# Research Notes

## Volume X

As usual these notes are not assumed correct, complete or chronological!

Contents

On simple modules for $SL(2,q)$

(This is joint work with D. Mason)

We are going to generalize the results of Mason from the case $p=2$ to arbitrary primes. Let us fix some notation. We set $G = SL(2,q)$, where $q = p^n$, and let $F$ be a splitting field of characteristic $p$ for $G$. Our main result is the following: We now assume $q > p$.

Theorem 1 If $V$ is an $FG$-module then the following statements are equivalent:

1) $V$ is simple of dimension a power of $p$;

2) There is an elementary abelian $p$-subgroup of $G$ such that the restriction of $V$ to this subgroup affords the regular representation.

We shall first prove that 1) implies 2) and then establish the reverse conclusion. Now $G$ acts on the algebra of polynomials in two variables, $X$ and $Y$, in the usual way: the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

maps $X$ to $aX + cY$ and $Y$ to $bX + dY$. The space $M$ of homogeneous polynomials of degree $p-1$ is a simple $FG$-module of dimension $p$ (the so-called basic Steinberg representation of $G$). Let $\sigma_1, \ldots, \sigma_r$ be distinct automorphisms of the field of $q$ elements so the tensor product

$$\sigma_1(M) \otimes \cdots \otimes \sigma_r(M)$$

of the twists of $M$ is a simple $FG$-module of dimension $p^r$; each simple module of dimension a power of $p$ is of this form so it suffices to produce an elementary abelian subgroup of $G$ of order $p^n$ such that the restriction of this tensor product to this subgroup is a projective module. Let $U$ be the subgroup of $G$ consisting of the lower unitriangular matrices so $U$ is a Sylow $p$-subgroup of $G$ and is elementary abelian of order $p^n$. We shall prove a preliminary result

<u>Lemma 2</u>. If $g_1, \dots, g_n$ are elements of $U$ then the tensor product is regular on $\langle g_1, \dots, g_n \rangle$ if, and only if,

$$\det(\sigma_i(\lambda_j)) \neq 0.$$

where $g_j = \begin{pmatrix} 1 & 0 \\ \lambda_j & 1 \end{pmatrix}$, $1 \leq j \leq n$.

Before proving this result, let us see that it provides all we require in order to establish that 1) implies 2) in the theorem. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the distinct automorphisms of the field of $q$ elements and let $x_1, \dots, x_n$ be generators of $U$ so that the $n$ lower left entries of the $x_i$ are a basis of the field of $q$ elements. The linear independence of automorphisms implies that

$$\det(\alpha_i(x_j)) \neq 0.$$

Hence, by the Laplace expansion of the determinant (using $n \times n$ minors instead of the usual expansion by row elements) for the $n$ rows given by $\sigma_1, \dots, \sigma_n$ there are $n$ of the $x_i$ such that the minor for these and $\sigma_1, \dots, \sigma_n$ is not zero. Hence, the lemma gives the desired projectivity. (This can also be done by complexity theory.)

We turn to the lemma. Let $D$ be the linear transformation of $M$ given by $D = Y\,\partial/\partial X$ so the kernel of $D$ is one-dimensional and is spanned by $Y^{p-1}$. Moreover, $D$ is an endomorphism of $M$

as a $FU$-module. Indeed, if

$$g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

is in $U$ then

$$g D(X^i Y^j) = g(i X^{i-1} Y^{j+1}) = i(X+\alpha Y)^{i-1} Y^{j+1},$$
$$D g(X^i Y^j) = D((X+\alpha Y)^i Y^j) = i(X+\alpha Y)^{i-1} Y^{j+1}.$$

Similarly, $D$ defines an endomorphism of the $FU$-module $\sigma(M)$ for an automorphism $\sigma$ of the field of $q$ elements: this is the same vector space but $g$ acts by sending $X$ to $\sigma(\alpha)X + Y$ and $Y$ to $Y$.

Finally, let $D_i$ be the endomorphism of

$$\sigma_1(M) \otimes \cdots \otimes \sigma_r(M),$$

as $FU$-module where

$$D_i(m_1 \otimes \cdots \otimes m_r) = m_1 \otimes \cdots \otimes Dm_i \otimes \cdots \otimes m_r,$$

with the obvious notation.

In the tensor product

$$\sigma_1(M) \otimes \cdots \otimes \sigma_r(M)$$

set

$$w_1 = XY^{p-2} \otimes Y^{p-1} \otimes \cdots \otimes Y^{p-1}$$
$$w_2 = Y^{p-1} \otimes XY^{p-2} \otimes Y^{p-1} \otimes \cdots$$

and so on for $w_3, \ldots, w_r$. Also let

$$w_0 = Y^{p-1} \otimes \cdots \otimes Y^{p-1}.$$

Thus, if

$$g = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in U$$

we have

$$g w_i = w_i + \sigma_i(\lambda) w_0, \qquad 1 \le i \le r$$
$$g w_0 = w_0$$

and $W = \langle w_0, w_1, \ldots, w_r \rangle$ is an $r+1$-dimensional $FU$-submodule of the tensor product.

Now suppose that $\det(\sigma_i(\lambda_j)) = 0$. Hence, there exist $\gamma_1, \dots, \gamma_r$ in $F$, not all zero, such that

$$\sum_i \gamma_i \sigma_i(\lambda_j) = 0$$

for $j = 1, \dots, r$. Then

$$\begin{aligned} g_j\left(\sum_{i=1}^{r} \gamma_i w_i\right) &= \sum \gamma_i (w_i + \sigma_i(\lambda_j) w_0) \\ &= \sum \gamma_i w_i + \left(\sum \gamma_i \sigma_i(\lambda_j)\right) w_0 \\ &= \sum \gamma_i w_i \end{aligned}$$

for each $j$. Thus, the tensor product has at least two dimensions of fixed points for $\langle g_1, \dots, g_r \rangle$ as $w_0$ is also left fixed so the tensor product, being of dimension $p^r$, cannot afford the regular representation.

Next, suppose that the tensor product was not afford the regular representation. Hence, there are at least two dimensions of fixed points for $\langle g_1, \dots, g_r \rangle$ and so there is a fixed point which is not a multiple of $w_0$. If we can produce such a fixed point that also has the property of being in $W$ then the above calculation shows that the determinant is indeed zero.

To do this we set $\mathcal{D}$ to be the algebra of linear transformations of the tensor product generated by the identity $I$ together with $D_1, \dots, D_r$.

<u>Lemma 3</u>. The socle of the tensor product, as a $\mathcal{D}$-module, is $Fw_0$ and the second socle is $W$.

Let's complete our proof before establishing the lemma. Let $f$ be a fixed point, $f \notin W$. Suppose that $f$ is not in the $k$-th socle but is in the $(k+1)$-st. Hence, there is an element of the radical of $D$ whose product with $f$ is in the $k$-th socle but not the $(k-1)$st and this product will again be a fixed point.

The lemma can be proved by an inspection: since $D$ is nilpotent on $M$ with one dimensional kernel there is a basis of $M$ of the form $m, Dm, \ldots, D^{p-1}m$ and we can use this to construct a basis of the tensor product which is easy to deal with. However, there is a short-cut. Let $E = \langle e_1, \ldots, e_r \rangle$ be an elementary abelian $p$-group of order $p^r$. Since $D_i^p = 0$, $D_i D_j = D_j D_i$, there is a homomorphism of $FE$ onto $\mathcal{D}$ sending each $e_i$ to $I + D_i$. Now, regarding the tensor product as an $FE$-module, there is only one dimension of fixed points, namely $Fw_0$, as this space is easily seen to be the common annihilator of $D_1, \ldots, D_r$. Hence, the tensor product is a free module of rank one for $FE$ so the map of $FE$ to $\mathcal{D}$ is an isomorphism. Hence, the second socle is of dimension $r+1$; but $W$ is included in this so $W/Fw_0$ is annihilated by all the $D_j$ so we are done.

We turn to the other half of our theorem. Let $V$ be an $FG$-module which affords the regular representation of the subgroup $U_0$ of $U$. Let $T$ be the natural two-dimensional module for $G$ restricted to $U$. Now if $U_0 = U$ then $V$ is projective of dimension $p^n$ so $V$ is simple as $q > p$ by known results (e.g. see Andersen et al). Hence, let $U_1$ be a subgroup of $U$ containing $U_0$ as a subgroup of index $p$.

The key step is the next result.

<u>Proposition 4.</u> If the fixed-point space of $U_0$ on $M \otimes V$ is not free as a module for $U_1/U_0$ then the restriction of $V$ to $U_1$ has a subquotient isomorphic with the restriction of $T$ to $U_1$.

Let us now prove the rest of the theorem assuming this proposition. We proceed by induction on $|U : U_0|$; we have already dealt with the case that this is 1. If the first possibility in the proposition holds then $M \otimes V$ affords the regular representation of $U_1$ so $M \otimes V$ is simple, by induction, so certainly $V$ is as well. Thus, we may assume that $T$ is a subquotient as stated. Furthermore, we can apply the same argument to each Galois conjugate of $M$ and so we may assume, as well, that each Galois conjugate of $T$ appears as a subquotient of the restriction of $V$ to $U_1$. We proceed to argue assuming all of this holds.

We first establish a preliminary result.

<u>Lemma 5</u> The common annihilator in $FU$ of $T$ and all its Galois conjugates is $\mathrm{rad}^2(FU)$.

Indeed, suppose $U = \langle x_1, \dots, x_n \rangle$ and set $X_i = x_i - 1 \in FU$ so $X_1, \dots, X_n$ give a basis of $\mathrm{rad}(FU)/\mathrm{rad}^2(FU)$. Since $T$ is two-dimensional and so $\mathrm{rad}^2(FU)$ annihilates it, we need only prove that if $\alpha_1, \dots, \alpha_n$ are in the field of $q^n$ elements then

$$\alpha_1 X_1 + \dots + \alpha_n X_n$$

does not annihilate $T$ and all its conjugates unless all the $\alpha_i$ are zero. However, suppose

$$x_i = \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix}$$

so $\alpha_1 X_1 + \dots + \alpha_n X_n$ is represented on $T$ by the matrix

$$\begin{pmatrix} 0 & 0 \\ \alpha_1\lambda_1 + \dots + \alpha_n\lambda_n & 0 \end{pmatrix}$$

and on the Frobenius of $T$ by

$$\begin{pmatrix} 0 & 0 \\ \alpha_1\lambda_1^p + \dots + \alpha_n\lambda_n^p & 0 \end{pmatrix}$$

and so on. Hence, we must see that the equations

$$\begin{aligned} \alpha_1\lambda_1 + \dots + \alpha_n\lambda_n &= 0 \\ \alpha_1\lambda_1^p + \dots + \alpha_n\lambda_n^p &= 0 \\ &\vdots \end{aligned}$$

are inconsistent. It suffices to prove that the determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^p & \lambda_2^p & \cdots & \lambda_n^p \\ & \vdots & & \end{vmatrix}$$

is non-zero. But it is easily seen to be the product of all factors of the form

$$\gamma_1\lambda_1 + \dots + \gamma_n\lambda_n$$

where the $\gamma_i$'s run over all $n$-tuples from $GF(p)$ (except the $n$-tuple of all zeroes). Since $\lambda_1, \dots, \lambda_n$ are linearly independent the lemma is proved.

Now let us continue the argument. We now have that the annihilator in $FU_1$ of $V$ is contained in $\mathrm{rad}^2(FU)$. This implies that the annihilator in $FU_1$ of $V$ is contained in $\mathrm{rad}^2(FU_1)$. Indeed, $\mathrm{rad}^2(FU) \cap (FU_1) \subseteq \mathrm{rad}^2(FU_1)$. For, if $U_1 = \langle x_1, \dots x_t \rangle$ with the above notation, then

$x_1, \dots, x_t$ are linearly independent modulo $\mathrm{rad}^2(FU)$ and give a basis of $\mathrm{rad}(FU_1)/\mathrm{rad}^2(FU_1)$. But now, $V$ is a free module of rank one for $FU_0$. Any element of $FU_1$ induces a linear transformation of $V$ commuting with $FU_0$, as $U$ is abelian, so agrees with an endomorphism of $V$ as $FU_0$-module. But again, $FU_0$ is commutative so this element of $FU_1$ agrees with an element of $FU_0$ on $V$, that is $FU_1$ is the sum of $FU_0$ and the annihilator in $FU_1$ of $V$. In particular,

$$FU_1 = FU_0 + \mathrm{rad}^2(FU_1).$$

Hence, as $\mathrm{rad}^2(FU_1) \cap FU_0 \leq \mathrm{rad}^2(FU_0)$, we have

$$\dim_F (FU_1/\mathrm{rad}^2 FU_1) \leq \dim (FU_0/\mathrm{rad}^2 FU_0),$$

a contradiction to $U_0 < U_1$.

We are now left with the proof of Proposition 4. Again we require some preliminary results.

<u>Lemma 6.</u> As a module for $U_1$, $M$ is uniserial and the top two-dimensional quotient of $M^*$ is isomorphic to the restriction of $T$ to $U_1$.

<u>Lemma 7.</u> There is an isomorphism of $\mathrm{Hom}_{FU_1}(W, M \otimes V)$ onto $\mathrm{Hom}_{FU_1}(W \otimes M^*, V)$, for any $FU_1$-module $W$, which takes $\varphi$ to $\Phi$ where

$$\varphi(w) = \sum m_i(w) \otimes v_i ,$$

for a basis $v_1, \dots, v_d$ of $V$ and $m_i(w) \in M$, and

$$\Phi(w \otimes \mu) = \sum \mu(m_i(w))\, v_i$$

for each $\mu \in M^*$, $w \in W$.

The fact that these vector spaces are isomorphic is standard; we require the details of the map.

Let's prove the proposition using the lemmas. We take $W$ to be the fixed-points of $U_0$ on $M \otimes V$. Let $\varphi$ be the inclusion of $W$ into $M \otimes V$. Since $W$ is not free as a module for $U_1/U_0$ its socle, as $U_1$-module, is of dimension at least two. Now $\mathrm{soc}(M) \otimes V \cong V$ as $FU_0$-module so its socle is one-dimensional. Hence, there exists $w \in \mathrm{soc}(W)$, $w \notin \mathrm{soc}(M) \otimes V$. Express, as in Lemma 7,

$$w = \sum m_i(w) \otimes v_i$$

so there is $m_j(w) \notin \mathrm{soc}(M)$. Hence, there is $\mu \in \mathrm{rad}(M^*)$ with $\mu(m_j(w)) \neq 0$. Thus, again with the notation of Lemma 7,

$$\Phi(w \otimes \mu) = \sum \mu(m_i(w)) \otimes v_i \neq 0$$

as $\mu(m_i(w)) \neq 0$ and the $v_i$ are linearly independent. In particular,

$$\Phi(\mathrm{soc}(W) \otimes \mathrm{rad}(M^*)) \neq 0.$$

Now $\mathrm{soc}(W) \otimes M^*$, as an $FU_1$-module, is isomorphic with a direct sum of copies of $M^*$. Hence, there is a summand of $\mathrm{soc}(W) \otimes M^*$ which is isomorphic with $M^*$ such that $\Phi$ does not annihilate its radical; lemma 6 concludes the proof.

We now prove Lemma 6. Since the standard two-dimensional module for $G$ is self-dual we need only prove the assertion for $M$. (The matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ conjugates each element of $SL(2,p)$ to the transpose of its inverse.) If $1 \neq g = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in U$ then the matrix of $g$ with respect to the basis $X^{p-1}, X^{p-2}Y, \ldots, Y^{p-1}$ of $M$ is

$$\begin{pmatrix} 1 & & & & \\ (p-1)\lambda & 1 & & & \\ & (p-2)\lambda & 1 & & \\ & & & \ddots & \\ & & & \lambda & 1 \end{pmatrix}$$

that is, lower unitriangular with subdiagonal entries $(p-1)\lambda, \ldots, \lambda$. Since none of these is zero the matrix is similar to a Jordan block so the vector space $M$ is uniserial with respect to this single element of $U$. To conclude the proof we need only see that the dual of the two-dimensional submodule of $M$ is isomorphic to the restriction of $T$ to $U$. But this two-dimensional submodule is clearly isomorphic with $T$ which is self-dual, as the two-dimensional standard module for $G$ is self-dual.

Last we prove Lemma 7. We shall show that there are two isomorphisms of $FU_1$-modules

$$Hom_F(W,M) \otimes V \cong Hom_F(W, M \otimes V)$$
$$Hom_F(W,M) \otimes V \cong Hom_F(W \otimes M^*, V)$$

which send, respectively, $\Psi \otimes v$, for $\Psi \in Hom_F(W,M)$, $v \in V$, to the maps which sends each $w \in W$ to $\Psi(w) \otimes v$ and to the maps which sends $w \otimes \mu$, for $\mu \in M^*$, to $\mu(\Psi(w)v)$. It is easy to see, by choosing bases for example, that these are such maps which are $F$-isomorphisms. It remains to see these maps commute with the action of any $g \in U_1$.

Now the first map sends $\Psi \otimes v$ to $\Psi(\ ) \otimes v$. Application of $g$ to $\Psi \otimes v$ yields $g(\Psi(g^{-1}(\ ))) \otimes gv$ which goes to $g(\Psi(g^{-1}(\ ))) \otimes gv$ as desired.

For the second, we send $\Psi \otimes v$ to the map $w \otimes \mu \to \mu(\Psi(w))v$

Hence, $g \cdot \Psi \otimes v = g \Psi g^{-1} \otimes g v$ goes to the map

$$w \otimes \mu \to \mu(g(\Psi(g^{-1}(w))))gv$$

while, on the other hand, the map $w \otimes \mu \to \mu(\Psi(w))v$ goes, under application of $g$, to, in three stages

$$w \otimes \mu \to g^{-1}w \otimes g^{-1}\mu \to (g^{-1}\mu)(\Psi(g^{-1}w))v \to g\left[(g^{-1}\mu)(\Psi(g^{-1}w))v\right]$$

which is

$$\mu(g(\Psi(g^{-1}(w)))gv$$

as desired.

Now let's go on to make some further observations. First, let's point out that we have proven a statement stronger than we have made explicit. We introduce some notation. If $V$ and $V'$ are simple $FG$-modules then we write $V/V'$ if $V'$ is the tensor product of $V$ and another simple module in the usual way. We have already established

<u>Proposition 8.</u> If $U_0 \leq U$ is regular on the simple $FG$-module $V$ and $U_0 \leq U_1$ another subgroup of $U$ then there is $V'$ with $V/V'$, such that $U_1$ is regular on $V'$.

Indeed, we have proven this for the case that the order of $U_1$ is $p$ times the order of $U_0$: to complete the proof we need only apply the obvious induction argument. This leaves unanswered the first conjecture of Buff's letter: if $U_0 \leq U_2$, $V_0 / V_2$ where $U_2$ is regular on $V_2$ find $V'$ with $V' / V_2$ as well.

Now let's state the new result:

<u>Proposition 9</u>. If $U_0$ is regular on the simple $FG$-module $V$ and $V/V'$ then there is $U_1$, $U_0 \le U_1 \le U$ with $U_1$ regular on $V'$.

This is, in fact, easier to prove than the previous result. The proof proceeds by using $U_1 \le U$ with $U_1$ regular on $V'$, $U_1$ <u>not</u> necessarily containing $U_0$, and then "changing" $U_1$. Hence, choose such a subgroup $U_1$ of $U$ which we have already proved to exist. Let $\mathfrak{A}$ be the annihilator of $V'$ in $FU$ so, as before, we have $FU = FU_1 + \mathfrak{A}$, a vector space direct sum. Next, observe that $FU/\mathfrak{A}$ is isomorphic with $V'$ as $FU$-modules: since $V'$ is regular for $FU_1$ it is certainly cyclic so there is an $FU$-homomorphism of $FU$ onto $V'$ and the kernel is $\mathfrak{A}$. Let $\pi$ be the projection of $FU$ onto $FU_1$ "along" $\mathfrak{A}$. Since $FU_1$ is now projective for $\pi(FU_0)$, we have that if $x_1, \dots, x_s$ are a basis of $U_0$ then the elements $\pi(x_i - 1)$, $1 \le i \le s$, are linearly independent modulo $\mathrm{rad}^2(FU_1)$ (by Theorem 6.2, p125, of J. Carlson's paper, The varieties and the cohomology ring of a finite groups, J. Algebra <u>85</u> (1983), 104–143). Hence, since using elements $g-1$, $g$ running over a basis of $U_1$, gives a basis of $\mathrm{rad}(FU_1)/\mathrm{rad}^2(FU_1)$, we can choose $x_{s+1}, \dots, x_t \in U_1$ so that the elements $\pi(x_j - 1)$, $1 \le j \le t$, are also a basis of this vector space. Again, this means that $FU_1$ is free (of rank one), for $\langle x_j \mid 1 \le j \le t \rangle$, by Theorem 6.2 again, so the proposition is proved using this group in place of $U_1$.

Now Geoff has made a second conjecture. Suppose $U_0$ is regular on $V$, $U_0 \le U_2$ with $U_2$ regular on $V''$, $V/V''$ and we have $V'$ with $V/V'$, $V'/V''$ then we can choose $U_1$, $U_0 \le U_1 \le U_2$ with $U_1$ regular on $V'$. Moreover we can do this by obtaining $U_1$ from $U_0$ by adding to $U_0$ some elements of any basis of $U_2$, fixed in advance, which contains a basis of $U_0$. We have shown above that this conjecture holds when $U = U_2$ as then $V''$ is the Steinberg module.

Let us proceed to prove the second conjecture. The set-up:

$$\begin{array}{ll} V'' & U_2 \\ V' & U_1\ ?? \\ V & U_0 \end{array}$$

(and we have a fixed basis of the elementary abelian group $U_2$ containing a basis of $U_0$). Choose $U_1 \le U$ that works for $V'$ so, as usual, $FU = FU_1 + \mathfrak{a}$. We will modify $U_1$ to get another "correct" $U_1$. Let $A = FU_1$ and let $B$ be the image of $FU_2$ in $A$ (using the usual projection). We assert that $B = A$; suppose, on the contrary, $B < A$. Now $B$ is also a local algebra so $\operatorname{rad} B < \operatorname{rad} A$, even the image of $\operatorname{rad} B$ in $\operatorname{rad} A/(\operatorname{rad} A)^2$ is proper, by the generation properties of $A$. Now $\operatorname{rad} A/(\operatorname{rad} A)^2$ and $\operatorname{soc}^2 A/\operatorname{soc} A$ are paired into $\operatorname{soc} A$ by multiplication so we deduce the existence of an ideal $I$ of $A$, $\operatorname{soc} A < I \le \operatorname{soc}^2 A$ with $\operatorname{rad} B . I = 0$. $\therefore \operatorname{rad} B . I V' = 0$, that is, $\operatorname{rad} B$ annihilates a space of at least two dimensions (we are using that $V'$ is a free module for $A$). Hence, $B = F1 + \operatorname{rad} B$ yields that $U_2$ fixes at least two dimensions of vectors of $V'$ and so the same holds for $V''$, as $V'/V''$, a contradiction.

Now $V'$ is a free module for $FU_0$ so the image of the part of the fixed basis lying in $U_0$, in $A$ consists of elements of the form $1+r$, $r \in \operatorname{rad} A$, where the elements $r$ are linearly independent modulo $\operatorname{rad}^2 A$. Since $B = A$ we can proceed as before to find a subset of the basis of $U_2$, containing the basis in $U_0$, whose images in $A$, when $1$ is subtracted from each, give a basis for $\operatorname{rad} A / \operatorname{rad}^2 A$. Hence, again quoting Carlson, we are done.

Next, we turn to the first conjecture. We begin with a generalization of Lemma 5:

<u>Lemma 10.</u> If $U' \leq U$, $V' = \sigma_1(M) \otimes \dots \otimes \sigma_r(M)$ (with the usual notation) and $V'$ affords the regular representation of $U'$ then the annihilator in $FU'$ of

$$\sigma_1(T) \oplus \dots \oplus \sigma_r(T)$$

is $\operatorname{rad}^2(FU')$.

To see this just copy the proof of Lemma 5 using $U'$ in place of $U$. The critical determinant that arises is then that of Lemma 2 !!

Now let's turn to the first conjecture. The set-up:

$$\begin{array}{lll} V'' & U_2 & \\ ??\; V' & U_1 & |U_1 : U_0| = p \\ V & U_0 & \end{array}$$

(the restriction $|U_1 : U_0| = p$ is no limitation). We must produce $V'$ of the form $\sigma(M) \otimes V / V''$.

If there is an automorphism $\sigma$ such that $\sigma(M) \mid V''$, $\sigma(M) \nmid V$ and $(\sigma(M) \otimes V)^{U_0}$ is free as a module for $U_1/U_0$ then $\sigma(M) \otimes V = V'$ clearly suffices. Hence, we assume otherwise: for each such $\sigma$, the module $(\sigma(M) \otimes V)^{U_0}$ is not free for $U_1/U_0$ so, by Proposition 4, $V_{U_1}$ has a subquotient isomorphic with $\sigma(T)_{U_1}$.

Now if $\tau$ is an automorphism so that $\tau(M) \mid V$ then the tensor product structure of $V$ also gives $V_{U_1}$ with a subquotient isomorphic with $\tau(T)_{U_1}$ (consider the submodule, for $U_1$ of the factor $\tau(M)$ tensored with the fixed points in the other factors). Hence, the annihilator in $FU_1$ of $V$ is contained in $\operatorname{rad}^2(FU_2)$, by Lemma 10. Now proceed exactly as before in the argument following the proof of Lemma 5 to reach a contradiction.

Hence, we have established this conjecture too!

However, this is all too heavy-handed: a direct determinantal argument applies! Consider the second conjecture:

$$\begin{array}{ccc} V'' & U_2 & \\ V' & & ?? \\ V & U_0 & \end{array}$$

with a fixed basis. Consider the non-zero determinant corresponding to $V''$ and $U_2$ so the rows correspond to certain automorphisms and the columns to the basis of $U_2$. Considering $V'$ means picking a subset of the rows, which will be of course linearly independent. The columns corresponding to $U_0$, restricted to these rows will also be linearly independent since they are already on the rows corresponding to $U_1$ by assumption. Hence looking at the matrix whose rows correspond to $V'$ (with all columns) and since its row rank equals its column rank we can supplement the columns corresponding to $V_0$ by other columns to get a square matrix of non-zero determinant and so conjecture 2 is proved! Arguing similarly for columns and rows gives conjecture 1. Note that the rank argument is an alternative to the Laplace expansion.

Virtual permutations

The study of the symmetric groups $\Sigma_X$ on an infinite set $X$ is quite old (R. Baer, Studia Math 5 (1934), 15-17; J. Schreier and S. Ulam, Studia Math 4 (1934), 131-41; Karrass and Solitar, Math Z. 66 (1956), 64-9). For example, there is no alternating group but the group of finitary permutations (almost everywhere the identity) is a normal subgroup; let $\overline{\Sigma}_X$ be the corresponding quotient. We wish to observe the fact that $\overline{\Sigma}_X$ has outer automorphisms (this is quite easy) and to interpret this event in a natural way.

Definition 1. A function $\alpha$ with domain $Y$ cofinite in $X$ and range $Z$ cofinite in $X$ which is one-to-one and onto from $Y$ to $Z$ is called an almost permutation of $X$.

If $\alpha$ and $\alpha'$ are almost permutations of $X$ we say they are equivalent if they agree on a cofinite subset of $X$ (that is, on a cofinite subset of the intersections of their domains of definition. This is easily seen to be an equivalence relation.

Definition 2 An equivalence class of almost permutations of $X$ is called a virtual permutation of $X$.

The virtual permutations form a group under multiplication. The product is well-defined, the inverse of a scan permutation containing the almost permutation $\alpha$ is the class of the inverse of $\alpha$, the function whose domain is the range of $\alpha$, whose range is the domain of $\alpha$ and which is inverse to $\alpha$ on these sets. Call this group $N_X$. We also have a natural homomorphism of $\Sigma_X$ to $N_X$ and the kernel is the finitary permutations so we have a natural embedding of $\bar{\Sigma}_X$ in $N_X$. Now let $\sigma$ be a one-to-one map of $X$ to $X$ with equal to $X$ less exactly one element so $\sigma$ is an almost permutation. Let $\sigma$ also denote the corresponding element of $N_X$.

<u>Theorem</u> The group $N_X$ is the semi-direct product of $\bar{\Sigma}_X$ by the infinite cyclic subgroup generated by $\sigma$.

After we establish this result, a little more work will show that $\sigma$ induces an automorphism of $\bar{\Sigma}_X$ which is <u>not</u> inner.

<u>Lemma 1</u>. If $f$ is an almost of $X$ then there exist permutations $\pi_1, \pi_2, \pi_3$ of $X$ and non-negative integers $m, n$ with

$$f = \pi_3 \circ \sigma^n \circ \pi_2 \circ \bar{\sigma}^m$$

Here $\bar{\sigma}$ is the inverse of $\sigma$ so is defined on $\sigma(X)$ and has range $X$.

<u>Proof</u>. Let $f = f_2 \circ f_1$ where $f_1$ has domain equal to that of $f$ but range $X$ while $f_2$ has domain $X$ with range equal that of $f$. Suppose that $|X - f_2(X)| = n$. Hence, there is a permutation $\alpha$ so that $\alpha \circ f_2(X) = \sigma^n(X)$ so there is another permutation $\beta$ with $\alpha \circ f_2 \circ \beta = \sigma^n$. Thus, $f_2 = \alpha^{-1} \circ \sigma^m \circ \beta^{-1}$. Similarly, suppose that the domain of $f_1$ has cocardinality $m$ so there is a permutation $\gamma$ so that

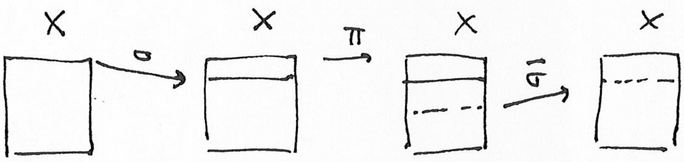
$f_1 \circ \gamma$ and $\bar{\sigma}^m$ have the same domain. Hence, there is another permutation $\delta$ with $\delta \circ f_1 \circ \gamma = \bar{\sigma}^m$ so $f_1 = \delta^{-1} \circ \bar{\sigma}^m \circ \gamma^{-1}$ and the lemma is proved.

Lemma 2 If $\pi$ is a permutation of $X$ then there is a permutation $\rho$ of $X$ such that $\sigma \circ \pi \circ \bar{\sigma}$ defines the same virtual permutation as does $\rho$.

Proof. $\sigma \circ \pi \circ \bar{\sigma}$ has domain and range of cocardinality one and it is one-to-one and onto from its domain to range.

Lemma 3. If $\pi$ is a permutation of $X$ then there is a permutation $\rho$ of $X$ such that $\bar{\sigma} \circ \pi \circ \sigma$ and $\rho$ define the same virtual permutation.

Proof. If $\pi$ preserves $\sigma(X)$ then $\bar{\sigma} \circ \pi \circ \sigma$ is a permutation. If $\pi$ does not preserve $\sigma(X)$ then we have the following picture:




We have that $\bar{\sigma} \circ \pi$ is defined on a set of cocardinality two so the range of $\bar{\sigma} \circ \pi$, which is the range of $\bar{\sigma} \circ \pi \circ \sigma$, is of cocardinality one, that is, the cocardinality of the domain of $\bar{\sigma} \circ \pi \circ \sigma$.

Proof of theorem. With the notation of the first lemma we have that $f$ and $\sigma^{n-m} f$ represent the same element for a suitable permutation $\pi$. The above two lemmas give the desired normality. Finally, different powers of $\sigma$ cannot "differ" by a permutation. Indeed, $\sigma^n$ and a permutation, for $n>0$ cannot represent the same class as they cannot agree on a cofinite set as the permutation does not map a cofinite set to a set of larger cocardinality.

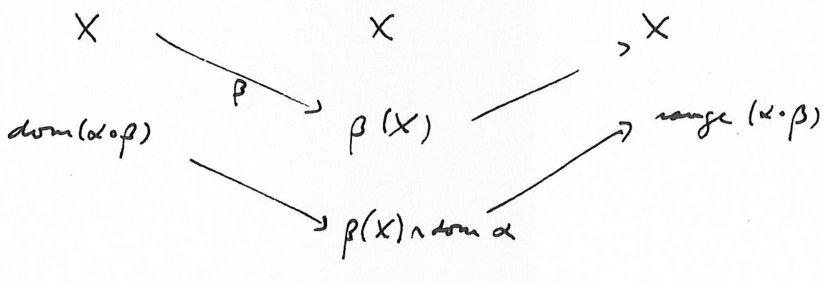
Remarks.

1. To finish the last argument introduce the following concept earlier. If $\alpha$ is an almost permutation define

$$\Delta(\alpha) = |X - \text{range}\,\alpha| - |X - \text{domain}\,\alpha|.$$

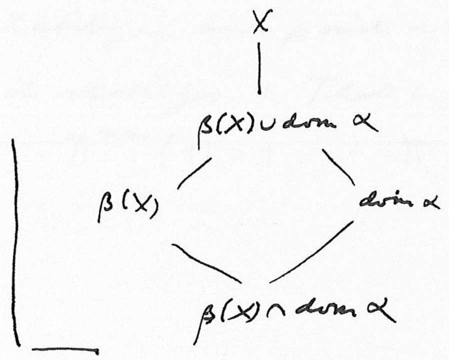
Then $\Delta(\alpha)$ depends only on its class as is easy to see. $\Delta(\sigma) = +1$, $\Delta(\pi) = 0$ if $\pi$ is a permutation and $\Delta(\pi\sigma^i) = i$, so in view of our results, $\pi$ is a homomorphism to $\mathbb{Z}^+$.

2. Seems easy to first prove directly that $\Delta$ is a homomorphism and so deduce our main result. Need only calculate $\Delta(\alpha\circ\beta)$ for almost permutations $\alpha, \beta$ where each has domain or range all of $X$. If each has range (or each has domain) all of $X$ this is easy. Say $\beta$ has domain $X$ and $\alpha$ has range $X$. Picture:



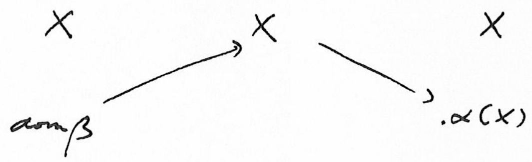

Hence, we have

$$\begin{aligned}
\Delta(\alpha\circ\beta) &= -|X - \mathrm{dom}(\alpha\circ\beta)| + |X - \mathrm{range}(\alpha\circ\beta)| \\
&= -|\beta(X) - \beta(X)\cap \mathrm{dom}\,\alpha| + |\mathrm{dom}\,\alpha - (\beta(X)\cap\mathrm{dom}\,\alpha)| \\
&= -|(\beta(X)\cup\mathrm{dom}\,\alpha) - \mathrm{dom}\,\alpha| + |(\beta(X)\cup\mathrm{dom}\,\alpha) - \beta(X)| \\
&= -\ |(\beta(X)\cup\mathrm{dom}\,\alpha) - \mathrm{dom}\,\alpha| \quad |X - (\beta(X)\cup\mathrm{dom}\,\alpha)| \\
&\qquad + |(\beta(X)\cup\mathrm{dom}\,\alpha) - \beta(X)| + |X - (\beta(X)\cup\mathrm{dom}\,\alpha)| \\
&= -|X - \mathrm{dom}\,\alpha| + |X - \beta(X)|
\end{aligned}$$




as required.

Similarly, suppose that $\beta$ has range $X$ and that $\alpha$ has domain $X$. Picture:




This is also easy.

Let's now show the automorphism given by $\sigma$ is outer. We can replace $\sigma$ by any other element in its coset (i.e. any element with $\Delta(\ ) = 1$). Assume $X = \{1, 2, \dots\} \cup F$ and let $\sigma : n \to n+1$, $f \to f$, all $f \in F$, all $n \in \mathbb{N}$. It suffices to show there is no virtual permutation represented by a permutation $\pi$ which is fixed by conjugation by $\sigma$, other than the identity. Suppose otherwise so $\bar{\sigma}\circ\pi\circ\sigma$ and $\pi$ agree on a cofinite set. But

$$\bar{\sigma}\circ\pi\circ\sigma(n) = \pi(n+1) - 1$$

so, for all large $n$, $\pi(n+1) = \pi(n) + 1$. Also, if $f \in F$ then $\pi\circ\sigma(f) = \sigma\circ\pi(f)$, almost always, that is, $\pi(f) = \sigma(\pi(f))$ so $\pi(f) \in F$. That is, the set $F_0$ of all $f \in F$ such that $\pi(f) \in F$ is cofinite in $F$. We can now get that $\Delta(\pi) \neq 0$

the module $\Pi$ affords the regular representation of the group
$$\left\langle \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix} ;\ 1 \leq i \leq r \right\rangle$$
of order $p^r$ containing $K$. Hence, $\Pi$ is certainly free as a module for $FK$.

Next, assume conversely that $\Pi$ is a free $FK$-module. It suffices to show that there is a subgroup $L$ of $U$ containing $K$ such that $L$ has order $p^r$ with $\Pi$ affording the regular representation of $L$. For then the $r$ by $s$ matrix under consideration will be an $r$ by $s$ submatrix of a non-singular $r$ by $r$ matrix, again by our previous theorem.

However, there is a subgroup $H$ of $U$, $H$ of order $p^r$ such that $\Pi$ affords the regular representation of $H$: the $n$ by $n$ matrix $(\sigma_i(\lambda_j))$ is non-singular so the $r$ by $n$ matrix corresponding to $\sigma_1, \ldots, \sigma_r$ has rank $r$ so it has $r$ linearly independent columns. Let $\mathfrak{a}$ be the annihilator of $\Pi$ in $U$, an ideal, so $FU = FH + \mathfrak{a}$, a vector space direct sum, since $FU$ induces endomorphisms of the regular $FH$-module $\Pi$ which are all induced by $FH$, as before in our results. Since $\Pi$ is cyclic we now have that $FU/\mathfrak{a} \simeq \Pi$ as $FU$-modules, so we can work with $FU/\mathfrak{a}$ instead of $\Pi$. We need only show that there is a subgroup $L$ of $U$ containing $K$ such that $FU/\mathfrak{a}$ is the regular $FL$-module.

Of course, $FU/\mathfrak{a} \simeq FH$ as $FH$-modules, so we can use the theory of the group algebra of an elementary abelian $p$-group. Indeed, if $H = \langle h_1, h_2, \ldots, h_r \rangle$ then $FU/\mathfrak{a}$ is a group algebra of the group with basis $h_i + \mathfrak{a}$, $1 \leq i \leq r$. Since $FK \cap \mathfrak{a} = 0$ (as $FU/\mathfrak{a}$ is free as a $FK$-module) we have a subgroup of the group of units of $FU/\mathfrak{a}$ generated by the elements $g_1 + \mathfrak{a}, \ldots, g_s + \mathfrak{a}$, this is groups is elementary abelian of order $p^s$ and (again) $FH/\Omega$ is a free module. Hence, we can apply a result we have mentioned before: Theorem 6.2, p125, of the paper by J. Carlson, entitled "The varieties and the cohomology rings of a finite groups," J. Algebra 85 (1983), 104-145. We deduce that the elements $g_i - 1 + \Omega$ of $FU/\Omega$ are in $\mathrm{rad}(FU/\Omega)$ and linearly independent modulo $\mathrm{rad}^2(FU/\Omega)$. Hence, after suitable renumbering, we have that the elements $g_i - 1 + \Omega$ and $h_j - 1 + \Omega$, $1 \le j \le s-r$ give a basis of $\mathrm{rad}(FU/\Omega)/\mathrm{rad}^2(FU/\Omega)$. Since these elements commute, their p-th powers are zero, they are a basis of the radical quotient given while $FU/\Omega$ has dimension $p^r$ it follows that $FU/\Omega$ is a group algebra on the subgroup $\langle g_1, \ldots, g_s, h_1, \ldots, h_{s-r} \rangle = L$. This is clearly the desired subgroup.

We now get a number of consequences just as in Geoff's letter!

Now let's turn to the question on $SL_2(q)$ of Geoff's about freeness on other simple modules. Let $V = T \otimes S$ be a simple with $S$ the "Steinberg factor" (usual tensor product of Steinberg's theorem, in this case Brauer-Nesbitt). Want freeness on $V$ to imply freeness on $S$. Need only show that there is an $SL_2(q)$ module $R$ with $k \mid R \otimes T$. By algebraic conjugacy only need to do this for $T$ one of the basic simple modules of dimensions $2, 3, \ldots; p-1$. Hence, only need splitting of Clebsch-Gordan sequence (where $V_i$ is of dimension $i$; the representation on homogeneous polynomials of degree $i-1$)

$$0 \to V_{i-1} \to V_2 \otimes V_i \to V_{i+1} \to 0$$

But the module $V_2 \otimes V_i$ is self-dual so we're O.K. (actually get $V_2$ self-dual then splitting of $V_2 \otimes V_2$ gives $V_3$ self-dual, etc....) For if non-split then $\mathrm{soc}(V_2 \otimes V_i) = V_{i-1}$, $\mathrm{soc}((V_2 \otimes V_i)^*) = V_{i+1}$.

Another approach is to get a backwards map

$$V_{i+1} \to V_2 \otimes V_i$$

(backwards to the multiplication map sending $V_2 \otimes V_i$ to $V_{i+1}$). Hence, we only need a commutative diagram

$$\begin{tikzcd}
V_{i+1} \arrow[r] & \overbrace{V_2 \otimes \cdots \otimes V_2}^{i} \arrow[dl] \\
V_2 \otimes V_i \arrow[u] &
\end{tikzcd}$$

where the undefined map is multiplication on the last $i-1$ factors and tracing $V_{i+1}$ all the way around gives the identity. But this is just the characteristic zero argument that the symmetric tensors, a subspace of a tensor power, are the symmetric power, a quotient space of tensor power, which follows using the central idempotent of the symmetric group for the principal character. This works here in this range, sending (if $V_1$ has basis $X, Y$)

$$X^a Y^b \to \frac{1}{i!} \sum \underbrace{X \otimes \cdots \otimes X}_{a} \otimes \underbrace{Y \otimes \cdots \otimes Y}_{b} \qquad (a+b=i)$$

where the sum means all results using place permutations of $\Sigma_i$.

Restricting the Steinberg module

We shall investigate Geoff Mason's factorization question. Let $G$ be a finite group of Lie type and characteristic $p$, $k$ an algebraically closed field of characteristic $p$, $P$ a parabolic subgroup of $G$ with Levi decomposition $P = L.U$. Let $St_G$ be the Steinberg $kG$-module ($St_L$ for $L$, etc.).

<u>Proposition 1</u>. $Res_P^G St_G \cong Ind_L^P St_L$.

<u>Remarks</u> 1. This gives the desired factorization in this case, as $Ind_L^P St_L \cong Ind_L^P((Res_L^P \hat{St}_L) \otimes k) \cong \hat{St}_L \otimes Ind_L^P k$ where $\hat{St}_L$ is the lift of $St_L$ to $P$.

2. Probably this result is well-known!

3. This should be easily provable by the alternating sum-building-formula for the Steinberg module: look at $GL(n,q)$ and parabolics containing a Borel subgroup in terms of flags, for example.

4. This raises a bevy of questions: What is the corresponding result for the basic and "partial" Steinberg modules? Is there an analogous result for the Steinberg complex of Webb and the Brown complex? Are these complexes related to the partial Steinberg modules? Is there a homotopy theoretic proof?

<u>Proof</u>. $Res_P^G St_G$ is a projective $kP$-module with the simple $kP$-module $\hat{St}_L$ as a submodule (by Steve Smith's theorem – see Geoff Mason's letter) so the injective envelope (= projective cover) of $\hat{St}_L$ is a summand of $Res_P^G St_G$. Hence, it suffices to prove that $Ind_L^P St_L$ is such a cover. However, $Ind_L^P St_L$ is projective, its dimension is $|P|_p$ so it is indecomposable and $\hat{St}_L$ is a homomorphic image since

$$\mathrm{Hom}_{kP}(\mathrm{Ind}_L^P St_L, \hat{St}_L) \cong \mathrm{Hom}_{kL}(St_L, \mathrm{Res}_L^P \hat{St}_L) \cong \mathrm{Hom}_{kL}(St_L, St_L) \neq 0.$$

Hence, the proposition is proved.

One part of the argument suggests a line to follow as a digression. Here let $G = L.U$, a semi-direct product of a normal $p$-subgroup $U$ and a complement $L$.

<u>Proposition 2.</u> If $S$ is a simple $kL$-module and $Q_S$ is its projective cover then $\mathrm{Ind}_L^G$ is the projective cover of the lift $\hat{S}$ of $S$ to $G$.

<u>Proof</u>. Let $S_1, S_2, \ldots, S_n$ be the distinct simple $kL$-modules and let $\hat{S}_i$ be the lift of $S_i$ to $G$. Let $Q_i$ be the projective cover of $S_i$. If $\dim_k S_i = d_i$ then, as $kL$-modules

$$kL \cong \bigoplus d_i Q_i$$

so

$$kG = \mathrm{Ind}_L^G kL \cong \bigoplus d_i \,\mathrm{Ind}_L^G Q_i .$$

But $\mathrm{Ind}_L^G Q_i$ is projective and has $\hat{S}_i$ as an image (as in the proof of Proposition 1) so the projective cover $P_i$ of $\hat{S}_i$ is a summand of $\mathrm{Ind}_L^G Q_i$ so $\bigoplus d_i P_i$ is a summand of $kG$. We must have equality so we have equality "term by term."

<u>Remark</u>. It is easy to see how to adapt this if $k$ is not algebraically closed using the dimension of the endomorphism algebra of $S_i$ which is the same as for $\hat{S}_i$. However, there is even a better result, an easy sort of "dual" to Green's induction theorem.

We no longer need $k$ algebraically closed. We keep $G = L.U$.

<u>Proposition 3.</u> If $X$ is an indecomposable $kL$-module then $\mathrm{Ind}_L^G X$ is an indecomposable $kG$-module.

<u>Proof.</u> First, we observe that if $Y$ is any $kG$-module then $\mathrm{rad}(kU).Y$ is a $kG$-submodule of $Y$ contained in $\mathrm{rad}(kG).Y$. For $\mathrm{rad}\, kU$ annihilates any simple $kG$-module, by Clifford's theorem, so $\mathrm{rad}\, kU \subseteq \mathrm{rad}\, kG$. If $g \in G$ then

$$g.\, \mathrm{rad}(kU)Y = g(\mathrm{rad}(kU))g^{-1} gY = \mathrm{rad}\, k(gUg^{-1}).\, Y = \mathrm{rad}(kU).Y$$

so our first claim is valid.

To prove the proposition it is therefore sufficient to see that if $Y = \mathrm{Ind}_L^G X$ then $\hat{X} \cong Y/\mathrm{rad}(kU).Y$, where $\hat{X}$ is the lift of $X$ to $G$. Indeed, if $Y = Y_1 \oplus Y_2$ then $Y/(\mathrm{rad}\, U)Y \cong Y_1/(\mathrm{rad}\, U)Y_1 \oplus Y_2/(\mathrm{rad}\, U)Y_2$, each summand being non-zero, if $Y_1$ and $Y_2$ are non-zero, as $\mathrm{rad}\, U \subseteq \mathrm{rad}\, kG$.

Now we have a vector space epimorphism $Y = kG \otimes_{kL} X \to \hat{X}$ sending each $g \otimes x$ to $x$ and this is clearly a $kG$-homomorphism. Since $U$ acts trivially on $\hat{X}$ it does on the quotient of $Y$ by the kernel, that is, if $u \in U$ then $u-1$ annihilates this quotient, that is, $\mathrm{rad}\, kU$ does so we have an epimorphism of $Y/(\mathrm{rad}\, kU)Y$ to $\hat{X}$. We just need a suitable inverse map! We have a vector space map of $\hat{X}$ to $Y$ sending each $x \in \hat{X}$ to $1 \otimes x \in Y$. We need only prove this is a $kG$-homomorphism when composed with the natural map of $Y$ onto $Y/\mathrm{rad}(kU).Y$, as the rest is clear. To check that this backwards map ($x \to 1 \otimes x$ composed with the natural map) commutes with the action of $G$ let $l \in L$, $u \in U$. We first must see that $u(1 \otimes x)$ is the image of $ux$, that is, $u(1 \otimes x) \equiv 1 \otimes ux = 1 \otimes x$. But $u(1 \otimes x) = u \otimes x = (1 + (u-1)) \otimes x \equiv 1 \otimes x$. Also $l(1 \otimes x) = l \otimes x = 1 \otimes lx$ the image of $lx$.

Remark. The proof shows that if $X_1 \neq X_2$ then $Y_1 \neq Y_2$ (with the obvious notation) so we have a sort of equivalence less than a Morita equivalence.

Remark. Proposition 1, in terms of characters, appears in Carter's latest book with a proof using the alternating sum due to Howlett.

Now let's return from the diversion. Let's consider the "factor" that comes in Proposition 1 (see Remark 1) and its structure in one case in the hope this leads to an insight. We consider the permutation module of $GL(n,p)$ over $\mathbb{F}_p$ acting on the vectors in the standard vector space for $GL(n,p)$, so this module is of dimension $p^n$. Our result generalizes a theorem Mike O'Nan told us years ago.

Proposition 3. The permutation module has a filtration the successive subquotients being the truncated symmetric powers of the standard vector space.

We mean the following. Let $V$ be the standard module. The truncated symmetric algebra on $V$ is the quotient of the symmetric algebra on $V$ by the $p$-th powers so this has homogeneous terms of degree $0, 1, \ldots, n(p-1)$ and total dimension $p^n$. (If $p=2$ this is also the exterior algebra - which is the case O'Nan told us about.)

To prove this we let $E$ be a (multiplicative) elementary abelian $p$-group of order $p^n$ so we can let $GL(n,p)$ "be" the automorphism group of $E$ so it acts as automorphisms of the group algebra $\mathbb{F}_p E$ and this is just our permutation module! Now $\mathbb{F}_p E$ is isomorphic with the truncated symmetric algebra in $n$ variables (generated by the elements $e_i - 1$ as $e_1, \dots, e_n$ is a basis of $E$). Hence, the successive quotients of the radical series of $\mathbb{F}_p E$ (the augmentation powers) are the homogeneous terms of the truncated symmetric algebra. It remains only to see that $GL(n,p)$ acts properly on $\mathrm{rad}(\mathbb{F}_p E)/\mathrm{rad}^2(\mathbb{F}_p E)$. But $e_1 - 1, \dots, e_n - 1$ give a basis for this space so the formula

$$xy - 1 = (x-1) + (y-1) + (x-1)(y-1)$$

is all that we need.

<u>Remarks.</u> 1. For $GL(n,q)$, $q = p^e$ we could get information using $GL(n,q) \leq GL(ne, p)$ about the standard module.

2. Question: Is there some sort of Steinberg factorization, at least up to composition factors that applies in the previous remark? (The case $GL(2,q)$ should be easily accessible.)

3. The number of orbits of a Sylow $p$-subgroup of $GL(n,p)$ on $V$ is easily seen to be $1 + (p-1)n$ so the fixed point space of this Sylow subgroup on the permutation module is of this dimension. If the permutation module were semisimple then, as each subquotient considered is simple, the fixed point space would be also $1 + n(p-1)$ so we can't easily get non-semisimplicity! But that should be the case. It looks like, for $p = 2$, after shedding the two trivial modules that the rest is uniserial and one should be able to prove this by considering the action of $GL(n,2)$ on fixed points of the Sylow $2$-subgroup.

Now let's try another approach. This doesn't do what is needed but it comes close. So now let $G$ be a universal Chevalley group over $k$ so $G$ is semisimple, simply connected and so on. Let $\alpha$ be a positive root so $L = \langle X_\alpha(t), X_{-\alpha}(t) \rangle = SL(2,k)$ is the derived group of the Levi complement of a (minimal) parabolic subgroup (put $\alpha$ as a basic root for the appropriate Borel subgroup). Let $St_G$ be the Steinberg module (in the sense of algebraic groups) for $G$ so it has dimension the appropriate power of $p$, restricts to the basic Steinberg for $G(q)$ and is an "induced" module (dual of a Weyl module) for the highest weight which is $p-1$ times the sum of the fundamental weights. (For the last, see Jantzen's book II 3.19 (4).) If $G = SL(2)$ then these induced modules are just the symmetric powers of the standard 2-dimensional module (again see Jantzen p209 or Green, LNM 830, 4.4 Example 1+(4.1c)

For the purposes of generalizing from $SL(2)$, Mason and I need certain connections to exist between $St_G|_L$ and $St_L$. For example, if the latter divided the former this would do. Or, if $St_G|_L$ had a filtration and $St_L$ divided each subquotient then we'd also be fine. This brings to mind the good filtration theory of Donkin + Mathieu (S. Donkin, Rational representations of algebraic groups, Springer Verlag, Lect. Notes in Math. v1140, 1985; Olivier Mathieu, Filtrations of G-modules, Annales Sci. L'Ecole Normal Superieure, v23 (4th series) 1990, 625-45). Suppose the factors of the good filtration of $St_G|_L$, which exists as part of the theory, were suitable. This doesn't occur as Donkin points out for $p=2$, $G = SL(3)$. However in that case we are still OK. For the Steinberg module for $SL(3)$

is eight-dimensional, can be realized as matrices of trace 0 under conjugation so it is easy to see that the restriction to $SL(2)$ is $E \oplus E^* \oplus (E \otimes E^*)$ $(\simeq E \oplus E \oplus (E \otimes E))$ when $E$ is the standard module for $SL(2)$. The picture:

$$E = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \qquad E \otimes E^* = \begin{pmatrix} -t & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad E^* = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

($t$ so trace is 0)

$$SL(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

[As a side remark note that lots of modules for $SL(2)$ are divisible by $St_L$. $St_L = V_p$ the $p$-dimensional symmetric power of degree $p-1$ and $V_p \otimes V_2^{(p)} \simeq V_{2p}$, by multiplication, where the upper $p$ denotes Frobenius. We also have the modular version of Clebsch-Gordan.]

It is also true that we would be fine if $St_L$ virtually divided $St_G|_L$ in the sense that there were modules $V_1$ and $V_2$ with $St_G|_L \oplus (V_1 \otimes St_L) \simeq V_2 \otimes St_L$. Applying the Frobenius, working in the Green ring and tensor the expressions for $St_G|_L$, $St_G^{(p)}|_L$, — and passing to the finite groups would give us what we need. [Also, it's more formal as $A \oplus B \simeq C$, $A' \oplus B' \simeq C'$ imply $A \otimes A' \oplus (B \otimes C') \oplus (B' \otimes C) \simeq (B \otimes B') \oplus (C \otimes C')$ if we have suitable cancellation. For, with obvious notation,

$$(aa' + bc' + b'c) + ab' = ac' + bc' + b'c = cc' + b'c$$
$$(bb' + cc') + ab' = cb' + cc'.]$$

However, Donkin has proven what we have just asked for and more. To state his result we have $G$ as above, and let $H$ be the derived group of a Levi factor of G (so $H = L$ is allowed). Then there are "tilting" modules for $H$ (in particular, modules; tilting, in this context is equivalent to having filtrations both by induced modules and by Weyl modules), $M_1$ and $M_2$ so that

$$St_G|_H \otimes (M_1 \otimes St_H) \simeq M_2 \otimes St_H.$$

This is an easy consequence of a (preliminary) preprint of his on tilting modules.

<u>Remarks</u> 1. Using Smith's result on other parabolics together with this result of Donkin, we can say more about comparing Steinberg modules and freeness properties, it would seem.

2. Perhaps some of this is relevant to characterizing the partial Steinberg modules (using Ronan-Smith "sheaf" methods?).

3. A number of questions face us. Is there a "semi-freeness" property relevant, in particular, for the root subgroups acting on partial Steinbergs? Is there an independence between different root subgroups? This might help inductively in building up a sheaf. Are there strengthened forms of Smith's theorem in our case so that we can give a filtration of the restriction of St, for example, to parabolics in terms of modules described by the St, of the Levi factor, as lifted to the parabolic that we are in the process of studying!

Let us study the "semi-free" question for root subgroups. Let $G$ be our universal Chevalley group over $k$, $X$ a root subgroup and $M$ a tensor product of $d$ conjugates of $St_1$. Let $Y \leq X$ be of order $p^d$ such that $St_1$ is a free $kY$-module. We want to know if $M^X = M^Y$? If the answer is positive and we pass to $G(q)$ get positive answer to the question of semi-freeness as we say a module $V$ for a $p$-gp $E$ is semi-free with respect to $F \leq E$ if $V^E = V^F$ and $V$ is a free $kF$-module. [note: In this case the socle series = radical series - we have $p$-groups so these are the same for free modules - of $V$ as $kF$-module equals the same series for $kE$. This is easy to prove as $\Delta E \cdot \Delta F = \Delta F \cdot \Delta E$ (augmentation ideals).]

Note that if this occurs and $Y' \leq X$ also of order $p^d$ and acting freely then $X$ is semi-free with respect to $Y'$, as $M^{Y'} \geq M^X$ and by dimension counting (which gives $\dim M^Y = \dim M^{Y'}$), and that $M^{Y'} = M^Y$. We next observe that it suffices to deal with the case $d = 1$. For say we have semi-freeness there and that, with the obvious notation,

$$M = S_1 \otimes \dots \otimes S_d ,$$

where $\dim S_i = p^N$ where $N$ is the number of positive roots. Now $\dim M^Y = p^{(N-1)d}$ $(\dim M / |Y|)$ so we need only show that $\dim M^X \geq p^{(N-1)d}$ and to do this it suffices to prove that $\dim S_i^X \geq p^{N-1}$. But each $y \in Y$, $y \neq 1$ acts freely on each $S_i$ (conjugacy, plus the field of $p$ elements and the projectivity of $St$ on $G(p)$) (We must work with points over the algebraic closure of the field of $p$ elements.) Hence we have $St^X = St^{\langle y \rangle}$ is of dimension $p^{N-1}$ as needed.

To deal with the case $d=1$ we need only consider $St_1$. Let's first consider type A so $G = SL_n$ for some $n \geq 2$. Let $X = X_{1,n}$ (matrix notation) so $G$ acts on the standard vector space with basis $x_1, \dots, x_n$, $X_{1,n}(t)\colon x_1 \to x_1 + t x_n$, $x_i \to x_i$, $i > 1$. If, as we remember, $St_1 \subseteq k[x_1, \dots x_n]$ then it suffices to prove that $k[x_1, \dots, x_n]^{X_{1,n}(1)} = k[x_1, \dots, x_n]^{X_{1,n}}$. But $k[x_1, \dots x_n]^{X_{1,n}(1)} = k[x_2, \dots, x_n] \otimes k[x_1, x_n]^{X_{1,n}(1)}$. A typical invariant is a Dickson invariant (a determinant as we have seen) $x_1^p x_n - x_1 x_n^p$. But this is not invariant under $X_{1,n}$. (There are, if memory serves, results that show this is the only sort of thing to worry about so presumably this leads to a proof of what the invariants of $X_{n,1}(1)$ are.) Hence, we must see that such invariants do not show up when dealing with $St_1$, as the hoped for equality is quite false.

Before attacking the case $d=1$ for type A by another means, let's reflect on what we are after. We will be looking at a module for an elementary abelian $p$-subgroup $X$ and $Y$ will be a subgroup of $X$. We will want that $M$ is a free module for $Y$ and $M^Y = M^X$ — i.e. semifreeness. One might ask for a stronger condition and ask that we have $kX = kY + \mathfrak{a}$, where $\mathfrak{a}$ is the annihilator of $M$ in $kX$, a condition that does not hold in the desired situation: consider $p=2$, $SL_3$ and use the restriction to $SL_2$ to get at $X$ (using previous discussions). So we will hope for an intermediate situation, something that will do later things we have in mind.

We return to $G = SL_n$ the universal Chevalley group of type $A_{n-1}$ over the algebraic closure $k$ of the field of $p$ elements, $X = X_{1,n}$, a root subgroup, $Y = X(p)$, the elements over the field of $p$ elements in $X$, $U$ the group of unitriangular (upper) matrices, $U(p)$ as you would guess. Let $M$ be the basic Steinberg module, $M = St_1$, of dimension $p^{\frac{n(n+1)}{2}}$. We want to show that $M$ as an $X$-module is semifree with respect to $Y$ (as this gives the desired semifreeness in general as we have seen above).

If $x \in X$ then the linear transformation $L_x$ induced by $x$ on $M$ is an endomorphism of $M$ considered as a module for $U$ since $x \in Z(U)$. Hence, as $L_x$ commutes with all endomorphisms, by definition, we have that $L_x$ is in the center of the algebra of $U$-endos of $M$. We <u>would like</u> that $L_x$ is in the center of the algebra of $U(p)$-endomorphisms of $M$! (For $L_x$ is certainly in this algebra). This would be the case if the two algebras coincide, that is, if the endomorphism algebra of $M$ as $U$-module is of dimension at least $p^{n(n+1)/2}$.

Let's <u>assume</u> that this <u>is</u> <u>the</u> case and proceed from there. The first consequence is that there is $z \in Z(kU(p))$ such that $x - z$ annihilates $M$. Indeed, choose a free basis of $M$ as $U(p)$-module: elements $m_u$, $u \in U(p)$ such that $u' m_u = m_{u'u}$ for any $u' \in U$. The endomorphisms consist of the space spanned by the maps $R_{u'}$, where $R_{u'} \cdot m_u = m_{uu'}$ so we can make sense out of all $R_a$, $a \in kU(p)$. The central ones, $a \in Z(k(U))$ then coincide with the (obviously defined) $L_a$ so $L_x = L_a$, $a \in Z(k(U))$ just as we desired. <u>Remark</u>. This is under an <u>assumption</u>.

Lemma. If $Q$ is a normal subgroup of the $p$-group $P$ then $kP$ has a $kP$-submodule which, as a $kQ$-module, is isomorphic with $kQ$.

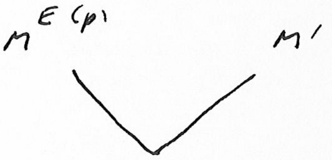
Proof. Let $N$ be a free $kQ$-module of rank one and let $Q \wr P/Q$ act on $N \oplus \dots \oplus N$ ( $|P/Q|$ copies) in the usual way. Now $P$ embeds in $Q \wr P/Q$ in the usual way, $Q$ embedding in the base group projecting onto each factor and the image of $P$ covering the quotient of $Q \wr P/Q$ by the base group. For these reasons, $P$ has just one dimension of fixed points on the sum (one in each summand under $Q$ and these permuted by $P$). By dimension counting we have a free module for $kP$. The "diagonal" copy of $N$ is invariant under $Q \wr P/Q$ so is a $kP$-module and the way $Q$ embeds in the base means there is one dimension of fixed points under $Q$ (the "diagonal" $N$ intersects the fixed points of $Q$ in a one-dimensional space, the fixed points of $Q$ being of dimension $|P/Q|$.

Next, let $U_0 \leq U$ be the subgroup with each first row and last column entry $0$ (except the diagonal). Let $E$ be the "complementary" subgroup so $U = U_0 E$ is a semi-direct product as is $U(p) = U_0(p) E(p)$ with $E(p)$ extra-special (we're assuming $n \geq 3$ as we may). Let $V, V_0$ be for the lower triangular group correspondingly so $V_0 E$ is another maximal unipotent subgroup and there is an element $w$ of the Weyl group with $wU_0w^{-1} = V_0$, $wEw^{-1} = E$, $w^2 = 1$. We have the "picture"

<u>Lemma</u> If $\alpha \in k[V_0(p)E(p)] - kE(p)$, $\beta \in kU(p) - kE$ and $\varepsilon \in kE_0(p)$ and $\alpha+\beta+\varepsilon$ annihilates $M$ then $\alpha=\beta=0$.

It suffices to prove that $\alpha=0$ and then to use the element $w$ just discussed to deduce, then, that $\beta=0$ as well.

<u>Proof</u>. Let $M'$ be a $kU_0(p)$-submodule of $M$ which is free, of rank one, as a $kE(p)$-module. Now $M^{E(p)}$ is, by Steve Smith's theorem as Geoff has shown, the basic Steinberg module for $\langle U_0(p), V_0(p)\rangle$ so the fixed point space of one of these Sylow subgroups on $M^{E(p)}$ is one-dimensional and this one-dimensional space generates $M^{E(p)}$ under the other, by Curtis-Richen theory, as $U_0, V_0$ are "opposites." In particular, $M^{E_0(p)} \cap M'$ is one-dimensional being $(M')^{E_0(p)}$ and we have the picture




Let $m$ be a non-zero vector in this intersection. We claim $m$ is a generator of $M$ as $V_0(p)E(p)$-module. Choose $a \in kV_0(p)$ so that $am$ and $m$ are linearly independent. Now $aM'$ is also a $kE(p)$-module, as $E(p)$ is normalized by $V_0(p)$ so $(aM')^{E(p)} \ni am$ so $aM' \cap M' = 0$ as else the intersection would have a non-zero vector fixed by $E(p)$. Continuing in this way, using a basis of $M^{E(p)}$ and the fact that $m$ generates $M^{E(p)}$ as $V_0(p)$-module (since $m$ is fixed by $U_0(p)$ as $V_0(p)$ certainly leaves invariant the $E$ fixed points on $M'$) we get a direct sum of "translates" of $M'$ under $kV_0(p)$ adding up to $M$, as the direct sum has the correct dimension. Thus, as $M$ is free as a module for $V_0(p)E(p)$, we have that the elements $ve.m$, $v\in V_0(p)$, $e\in E(p)$ are a basis of $M$.

Now consider $\alpha+\beta+\varepsilon$. The elements $\beta$, $\varepsilon$ leave $M'$ invariant, $M'$ free for $E(p)$. Hence the elements $ve.m$, $v\neq 1$ are linearly independent and with the elements $e.m$ are the basis of $M$ we are using. Now $\alpha m$ is a linear combination of the elements $ve.m$, $e\neq 1$ as $\alpha$ is such a linear combination of elements $ve$, so $(\alpha+\beta+\varepsilon)m=0$ implies $\alpha m=0$ so $\alpha=0$.

Let's return to the situation we had: $x-z$ annihilates $M$ where $z\in Z(kU(p))$. Write $z=\beta+\varepsilon$ where $\beta\in kU(p)-kE(p)$, $\varepsilon\in kE(p)$. Then $w(x-z)w^{-1}$ also annihilates $M$, $wxw^{-1}=x$ by definition of $x$, so $z-wzw^{-1}$ annihilates $M$. But $wzw^{-1}=w\beta w^{-1}+w\varepsilon w^{-1}$ and $w\beta w^{-1}$ is in $kV_0(p)E(p)-E(p)$ so we deduce that $w\beta w^{-1}=0$ so $\beta=0$ and hence $z\in kE(p)$, $z\in Z(kU(p))$. Thus $z$ is a linear combination of $U(p)$ class sums lying in $E(p)$.

To get at semi-freeness we need to see how such elements act on $M^{X(p)}$. Let $x_0$ be a generator for $X(p)$. The elements in $X(p)$ are, of course, no problem. Any element of $E(p)-X(p)$ is conjugate, in $E(p)$, to the product of itself by $x_0$ (or itself by any power of $x_0$). Hence, let $\sigma$ be a class sum of $E(p)-X(p)$ so $\sigma x_0=\sigma$ as multiplication by $x_0$ permutes the elements of the class. Thus $\sigma(x_0-1)=0$. $\therefore$ $\sigma$ annihilates $(x_0-1)M$ which certainly contains $M^{X(p)}$. Thus, $z$ must act on $M^{X(p)}$ by scalar multiplication and $x$ is unipotent.

Remarks. 1. Above goes over for $G(q)$ and the appropriate Steinberg.

2. Some other goals: characterizations starting perhaps with $St_1$; a proof of Steve Smith's theorem in our case via a characterization and strengthening his result via filtrations using some kind of semifreeness for unipotent radicals; semifreeness for more than one root group to be used in characterizations and strengthening.

3. The following can't be used for semifreeness on tensor products but is indicative.

*Lemma.* If $G$ is a group, $U, V$ are $kG$-modules then

$$\operatorname{codim}(\operatorname{ann}_{kG} U)\cdot \operatorname{codim}(\operatorname{ann}_{kG} V) \geq \operatorname{codim}(\operatorname{ann}_{kG} U\otimes V).$$

*Proof.* Say the first two codimensions are $m, n$ and $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are in $G$ and give bases for $kG$ modulo these annihilators. Let $X_i$ be the linear transformation induced by $x_i$ on $U$, $Y_j$ for $y_j$ on $V$. Say $g\in G$ so there are scalars $\lambda_1, \ldots, \lambda_m$ such that

$$gu = \sum \lambda_i X_i u$$

for all $u\in U$, and scalars $\mu_1, \ldots, \mu_n$ so that

$$gv = \sum \mu_j Y_j v$$

for all $v\in V$. Hence

$$g.u\otimes v = \sum \lambda_i \mu_j (X_i\otimes Y_j)(u\otimes v)$$

and the proof is complete (tensor product linear transformation acts as usual).

4. Seem only to need that any $kU(p)$ endomorphism of $St_1$ is a $kU(p)X$ endomorphism.

Let $G(q)$, $q = p^e$, $U(q)$, etc... be as above with the obvious notation. We present an equivalent formulation of the result we need, the "if" above, to get the semifreeness for root subgroups.

<u>Proposition</u> The following assertions are equivalent:

1) Every $U(p)$ endomorphism of $St_1$ is a $U(q)$-endomorphism;

2) Every $U(q)$ endomorphism of $St_1 \otimes St_1^{(p)} \otimes \cdots \otimes St_1^{(p^{e-1})}$ is a linear combination of tensor products of $U(q)$ endomorphisms of $St_1, St_1^{(p)}, \ldots, St_1^{(p^{e-1})}$.

We can express the latter statement by
$$\mathrm{End}_{U(q)}(St_1 \otimes \cdots \otimes St_1^{p^{e-1}}) = \mathrm{End}_{U(q)}(St_1) \otimes \cdots \otimes \mathrm{End}_{U(q)}(St_1^{(p^{e-1})}).$$
The key step is the following calculation: Here $St = St_1 \otimes \cdots \otimes St_1^{p^{e-1}}$, the "usual" Steinberg on $G(q)$.

<u>Lemma</u>. $\dim_k \mathrm{End}_{kU(q)}(St) = q^{n(n+1)/2}$.

<u>Proof</u>. Let $B(q)$ be the Borel corresponding to $U(q)$, $B(q) = T(q)U(q)$, $T(q)$ the torus. Let $St(B)$, $St(T)$ be the usual Steinberg modules for this parabolic and Levi complement. Then
$$\begin{aligned}
\mathrm{Hom}_{kU(q)}(St, St) &\simeq \mathrm{Hom}_{kB(q)}(\mathrm{Res}^{G(q)}_{B(q)} St, \mathrm{Ind}^{B(q)}_{U(q)} St) \\
&\simeq \mathrm{Hom}_{kB(q)}(St(B) \otimes \mathrm{Ind}^{B(q)}_{T(q)} k, \mathrm{Ind}^{B(q)}_{U(q)} St) \\
&\simeq \mathrm{Hom}_{kB(q)}(\mathrm{Ind}^{B(q)}_{T(q)} St(T), \mathrm{Ind}^{B(q)}_{U(q)} St) \\
&\simeq \mathrm{Hom}_{kT(q)}(St(T), \mathrm{Res}^{B(q)}_{T(q)} \mathrm{Ind}^{B(q)}_{U(q)} St) \\
&\simeq \mathrm{Hom}_{kT(q)}(St(T), \mathrm{Ind}^{T(q)}_{T(q)\cap U(q)} \mathrm{Res}^{U(q)}_{T(q)\cap U(q)} St)
\end{aligned}$$
by Mackey's theorem, as $B(q) = T(q)\,U(q)$,
$$\simeq \mathrm{Hom}_{kT(q)}(St(T), \mathrm{Ind}_1^{T(q)}(\underbrace{k \oplus \cdots \oplus k}_{\dim St}))$$
so the lemma holds as the right hand side is $\dim St$ copies of the regular representation and $kT(q)$ is semisimple.

<u>Remark</u>: Easier proof! $St$ is free of rank one for $kU(q)$.

Now suppose that 1) holds. Then $\dim \mathrm{End}_{U(q)}(St_1) = p^{n(n+1)/2}$ so $\dim \mathrm{End}_{U(q)}(St_1^{(p^i)}) = p^{n(n+1)/2}$ (as $St_1$ and $St_1^{(p^i)}$ are conjugate $U(q)$ modules) so

$$\dim\left(\mathrm{End}_{U(q)}(St_1) \otimes \dots \otimes \mathrm{End}_{U(q)}(St_1^{(p^{e-1})})\right) = \overbrace{p^{n(n+1)/2} \times \dots \times p^{n(n+1)/2}}^{e}$$
$$= q^{n(n+1)/2}$$
$$= \dim \mathrm{End}_{U(q)}(St),$$

by the Lemma, so 2) holds.

Finally, suppose that 2) is valid. Then

$$q^{n(n+1)/2} = \dim \mathrm{End}_{U(q)}(St) = \dim(\mathrm{End}_{U(q)} St_1) \times \dots \times \dim(\mathrm{End}_{U(q)} St_1^{(p^{e-1})})$$
$$= (\dim(\mathrm{End}_{U(q)} St_1))^e$$

so $\dim(\mathrm{End}_{U(q)} St_1) = p^{n(n+1)/2} = \dim(\mathrm{End}_{U(p)} St_1)$ so 1) holds too!

This leaves us with the problem of establishing 2). One thing (and further calculations of this sort) is that

$$\mathrm{Hom}_{U(q)}(St_1, St_1^{(p^i)}) \cong k$$

if $0 < i < e$. For

$$\mathrm{Hom}_{U(q)}(St_1, St_1^{(p^i)}) \cong (St_1^* \otimes St_1^{(p^i)})^{U(q)}$$
$$\cong (St_1 \otimes St_1^{(p^i)})^{U(q)}$$

which is the fixed points of $U(q)$ on the restriction of a simple module to $U(q)$, so this is one-dimensional by Curtis' Theorem. In fact, this gives us that each of the modules for $U(q)$

$$St_1 \otimes St_1^{(p)} \otimes \dots \otimes St_1^{(p^i)}$$

has the property that it has a simple socle and also a simple radical quotient and there are no maps between this and $St_1^{(p^{i+1})}$, in either direction except the ones with image the socle and kernel the radical. This is the key. The next result then finishes this stage?

A good bet now is that this is not going to work. If it did then $\mathrm{End}_{kU(q)}(St_1) \simeq kU(p)$. Since the endomorphism algebra is a group algebra we will then have $\mathrm{End}_{kU(q)}(St_1^{(p)}) \simeq kU(p)$ as well. (Look at it in terms of intertwining matrices. A group basis goes to a group basis under field automorphisms.) Then, we would obtain

$$
\begin{aligned}
kU(q) &\simeq \mathrm{End}_{kU(q)}(St) \\
&= \mathrm{End}_{kU(q)}(St_1 \otimes \cdots \otimes St_1^{(p^{e-1})}) \\
&\simeq \mathrm{End}_{kU(q)}(St_1) \otimes \cdots \otimes \mathrm{End}_{kU(q)}(St_1^{(p^{e-1})}) \\
&\simeq kU(p) \otimes \cdots \otimes kU(p) \\
&\simeq k[U(p) \times \cdots \times U(p)]
\end{aligned}
$$

a most unlikely isomorphism.

Perhaps one can look at $SL(3,4)$, with $k$ the field of four elements. This argument seems to carry over there.

Tilting modules for $SL_2$ in characteristic two.

We shall work with the algebraic group $SL_2(k)$, $k$ an algebraically closed field of characteristic two. We use the usual notation we have used (for $SL_2(2^n)$) so $V_1 = St_1$ is the standard 2-dimensional module and $V_i$ is the $(i-1)$st Frobenius twist, $V_I = \bigotimes_{i\in I} V_i$, where $I \subseteq \mathbb{N} = \{1, 2, \dots\}$ and $V_\emptyset = k$. If $S$ is an initial segment of $\mathbb{N}$, so $S = \{1, 2, \dots, s\}$ for some $s \in \mathbb{N}$ and $I \subseteq S$ then $V_{I,S} = V_I \otimes V_S$ (as in our previous study of simply generated modules for $SL_2(2^n)$. Recall that a tilting module is one that has a filtration by "induced" modules (that is, symmetric powers of $V_1$) and has a filtration by Weyl modules (the duals of the induced modules)

Theorem. The modules $V_{I,S}$ are indecomposable, self-dual, tilting with a simple socle ($V_{S-I}$ for $V_{I,S}$). They are exactly the indecomposable summands of the tensor powers of $V_1$.

Proof. Suppose $I \subsetneqq S$. Now $V_{I,S} = V_I \otimes V_I \otimes V_{S-I}$ so as $k$ is a submodule of $V_I \otimes V_I$ we certainly have $V_{S-I}$ in the socle. However, this is the socle on restriction to $SL_2(2^s)$, as $I \subsetneqq S$, the restriction being the projective cover, so this is the socle and so is indecomposable as well (and self dual as each $V_i$ and $V_I$ is too). If $I = S$ this last part fails but comparing $V_{S,S}$ and $V_{S, S\cup\{s+1\}} \cong V_{S,S} \otimes V_{s+1}$ completes this part of the argument. To see these are tilting modules we need only prove the second statement, as $V_1$ is certainly alright, by work of Donkin on filtrations of tensor products.

However, the second part is just an old analysis (see my paper on $SL(2,2^n)$) and can be done by induction. We just illustrate the first cases.

$$
\begin{array}{ll}
V_1 & V_{\emptyset,\{1\}} \\
V_1 \otimes V_1 & V_{\{1\},\{1\}} \\
V_1 \otimes V_1 \otimes V_1 = V_1 \oplus V_1 \oplus V_{12} & \text{new: } V_{12} = V_{\emptyset,\{1,2\}} \\
V_1 \otimes V_{12} & V_{\{1\},\{1,2\}} \\
V_1 \otimes V_1 \otimes V_{12} = (V_1 \oplus V_1 \oplus V_{12}) \otimes V_2 & \text{new: } V_{12} \otimes V_2 = V_{\{2\},\{1,2\}} \\
V_1 \otimes V_{12} \otimes V_2 & V_{\{1,2\},\{1,2\}}
\end{array}
$$

$$
\begin{aligned}
V_1 \otimes V_{12} \otimes V_{12} &= (V_1 \otimes V_1 \otimes V_1) \otimes (V_2 \otimes V_2) \\
&= (V_1 \oplus V_1 \oplus V_{12}) \otimes (V_2 \otimes V_2) \\
&= V_{\{2\},\{1,2\}} \oplus V_{\{2\},\{1,2\}} \oplus V_1 \otimes V_2 \otimes V_2 \otimes V_2
\end{aligned}
$$

$$+ .\ V_1 \otimes V_2 \otimes V_2 \otimes V_2 = V_1 \otimes (V_2 \oplus V_2 \oplus V_{23}) \qquad \text{new: } V_{123} = V_{\emptyset,\{1,2,3\}}$$

etc.

Examples. Let's start by describing the "induced" modules, by "degree" (of symmetric power) heuristically using Clebsch-Gordan (though Carter-Cline or later should prove it).

| Degree | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| module | 0 | 1 | 0<br>2 | 12 | 2<br>0<br>3 | 1<br>13 | 0<br>3<br>23 | 123 | 23<br>3<br>0<br>4 | 13<br>1<br>14 |

| Degree | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|
| module | $\begin{array}{ccc} 3 & & \\ \vert & & \\ 0 & - & 2 \\ \vert & & \vert \\ 4 & - & 24 \end{array}$ | 12<br>124 | $\begin{array}{ccc} 2 & - & 24 \\ \vert & & \vert \\ 0 & - & 4 \\ & & \vert \\ & & 34 \end{array}$ | 1<br>14<br>134 | 0<br>4<br>34<br>234 | 1234 | 234<br>34<br>4<br>0<br>5 | 134<br>14<br>1<br>15 |

Now some of our modules with filtrations suggested.

$$1 \otimes 1 \otimes 2 = \begin{array}{c} 2 \\ 0 \\ 3 \\ \hline 0 \\ 2 \end{array}$$

$$1 \otimes 1 \otimes 2 \otimes 2 = \begin{array}{c} 0 \\ 2 \\ 0 \end{array} \otimes \begin{array}{c} 0 \\ 3 \\ 0 \end{array} = \begin{array}{ccc} 0 - & 3 - & 0 \\ 2 & 23 & 2 \\ 0 - & 3 - & 0 \end{array}$$

$$1 \otimes 1 \otimes 2 \otimes 2 \otimes 3 = \begin{array}{ccc|cc} 3 & 0 & 4 & 0 & 3 \\ 23 & 2 & 24 & 2 & 23 \\ 3 & 0 & 4 & 0 & 3 \end{array}$$

<u>Theorem</u>. Every tilting module is isomorphic with a direct sum of various $V_{I,S}$.

<u>Proof</u>. The highest weight for $V_1$ is $1.\lambda$ and each tensoring with $V_1$ produces a module with highest weight one more multiple of $\lambda$ and produces one "new" indecomposable summand so the i-th power of $V_1$ gives the i-th $V_{I,S}$ and it has highest weight $i\lambda$. The result of Donkin finishes the proof.

Let's produce a table listing the indecomposable tilting modules first, the degree, i.e. multiple that occurs in the highest weight, and the corresponding simple composition factors of the indecomposable module. The pattern is clear and should be easily made formal, if desired.

| $V_{S,I}$ | Degree | Simple |
| --- | --- | --- |
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| $1 \otimes 1$ | 2 | 2 |
| $1 \otimes 2$ | 3 | 12 |
| $1 \otimes 1 \otimes 2$ | 4 | 3 |
| $1 \otimes 2 \otimes 2$ | 5 | 13 |
| $1 \otimes 1 \otimes 2 \otimes 2$ | 6 | 23 |
| $1 \otimes 2 \otimes 3$ | 7 | 123 |
| $1 \otimes 1 \otimes 2 \otimes 3$ | 8 | 4 |
| $1 \otimes 2 \otimes 2 \otimes 3$ | 9 | 14 |
| $1 \otimes 1 \otimes 2 \otimes 2 \otimes 3$ | 10 | 24 |
| $1 \otimes 2 \otimes 3 \otimes 3$ | 11 | 124 |
| $1 \otimes 1 \otimes 2 \otimes 3 \otimes 3$ | 12 | 34 |

For odd characteristic the same technique works: the tensor powers of the two-dimensional module keep producing tilting modules and a unique new highest weight – hence a unique new indecomposable summand which then must involve this new weight. Presumably the structures can be worked out here as well.

Of course we can carry on to $SL_n$, e.g., or any situation where the Weyl modules for the fundamental are closed – as a set – under the taking of duals.

We can also give a slightly different proof of weak – i.e. virtual – divisibility for $SL_2$. For characteristic two all the tilting modules are divisible by the (basic) Steinberg (so Krull-Schmidt gives divisibility, in fact) while for odd characteristic the tilting modules for weights $a\lambda_1$, $a < p-1$ are of dimension $a+1$ and not of dimension divisible by $p$ while the rest have dimension divisible by $p$ and, inductively are virtually divisible by the Steinberg. The divisibility of the dimension by $p$ follows from restriction to $SL_2(p)$ and the projectivity of the Steinberg there and from the tensor product construction, using the Steinberg and other summands to get tilting modules.

This last discussion gives the splitting of the "Clebsch-Gordan" sequence for $SL_2$. That is, if $V_i$ is the $i$-dimensional induced module, i.e. polynomials of degree $i-1$, then the exact sequence

$$0 \to V_{i-1} \to V_2 \otimes V_i \to V_{i+1} \to 0$$

splits if $i < p$. For the extension, if non-split is not self-dual so get two tilting modules with the same highest weight.

Tilting modules for $SL_4$ in characteristic two

We shall use the tensor product method (also described by Donkin in an early paper) to do a few cases and apply the results to the semi-free question. Let $k$ have characteristic two, $G = SL_4(k)$, $V$ be the standard four-dimensional module, $V^{(2)}$ the first "twist" (and use similar notation elsewhere) so $V$, $V \wedge V$, $V \wedge V \wedge V = V^*$ are the fundamental modules.

<u>Lemma</u> $V \otimes V$ is uniserial with composition factors $V \wedge V$, $V^{(2)}$, $V \wedge V$ in that order.

<u>Proof</u>. Let $X_1, X_2, X_3, X_4$ be a basis of $V$ so $V \otimes V$ has a submodule isomorphic with $V \wedge V$ with quotient the homogeneous polynomials in the $X_i$ of degree two. The latter has a submodule of squares, $X_i^2$, $1 \le i \le 4$, so it is clear that the composition factors are as asserted. Now

$$\mathrm{Hom}_{kG}(V \otimes V, V^{(2)}) \simeq \mathrm{Hom}_{kG}(V, V^* \otimes V^{(2)})$$

and $V^* \otimes V^{(2)}$ is simple, by the Steinberg tensor product theorem, so $V \otimes V$ has no quotient isomorphic with $V^{(2)}$. Similarly, it has no submodule isomorphic with $V^{(2)}$ so the lemma is valid.

Notice that we similarly have a structure for $V^* \otimes V^*$ in terms of $V \wedge V$, $(V^*)^{(2)}$, $V \wedge V$, as $V \wedge V = V^* \wedge V^*$. (We thus have the structure of the tilting modules for highest weights $2\omega_1$, $2\omega_3$.) (Next, we start an analysis for the tilting module for $\omega_1 + \omega_2$ which we do not conclude.)

<u>Lemma</u>. The module $V \otimes (V \wedge V)$ has $V^*$ as a summand.

<u>Proof</u>. Multiplication in the exterior algebra on $V$ gives an epimorphism of $V \otimes (V \wedge V)$ to $V \wedge V \wedge V \cong V^*$. We shall give a suitable "backwards" map. If $u, v, w \in V$ we map

$$u \wedge v \wedge w \to u \otimes (v \wedge w) + v \otimes (w \wedge u) + w \otimes (u \wedge v).$$

Does this make sense? (If it is then it is certainly a $kG$-homomorphism and composition with multiplication by three, that is, by one!) Since we do have a map from $V \otimes V \otimes V$ we need only see that if any two of $u, v, w$ coincide then the right hand side is zero and that the right hand side is skew symmetric with respect to transpositions among $u, v, w$. This is all easy to check.

Note that we similarly have $V \mid V^* \otimes (V \wedge V)$ $(\cong V^* \otimes (V^* \wedge V^*))$.

<u>Lemma</u> $V \otimes (V \wedge V) \otimes (V \wedge V \wedge V) \cong (V \otimes V) \oplus (V^* \otimes V^*) \oplus St_1(G)$.

<u>Proof</u>. We have $V^* \mid V \otimes (V \wedge V)$ so $V^* \otimes V^* \mid V \otimes (V \wedge V) \otimes V^*$. Similarly, $V \otimes V \mid V \otimes ((V \wedge V) \otimes V^*)$ so the indecomposable modules $V \otimes V$ and $V^* \otimes V^*$ are summands. But $St_1$ is also a summand - it's the tilting module for weight $\omega_1 + \omega_2 + \omega_3$ so we're done by dimension considerations.

Now let $L = SL_2(k) \leq G = SL_4(k)$, the "upper left hand corner." The natural 2-dimensional module for $L$ is just $St_1(L)$. We easily see that $V|_L \cong St_1(L) \oplus k \oplus k$, $V^*|_L \cong St_1(L) \oplus k \oplus k$ and $V \wedge V|_L \cong St_1(L) \oplus St_1(L) \oplus k \oplus k$. Hence, we have directly

<u>Lemma</u> We have the isomorphism

$$St_1(G)|_L \simeq 8\, St_1(L) \oplus 8(St_1(L) \otimes St_1(L)) \oplus 2(St_1(L) \otimes St_1(L) \otimes St_1(L)).$$

This is an easy calculation and application of the Krull-Schmidt theorem. Now $St_1(L) \otimes St_1(L)$ is uniserial with composition factors $k$, $St_1(L)^{(2)}$, $k$ and

$$St_1(L) \otimes St_1(L) \otimes St_1(L) \simeq St_1(L) \oplus St_1(L) \oplus (St_1(L) \otimes St_1(L)^{(2)})$$

Hence, semipureness is not valid in the sense we have been using - though the summands each is semipure.

Let's also examine the restriction to $H = SL_3(k)$. It has the standard module $W$ of dimension three. Then

$$V|_H \cong W \oplus k$$
$$V \wedge V|_H \simeq W \oplus W^*$$
$$V^*|_H \simeq W^*$$

as is easy to see. (To see the second isomorphism use the fact that $V \wedge V$ is self-dual). Then, it is easy to calculate that

$$St_1(G)|_H \simeq W \otimes St_1(H) \oplus W^* \otimes St_1(H) \oplus St_1(H) \oplus St_1(H).$$

On the Steinberg module for $SL_3$.

We are going to determine the restriction of $St_1(G)$, $G = SL_3(k)$, $k$ algebraically closed of characteristic $p$, to $L = SL_2(k)$. Let $V_1 \simeq k, V_2, \dots, V_p \simeq St_1(L)$ be the standard modules, with $\dim_k V_i = i$. Our result is

<u>Proposition</u>. We have the isomorphism

$$St_1(G)|_L \simeq V_p \otimes V_p \oplus 2(V_1 \oplus \dots \oplus V_{p-1}) \otimes V_p.$$

The key step is the next result.

<u>Lemma</u>. We have the isomorphism of $kG$-modules

$$S^{p-1}(V) \otimes S^{p-1}(V^*) \simeq S^{p-2}(V) \otimes S^{p-2}(V^*) \oplus St_1(G)$$

Here $V$ is the standard three dimensional module for $G$. Let's first observe that the proposition does follow from the lemma. Consider $S^n(V)$ in the usual way so that it has a basis consisting of all monomials $X^i Y^j Z^k$, $i+j+k = n$. We may assume that $L$ acts on $X, Y$ fixing $Z$ so $S^n(V)$ has a direct decomposition according to powers of $Z$, as an $L$-module. Hence, if $W = V_2$ we have

$$S^n(V)|_L \simeq S^n(W) \oplus \dots \oplus S^1(W) \simeq V_{n+1} \oplus \dots \oplus V_1.$$

Now $V_L \simeq W \oplus k$ so $V_L$ is self dual so $S^n(V^*)|_L \simeq S^n(V)|_L$. The lemma now gives the proposition by a direct calculation using the Krull-Schmidt theorem. It remains now to establish the lemma.

First, there are $(p-1)^2$ "contractions"

$$V^{\otimes(p-1)} \otimes (V^*)^{\otimes(p-1)} = V\otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^* \to V^{\otimes(p-2)} \otimes (V^*)^{\otimes(p-2)}$$

(sending a factor $V \otimes V^* \to k$) so let $\sigma$ be their sum so $\sigma$ is a $kG$-homomorphism. It's definition means it defines a $kG$-homomorphism

$$S^{p-1}(V) \otimes S^{p-1}(V^*) \to S^{p-2}(V) \otimes S^{p-2}(V^*).$$

It suffices to see that this is surjective! Indeed, suppose we have this established. By tilting theory

$$V^{\otimes(p-1)} \otimes (V^*)^{\otimes(p-1)} = St_1 \oplus \dots$$

by highest weights (as $(p-1)(\lambda_1+\lambda_2)$ is highest and unique and $St_1$ is the corresponding tilting module as it is self-dual - so is "Weyl" and "induced"). Now $St_1$ is simple and the highest weight $(p-1)\rho = (p-1)(\lambda_1+\lambda_2)$ "lives" in $S^{p-1}(V) \otimes S^{p-1}(V^*)$, as is easy to see, so this tensor product has $St_1$ as a submodule. But $(p-1)\rho$ does not "live" in $S^{p-2}(V) \otimes S^{p-2}(V^*)$ and the difference in the dimensions between $S^{p-1}(V) \otimes S^{p-1}(V^*)$ and $S^{p-2}(V) \otimes S^{p-2}(V^*)$ is the dimension of $St_1$. Hence, the surjectivity is all we need, as claimed.

As above let $X, Y, Z$ be a basis of $V$ with $X$ a generator under the "upper" Sylow subgroup so if $X^*, Y^*, Z^*$ is the dual basis then $X^*$ is fixed under the upper Sylow while the lower fixes $X$ and $X^*$ is a generator under it. The same holds for $X^{p-1}$ and $(X^*)^{p-1}$ since we're in characteristic $p$ and degree $< p$. Hence, it follows that $X^{p-1} \otimes X^{*\,p-1}$ generates $S^{p-1}(V) \otimes S^{p-1}(V^*)$. Similarly, $X^{p-2} \otimes X^{*\,p-2}$ generates $S^{p-2}(V) \otimes S^{p-2}(V^*)$. But $X^{p-1} \otimes X^{*\,p-1}$ is mapped to $(p-1)^2 X^{p-2} \otimes X^{*\,p-2}$ so we're done.

## Strongly embedded subgroups

Let $G$ be a finite group and $k$ a field of characteristic $p$.

Proposition. If $H$ is a proper subgroup of $G$ then the following are equivalent:

1) $H$ is strongly $p$-embedded;

2) For every $kG$-module $M$ and every positive integer $n$, the restriction map

$$\hat{H}^n(G, M) \to \hat{H}^n(H, M)$$

is an isomorphism.

(This remark is motivated by a result of D. Mislin in Comm. Math. Helv. 65 (1990), 454–61).

Proof. We may assume $p \mid |G|$ so also $p \mid |H|$ by hypothesis. (For 1) $\Rightarrow$ 2) is standard by stable equivalence so we need only show 2) $\Rightarrow$ 1).) Let $Q$ be a $p$-subgroup of $H$ so

$$\begin{aligned} H^*(Q, k) &\simeq H^*(G, \operatorname{Ind}_Q^G k) = H^*(H, \operatorname{Res}_H^G \operatorname{Ind}_Q^H k) \\ &\simeq \bigoplus_{H \backslash G / Q} H^*(H, \operatorname{Ind}_{tQt^{-1} \cap H}^H k) \\ &= H^*(H, \operatorname{Ind}_Q^H k) \oplus \ldots \oplus H^*(H, \operatorname{Ind}_{tQt^{-1} \cap H}^H k) \oplus \ldots \end{aligned}$$

But $H^*(H, \operatorname{Ind}_Q^H k) \simeq H^*(Q, k)$ so we deduce that if $t \notin H$

$$H^*(H, \operatorname{Ind}_{tQt^{-1} \cap H}^H k) = 0.$$

If $t \in N(Q)$, $t \notin H$ then

$$H^*(H, \operatorname{Ind}_{tQt^{-1} \cap H}^H k) \simeq H^*(Q, k) \neq 0,$$

a contradiction, so $N(Q) \subseteq H$, as required.

Further questions (when $H \leq G$, $k$ as above).

1. What happens if we assume just
$$H^*(G, M) \xrightarrow{\text{res}} H^*(H, M)$$
to be an isomorphism when $M \in B_0(G)$?
2. Or, assume $\mathrm{Ext}^*(M_1, M_2) \xrightarrow{\text{res}} \mathrm{Ext}^*(M_1, M_2)$ always an isomorphism when $M_1, M_2 \in B_0(G)$.
3. Or further assume that restriction gives a stable equivalence for $B_0(G)$.

Notice that there are two important cases for 3. There is $H$ weakly $p$-embedded, by a theorem of Broué, and there is the isomorphic block situation. (The latter is not included in the former: Let $H = GL(3, 11^2)$, $G = H\langle \varphi \rangle$ with $\varphi$ the Frobenius of order two and let $p = 5$ so the Sylow 5-subgroups of $H$ are in $GL(3, 11)$ and fixed by $\varphi$. The normalizer = centralizer (by eigenvalues) of
$$\left\langle \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\rangle$$
where $\alpha$ is a (primitive) fifth root of unity is a $p$-local subgroup that has the desired (non-)properties.)

We want to indicate a previous, longer, proof of the proposition. Apply the argument to $N_G(P)$ instead of $G$ where $P$ is a Sylow $p$-subgroup of $H$ and so of $G$, by Nishida's theorem (as that gives $H$ controlling strong fusion on a weaker hypothesis) and get that $P$ intersects any Sylow $p$-subgroup of $G$ not in $H$, in the identity. The stabilizer $K$ of the set of Sylow $p$-subgroups of $H$ is then strongly $p$-embedded and
$$K \geq H \geq O^{p'}(K) = O^{p'}(H).$$

Next, $K = H\,C_K(P)$, by the Frattini argument and the strong control (even though $H$ need not be normal!). Finally, apply the double coset argument to $H^*(H,k)$ using elements of $C_K(P)$ as double coset representatives!

One more comment, a question suggested by discussion with D. Benson. Let $P$ be a Sylow $p$-subgroup of $G$. If a family of subgroups of $P$ "tests" $H^*(G,k)$ (i.e. $\alpha \in H^*(P,k)$ comes from $H^*(G,k)$ if, and only if, $\alpha_Q$ comes from $N_G(Q)$ for all $Q$ in the family) then does this family control strong fusion?

Partial Steinberg modules and complexity.

First, let's review the result for $SL(2, \mathbb{F})$ relating freeness and the rank of a certain Van der Monde type determinant. As usual

$$x(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and the matrix representing $X(\lambda)$ on $V_p = St_1$ is

$$\begin{pmatrix} 1 & \lambda & & & \\ & 1 & 2\lambda & & \\ & & 1 & \ddots & \\ & & & & (p-1)\lambda \\ & & & & 1 \end{pmatrix}$$

Hence, the shifted element

$$1 + \alpha_1 ( X(\lambda_1) - 1) + \dots + \alpha_r ( X(\lambda_r) - 1)$$

is represented by

$$\begin{pmatrix} 1 & \alpha_1\lambda_1 + \dots + \alpha_r\lambda_r & & & \\ & 1 & 2(\alpha_1\lambda_1 + \dots + \alpha_r\lambda_r) & & \\ & & 1 & \ddots & \\ & & & & (p-1)(\alpha_1\lambda_1 + \dots + \alpha_r\lambda_r) \\ & & & & 1 \end{pmatrix}$$

so it acts freely if, and only if, $\alpha_1\lambda_1 + \dots + \alpha_r\lambda_r \neq 0$ (where $(\alpha_1, \dots, \alpha_r) \neq (0, \dots, 0)$ is assumed. Hence, by the intersection theorem for rank varieties, the same element acts freely on

$$St_1^{(p^{e_1})} \otimes \dots \otimes St_1^{(p^{e_s})}$$

if, and only if, for some $j$, $1 \leq j \leq s$,

$$\alpha_1 \lambda_1^{p^{e_j}} + \dots + \alpha_r \lambda_r^{p^{e_j}} \neq 0.$$

(That is, the element fails to be on the intersection of the rank varieties we have just calculated.) Hence, by Dade's theorem on shifted elements, the whole subgroup $\langle X(\lambda_1), \dots, X(\lambda_R) \rangle$ is free if, and only if, the $s$ equations in $r$ unknowns

$$\lambda_1^{p^{e_1}} x_1 + \dots + \lambda_r^{p^{e_1}} x_r = 0$$

$$\vdots$$

$$\lambda_1^{p^{e_s}} x_1 + \dots + \lambda_r^{p^{e_s}} x_r = 0$$

have only the trivial solution, that is the corresponding matrix has full rank. (all of this over the field of $q$ elements, of course).

Now we wish to turn our thoughts to generalizing the characterization of partial Steinberg modules for $SL(3,q)$ to bigger groups (including $SL(n,q)$, e.g.). Our first goal is a recognition lemma:

<u>Lemma</u> Let $G$ be a finite group of Lie type of characteristic $p$ and $k$ an algebraically closed field of characteristic $p$. Let $S$ and $V$ be $kG$-modules with $S$ simple and $\dim_k S = \dim_k V$. Furthermore, assume that whenever $P$ is a parabolic subgroup of $G$ with unipotent radical $U$ then $S^U$ and $V^U$ are isomorphic $kP$-modules. It follows that $S \simeq V$.

<u>Remark</u>: S. Smith has proved a similar result where the hypothesis on $\dim_k V$ is replaced by a "generation" hypothesis, namely, that $V$ is generated by all the subspaces $V^U$ as $U$ runs over all such radicals.

<u>Proof</u>. Let $S'$ be a simple submodule for $V$. With the given notation, $S^U = V^U \supseteq S'^U \neq 0$ so $S^U \simeq S'^U$ as $S^U$ is simple by S. Smith's theorem. We now shall prove that $S, S'$ have the same Curtis-Richen weight in Curtis' sense (cf end of his article in LNM #131). But it is easy to see this weight is determined by the largest parabolic subgroup, containing the Borel subgroup $B$, for which the S. Smith corresponding simple module is one-dimensional. For the unipotent radicals are included opposite to the inclusion of parabolics so fixed point spaces of unipotent radicals "go up."

Now, with J. Carlson's work on $SL(2,q)$ using complexity in mind, we can formulate a conjecture. Namely, a $kG$-module is a partial Steinberg (and say $G$ is of type $A$, $D$ or $E$ if necessary) if

$$p^{C^*(V)} \dim V = |G|_p$$

where $C^*(V)$ is a new invariant, a variation of complexity – a sort of layered complexity.

Let's give an example of one of these new invariants: Say $Q \supset R$ are $p$-groups and $M$ is a $kQ$-module. Then

$$H^*(Q/R, H^*(R, M))$$

is a graded commutative algebra so has a growth! We can also use more subgroups and consider

$$H^*(Q/R, H^*(R/S, H^*(S, M))),$$

presumably the $E^2$ term of a triple complex spectral sequence!!

In our case above, perhaps we want to use filtrations of the Sylow $p$-subgroup by unipotent radicals of parabolic subgroups. In fact, let's propose a conjecture based on the (usual) complexity. First, a couple of definitions, the first "standard" from Mason's letters.

Def. A module $V$ for an elementary abelian $p$-group $E$ is semi-free if there is a subgroup $F$ of $E$ so that $V_F$ is a free $kF$-module and $V^F = V^E$. In this case, the breadth of $V$ is the dimension of $V^F$ and $|F|$ is the depth of $V$ (so $\dim_k V$ is their product).

Def. A $kG$-module $V$ is semi-free with respect to the collection $\mathcal{H}$ of subgroups of $G$ if $V^{H_2}$ is a semi-free $H_1/H_2$ module whenever $H_1, H_2 \in \mathcal{H}$, $H_2 \lhd H_1$ and $H_1/H_2$ is elementary abelian.

<u>Conjecture</u>. If $V$ is a partial Steinberg module for $SL(n,q)$, $q=p^e$, the $t$ fold tensor product of conjugates of $St_1$, then $V$ is semi-free with respect to the radicals of parabolic subgroups. Moreover, if $U_1, U_2$ are such subgroups, $U_2 \lhd U_1$ and $U_1/U_2$ is elementary abelian, say, of order $q^w$, then $V^{U_2}$ is semi-free of depth $p^{wt}$ for $U_1/U_2$.

Conversely, if $V'$ is a $kSL(n,q)$-module, $q>p$, $\dim V' = p^{t\frac{n(n-1)}{2}}$ while $V'$ has the semi-free properties of the preceding paragraph, then $V'$ is a ($t$ fold) partial Steinberg module.

The converse part seems to be easier! Namely, we should be able to use the above lemma and the first Alperin-Mason paper. For that paper should be enough to keep track of all the Curtis weights. We need to go between the usual Lie weights and the Curtis weights and can probably refer to Nalsey Trinberg's paper, "Weights and admissible pairs," Comm. in Alg. v 12 (1984), 1257-1263, as phrased nicely. I believe the parabolic belonging to $St_1$, the basic Steinberg, is the Borel subgroup (in the Curtis-Richen sense) and the linear character is easy to guess. Let

$$\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \in SL(n,q)$$

so we can determine the linear characters of $B$ corresponding to the fundamental representations: $V$, $V\wedge V$, $V\wedge V\wedge V$, ... (where $V$ is the standard module. We get $\lambda_1$ corresponds to $\alpha_n$ (i.e. the value on the diagonal matrix is $\alpha_n$), $\lambda_2$ to $\alpha_{n-1}\alpha_n$ etc.— so $St_1$ corresponds to

$$\left((\alpha_2 \cdots \alpha_n)(\alpha_3 \cdots \alpha_n) \cdots \alpha_n\right)^{p-1},$$

that is

$$\left(\alpha_2^{\,1}\alpha_3^{\,2} \cdots \alpha_{n-1}^{\,n-2}\alpha_n^{\,n-1}\right)^{p-1}.$$

The first part of the conjecture seems harder. But if we work in a "shifted" sense then things are much simpler. So let's make the definitions of <u>shifted semi-free</u> and <u>shifted semi-free with respect to a collection of subgroups</u> just as above where we allow $F$ to be a shifted subgroup of $kE$. (Note: there are two senses of "shifted" which doesn't matter. Evens, in his book allows any image of a subgroup of $E$ under any algebra automorphism of the algebra $kE$.) Then we have the <u>Shifted conjecture</u> which seems more amenable. The second part should go as above since Carlson proves the shifted form of the theorem of characterization in the first Alperin-Mason paper.

Here we should review the connection with complexity.

<u>Lemma</u> If $V$ is a $kE$-module for an elementary abelian group $E$ of order $p^n$ then $V$ is shifted semi-free if, and only if,

$$p^{c_E(V)} \dim_k V = p^n \dim_k V^E.$$

<u>Proof</u>. Now there is a shifted subgroup $F$ of order $p^{n-c_E(V)}$ by the usual theory, and $V^F \supseteq V^E$ so if the equality holds then $V$ is certainly shifted semifree. On the other hand, suppose $V$ is such for a subgroup $F$, $|F| = p^f$. Then the extension spectral sequence (of Lyndon-Hochschild-Serre) yields $H^*(E, V) \cong H^*(E/F, V^F)$ so $c_E(V) = n - f$ so $\dim V = p^{n-c_E(V)} \dim V^E$ as required. (Remember, $\dim V = |F| \dim V^F$.)

Now, we want to turn to the first part of the twisted conjecture. We shall describe what to do in the easiest case, that when we consider the unipotent radical of a maximal parabolic, which is here elementary abelian. The general case should go by induction and by consideration of direct products of special linear groups, we hope!

So we have a partial Steinberg module $V$ of dimension $p^{t\left(\frac{n(n-1)}{2}\right)}$ the $t$ fold tensor product. We let $E$ be the abelian unipotent radical of order $q^w$, as just above. We have $q = p^e$. Our problem is to see that

$$C_E(V) = ew - tw.$$

We proceed by eliminating the possibilities that $C_E(V)$ is too small or too big. The previous lemma then means we are done. Now, if $C_E(V) < ew - tw$ then there is a shifted subgroup $F$, $|F| > p^{tw}$ which is free on $V$ so $\dim V^F < p^{t\frac{n(n-1)}{2} - tw}$ which means $V^E$ is no bigger and this is too small by Smith's theorem (presumably it gives a partial Steinberg, $t$ fold product, for $N(E)/E$). (This is O.K. e.g. if $E$ is the "last column.")

Now, let's first finish all the argument when $t=1$. We must eliminate the possibility that $C_E(V) > (e-1)w$. But then the same holds for the $e$ conjugates of $V$ and so the complexity of their tensor product is not zero, it is too big as tensor products have the intersection variety and this will be of positive dimension.

Now for general $t$, if $C_E(V) > ew - tw$ we can take the tensor product with the "remaining" $e-t$ basic Steinberg twists and use the fact that we know the dimension of the variety when $t=1$ to again contradict the projectivity of the tensor product of all $e$ twists of $St_1$.

This approach seems quite hopeful! Perhaps we can even calculate the complexity of $St_1$! We don't know off-hand what are the maximal elementary abelian $p$-subgroups of $SL(n,q)$ but the normal such subgroups (normal in a Sylow $p$-subgroup) are known by work of Weir. Hence, we guess that if $q = p^e$ as above then

$$C_{SL(n,q)}(St_1) = (e-1)[n^2/4].$$

Let's turn to the first part of our first (non-shifted) conjecture. Let $E$ be a unipotent radical of order $q^w$ as before, $E$ a product of $w$ root subgroups. We want to know the variety of the $kE$-module $(St_1)_E$. First, let's observe what is the variety of the $kX$-module $(St_1)_X$ where $X$ is a root subgroup. By Smith's theorem, applied to the right Levi complement, $(St_1)_X$ is just the restriction of the $SL(2,q)$-module $V_p = St_1(SL(2,q))$ to $X$ plus another summand. Hence, a non-free shifted element on $V_p$ is still non-free on $St_1$. On the other hand, the Lin-Lin theorem implies that free on $V_p$ is free on $St_1$ (using the connection between tensor products and intersections of varieties.

Now we can derive the result for any partial Steinberg again using the tensor product – intersection of varieties connection. This gives the Theorem B of Alperin-Mason. The variety is a subspace! Now to get the first part of the conjecture it would also be enough to see that the variety of $St_1$ on $E$ is a subspace for then, by dimension considerations, it will be the subspace generated by the subspaces for each of the root subgroups. The linearity of the variety easily gives the conjecture as we can choose the desired subgroup $F$ as a direct product of groups one from each root subgroup.

We turn to the question of semi-freeness (as opposed to shifted semi-freeness). Let $R$ be an abelian radical for $SL(n,k)$, $k$ algebraically closed of characteristic $p$, $R$ the direct product of root groups $X_1, \dots, X_r$. Let $V = St_1^{(p^{e_1})} \otimes \dots \otimes St_1^{(p^{e_s})}$ where $0 < e_1 < \dots < e_s < e$ where $q = p^e$ and the finite group being studied is $SL(n,q)$.
Let $t = rs$ and suppose

$$g_i = X_1(\lambda_{1i})\, X_2(\lambda_{2i}) \cdots X_r(\lambda_{ri}),$$

$1 \le i \le t$, are elements of $R(q) \le SL(n,q)$. We wish to know necessary and sufficient conditions are that $\langle g_1, \dots, g_t \rangle$ be of order $p^t$ and act freely on $V$.

<u>Theorem</u>. If the rank variety of $R(q)$ on $V$ is the product of the rank varieties of the $X_i(q)$ then the following two conditions are equivalent:

1) $\langle g_1, \dots, g_t \rangle$ is of order $p^t$ and free on $V$;

2) the $t \times t$ matrix

$$\begin{pmatrix} \lambda_{11}^{p^{e_1}} & \lambda_{12}^{p^{e_1}} & \cdots & \lambda_{1t}^{p^{e_1}} \\ \vdots & & & \\ \lambda_{11}^{p^{e_s}} & \lambda_{12}^{p^{e_s}} & \cdots & \lambda_{1t}^{p^{e_s}} \\ \lambda_{21}^{p^{e_1}} & \lambda_{22}^{p^{e_2}} & \cdots & \lambda_{2t}^{p^{e_2}} \\ & & \vdots & \\ \lambda_{r1}^{p^{e_s}} & \lambda_{r2}^{p^{e_s}} & \cdots & \lambda_{rt}^{p^{e_s}} \end{pmatrix}$$

is non-singular.

For $g \in R(g)$ let $\bar{g} = g - 1 + \mathrm{rad}^2 kR(g)$, so $\overline{gh} = \bar{g} + \bar{h}$, as usual. Assume 1) holds. The freeness holds, by Dade's theorem on shifted elements, exactly when all the shifted elements

$$\alpha_1 \bar{g}_1 + \dots + \alpha_t \bar{g}_t$$

represent shifted elements which act freely on $V$ (i.e. pull each modulo $\mathrm{rad}^2(kR(g))$ and add 1 to get a shifted element - in the broad sense of shifted element a la Evens' book) whenever $(\alpha_1, \dots, \alpha_t) \neq (0, \dots, 0)$ is in $k^t$. But

$$\bar{g}_i = \overline{X_1(\lambda_{1i})} + \dots + \overline{X_r(\lambda_{ri})}$$

so, by our results on root groups and by our hypothesis on direct products, this freeness is equivalent to not all

$$\alpha_1 \overline{X_j(\lambda_{j1})} + \alpha_2 \overline{X_j(\lambda_{j2})} + \dots + \alpha_t \overline{X_j(\lambda_{jt})}$$

being in the $j$-th variety (i.e. given $(\alpha_1, \dots \alpha_t)$ there is such a $j$). That is, by the first page of this section, exactly when not all the numbers

$$\alpha_1 \lambda_{j1}^{p^{e_1}} + \dots + \alpha_t \lambda_{jt}^{p^{e_1}}$$
$$\vdots$$
$$\alpha_1 \lambda_{j1}^{p^{e_s}} + \dots + \alpha_t \lambda_{jt}^{p^{e_s}}$$

are zero. Hence, we deduce that if 1) holds then the product of the matrix in 2) and the vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix}$ is not zero whenever some $\alpha_j$ is not zero. Thus 2) holds.

Conversely, all goes right through, except we must see that the condition 2) implies that the order of $\langle g_1, \dots, g_t \rangle$ is $p^t$ and not smaller. Hence, suppose

$$g_1^{a_1} \cdots g_t^{a_t} = 1,$$

$a_i$ in the field of $p$ elements. We wish to deduce that $a_1 = \dots = a_t = 0$. We do have immediately that

$$a_1 \bar{g}_1 + \dots + a_t \bar{g}_t = 0$$

so that for each $i$, $1 \le i \le r$

$$a_1 \lambda_{i1} + \dots + a_t \lambda_{it} = 0$$

Since each $a_j$ is in the field of $p$ elements this implies that

$$a_1 \lambda_{i1}^{p^{e_u}} + \dots + a_t \lambda_{it}^{p^{e_u}}$$

whenever $1 \le u \le s$. Hence, $\begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix}$ is in the kernel of the matrix in 2) and we have the desired conclusion.

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Next, we turn to the question of the rank variety in the case $p=2$, $n=3$. Let

$$X = X_{13} = \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} \qquad Y = X_{23} = \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix}$$

and $R = \langle X, Y \rangle$ a "radical." Let

$$g = X(\lambda)Y(\mu) = \begin{pmatrix} 1 & & \lambda \\ & 1 & \mu \\ & & 1 \end{pmatrix}$$

The action of $g$ on $V \otimes V^*$ is as follows (remember $p=2$):

$$\begin{pmatrix} 1 & 0 & \lambda \\ & 1 & \mu \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ 0 & 1 & \\ \lambda & \mu & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & & & & \lambda & & \\ 0 & 1 & 0 & & & & & \lambda & \\ \lambda & \mu & 0 & & & & \lambda^2 & \lambda\mu & \lambda \\ & & & 1 & 0 & 0 & \mu & & \\ & & & 0 & 1 & 0 & & \mu & \\ & & & \lambda & \mu & 1 & \lambda\mu & \mu^2 & \mu \\ & & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & 0 \\ & & & & & & \lambda & \mu & 1 \end{pmatrix}$$

Hence, $\sum \alpha_i (X(\lambda_i) - 1) + \sum \beta_j (Y(\mu_j) - 1)$ is represented by (with $L = \sum \alpha_i \lambda_i$, $M = \sum \beta_j \mu_j$)

$$\begin{pmatrix} 0 & 0 & 0 & & & & L & & \\ 0 & 0 & 0 & & & & & L & \\ L & M & 0 & & & & * & * & L \\ & & & 0 & 0 & 0 & M & & \\ & & & 0 & 0 & 0 & & M & \\ & & & L & M & 0 & * & * & M \\ & & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & L & M & 0 \end{pmatrix}$$

We want: this matrix has rank $< 4$ if, and only if, $L = M = 0$. For $V \otimes V^* \simeq St_1 \oplus k$ (is eight-plus one-dimensional) so this condition of rank less than four is the condition for being in the variety. But if $L = M = 0$ this is clear while if $L \neq 0$ or $M \neq 0$ it is easy to get four linearly independent rows.

Conclusion:

<u>Prop</u> The rank varieties are as conjectured in $SL(3, 2^e)$.