_Q^K is nilpotent if, and only if, $K/QC(Q)$ is a p-group.

<u>Proof</u>. If b_Q^K is nilpotent then K induces only p-automorphisms on Q, since (Q, b_Q) is now a b_Q^K-subpair, so certainly $K/QC(Q)$ is a p-group (as we have just seen that $K/C(Q)$ is a p-group).

Conversely, suppose that $K/QC(Q)$ is a p-group. Let (R, b_R) be a b_Q^K-subpair so R is a subgroup of a defect group of b_Q^K. After taking a suitable conjugate, we may assume — see Alperin-Broué again, for example — that this defect group intersects $QC(Q)$ in a defect group of b_Q, that is, in Q. Hence, $R \cap QC(Q) \le Q$. Thus if $x \in N_K(Q)$ then x induces a p-automorphism on $R \cap QC(Q)$ as it does on Q. It also induces a p-automorphism on $R/R \cap QC(Q)$ as $K/QC(Q)$ is a p-group. Hence, we have shown that b_Q^K satisfies the defining property of nilpotent blocks.

This suggests the following extension and improvement of earlier attempts to define parabolics for blocks. Suppose that B is a block of G and (Q, b_Q) is a Brauer B-subpair such that $O_p(N(Q, b_Q)/QC(Q)) = 1$. We then say that the B-local $N(Q, b_Q)$ is a <u>B-parabolic</u> (and it should really be a pair, $(N(Q, b_Q), b_Q^{N(Q, b_Q)})$).

An example of these are the normalizer of the "essential" subpairs — see Broué's article in the Santa Cruz Proceedings. Another example, by the First Main Theorem, is $N(D, b_D)$ for a Sylow B-subpair (D, b_D). One should now check whether one can carry over

Puig's work so that one would have $N(D, b_D)$ controlling strong fusion if, and only if, it is the only B-parabolic, up to conjugacy.

Of course, in the case of the principal p-block $B_0(G)$, we get exactly what we want: a p-local subgroup L is a $B_0(G)$-parabolic (call this a <u>parabolic</u>) if, and only if, $L = N_G(O_p(L))$ and $O_p(L)$ is a Sylow p-subgroup of $O_{p',p}(L)$ and L is p-constrained. For L is $B_0(G)$-parabolic if, and only if, there is a non-identity p-subgroup Q such that Q is a Sylow p-subgroup of $QC(Q)$ (so $QC(Q) = QO_{p'}(L)$) and $O_p(L/QC(Q)) = 1$.

This suggests the following question: If L (with the block b of L in mind) is a B-parabolic of G then can we define $O_{b',b}(L)$? Yes, just take $QC(Q)$ where $L = N(Q, b_Q)$, $b = b_Q^L$. This is well-defined as $Q = O_p(L)$ and b_Q is then determined by b, as we have seen above.

What about doing this for more general b-local subgroups? Say $L = N(Q, b_Q)$, (Q, b_Q) is Brauer and $b = b_Q^L$. If we set $O_{b',b}(L) = N$ where $N/QC(Q) = O_p(L/QC(Q))$ is this well-defined? That is, if $L = N(R, b_R)$, (R, b_R) is Brauer and $b_R^L = b$ and we let $K/RC(R) = O_p(L/RC(R))$ is $K = N$? The answer is <u>yes</u>, but let's develop this more fully. The group G containing L is irrelevant, in fact.

Hence, let us fix a group L and a block b of L. If (Q, b_Q) is a normal Brauer b-subpair of L let K_Q be the subgroup of L containing $QC(Q)$ such that

$K_Q / Q\,C(Q) = O_p(L / Q\,C(Q))$; the characterization of Lemma 2 still applies.

<u>Proposition 3</u> If (Q, b_Q) and (R, b_R) are normal Brauer b-subpairs of L then $K_Q = K_R$.

We proceed by establishing some easy lemmas.

<u>Lemma 4.</u> If $(Q, b_Q) \triangleleft (R, b_R)$ are b-subpairs and (Q, b_Q) is Brauer then so is (R, b_R).

<u>Proof.</u> Let $\beta = b_R^{\,R\,C(Q)} = b_Q^{\,R\,C(Q)}$ (which is possible by assumption). Let D be a defect group of β containing R — it exists as $b_R^{\,R\,C(Q)} = \beta$. Since $Q\,C(Q) \triangleleft R\,C(Q)$ there is a conjugate of D intersecting $Q\,C(Q)$ in the defect group Q of b_Q; hence, as $Q \triangleleft R\,C(Q)$, $D \cap Q\,C(Q) = Q$. Now this implies that $D = R$: if $D \supsetneqq R$ then $D \cap C(Q)$ properly contains $R \cap C(Q)$ (consider $D \cap R\,C(Q)$) so also $D \cap C(Q) \supsetneqq Q \cap C(Q)$, a contradiction. Hence, R is also a defect group of b_R as D contains a defect group of b_R.

Lemma 5. If (Q, b_Q) and (R, b_R) are normal Brauer b-subpairs and $Q \subseteq R$ then $K_Q = K_R$.

<u>Proof.</u> $R\,C(Q) / Q\,C(Q)$ is a p-group so $K_Q \supseteq R\,C(Q)$. Since $C(R) \subseteq C(Q)$ we have the following picture:

$$\begin{tikzcd}
 & K_Q \arrow[d, dash] & \\
 & RC(Q) \arrow[dl, dash] \arrow[dr, dash] & \\
QC(Q) & & RC(R)
\end{tikzcd}$$

Now $b_Q^{K_Q} = \left(b_Q^{RC(Q)}\right)^{K_Q} = \left(b_R^{RC(Q)}\right)^{K_Q} = b_R^{K_Q}$ (this is also clear by stability of b_Q, b_R in L, as this leads to unique blocks covered by b). Hence, K_Q is normal and $b_R^{K_Q}$ is nilpotent, so $K_Q \subseteq K_R$ by Lemma 2. But then $K_R \supseteq QC(Q)$ so, by a similar argument, we get $K_R \subseteq K_Q$.

We can now prove the proposition. If (Q, b_Q) is a normal Brauer b-subpair then set $S = O_p(L)$ and let (S, b_S) be a b-subpair. Since $Q \subseteq S$ we have that (S, b_S) is Brauer, by Lemma 4, so Lemma 5 applies and $K_Q = K_S$. But this is also true for any other normal Brauer b-subpair (R, b_R): $K_R = K_S$.

Blocks with one simple

We wish to consider the possibility of a generalization of nilpotent blocks. Suppose that B is a p-block of G, (D, b_D) is a Sylow b-subgroup of G and $\ell(b)=1$, where $b = b_D^N$ and $N = N(D, b_D)$. If N controls strong fusion in (D, b_D) then our projective weight question suggests that $\ell(B)=1$. Could this be proved?

Let T be the simple b-module (so $T_{DC(D)}$ is a direct sum of copies of S, the canonical module). We guess, and one has to use Dade's theory to check this, that if U is an indecomposable b-module then $U_D \cong V \oplus \cdots \oplus V$ is the direct sum of copies of an indecomposable kD-module V and that this gives a one-to-one correspondence between indecomposable B-modules and indecomposable kD-modules. Suppose that we can also establish that a stable equivalence between b and B which preserves Ext^1 (this may be redundant). We can then generalize our argument about p groups and stable equivalence to prove that $\ell(B) = 1$.

Indeed, it suffices to show that if U and W are indecomposable b-modules corresponding with distinct simple B-modules then $\mathrm{Ext}^1(U, W) = 0$. However, $U \otimes T$ and $(\dim T)\, U$ have the same restriction to D so our "guess" above implies that $\overline{\mathrm{Hom}}(U \otimes T, W) = 0$ as $b \cdot U \otimes T$ is a direct sum of copies of U and $\overline{\mathrm{Hom}}(U, W) = 0$. Thus $\overline{\mathrm{Hom}}(T, U^* \otimes W) = 0$. But this works for any simple kN-module, not just T! Hence $U^* \otimes W$ is injective so $\mathrm{Ext}^1(k, U^* \otimes W) = 0$ and $\mathrm{Ext}^1(U, W) = 0$.

Weights and Engel conditions

Motivated by a desire to study the n-th Engel condition for large primes, we can study free Lie algebras and even complex free Lie algebras. Let L be such on two generators x, y. Let L_d be the homogeneous component of degree d. In the obvious way, L is a module for $GL(2, \mathbb{C})$. The weight spaces in the sense of representation theory, for the diagonal subgroup of $GL(2,\mathbb{C})$, are the spaces spanned by the "words" in x and y of a fixed weight in x and in y. We know that the irreducible polynomial representations of $GL(2,\mathbb{C})$ are tensor products of a power of the determinant representation and a symmetric power of the standard representation. All the weight spaces are one-dimensional and we shall consider highest weights as the ones fixed by the transformation $T \in GL(2, \mathbb{C})$ such that $T(x) = x$, $T(y) = x+y$. In terms of L it is easy to see that this means an irreducible summand of L_d will have a weight space which is weight i (in the other sense) in x, j in y where $i \geq j$, $d = i+j$, one of weights $i-1$, $j+1$ and so on up to (j, i) and that the first one is the highest weight.

Let $D = D_y^x$ be the derivation of L such that $D(x) = 0$, $D(y) = x$. Since D is nilpotent on each L_d, we have that e^D is defined and is an automorphism of L. But, as is trivial to calculate, $e^D(x) = x$, $e^D(y) = x+y$ so $e^D = T$ (identifying $GL(2, \mathbb{C})$ with its action on L). Therefore $D = \log T$

so on each L_d, D is a polynomial in T and so leaves invariant each irreducible submodule. Now D visibly maps the weight space of L of elements of weight r in x and s in y to the $(r+1, s-1)$ space. Hence, in each irreducible the space annihilated by D is the highest weight space. On the other hand, if w is a weight vector in L_d and $Dw = 0$ then $Tw = w$, as $T = e^D$, so by the representation theory of Lie groups, w is a highest weight vector of an irreducible summand of L_d.

We are interested in calculating consequences of an Engel identity in L as much as possible by just using highest weight vectors. We begin with the following question: if w is a highest weight vector in L_d, belonging to the irreducible module V then VL_1 is a submodule of L_{d+1} and what are its highest weight vectors?

Let $E = D_x^y$, the derivation "pushing the other way" associated to the transformation U with $U(x) = x+y$ $U(y) = y$. With this notation we have the

<u>Lemma 1</u> If w is a highest weight vector in V and V has dimension t then wx and $(Ew)x - (t-1)wy$ are the highest weight vectors (when non-zero) for VL_1.

We can work in $SL(2, \mathbb{C})$ and see what happens in $V \otimes L_1$ and take VL_1 as a homomorphic image. Since we are working with $SL(2, \mathbb{C})$ and V has dimension t we may identify V and the polynomials spanned by

$y^{t-1}, y^{t-2}x, \ldots, x^{t-1}$ with w identified with x^{t-1} (as identifications can vary by a scalar multiple. Of course, D, E carry over. Clearly $D(x^{t-1} \otimes x) = 0$. Moreover,

$$\begin{aligned} D(&Ex^{t-1} \otimes x - (t-1)\, x^{t-1} \otimes y) \\ &= D((t-1)\, x^{t-2} y \otimes x) - D((t-1)\, x^{t-1} \otimes y) \\ &= (t-1)\, x^{t-1} \otimes x - (t-1)\, x^{t-1} \otimes x \\ &= 0. \end{aligned}$$

The Clebsch-Gordon theorem now concludes the proof as $x^{t-1} \otimes x$ and $Ex^{t-1} \otimes x - (t-1)\, x^{t-1} \otimes y$ are weight vectors.

Our next question deals with the problem of verbal consequences of the Engel conditions. Suppose w is a weight vector in W_d. How do we determine the highest weight vectors in the irreducible submodules of the cyclic module generated by w? The following observation makes this easy.

<u>Lemma 2</u> The submodule generated by w is multiplicity-free.

Indeed, if this were false then there would be an irreducible module S for $GL(2, \mathbb{C})$ such that $S \oplus S$ was generated by a weight vector. But if $(w_1, 0)$ and $(0, w_2)$ are weight vectors for the same weight then (w_1, w_2) cannot generate $S \oplus S$ since there is a scalar multiplication on S carrying w_1 to w_2, by the fact that weights occur with multiplicity one in irreducible modules for $GL(2, \mathbb{C})$.

Let's use this to answer the last question asked. Let w be a weight vector in L_d so w generates the submodule

$$S_1 + \cdots + S_r$$

which is the direct sum of the irreducible modules $S_1, \ldots, S_r$, no two of which are isomorphic. Express

$$w = w_1 + \cdots + w_r$$

where w_i is a weight vector in S_i, $1 \leq i \leq r$. Suppose that $D^{n_i} w_i = 0$, $D^{n_i - 1} w_i = v_i \neq 0$. The vectors $v_1, \ldots, v_r$ are the ones we are after. Suppose, by choice of notation, that $n_1 \geq n_2 \geq \cdots \geq n_r$. Then $n_1 > n_2 > \cdots > n_r$: if $n_i = n_{i+1}$ then w_i and w_{i+1} are weight vectors for the same weight (the weight of w) and then so are Dw_i and Dw_{i+1} and so are v_i, v_{i+1}, so $S_i \simeq S_{i+1}$, a contradiction. In particular, $D^{n_1 - 1} w = v_1 \neq 0$ and $D^{n_1} w = 0$. This picks out v_1. But then w_1 is a scalar multiple (easily calculated by looking at the symmetric algebra) of $E^{n_1-1} v_1 = E^{n_1 - 1} D^{n_1 - 1} w$ so we can now pass to the weight vector $w - w_1$ and continue in this fashion.

We wish to append a remark to Lemma 1. The vector wx is never zero. Indeed, multiplication by x gives a one-to-one map of L into itself. To see this let A be the free associative algebra on x, y with L embedded in the usual way. It suffices to see that the polynomials in x are exactly the elements of the centralizer of x, i.e. the kernel of $[\,,x]$.

But if u centralizes x, so $xu = ux$ then the monomials in xu, ux beginning with the largest number of x's are different unless u contains a power of x. Continuing in this way our assertion is proved.

Now consider the n-th Engel and $y\overbrace{x\cdots x}^{n} \in L_{n+1}$. It is a highest weight vector for an irreducible module E_n and $\dim E_n = n$. There is only one basic commutator of weight $n+1$ which has just one y so all other highest weight vectors in L_{n+1} have more y's and so generate smaller dimensional modules. Thus, $L_{n+1} = E_n + M_{n+1}$ a canonical direct sum. We assert that $M_{n+1} = L_{n+1} \cap L''$. This follows from structures for metabelian Lie algebras. The images in L/L'' of the images of $yx\cdots x$ under $GL(2, \mathbb{C})$ span $L_{n+1} + L''/L''$. Hence, Kostrikin's theorem shows that the verbal and ideal closure of the weight vector $y\overbrace{x\cdots x}^{n}$ is of finite codimension but now we have that any other highest weight vector in L_{n+1} does not have this property.

Could it be that if w is a highest weight vector in L_{n+1}, with r y's involved in w then the verbal and ideal closure of w eventually sweeps out, in high enough degrees, all highest weight vectors with r or more y's?

Completely Conjugate Groups

Remember the old conjecture: if any two elements of the finite group G which are of the same order are conjugate then $G \cong \Sigma_1, \Sigma_2$ or Σ_3. Let's call such a group "completely conjugate."

Let's assume that there are no (non-abelian) simple groups which are completely conjugate, i.e. the conjecture holds for simple groups. We shall now prove that the conjecture is a consequence. Hence, assume that G is a completely conjugate finite group and $|G| > 2$.

<u>Lemma 1</u>. If N is a normal subgroup of G then G/N is a completely conjugate group.

<u>Proof</u>. Let $x, y \in G$ such that $\bar{x} = Nx$, $\bar{y} = Ny$ have the same order in $\bar{G} = G/N$ and that this order is a π-number, $\pi = \{p_1, \dots, p_r\}$. We can assume that x and y are also π-numbers. Suppose that the common order of $\bar{x}$ and $\bar{y}$ is $p_1^{c_1} \cdots p_r^{c_r}$ while x, y have orders, respectively, $p_1^{a_1} \cdots p_r^{a_r}$ and $p_1^{b_1} \cdots p_r^{b_r}$ so we can assume $a_i \geq c_i > 0$, $b_i \geq c_i > 0$, $1 \leq i \leq r$. It suffices therefore to prove that $a_i = b_i$ for all i, so suppose that $a_i > b_i$ for some i. Thus,

$$|\langle x \rangle \cap N| = \langle p_1^{a_1 - c_1} \cdots p_r^{a_r - c_r} \rangle$$

$$|\langle y \rangle \cap N| = \langle p_1^{b_1 - c_1} \cdots p_r^{b_r - c_r} \rangle.$$

Hence, the subgroup of order $p^{b_i - c_i + 1}$ in $\langle y \rangle$ is in N while the subgroup of order $p^{a_i - c_i}$ in $\langle x \rangle$ is not in N.

But $p^{a_i - c_i} \geq p^{b_i - c_i + 1}$, contradicting the complete conjugacy of G.

Lemma 2. There is a unique maximal normal subgroup of G and its index in G is two.

Proof. If M is a maximal normal subgroup then G/M is simple and completely conjugate so $|G/M| = 2$. If M is not unique then G has a homomorphic image isomorphic with $Z_2 \times Z_2$, which contradicts Lemma 1.

Let M be this unique maximal normal subgroup.

Lemma 3. There is an involution $t \in G$, $t \notin M$.

Proof. Let t be a 2-element of G not in M of least possible order subject to these conditions. Assume that t has order greater than two. Therefore, as G is completely conjugate, there is a 2-element s such that

$$sts^{-1} = t^{-1}.$$

We can assume that $s \in M$; for if $s \notin M$ we can replace s by st. Now, since G is completely conjugate, it follows that s has order less than the order of t. But

$$\begin{aligned} ts^{-1}ts^{-1} &= (ts^{-1}t)s^{-1} \\ &= s^{-1}s^{-1} \\ &= s^{-2} \end{aligned}$$

so ts^{-1} is also a 2-element, $ts^{-1} \notin M$ and ts^{-1} has the same order as s. But this contradicts our choice of t as s has order smaller than the order of t.

Lemma 4. M is a 3-group.

Proof. If this is not the case then let $p \mid |M|$, $p \neq 3$; hence $p > 3$ as $p \neq 2$ by Lemma 3 and the complete conjugacy of G. Let Z be a subgroup of M of order p so $N_G(Z)/C_G(Z) \simeq Z_{p-1}$. Now M is of odd order so $4 \nmid |G|$ so $4 \nmid p-1$ so $p-1$ is twice an odd number. Hence, choosing a suitable cyclic subgroup of $N_G(Z)$ (of order a multiple of $p-1$) we deduce that there is an element y of odd prime power order such that $2 \nmid |C(y)|$. But $2 \mid N_G(\langle y\rangle)/C_G(\langle y\rangle)$ so $4 \mid N_G(\langle y\rangle)$, a contradiction.

Lemma 5. $|M| = 3$.

Proof. Apply Lemma 4 to $G/\Phi(M)$ and deduce that M is cyclic and finish by inspection.

A Puigian Complex?

Let's first specialize the Puig theory of pointed groups to the case of the G-algebra $\mathrm{End}_k(U)$, where U is an indecomposable kG-module, usual assumptions. The "points" are now pairs (Q, V) where Q is a p-subgroup of G, V is an indecomposable kQ-module with vertex Q and $V \mid U_Q$.

Puig points out, as it applies to this case, that there is a vertex R of U and a source S of U such that $Q \subseteq R$ and $V \mid S_Q$. Indeed, this is easy to see. Let R and S be any vertex and source for U (S a kR-module, of course). Hence,

$$V \mid U_Q \mid (S^G)_Q$$

so

$$V \mid \left(t(S)_{tRt^{-1} \cap Q}\right)^Q$$

for some $t \in G$ so $tRt^{-1} \cap Q = Q$, i.e. $tRt^{-1} \supseteq Q$ and $V \mid t(S)_Q$ so we are done.

The "points" as above form a partially ordered set under the inclusion

$$(Q_1, V_1) \subseteq (Q_2, V_2)$$

when

$$Q_1 \subseteq Q_2$$
$$V_1 \mid (V_2)_{Q_1}$$

Let $\mathcal{P}(U)$ be the corresponding complex over k. If $U = k$ then the points are the pairs (Q, k), where Q

is a p-subgroup so $\mathcal{P}(k)$ is the Brown complex. Does $\mathcal{P}(U)$ have interesting properties? Can we use it, a la Weil, to calculate

$$\mathrm{Ext}^*(U, -)$$

locally?

A complex for A_4

Benson informed us that he has constructed a free $\mathbb{Z}A_4$-complex with the homology the square of a sphere. We thought this wouldn't happen for kA_4, k a splitting field of characteristic two. We were wrong as the following shows, usual notations. (We're adding to a suitable projective complex!)

C_2	C_1	C_0
$\begin{array}{ccccccc} & \bar\omega & & 1 & & \omega & \\ 1 & \omega & = & \omega\ \bar\omega & = & \bar\omega & 1 \\ & \bar\omega & & 1 & & \omega & \end{array}$	$\begin{array}{ccccc} & \omega & & \bar\omega & \\ \bar\omega & 1 & = & 1 & \omega \\ & \omega & & \bar\omega & \end{array}$	$\begin{array}{c} \boxed{1} \\ \omega\ \bar\omega \\ 1 \end{array}$ ← homology
$\xrightarrow{\simeq} P_\omega \oplus P_{\bar\omega} \oplus P_\omega \oplus P_{\bar\omega}$	$P_\omega \oplus P_{\bar\omega} \xrightarrow{\simeq}$	$P_\omega \oplus P_{\bar\omega}$
$P_1 \oplus P_1 \xrightarrow{\simeq}$	$P_1 \oplus P_1$	

C_4	C_3
$\begin{array}{ccccccccccc} & \omega & & \bar\omega & & 1 & & \omega & & \bar\omega & \\ \bar\omega & 1 & = & 1\ \omega & = & \omega\ \bar\omega & = & \bar\omega\ 1 & = & \omega & 1 \\ & \omega & & \bar\omega & & \bar\omega & & \omega & & \bar\omega & \end{array}$	$\begin{array}{ccccccccc} & 1 & & \omega & & \bar\omega & & 1 & \\ \omega & \bar\omega & = & \bar\omega\ 1 & = & 1\ \omega & = & \omega & \bar\omega \\ & 1 & & \omega & & \bar\omega & & 1 & \end{array}$
	$P_\omega \oplus P_{\bar\omega} \oplus P_\omega \oplus P_{\bar\omega} \xrightarrow{\simeq}$
$P_1 \xrightarrow{\simeq}$	P_1

C_5

$$\begin{array}{cccccccc} & \bar\omega & 1 & \omega & \bar\omega & 1 & \omega & \\ \text{homology} \to \boxed{1} & \omega = & \omega\ \bar\omega = & \bar\omega\ 1 = & 1\ \omega = & \omega\ \bar\omega = & \bar\omega & \boxed{1} \leftarrow \text{homology} \\ & \bar\omega & 1 & \omega & \bar\omega & 1 & \omega & \end{array}$$

C_7

$$\begin{array}{cccc} 1 & \omega & \bar\omega & 1 \\ \omega\ \bar\omega = & \bar\omega\ 1 = & 1\ \omega = & \omega\ \bar\omega \\ 1 & \omega & \bar\omega & 1 \end{array}$$

C_6

$$\begin{array}{ccccc} \omega & \bar\omega & 1 & \omega & \bar\omega \\ \bar\omega\ 1 = & 1\ \omega = & \omega\ \bar\omega = & \bar\omega\ 1 = & 1\ \omega \\ \omega & \bar\omega & 1 & \omega & \bar\omega \end{array}$$

$P_1 \xrightarrow{\simeq} P_1$

$\xrightarrow{\simeq} P_\omega \oplus P_{\bar\omega} \oplus P_\omega \oplus P_{\bar\omega}$

C_{10}

$$\begin{array}{c} 1 \\ \omega\ \bar\omega \\ \boxed{1} \end{array}$$

homology

C_9

$$\begin{array}{cc} \omega & \bar\omega \\ \bar\omega\ 1 = & 1\ \omega \\ \omega & \bar\omega \end{array}$$

C_8

$$\begin{array}{ccc} \bar\omega & 1 & \omega \\ 1\ \omega = & \omega\ \bar\omega = & \bar\omega\ 1 \\ \bar\omega & 1 & \omega \end{array}$$

$P_\omega \oplus P_{\bar\omega} \xrightarrow{\simeq} P_\omega \oplus P_{\bar\omega}$ $\qquad P_\omega \oplus P_{\bar\omega} \oplus P_\omega \oplus P_{\bar\omega} \xrightarrow{\simeq}$

$P_1 \oplus P_1 \xrightarrow{\simeq} P_1 \oplus P_1$

Remarks on the Brown complex

We are speaking of the complex formed from the poset of p-subgroups of a finite group G.

Remark 1. Strongly p-embedded subgroups.

Suppose that $G \supsetneq M$ the stabilizer of a component of the Brown complex. Note that this is equivalent to the existence of a strongly p-embedded subgroup since a component of the complex is the complex corresponding with a component of the poset, that is, with respect to the inclusion relation for p-subgroups. We wish to point out, as an alternative argument to Webb's, that in Ext calculations we can work in M and ignore G. Indeed, if P is a Sylow p-subgroup of G with $P \subseteq M$ then P permutes the other components freely so the homology of the other components - all added up - is projective as an M-module.

Remark 2. Graph case of the Brown complex

We wish to show that Webb's latest results on Ext imply his earlier result in the graph case of the Brown complex. We assume we're in the connected case. The picture:

$$\begin{tikzcd}
C_1 \arrow[dr, "\partial"] & C_0 = Z_0 \\
Z_1 & B_0 \\
B_1 = 0 & 0
\end{tikzcd}
\qquad \left.\begin{array}{c} C_0 = Z_0 \\ B_0 \end{array}\right\} \cong k$$

Which shows for any kG-module M,

$$0 \to \mathrm{Ext}^n(k, M) \to \mathrm{Ext}^n(C_0, M) \to \mathrm{Ext}^n(C_1, M) \to 0$$

But $C_0 \cong k \oplus B_0$ and so

$$\mathrm{Ext}^n(B_0, M) \overset{\partial^*}{\cong} \mathrm{Ext}^n(C_1, M)$$

that is,

$$\mathrm{Ext}^n(C_1/Z_1, M) \cong \mathrm{Ext}^n(C_1, M)$$

so

$$\mathrm{Ext}^n(Z_1, M) = 0$$

and Z_1 is projective, as desired.

Remark 3. Bouc complex

Here we are referring to the complex corresponding with the p-subgroups P such that $P = O_p(N_G(P))$. This gives the same homology as the Brown complex. Question: Can we get the same just using subgroups P which also have the property that P is a Sylow p-subgroup of $O_{p',p}(N_G(P))$ and $N_G(P)$ is p-constrained, that is, the $B_0(G)$-parabolics "p-subgroups."

Answer: No. Take $G = \Sigma_5$, $p = 2$. The 2-subgroups are $E_4 = \langle (12)(34), (13)(24) \rangle$, D_8 up to conjugacy. The complex

Degree 0	D_8, E_4	$15+5$	(dimensions)
1	$E_4 \subseteq D_8$	15	(")

so Euler characteristic is $\neq 1$.

Projective Complexes

We are interested in the category of complexes.

Theorem. If R is a ring and $n \geq 0$ then the projectives in the category of n-complexes are the projective resolutions of projective modules.

That is, the complex

$$M_n \to \cdots \to M_0$$

is projective if, and only if, each M_i is projective, the homology in dimension zero is projective and the homology vanishes in positive dimensions.

Proof. First, assume we have a projective complex; if $n=0$ then all is clear. We move on to the case $n=1$ so we have a projective 1-complex

$$M_1 \xrightarrow{\partial} M_0 .$$

First, we assert that M_1 is projective. Indeed, suppose we have a commutative diagram

$$\begin{tikzcd} & M_1 \arrow[d, "\psi"] \\ U \arrow[r, "\varphi"] & V \end{tikzcd}$$

with φ an epimorphism; we then have

$$\begin{tikzcd} M_1 \arrow[r, "\partial"] \arrow[d, "\psi"'] & M_0 \arrow[d] \\ V \arrow[r] & 0 \\ U \arrow[u, "\varphi"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

and the projectivity of the complex gives us the map of M_1 to U that shows M_1 to be projective. Next, setting $B_0 = \partial(M_1) \subseteq M_0$ we claim that M_0/B_0 is projective. Suppose we have a commutative diagram

$$\begin{array}{ccc} & & M_0/B_0 \\ & & \downarrow{\scriptstyle \psi_0} \\ U_0 & \xrightarrow{\varphi_0} & V_0 \end{array}$$

with φ_0 an epimorphism so we have maps of complexes

$$\begin{array}{ccc} M_1 & \xrightarrow{\partial} & M_0 \\ \downarrow & & \downarrow{\scriptstyle \psi_0} \\ 0 & \longrightarrow & V_0 \\ \uparrow & & \uparrow{\scriptstyle \varphi_0} \\ 0 & \longrightarrow & U_0 \end{array}$$

(remember $\psi_0 \, \partial(M_1) = 0$ so ψ_0, as a map of M_0, has B_0 in its kernel). Thus, there is

$$\begin{array}{ccc} M_1 & \xrightarrow{\partial} & M_0 \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ 0 & \longrightarrow & U_0 \end{array}$$

commuting with the above. Hence, we have a commutative diagram

$$\begin{array}{ccc} & & M_0 \\ & {\scriptstyle \rho_0}\swarrow & \downarrow{\scriptstyle \psi_0} \\ U_0 & \xrightarrow[\varphi_0]{} & V_0 \end{array}$$

and $\rho_0(B_0) = 0$ as $\rho_0(B_0) = \rho_0(\partial M_1) = \partial \rho_1(M_1) = 0$. Therefore, the complex

$$0 \longrightarrow M_0/B_0$$

is a direct summand of our original complex.

Hence to conclude, we may assume $\partial(M_1) = M_0$. We must show that ∂ is an isomorphism of M_1 onto M_0, since we already have that M_1 is projective. However, it is easy to see that M_0 is isomorphic with a summand of M_1: we have the map of complexes

$$\begin{array}{ccc} M_1 & \xrightarrow{\partial} & M_0 \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 1} \\ M_0 & \xrightarrow{1} & M_0 \end{array}$$

which is surjective so there is a backwards map from M_0 to M_1. Thus, M_0 is also projective, so if $Z_1 = \ker(\partial)$ then our complex is the direct sum of

$$M_0 \xrightarrow{1} M_0$$
$$Z_1 \to 0$$

(up to isomorphism). Hence, as the second is projective, we need only prove that $Z_1 = 0$ to finish up the case $n = 1$. But we have the epimorphism

$$\begin{array}{ccc} Z_1 & \xrightarrow{1} & Z_1 \\ \downarrow{\scriptstyle 1} & & \downarrow \\ Z_1 & \xrightarrow{\partial} & 0 \end{array}$$

which is not split if $Z_1 \neq 0$.

Now, let us assume that $n > 1$ and we have

$$M_n \to M_{n-1} \to \cdots \to M_0$$

as a projective complex, $n-1 > 0$. As above, we can argue to see that M_n is projective. Similarly, using complexes

$$U_n \to U_{n-1} \to 0 \to \cdots \to 0$$

we get that $M_n \to M_{n-1}$, a 1-complex, is a projective complex. Thus the complex

$$M_n \to M_n$$

is a summand, the other summand being

$$0 \to M_{n-1}/\partial(M_n) \to M_{n-2} \to \cdots \to M_0$$

and it is easy to see that

$$M_{n-1}/\partial(M_n) \to \cdots \to M_0$$

is projective in the category of $(n-1)$-complexes, by working with complexes of the form

$$0 \to U_{n-1} \to \cdots \to U_0 .$$

Hence, the proof is complete.

Conversely, suppose that we have an n-complex of the desired sort. It is therefore a direct sum of complexes of the form

$$0 \to 0 \to \cdots \to 0 \to P$$

and

$$0 \to \cdots \to 0 \to P \xrightarrow{=} P \to 0 \to \cdots \to 0$$

where P is a projective module, so we need only see that these types of complexes are indeed projective. Suppose we have a surjection

$$\begin{array}{c} C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \\ \phantom{C_n \to C_{n-1} \to \cdots \to C_1 \to{}} \downarrow \\ 0 \to 0 \to \cdots \to 0 \to P \end{array}$$

so the map of C_0 to P is any surjection of C_0 to P. The "backwards" map of P to C_0 gives us a backwards map of complexes. Suppose that we have a surjection

$$\begin{array}{ccccccccccccc}
C_n & \to & \cdots & \to & C_{i+1} & \xrightarrow{\partial} & C_i & \xrightarrow{\partial} & C_{i-1} & \xrightarrow{\partial} & C_{i-2} & \to \cdots \to & C_0 \\
\downarrow{\scriptstyle \pi_n} & & & & \downarrow{\scriptstyle \pi_{i+1}} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i-1}} & & \downarrow{\scriptstyle \pi_{i-2}} & & \downarrow{\scriptstyle \pi_0} \\
0 & \to & \cdots & \to & 0 & \to & P & \xrightarrow{1} & P & \to & 0 & \to \cdots \to & 0
\end{array}$$

Let η be a backwards map to π_i:

$$\begin{array}{c} C_i \\ {\scriptstyle \eta}\uparrow\downarrow{\scriptstyle \pi_i} \\ P \end{array}$$

and consider the backwards map

$$\begin{array}{ccccccccccccc}
C_n & \to & \cdots & \to & C_{i+1} & \to & C_i & \xrightarrow{\partial} & C_{i-1} & \to & C_{i-2} & \to \cdots & \to 0 \\
\uparrow & & \cdots & & \uparrow & & \uparrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \partial\circ\eta} & & \uparrow & \cdots & \\
0 & \to & \cdots & \to & 0 & \to & P & \xrightarrow{1} & P & \to & 0 & \to \cdots & \to 0
\end{array}$$

This is a map of complexes, as is easily checked. Is it a backwards map? That is, is

$$P \xrightarrow{\partial\circ\eta} C_{i-1} \xrightarrow{\pi_{i-1}} P$$

the identity on P? But $\pi_{i-1} \circ \partial \circ \eta = (\pi_{i-1} \circ \partial)\, \eta = \pi_i \eta = 1_P$, by definition of η.

Groups acting on surfaces

We're interested in the following result of Browder-Ewing: If G is cyclic of prime order p and acts on a surface M preserving orientation then there is a fixed point if, and only if, $H^2(G, H_1(M)) = 0$.

Let's first give another proof of half: assume that there are no fixed points and that $H^2(G, H_1(M)) = 0$ and derive a contradiction. We have the chain complex

$$C_2 \to C_1 \to C_0$$

where, as usual

$$C_i \supseteq Z_i \supseteq B_i$$

so $C_0 = Z_0$, $C_0/B_0 \simeq \mathbb{Z}$, $Z_2 \simeq \mathbb{Z}$ and $Z_1/B_1 \simeq H_1(M)$ is finitely generated and free abelian. Hence if $n \geq 0$, - and dropping "G" from all the cohomology,

$$H^n(\mathbb{Z}) \simeq H^{n+1}(B_0) \simeq H^{n+1}(C_1/Z_1) \simeq H^{n+2}(Z_1)$$

by dimension shifting, as all the C_i are free $\mathbb{Z}G$-modules and $B_0 \simeq C_1/Z_1$. We have the exact sequence

$$0 \to B_1 \to Z_1 \to H_1(M) \to 0$$

so

$$H^{n+1}(H_1(M)) \to H^{n+2}(B_1) \to H^{n+2}(Z_1) \to H^{n+2}(H_1(M))$$

is exact. But G is cyclic so if n is <u>even</u>

$$H^{n+2}(H_1(M)) \simeq H^2(H_1(M)) = 0$$

while

$$H^{n+2}(B_1) \simeq H^{n+2}(C_2/Z_2) \simeq H^{n+3}(Z_2) \simeq H^{n+3}(\mathbb{Z})$$
$$H^{n+2}(Z_1) \simeq H^{n+1}(C_1/Z_1) \simeq H^{n+1}(B_0) \simeq H^n(C_0/B_0) \simeq H^n(\mathbb{Z})$$

so

$$H^{n+3}(\mathbb{Z}) \to H^n(\mathbb{Z}) \to 0$$

is exact. But n is even so $H^{n+3}(\mathbb{Z}) = 0$, $H^n(\mathbb{Z}) \neq 0$ and we have a contradiction.

Next, we want to describe the complex. We'll work over the p-adic integers R. We <u>guess</u>, that "neglecting" summands of the following types

$$
\begin{array}{ccccc}
U_2 & & U_1 & & U_0 \\
\hline
0 & & RG & & 0 \\
RG & \xrightarrow{\cong} & RG & & 0 \\
 & & RG & \xrightarrow{\cong} & RG
\end{array}
$$

that we have, with x a generator of G,

$$
\begin{array}{ccccc}
C_2 & & C_1 & & C_0 \\
\hline
 & & RG & \xrightarrow{1-x} & RG \\
 & & \oplus & & \\
RG & \xrightarrow{1-x} & RG & &
\end{array}
$$

in the fixed-point free case and, in the case there are m fixed points

$$
\begin{array}{ccccc}
C_2 & & C_1 & & C_0 \\
\hline
 & & & & R \\
 & & & & \oplus \\
 & & RG & \xrightarrow{\varepsilon} & R \\
 & & \oplus & & \oplus \\
 & & \vdots & & \vdots \\
 & & \oplus & & \oplus \\
 & & RG & \xrightarrow{\varepsilon} & R \\
 & & \oplus & & \\
RG & \xrightarrow{1-x} & RG & \xrightarrow{\varepsilon} & R
\end{array}
\quad \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} m-2
$$

ε = augmentation

Structure of complexes

Let k be a field, G a finite group and assume all complexes of kG-modules are (totally) finite-dimensional.

__Proposition 1.__ If C and D are kG-complexes and there are chain maps from C to D and D to C inducing isomorphisms on homology then there are direct decompositions, as kG-complexes,

$$C = C' + C^a$$
$$D = D' + D^a$$

such that $C' \cong D'$ while C^a and D^a are acyclic.

__Proof.__ Let $f : C \to D$, $g : D \to C$ be chain maps as hypothesized. Then $f \circ g \circ f : C \to D$ and $g \circ f \circ g : D \to C$ have the same properties and their images have dimensions at most the dimensions of the images of f and g. Continuing in this way, as the complexes are finite-dimensional we reach chain maps $F : C \to D$, $G : D \to C$ which induce isomorphisms and homology and satisfy

$$F \circ G \circ F(C) = F(C)$$
$$G \circ F \circ G(D) = G(D).$$

Hence, $F \circ G$ must induce an isomorphism of $F(C)$ onto itself and $G \circ F$ must induce an isomorphism of $G(D)$ onto itself. Now $G(F(C)) = G(D)$, as

$$G(D) \supseteq G(F(C)) \supseteq G(F(G(D))) = G(D)$$

and similarly $F(G(D)) = F(C)$ so if we set

$$C' = G(D)$$
$$D' = F(C)$$

so then F and G induce isomorphisms between these complexes and isomorphisms on homology between C and D. Next, let

$$C^a = \mathrm{Ker}(F)$$
$$D^a = \mathrm{Ker}(G)$$

so we have the picture

$$\begin{array}{ccc} C & \overset{F}{\underset{G}{\rightleftarrows}} & D \\ C' \quad C^a & & D' \quad D^a \end{array}$$

Then $C' \cap C^a = 0$ as F is one-to-one on C' and $C' + C^a = C$ as $F(C') = F(C)$. Hence, we have direct sum decompositions

$$C = C' + C^a$$
$$D = D' + D^a$$

and since, by construction, C' and D' have the same homology as C and D we must have that C^a and D^a are acyclic.

<u>Proposition 2</u>. If C is a complex of kG-modules, C^* is the dual complex then

$$B_n \simeq (C^n/Z^n)^*$$
$$H(C)_n \simeq H(C^n)^*$$
$$C_n/Z_n \simeq (B^n)^*$$

The notation is standard so $B_n = \partial(C_n)$, Z_n is the kernel of ∂ on C_n and B^n, Z^n, C^n apply to C^*. Now the modules C_n and C^n are "paired" since $C^n = \mathrm{Hom}_k(C_n, k)$. We have

$$C_n \supseteq Z_n \supseteq B_n \supseteq 0$$
$$C^n \supseteq Z^n \supseteq B^n \supseteq 0$$

so it suffices to prove that Z^n and B_n are mutual annihilators as are Z_n and B^n. But if $\varphi \in Z^n$ then $\partial^* \varphi = 0$, by definition of Z^n, that is, $\varphi(\partial(c)) = 0$ for all $c \in C_{n+1}$ so $\varphi(B_n) = 0$ and conversely, clearly, if φ annihilates B_n then $\varphi \in Z^n$. Next, if $\varphi \in B^n$ then $\varphi = \partial^*(\psi)$, $\psi \in C^{n-1}$ so for $z \in Z_n$,

$$\varphi(z) = (\partial^*(\psi))(z) = \psi(\partial(z)) = \psi(0) = 0 .$$

Conversely, if $c \in C_n$ and $\varphi(c) = 0$ for all $\varphi \in B^n$ then $(\partial^*(\psi))(c) = 0$ for all $\psi \in C^{n-1}$ so $\psi(\partial c) = 0$ for all $\psi \in C^{n-1}$.

Extending Characters

<u>Theorem</u>. Let b be a p-block of defect zero of the normal subgroup N of the group G. If χ is the unique character in b and φ is the unique Brauer character in b then χ extends to G if, and only if φ extends to G.

Half is clear: we need only show that if φ extends then so does χ.

Let's begin with a review and let F be any algebraically closed field. Let ρ be an irreducible representation of N over F stable in G. For each $\alpha \in G/N$ let $t_\alpha \in \alpha$, i.e. t_α is a coset representative of the coset α, $\alpha = N t_\alpha$. Hence, if $\alpha, \beta \in G/N$

$$t_\alpha t_\beta = n_{\alpha,\beta} t_{\alpha\beta}$$

for $n_{\alpha,\beta} \in N$. Hence, as usual, for $\alpha, \beta, \gamma \in G/N$

$$t_\alpha n_{\beta,\gamma} t_\alpha^{-1} n_{\alpha,\beta\gamma} = n_{\alpha,\beta} n_{\alpha\beta,\gamma}$$

by looking at $(t_\alpha t_\beta) t_\gamma = t_\alpha (t_\beta t_\gamma)$. Since ρ is stable it follows that for each $\alpha \in G/N$ there is a non-singular matrix M_α such that

$$M_\alpha \rho(n) M_\alpha^{-1} = \rho(t_\alpha n t_\alpha^{-1})$$

for all $n \in N$. Now, what about extending ρ to G? If we can extend ρ then t_α must be mapped to a non-zero scalar multiple of M_α, since the conjugation determines M_α up to a scalar by Schur's lemma. What we need is to be able to choose the M_α (i.e.

modify by multiplication by a scalar) so that

$$M_\alpha M_\beta = \rho(n_{\alpha,\beta}) M_{\alpha\beta}$$

as is easy to see (the conjugation and multiplication relations suffice). However, the conjugations by $t_\alpha t_\beta$ and $n_{\alpha,\beta} t_{\alpha\beta}$ are certainly equal so there is $\lambda_{\alpha,\beta} \in F^*$ such that

$$M_\alpha M_\beta = \lambda_{\alpha,\beta}\, \rho(n_{\alpha,\beta}) M_{\alpha\beta}.$$

Thus, ρ extends if, and only if, we can replace each M_α by a scalar multiple so that the "$\lambda_{\alpha\beta}$ term" disappears. We claim that $\lambda_{\alpha,\beta} \in Z^2(G/N, F^*)$, i.e. is a 2-cocycle. Indeed,

$$\begin{aligned}(M_\alpha M_\beta) M_\gamma &= \lambda_{\alpha,\beta}\, \rho(n_{\alpha,\beta}) M_{\alpha\beta} M_\gamma \\ &= \lambda_{\alpha,\beta}\, \rho(n_{\alpha\beta})\, \lambda_{\alpha\beta,\gamma}\, \rho(n_{\alpha\beta,\gamma}) M_{\alpha\beta\gamma}\end{aligned}$$

while

$$\begin{aligned}M_\alpha(M_\beta M_\gamma) &= M_\alpha \lambda_{\beta,\gamma}\, \rho(n_{\beta,\gamma}) M_{\beta\gamma} \\ &= \lambda_{\beta\gamma} M_\alpha \rho(n_{\beta,\gamma}) M_\alpha^{-1} M_\alpha M_{\beta\gamma} \\ &= \lambda_{\beta,\gamma}\, \rho(t_\alpha n_{\beta\gamma} t_\alpha^{-1}) \cdot \\ &\qquad \lambda_{\alpha,\beta\gamma}\, \rho(n_{\alpha,\beta\gamma}) M_{\alpha\beta\gamma} \\ &= \lambda_{\beta,\gamma} \lambda_{\alpha,\beta\gamma}\, \rho(t_\alpha n_{\beta,\gamma} t_\alpha^{-1} n_{\alpha,\beta\gamma}) M_{\alpha\beta\gamma}\end{aligned}$$

so using the relation already derived for the $n_{*,*}$, we deduce that

$$\lambda_{\alpha,\beta} \lambda_{\alpha\beta,\gamma} = \lambda_{\beta,\gamma} \lambda_{\alpha,\beta\gamma}$$

as claimed. Moreover, how does $\lambda_{\alpha,\beta}$ change if M_α is multiplied by a non-zero scalar? We have, if the multiple is μ_α, that $\lambda_{\alpha,\beta}$ is multiplied by $\mu_\alpha \mu_\beta \mu_{\alpha\beta}^{-1}$.

Hence, we have a well determined element of $H^2(G/N, F^*)$ which vanishes if, and only if, ρ extends.

Now let K be the algebraic closure of $\mathbb{Q}$, R the ring of algebraic integers and $\mathfrak{p}$ a prime ideal of R over p so $R/\mathfrak{p}$ is the algebraic closure of a prime field of p elements. The group K^*/R^* is torsion-free and divisible so the map

$$H^2(G/N, R^*) \to H^2(G/N, K^*)$$

is an isomorphism onto. Moreover, if U is the group of roots of unity (in K) then similarly

$$H^2(G/N, U) \simeq H^2(G/N, R^*).$$

Now there is a natural homomorphism of R onto $R/\mathfrak{p} = k$ so we have a map

$$H^2(G/N, U) \to H^2(G/N, k^*).$$

This map is onto and the kernel is the Sylow p-subgroup, for U is the direct product of the roots of unity of order a power of p which goes to 1 in k^* and the p'-roots of unity which are mapped isomorphically onto k^*.

Suppose that M is an RN-module so that $K \otimes M$ is an irreducible KN-module and $\bar{M}$ is the reduction of M modulo p, the kN module $M/\mathfrak{p}M$. Claim: the obstruction (above) to the extension of M to G is mapped to the obstruction to the extension of $\bar{M}$ to G. Indeed, the isomorphism $H^2(G/N, R^*) \simeq H^2(G/N, K^*)$ means we can assume that all the $\lambda_{\alpha,\beta} \in R$. Hence, $\bar{\lambda}_{\alpha,\beta} \in k$ work there: use $\bar{M}_\alpha$ in place of M, etc.

Now let us prove the theorem. Let $\lambda_{\alpha,\beta}$ be a cocycle obstructing the extension of χ to G. Since φ extends to G we know that $\bar{\lambda}_{\alpha\beta}$ represents the zero cohomology class. Hence, the cohomology class of $\lambda_{\alpha\beta}$ has p-power order so, if H/N is a Sylow p-subgroup of G/N, we only need to see that the class of the restriction of $\lambda_{\alpha\beta}$ to H/N is zero, that is, to see that χ extends to H.

Let S be a kN-module affording φ and let T be an extension of φ to H. Let B be the block containing T so B covers b. Hence, the defect groups of B intersects N in 1. Now H/N is a p-group so now if E is a p-subgroup of the defect group of B then $N_H(E)/C_H(E)$ is a p-group. Thus, B is nilpotent. Hence, the decomposition matrix of B is a column consisting of the degrees of the characters of the defect group. Since 1 is a degree there is an irreducible character Ψ of H whose "reduction modulo p" has Brauer character an extension of φ so Ψ_H extends χ since Ψ_N reduces modulo p to φ so Ψ_N is in b so $\Psi_N = \chi$.

A Clifford theory result

We let H be a normal subgroups of the finite groups G and shall give a quick proof of a very special case of general results. Let k be an algebraically closed field of prime characteristic p, S a simple kH-module which extends to a kG-module T.

<u>Theorem</u>. The category of kG-modules whose restriction to H is a direct sum of modules each isomorphic with S is equivalent with the category of $k[G/H]$-modules.

<u>Proof</u>. If U is a $k[G/H]$-module then $T \otimes U$ is a module which is "S-homogeneous" in that $(T\otimes U)_H$ is isomorphic with a direct sum of copies of S. Thus, we have defined (as it is clear what to do with maps) a functor from the category of $k[G/H]$-modules to the category of S-homogeneous kG-modules. Moreover, if V is another $k[G/H]$-module then

$$Hom_{kG}(T\otimes U, T\otimes V) \simeq Hom_{kG}(U, T^* \otimes T \otimes V)$$

$$\simeq Hom_{kG}(U, C_H(T^*\otimes T \otimes V),$$

where C_H means H fixed points and this is so as H acts trivially on U,

$$\simeq Hom_{kG}(U, C_H(T^*\otimes T)\otimes V),$$

as $C_H(V) = V$,

$$\simeq Hom_{kG}(U, k \otimes V),$$

as $C_H(T^*\otimes T) \simeq Hom_{kH}(T, T) = k$,

$$\simeq \mathrm{Hom}_{k[G/H]}(U,V).$$

Hence, since our functor certainly sends a non-zero map to a non-zero map, we have that the category of $k[G/H]$-modules is equivalent with the image category.

It therefore remains to see that if W is a kG-module which is S-homogeneous then $W \simeq T \otimes U$ for some $k[G/H]$-module U. However to show this, it suffices to prove that every S-homogeneous module is a homomorphic image of such a module $T \otimes U$. Indeed, suppose we have an exact sequence

$$0 \to Z \to T \otimes U \to W \to 0.$$

It then follows that Z is also S-homogeneous, as $T \otimes U$ is, so we will also have an exact sequence

$$T \otimes V \to T \otimes U \to W \to 0$$

where V is a $k[G/H]$-module. However, the map from $T \otimes V$ to $T \otimes U$ comes from, by what we have already shown, a map of V to U, that is, an exact sequence

$$V \to U \to X \to 0$$

where X is also a $k[G/H]$-module. Hence, we must have $W = T \otimes X$ as W and $T \otimes X$ are each the cokernel of the map

$$T \otimes V \to T \otimes U.$$

We shall now prove the statement on homomorphic images. Now

$$S^G = (T_H)^G \simeq (T_H \otimes k)^G \simeq T \otimes (k_H)^G$$

so S^G is S-homogeneous. Suppose that W is S-homogeneous,

$$W_H \simeq S \oplus \dots \oplus S \quad (n \text{ copies}).$$

Hence, there is a kH-map, an epimorphism

$$\overbrace{S\oplus\cdots\oplus S}^{n} \longrightarrow W$$

so, by properties of induction, we have a commutative diagram.

$$\begin{tikzcd}
S^G\oplus\cdots\oplus S^G \arrow[drr] & & \\
S\oplus\cdots\oplus S \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr] & & W
\end{tikzcd}$$

where the new map is a kG-homomorphism. The theorem is proved.

<u>Corollary</u> (Berger-Harris) If S is also projective then the summand A of kG, consisting of the blocks covering the block of kH whose simple module is S, satisfies

$$A \simeq \mathcal{M}_{\dim S}\left(k[G/H]\right)$$

Proof. Any A-module has a restriction to H which is S-homogeneous since S is projective and simple. The equivalence we have given sends a simple module U of G/H to the simple A-module $T\otimes U$ which has dimension

$$\dim T \dim U = \dim S \dim U$$

so we have the result by the Morita theorems in this case.

Question: What about the p-adic result of Berger-Harris? Or what about a p-adic version of the theorem?

Answer: Seems to work. Just need that if U_1 is of right type, $U_2 \subseteq U_1$, U_1/U_2 right then so is U_2. But we haven't checked this in detail.