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Sylow Intersections and Fusion

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1. INTRODUCTION

It is common in mathematics for a subject to have its local and global aspects; such is the case in group theory. For example, the structure and embedding of subgroups of a group G may be usefully thought of as part of the local structure of G while the normal subgroups, quotient groups and conjugacy classes are relevant to the global structure of G. Furthermore, the connections between local and global structure are very important. In the study of these relations, the methods of representation theory and transfer are very useful. The application of these techniques is often based upon results concerning the fusion of elements. (Recall that two elements of a subgroup H of a group G are said to be *fused* if they are conjugate in G but not in H.) Indeed, the formula for induced characters clearly illustrates this dependence. However, more pertinent to the present work, and also indicative of this connection with fusion, is the focal subgroup theorem [8]: *if P is a Sylow p-subgroup of a group G then $P \cap G'$ is generated by all elements of the form $a^{-1}b$, where a and b are elements of P conjugate in G.* Hence, this result, an application of transfer, shows that the fusion of elements of P determines $P \cap G'$ and thus $P/P \cap G'$ which is isomorphic with the largest Abelian p-quotient group of G.

It is the purpose of this paper to demonstrate that the fusion of elements of a Sylow subgroup P is completely determined by the normalizers of the nonidentity subgroups of P. Therefore, $P/P \cap G'$, a global invariant of G, is completely described by the local structure of G. A weak form of our main result is as follows: *if a and b are elements of a Sylow subgroup P of the group G and a and b are conjugate in G, then there exist elements $a_1, \ldots, a_m$ of P and subgroups $H_1, \ldots, H_{m-1}$ of P such that $a = a_1$, $b = a_m$ and a_i and a_{i+1} are contained in H_i and conjugate in $N(H_i)$, $1 \leqslant i \leqslant m-1$.* We shall strengthen

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this theorem in several ways. First, the result applies to arbitrary subsets of P as well as to elements. Second, only very special types of subgroups need appear as the H_i and, finally, the elements which conjugate a_i into a_{i+1} are not of an arbitrary nature. These refinements will be useful in the derivation of many of the classical theorems of Frobenius, Burnside, and Grun on p-quotient groups. Furthermore, this theorem is quite relevant to the eighth of Brauer's problems [*3*]. Indeed, given a p-group P, the question is to determine which "patterns of fusion" may occur in P when P is a Sylow p-subgroup of a group. Our results give some limitations on the number of possibilities.

At this point we should like to give an example of the phenomenon just described above. Let p be a prime and let $G = GL(3, p)$, the 3-dimensional general linear group over the field of p elements. Let P be the subgroup of upper triangular matrices whose main diagonal have only the entry "1" appearing so that P is Sylow p-subgroup of G. Let

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that a, b, and c are elements of P. Since these matrices have the same Jordan canonical form (a 2-dimensional and a 1-dimensional block for the eigenvalue "1") they are conjugate in G (matrix similarity simply being the conjugacy relation). However, a and b generate P and are not conjugate in $N(P)$, the group of all upper triangular matrices. Thus, there is no single subgroup H of P which contains a, and b is such that a and b are conjugate in $N(H)$. However, if we let $a_1 = a$, $a_2 = c$, and $a_3 = b$, let H_1 be the subgroup of P generated by a_1 and a_2, and let H_2 be the subgroup of P generated by a_2 and a_3, then a_1 and a_2 are conjugate in $N(H_1)$ while a_2 and a_3 are conjugate in $N(H_2)$.

The method employed in the proof of our main theorem is classical and is simply a sequence of applications of Sylow's theorem to the normalizers of p-subgroups. Our proof is based upon Sylow's theorem and the fact that proper subgroups of p-groups are proper subgroups of their normalizers. For this reason and in view of a theorem of Wielandt [*10*], all our results are easily generalized to theorems about nilpotent Hall subgroups.

In order to understand and motivate what is to follow, it is convenient to use a theorem of Burnside as a model. Burnside showed ([*4*], p. 327) that if P is a Sylow p-subgroup of a group G and P is in the center of $N(P)$ then G has a normal p-complement (that is, a normal subgroup N such that $P \cap N = 1$ and $G = PN$). The proof of this result falls into three parts. First, Sylow's theorem is required. Second, an application of this result to the centralizers of elements of P shows that two elements of P are conjugate in G if and only if they are conjugate in $N(P)$. Thus, a theorem on fusion is a consequence of Sylow's theorem. Finally, transfer yields the desired conclusion from this fusion result. We shall proceed in a similar fashion. The key lemma is, in fact, a strengthening of Sylow's theorem and we derive our general result on fusion from it. Finally, by means of transfer, in the form of the focal subgroup theorem, we shall derive many corollaries.

In addition to proving standard theorems by our methods, we shall demonstrate a few entirely new results. To describe these, let P be a Sylow p-subgroup of the group G and let P^* be a proper subgroup of $P \cap G'$. The following assertions then hold: *$P \cap G'$ is generated by the subgroups $[H, N(H)]$ as H runs over all the subgroups of P. Thus, there exists a subgroup H of P such that $[H, N(H)] \nleqslant P^*$. Furthermore, $P \cap G'$ is generated by the subgroups $P \cap N(H)'$ as H runs over all the nonidentity subgroups of P.* In general, one would like to improve this last result by choosing a nonidentity subgroup H of P such that $P \cap G' = P \cap N(H)'$. For example, Grun's second theorem gives a sufficient condition for the center of P to be such a subgroup. A general theorem of this nature, which appears in [2], is also dependent on the techniques we shall now develop.

The organization of the paper is as follows. Section 2 contains a series of technical lemmas on the conjugacy of Sylow subgroups as well as several useful concepts of subordinate interest. However, that section contains the heart of the proof of our main theorem. The third part is devoted to the statement and proof of that result, while Section 4 contains the derivations of many old and new theorems from the Main Theorem. The next part contains a generalization and axiomization of the methods of Section 2. This will be used to prove several versions of our conjugacy result as stated above, one of which will then lead to a short new proof of Grun's second theorem. The last section is devoted to some concluding remark and questions.

We shall now summarize our notation, all of which is standard. If G is a group, then G' will denote the derived group of G and $Z(G)$ the center of G. If H and K are subgroups or subsets of G, then $[H, K]$ is the subgroup of G generated by all the elements $[h, k] = h^{-1}k^{-1}hk$ as h and k run over H and K, respectively. Furthermore, if x and y are elements of G then $x^y = y^{-1}xy$ so that $[x, y] = x^{-1}x^y$. We shall denote $N_H(K)$ and $C_H(K)$ for the normalizer and centralizer of K in H while $N(K)$ and $C(K)$ will be the normalizer and centralizer of K in G. Thus, if $x \in N(H)$ and $y \in C(H)$, then $[H, xy] = [H, x]$. If p is a prime and x is an element of G then x is said to be a p-element provided that x has order a power of p. A subgroup H of a subgroup K of G is said to be weakly closed in K (with respect to G) if the only conjugate of H contained in K is H itself. Throughout the paper all groups mentioned are assumed to be finite. The order of a group G is denoted by $|G|$ and the index of a subgroup H is $|G : H|$. Furthermore, we shall signify the relation of containment by $H \leqslant G$ and reserve $H < G$ for proper containment.

2. Sylow Intersections

The most important part of the proofs of our results will be established in this section in a series of five lemmas, the last of which is the culmination of the preceding ones and the only one needed later in the paper. We shall begin by introducing several concepts and giving some examples. Throughout this section we will let G denote a fixed finite group, p a prime divisor of $|G|$ and P a fixed Sylow p-subgroup of G.

Definition 2.1. If Q and R are Sylow p-subgroups of G then we shall say that $Q \cap R$ is a *tame intersection* provided that $N_Q(Q \cap R)$ and $N_R(Q \cap R)$ are Sylow p-subgroups of $N(Q \cap R)$.

Note that this definition depends not just on the subgroups $Q \cap R$ but on the particular way it is described. Thus, when we say that $Q \cap R$ is a tame intersection we shall mean with respect to Q and R, and not necessarily with respect to two other Sylow p-subgroups which also intersect in $Q \cap R$.

Each Sylow p-subgroup is itself a tame intersection; indeed, if Q is such a subgroup then $Q = Q \cap Q$ demonstrates this at once. However, there are other tame intersections. Indeed, suppose that R is a Sylow p-subgroup other than Q such that $Q \cap R$ is a *maximal Sylow intersection with* Q; that is, $Q \cap R \leqslant Q \cap S$, for a Sylow p-subgroup S, implies that $Q = S$ or $Q \cap R = Q \cap S$. In this case $Q \cap R$ is a tame intersection. Indeed, we may see this as follows: If $N_Q(Q \cap R)$ is not a Sylow p-subgroup of $N(Q \cap R)$ then let T be a Sylow p-subgroup of $N(Q \cap R)$ containing $N_Q(Q \cap R)$ and let U be a Sylow p-subgroup of G containing T. Thus, $T \nleqslant Q$ so $U \neq Q$ and $Q \cap U$ properly contains $Q \cap R$, a contradiction. Hence, we will have shown that $Q \cap R$ is a tame intersection once we establish that $N_R(Q \cap R)$ is a Sylow p-subgroup of $N(Q \cap R)$. But this will follow in a similar fashion once we prove that $Q \cap R$ is a maximal Sylow intersection with R. However, as we shall see in Section 4, $Q \cap R$ is, in fact, a *maximal Sylow intersection*; that is, if $Q \cap R \leqslant S \cap T$, where S and T are Sylow p-subgroups of G, then $Q \cap R = S \cap T$ or $S = T$.

We now turn to the critical definition of this section.

Definition 2.2. If R and Q are Sylow p-subgroups of G, then we shall write $R \sim Q$ provided there exist Sylow p-subgroups $Q_1, \ldots, Q_n$ and elements $x_1, \ldots, x_n$ of G such that

(a) $P \cap Q_i$ is a tame intersection, $1 \leqslant i \leqslant n$,

(b) x_i is a p-element of $N(P \cap Q_i)$, $1 \leqslant i \leqslant n$,

(c) $P \cap R \leqslant P \cap Q_1$ and $(P \cap R)^{x_1 \cdots x_i} \leqslant P \cap Q_{i+1}$, $1 \leqslant i \leqslant n-1$,

(d) $R^x = Q$, where $x = x_1 \cdots x_n$.

If we wish to refer to the conjugating element x we shall say that $R \sim Q$ via x.

Thus, the relation $R \sim Q$ signifies that R is conjugate to Q in a very special manner. *Note that the fixed Sylow subgroup P is involved in the definition in an important way.* (We should perhaps even write $R \sim_P Q$ instead of $R \sim Q$.) In fact, the key idea of this concept is the "keeping track" of $P \cap R$ in a series of conjugations. Indeed, we first pass from R to R^{x_1} where x_1 normalizes $P \cap Q_1$ which in turn contains $P \cap R$. This is followed by passage to $R^{x_1 x_2}$ where x_2 normalizes $P \cap Q_2$ which contains $(P \cap R)^{x_1}$, the image of $P \cap R$ under the first conjugation. In particular, if $R \sim Q$ then $| P \cap R | \leqslant | P \cap Q |$ so that this relation is *not*, in general, symmetric. However, it is reflexive, as $R \sim R$ for all Sylow p-subgroups R, and the first lemma we shall prove states that this relation is transitive. The purpose of this part of the paper is to establish that $Q \sim P$ for all Sylow p-subgroups Q of G. This will yield our main theorem almost immediately.

LEMMA 2.1. *If $S \sim R$ and $R \sim Q$, where S, R, and Q are Sylow p-subgroups of G, then $S \sim Q$.*

Proof. Suppose that y_i, T_i, $1 \leqslant i \leqslant s$ and z_i, U_i, $1 \leqslant i \leqslant r$, are elements and Sylow p-subgroups of G realizing the relations $S \sim R$ and $R \sim Q$, respectively. In particular, we have

$$P \cap S \leqslant P \cap T_1, \qquad (P \cap S)^{y_1 \cdots y_i} \leqslant P \cap T_{i+1}, \qquad 1 \leqslant i \leqslant s-1, \quad (1)$$

$$P \cap R \leqslant P \cap U_1, \qquad (P \cap R)^{z_1 \cdots z_i} \leqslant P \cap U_{i+1}, \qquad 1 \leqslant i \leqslant r-1. \quad (2)$$

Let $n = s + r$ and define

$$x_i = \begin{cases} y_i \\ z_{i-s} \end{cases}, \qquad Q_i = \begin{cases} T_i & 1 \leqslant i \leqslant s, \\ U_{i-s}, & s+1 \leqslant i \leqslant n. \end{cases} \quad (3)$$

We shall establish that the elements x_i and Sylow p-subgroups Q_i, $1 \leqslant i \leqslant n$, give the relation $S \sim Q$. All the requirements, except condition (c) of the definition, are clearly satisfied. Indeed, $P \cap Q_i$ is a tame intersection and x_i is a p-element of its normalizer, for all i, as corresponding assertions hold for $P \cap T_i$, $P \cap U_i$, y_i, and z_i. Furthermore, if $x = x_1 \cdots x_n$, $y = y_1 \cdots y_s$, and $z = z_1 \cdots z_r$, then $S^x = S^{yz} = R^z = Q$. Thus, we need only verify condition (c).

However, $P \cap Q_1 = P \cap T_1$ while $P \cap S \leqslant P \cap T_1$ by (1), so that $P \cap S \leqslant P \cap Q_1$. It remains to demonstrate the last part of the statement of (c) and we shall do this in several steps. First, suppose that $1 \leqslant i \leqslant s-1$. In this case, we have by (1) and (3),

$$(P \cap S)^{x_1 \cdots x_i} = (P \cap S)^{y_1 \cdots y_i} \leqslant P \cap T_{i+1} = P \cap Q_{i+1}.$$

Furthermore, as $(P \cap S)^{y_1 \cdots y_{s-i}} \leqslant P \cap T_s$ and $y_s \in N(P \cap T_s)$ we have $(P \cap S)^y \leqslant P$. But $(P \cap S)^y \leqslant S^y = R$, so $(P \cap S)^y \leqslant P \cap R$, which is

$$(P \cap S)^{x_1 \cdots x_s} = (P \cap S)^y \leqslant P \cap R. \tag{4}$$

Hence, by (2) and (3),

$$(P \cap S)^{x_1 \cdots x_s} \leqslant P \cap R \leqslant P \cap U_1 = P \cap Q_{s+1}.$$

Finally, if $s + 1 \leqslant i \leqslant n - 1$, then

$$\begin{aligned}(P \cap S)^{x_1 \cdots x_i} &= ((P \cap S)^{x_1 \cdots x_s})^{x_{s+1} \cdots x_i} \\ &\leqslant (P \cap R)^{x_{s+1} \cdots x_i} && \text{[by (4)]} \\ &= (P \cap R)^{z_1 \cdots z_{i-s}} && \text{[by (3)]} \\ &\leqslant P \cap U_{i+1-s} && \text{[by (2)]} \\ &= P \cap Q_{i+1} && \text{[by (3)]}\end{aligned}$$

and the lemma is proved.

LEMMA 2.2. *If Q and R are Sylow p-subgroups of G such that $P \cap R \geqslant P \cap Q$, $R \sim P$ via x and $Q^x \sim P$, then $Q \sim P$.*

Proof. By the preceding lemma, insomuch as $Q^x \sim P$, it will suffice to show that $Q \sim Q^x$. Let $x_1, \ldots, x_n$ and $Q_1, \ldots, Q_n$ be the elements and Sylow p-subgroups giving $R \sim P$ such that $x = x_1 \cdots x_n$. We claim that $Q \sim Q^x$ is also given by these same elements and Sylow p-subgroups. In fact, all the conditions required are clear except (c). However, $P \cap Q \leqslant P \cap R$, by hypothesis, and $P \cap R \leqslant P \cap Q_1$ as $R \sim Q$ so that $P \cap Q \leqslant P \cap Q_1$. Furthermore, if $1 \leqslant i \leqslant n - 1$, then we have

$$(P \cap Q)^{x_1 \cdots x_i} \leqslant (P \cap R)^{x_1 \cdots x_i} \leqslant P \cap Q_{i+1},$$

and the proof is complete.

LEMMA 2.3. *Let Q and R be Sylow p-subgroups of G with $R \cap Q > P \cap Q$ and $R \sim P$. If, in addition, $S \sim P$ for all Sylow p-subgroups S of G with $| P \cap S | > | P \cap Q |$, then $Q \sim P$.*

Proof. Suppose that $R \sim P$ via x so that $R^x = P$. We first show that $Q^x \sim P$. Indeed,

$$P \cap Q^x = R^x \cap Q^x = (R \cap Q)^x$$

so that

$$| P \cap Q^x | = | (R \cap Q)^x | = | R \cap Q | > | P \cap Q |.$$

Thus, $Q^x \sim P$, by our hypothesis with $S = Q^x$. However, we have assumed that $R \sim P$, so that the result will follow from the preceding lemma once we have proved that $P \cap R \geqslant P \cap Q$. But

$$P \cap R \geqslant P \cap (R \cap Q) \geqslant P \cap (P \cap Q) = P \cap Q$$

and the lemma is proved.

LEMMA 2.4. *If Q is a Sylow p-subgroup of G, $P \cap Q$ is a tame intersection, and $S \sim P$ for all Sylow p-subgroups S with $| P \cap S | > | P \cap Q |$, then $Q \sim P$.*

Proof. If $Q = P$ then $Q \sim P$ so that we may assume $Q \neq P$. Hence, if $P_1 = N_P(P \cap Q)$ then $P \cap Q < P_1$. Furthermore, if $Q_1 = N_Q(P \cap Q)$ then P_1 and Q_1 are Sylow p-subgroups of $N(P \cap Q)$. Let K be the subgroup of $N(P \cap Q)$ generated by all the p-elements of $N(P \cap Q)$ so that P_1 and Q_1 are Sylow p-subgroups of K. Thus, there exists $x \in K$ such that $Q_1{}^x = P_1$. Let $x = x_1 \cdots x_n$ where each x_i is a p-element of $N(P \cap Q)$. If we set $Q_i = Q$, $1 \leqslant i \leqslant n$, then certainly $P \cap Q_i$ is a tame intersection. Furthermore, $P \cap Q = P \cap Q_1$ and $(P \cap Q)^{x_1 \cdots x_n} = P \cap Q = P \cap Q_{i+1}$, a $\leqslant i \leqslant n - 1$, so that it is immediate from Definition 2.2 that $x_1, ..., x_n$ and $Q_1, ..., Q_n$ yield that $Q \sim Q^x$. Moreover,

$$P \cap Q^x \geqslant P \cap Q_1{}^x = P \cap P_1 = P_1 > P \cap Q$$

so that $Q^x \sim P$ by our hypothesis with $S = Q^x$. Hence, $Q \sim P$ by Lemma 2.1 and the lemma is proved.

We now have arrived at the goal of this section, a result which is in fact a strengthening of the conjugacy part of Sylow's theorem.

LEMMA 2.5. *If Q is a Sylow p-subgroup of G then $Q \sim P$.*

Proof. We shall proceed by induction on the index $| P : P \cap Q |$. If this index is one then $P = Q$ and certainly $Q \sim P$. We may now assume that $Q \neq P$ and that $S \sim P$ for all Sylow p-subgroups S of G with $| P \cap S | > | P \cap Q |$. Therefore, part of the hypotheses of Lemmas 2.3 and 2.4 are already verified.

Let S be a Sylow p-subgroup of $N(P \cap Q)$ containing $N_P(P \cap Q)$ and let R be a Sylow p-subgroup of G which contains S. Therefore,

$$P \cap R \geqslant P \cap S \geqslant N_P(P \cap Q) > P \cap Q,$$

since $P \cap Q < P$, so that $R \sim P$ via x. Thus, since $P \cap R \geqslant P \cap Q$ and $R \sim P$ via x, Lemma 2.2 shows that we need only establish that $Q^x \sim P$ in order to deduce that $Q \sim P$. This we shall now do.

First, $(P \cap Q)^x \leqslant S^x \leqslant P$ so that

$$P \cap Q^x \geqslant P \cap (P \cap Q)^x = (P \cap Q)^x.$$

However, if $|P \cap Q^x| > |P \cap Q|$ then $Q^x \sim P$ by induction. Thus, we may assume that $|P \cap Q^x| = |P \cap Q|$ so that $|P \cap Q^x| = |(P \cap Q)^x|$ and $P \cap Q^x = (P \cap Q)^x$.

We now claim that $N_P(P \cap Q^x)$ is a Sylow p-subgroup of $N(P \cap Q^x)$. Indeed, S is a Sylow p-subgroup of $N(P \cap Q)$ so that S^x is a Sylow p-subgroup of $(N(P \cap Q))^x$. Hence, S^x is a Sylow p-subgroup of

$$N((P \cap Q)^x) = N(P \cap Q^x).$$

However, $S^x \leqslant R^x = P$ so that $S^x \leqslant N_P(P \cap Q^x)$. But $N_P(P \cap Q^x)$ is a p-subgroup of $N(P \cap Q^x)$ containing the Sylow p-subgroup S^x of $N(P \cap Q^x)$, so $S^x = N_P(P \cap Q^x)$, and our assertion holds.

Let T be a Sylow p-subgroup of $N(P \cap Q^x)$ containing $N_{Q^x}(P \cap Q^x)$ and let U be a Sylow p-subgroup of G which contains T. We claim that it suffices to show that $U \sim P$ to complete the proof. In fact, $P \cap Q^x < Q^x$ as $|P \cap Q^x| = |P \cap Q|$ so that

$$U \cap Q^x \geqslant N_{Q^x}(P \cap Q^x) > P \cap Q^x.$$

Thus, once we show that $U \sim P$ we may apply Lemma 2.3, with U and Q^x in place of R and Q of that lemma, and deduce that $Q^x \sim P$.

However, $P \cap U \geqslant P \cap T \geqslant P \cap Q^x$ so if $P \cap U > P \cap Q^x$ we are done by induction. Hence, we may assume that $P \cap U = P \cap Q^x$. In this case we shall conclude with an application of Lemma 2.4. First, $T = N_U(P \cap Q^x)$ by the choice of T and U. Thus, since $P \cap Q^x = P \cap U$, we have that $T = N_U(P \cap U)$ is a Sylow p-subgroup of $N(P \cap U)$. On the other hand, $N_P(P \cap U) = N_P(P \cap Q^x)$ is a Sylow p-subgroup of $N(P \cap U) = N(P \cap Q^x)$, as we have seen above. Thus, $P \cap U$ is a tame intersection. Finally, $|P \cap U| = |P \cap Q|$ so that $S \sim P$ for all Sylow p-subgroups S with $|P \cap S| > |P \cap U|$ and Lemma 2.4 is applicable and $U \sim P$. This completes the proof.

3. The Fusion Theorem

We shall state and prove the following.

Main Theorem. *If A and B are nonempty subsets of the Sylow p-subgroup of the group G and $B = A^g$ for some g in G, then there exist elements $x_1, \ldots, x_n$ and Sylow p-subgroups $Q_1, \ldots, Q_n$ of G and an element y of $N(P)$ such that*

(1) $g = x_1 \cdots x_n y$,

(2) *$P \cap Q_i$ is a tame intersection*, $1 \leqslant i \leqslant n$,

(3) *x_i is a p-element of $N(P \cap Q_i)$*, $1 \leqslant i \leqslant n$,

(4) *A is contained in $P \cap Q_1$ while $A^{x_1 \cdots x_i}$ is a subset of $P \cap Q_{i+1}$*, $1 \leqslant i \leqslant n-1$.

Note that in addition to (4) we have $A^{x_1 \cdots x_n} \leqslant P$ as $x_n \in N(P \cap Q_n)$ and $A^{x_1 \cdots x_{n-1}} \leqslant P \cap Q_n$. This result describes in detail how the conjugation of A into B by g may be accomplished in a sequence of steps. Moreover, we not only have $A^g = A^{x_1 \cdots x_n y}$ but even the equality (1). The theorem mentioned in the introduction may be easily derived from this result (and we shall do so in the next section) by taking A and B to be one-element subsets and ignoring other refinements.

We should like to make two other observations before beginning the proof. First, only the element y need not be a p-element. Second, each of the tame intersections $P \cap Q_i$ is not the identity subgroup unless the subset A consists only of the identity element of P, in which case, the theorem is of no interest. (It specializes to the Frattini argument applied to the subgroup of G generated by all the p-elements.) Thus, we only require the special case of Lemma 2.5 in which $P \cap Q \neq 1$. This in turn follows from the form of Lemma 2.4 in which $P \cap Q \neq 1$. However, the fact that we need only these slightly weakened versions would shorten none of the proofs.

Proof. The containment $A \leqslant P$ implies that $A^g \leqslant P^g$ so $A^g \leqslant P \cap P^g$ as $A^g = B \leqslant P$. Hence, $A \leqslant P^{g^{-1}} \cap P$. By Lemma 2.5 there is x in G such that $P^{g^{-1}} \sim P$ via x. (Here we are letting P be the Sylow p-subgroup P fixed in the preceding section.) Let $x_1, \ldots, x_n$ and $Q_1, \ldots, Q_n$ be elements and Sylow p-subgroups of G giving $P^{g^{-1}} \sim P$ such that $x = x_1 \cdots x_n$. Thus, $P^{g^{-1}x} = P$, as $P^{g^{-1}} \sim P$ via x, so that if we set $y = x^{-1}g$ then $y \in N(P)$. We now claim that the elements $x_1, \ldots, x_n$ and y and Sylow p-subgroups $Q_1, \ldots, Q_n$ suffice for the proof.

First, $g = x(x^{-1}g) = x_1 \cdots x_n y$ so (1) is fulfilled. The next two conditions are also satisfied by choice of the elements $x_1, \ldots, x_n$ and Sylow p-subgroups $Q_1, \ldots, Q_n$. Finally, $A \leqslant P \cap P^{g^{-1}}$ and $P \cap P^{g^{-1}} \leqslant P \cap Q_1$ since $P^{g^{-1}} \sim P$ via x. Similarly,

$$A^{x_1 \cdots x_i} \leqslant (P \cap P^{g^{-1}})^{x_1 \cdots x_i} \leqslant P \cap Q_{i+1}$$

for $1 \leqslant i \leqslant n-1$ and the proof is complete.

The idea of this proof may serve as further motivation. Suppose that g is some element of G. Then $P^{g^{-1}} \cap P$ and $P \cap P^g$ are conjugate subgroups of P which are clearly conjugate in G but are not obviously conjugate in a series of steps of the type we desire. The proof shows that this may be accomplished by conjugating $P^{g^{-1}}$ into P in a prescribed manner and that the general case of two conjugate subsets A and B then follows.

4. Applications

First, we shall prove the weak form of the Main Theorem mentioned in the introduction.

Theorem 4.1. *If a and b are elements of the Sylow p-subgroup P of the group G and are conjugate in G then there exist elements $a_1, \ldots, a_m$ and subgroups $H_1, \ldots, H_{m-1}$ of P such that $a = a_1$, $b = a_m$ while a_i and a_{i+1} are contained in H_i and conjugate in $N(H_i)$, $1 \leqslant i \leqslant m - 1$.*

Proof. We shall apply the main theorem with $A = a$ and $B = b$. Suppose that $b = a^g$ for g in G and choose $x_1, \ldots, x_n$, y and $Q_1, \ldots, Q_n$ in accordance with the main theorem. Let $m = n + 2$ and define

$$a_i = \begin{cases} a & i = 1 \\ a^{x_1 \cdots x_{i-1}} & 2 \leqslant i \leqslant n + 1 \\ a^{x_1 \cdots x_n y} & i = n + 2 \end{cases}$$

$$H_i = \begin{cases} P \cap Q_i & 1 \leqslant i \leqslant n \\ P & i = n + 1. \end{cases}$$

We claim that these elements and subgroups suffice for the proof. In fact, $a = a_1$ while $b = a^g = a^{x_1 \cdots x_n y} = a_{n+2} = a_m$. Furthermore, if $1 \leqslant i \leqslant m - 2$ then a_i and a_{i+1} are contained in H_i and are conjugate in $N(H_i)$ by condition (4) of the Main Theorem. Finally, a_{m-1} and a_m are contained in $H_{m-1} = P$ and are conjugate in $N(P)$.

We now turn to a purely local characterization of the focal subgroup.

Theorem 4.2. *If P is a Sylow p-subgroup of the group G then $P \cap G'$ is generated by $[P, N(P)]$ together with all the subgroups $[H, x]$ as H ranges over all tame intersections of P with the Sylow p-subgroups of G and x runs over all p-elements of $N(H)$.*

Proof. Let P^* be the subgroup of P generated by the subgroups described in the statement of the theorem so that P^* is certainly contained in $P \cap G'$. Indeed, $[P, N(P)] \leqslant P \cap [G, G] = P \cap G'$ while $[H, x] \leqslant H \cap N(H') \leqslant P \cap G'$. The focal subgroup theorem described in the introduction shows that $P \cap G'$ is generated by all elements of the form $a^{-1}b$, where $a, b \in P$ and a and b are conjugate in G. Thus, we need only show that each of these elements $a^{-1}b$ lies in P^*.

However, suppose that $b = a^g$ for $g \in G$ and choose $x_1, \ldots, x_n$, y and $Q_1, \ldots, Q_n$ in accordance with the Main Theorem. Let m, a_i, $1 \leqslant i \leqslant m$, and H_j, $1 \leqslant j \leqslant m - 1$ be defined as in the proof of the previous result. Thus

$$a^{-1}b = (a_1^{-1}a_2)(a_2^{-1}a_3) \cdots (a_{m-1}^{-1}a_m).$$

We shall conclude the proof by establishing that $a_i^{-1}a_{i+1}$, $1 \leqslant i \leqslant m - 1$, lies in P^*. Indeed, if $1 \leqslant i \leqslant m - 2$, then

$$a_i^{-1}a_{i+1} = a_i^{-1}a_i^{x_i} = [a_i, x_i] \in [H_i, x_i] \leqslant P^*$$

as x_i is a p-element of $N(H_i)$ and $H_i = P \cap Q_i$ is a tame intersection. Finally,

$$a_{m-1}^{-1}a_m = a_{m-1}^{-1}a_{m-1}^{y} = [a_{m-1}, y] \in [P, N(P)] \leqslant P^*.$$

An immediate consequence of the theorem is the following:

COROLLARY 4.3. *If P is a Sylow p-subgroup of the group G and P^* is a proper subgroup of $P \cap G'$ then there is a subgroup H of P such that $[H, N(H)] \nleqslant P^*$.*

Several weak forms of the above theorem are also of interest.

COROLLARY 4.4. *If P is a Sylow p-subgroup of the group G then $P \cap G'$ is generated by all the subgroups $P \cap N(H)'$ as H ranges over all the non-identity subgroups of P.*

Proof. Let P^* be the subgroup of P generated by all the subgroups $P \cap N(H)'$ described in the statement of this corollary. Certainly we have $P^* \leqslant P \cap G'$ as $P \cap N(H)' \leqslant P \cap G'$ for each subgroup H. On the other hand, Theorem 4.2 gives us a set of subgroups which generate $P \cap G'$ so that it will suffice to show that each of these is contained in P^*. But $[P, N(P)] \leqslant P \cap N(P)'$ while if $H = P \cap Q$ is a tame intersection, where Q is a Sylow p-subgroup of G, and x is a p-element of $N(H)$, then $[H, x] \leqslant H \cap N(H)'$. Thus, if $H = 1$ then $H \cap N(H)' = 1$ and if $H \neq 1$ then $H \cap N(H)' \leqslant P \cap N(H)' \leqslant P^*$.

COROLLARY 4.5 (Grun's first theorem). *If P is a Sylow p-subgroup of the group G then $P \cap G'$ is generated by $P \cap N(P)'$ together with all the subgroups $P \cap Q'$ as Q ranges over all the Sylow p-subgroups of G.*

Proof. We proceed as in the above argument. Again $P \cap N(P)'$ and $P \cap Q'$, for any Sylow p-subgroup Q of G, are contained in $P \cap G'$. Thus, it suffices to show that each subgroup $[H, x]$, where H and x are as in the statement of Theorem 4.2, is contained in one of the subgroups $P \cap Q'$.

However, the subgroup R of $N(H)$ generated by H and x is a p-subgroup, as x s a p-element of $N(H)$. If Q is a Sylow p-subgroup of G which contains R, then $[H, x] \leqslant H \cap R' \leqslant P \cap Q'$ and the lemma is proved.

Our next application is the proof of a strong form of one of the standard results which gives conditions sufficient for the existence of a normal p-complement.

THEOREM 4.6. *If P is a Sylow p-subgroup of the group G and $N(H)/C(H)$ is a p-group, for each nonidentity tame intersection $H = P \cap Q$ of P and a Sylow p-subgroup Q of G, then G has a normal p-complement.*

This has an immediate consequence:

COROLLARY 4.7 (Frobenius). *If P is a Sylow p-subgroup of the group G and $N(H)/C(H)$ is a p-group for each nonindentity subgroup H of P, then G has a normal p-complement.*

The author has been informed by H. Wielandt that he has obtained this result, and the above ones on p-quotient groups, by other methods (in unpublished work).

Proof. In view of Tate's theorem [9], it suffices to show that $P \cap G' \leqslant P'$. By Theorem 4.2, it will be enough to show that $[H, N(H)] \leqslant P'$ for all subgroups H of P which are tame intersections of P and a Sylow p-subgroup of G. (Recall that P itself is such a subgroup so that we will also have $P \cap N(P)' \leqslant P'$.) However, if H is such a subgroup, then $N_P(H)$ is a Sylow p-subgroup of $N(H)$, by the definition of tame intersection. Since $N(H)/C(H)$ is a p-group and $N_P(H)\,C(H)/C(H)$ is a Sylow p-subgroup of $N(H)/C(H)$ it follows that $N(H) = N_P(H)\,C(H)$. Hence

$$[H, N(H)] = [H, N_P(H)\,C(H)] = [H, N_P(H)] \leqslant [H, P] \leqslant P',$$

and the proof is complete.

We now turn to two theorems of Burnside [4], the first of which we mentioned above as motivation for the method of proof of our main theorem. Even though Burnside's result is very simple to prove directly, we shall show that it does follow from our general methods.

THEOREM 4.8 (Burnside). *Let A and B be normal subsets of the Sylow p-subgroup P of G. If A and B are conjugate in G then they are conjugate in $N(P)$.*

Proof. Suppose that $B = A^g$ for $g \in G$. Thus, as A is normal in P, we have that B is normal in P^g as well as in P so that P and P^g are Sylow p-subgroups of $N(B)$. Applying Lemma 2.5 to $N(B)$ we have $P^g \sim P$, where the relation is relative to P. In particular, it follows that $P^{gx} = P$ for some $x \in N(B)$ so that $gx \in N(P)$ and $A^{gx} = B^x = B$ and the lemma is proved.

This proof does no more than replace, in the usual argument, the use of the conjugacy part of Sylow's theorem by Lemma 2.5.

The other result of Burnside that we shall derive is

THEOREM 4.9 (Burnside). *If K is a normal subgroup of the Sylow p-subgroup P of the group G and K is a non-normal subgroup of some other Sylow p-subgroup of G, then there is a subgroup H of P containing K such that K is not normal in $N(H)$ and the number of conjugates of K in $N(H)$ in relatively prime to p.*

Proof. Suppose that K is a non-normal subgroup of the Sylow p-subgroup Q of G and that $Q = P^{g^{-1}}$ for $g \in G$. Therefore, K^g is a non-normal subgroup of $Q^g = P$. Hence, K and K^g are subgroups of P conjugate in G so that we may choose elements $x_1, \dots, x_n$, y and Sylow p-subgroups $Q_1, \dots, Q_n$, in accordance with the Main Theorem with $A = K$ and $B = K^g$, such that $g = x_1 \cdots x_n y$. But K^g is not a normal subgroup of P; so neither is $K^{gy^{-1}} = K^{x_1 \cdots x_n}$, since $y \in N(P)$. Thus, if we define $K_0 = K$ and $K_j = K^{x_1 \cdots x_j}$, $1 \leqslant j \leqslant n$, then we may choose an integer i, $1 \leqslant i < n$, such that K_i is normal in P and K_{i+1} is not. Let $H_i = P \cap Q_i$ so that H is a tame intersection, $x_{i+1} \in N(H_i)$ and $K_i^{x_{i+1}} = K_{i+1}$. We claim that K_i is not normal in $N(H_i)$ and that the number of conjugates of K_i in $N(H_i)$ is relatively prime to p. In fact, K_i and K_{i+1} are conjugate in $N(H_i)$ and $K_i \neq K_{i+1}$, as one of these subgroups is normal in P and the other is not, so K_i is not normal in $N(H_i)$. Furthermore, K_i is normal in P so K_i is normal in $N_P(H_i)$ which is a Sylow p-subgroup of $N(H_i)$, since $H_i = P \cap Q_i$ is tame. Thus, the index in $N(H_i)$ of the normalizer in $N(H_i)$ of K_i relatively prime to p and we have established our assertion.

Finally, K and K_i are normal subgroups of P conjugate in G so, by Theorem 4.8, there exists $u \in N(P)$ such that $K_i^u = K$. Let $H = H_i^u$ so that $K \leqslant H \leqslant P$, K is not normal in $N(H)$ and the number of conjugates of K in $N(H)$ is relatively prime to p.

The last theorem of Burnside has been strengthened in several ways (see [7], p. 46 and also [5]). However, we have been unable, as yet, to obtain these stronger forms from our methods. Such an accomplishment might lead to some interesting results.

The next theorem is also easily proven directly but we shall derive it from our general theorem.

THEOREM 4.10. *If P is a Sylow p-subgroup of the group G and P intersects any other Sylow p-subgroup in the identity then any two elements of P which are conjugate in G are conjugate in $N(P)$.*

Proof. Let a and b be elements of P with $b = a^g$ gor some g in G. Choose elements $x_1, ..., x_n$, y and Sylow p-subgroups $Q_1, ..., Q_n$ as in the Main Theorem with $A = a$ and $B = b$, so that $g = x_1 \cdots x_n y$. It is clear that we may assume $a \neq 1$ so that each intersection $P \cap Q$, $1 \leqslant i \leqslant n$, is not the identity, in as much as $a \in P \cap Q_1$ and $a^{x_1 \cdots x_{i-1}} \in P \cap Q_i$, $2 \leqslant i \leqslant n$. Thus, $P = Q_i$, by our hypothesis. However, the p-elements of $N(P \cap Q_i) = N(P)$ are precisely the elements of P so $x_i \cap P$, $1 \leqslant i \leqslant n$. Hence, as $y \in N(P)$, it follows that $g = x_1 \cdots x_n y \in N(P)$. This completes the proof.

Our last application is the following:

THEOREM 4.11. *If P and Q are distinct Sylow p-subgroups of the groups G and $P \cap Q$ is a maximal Sylow intersection with P, then any conjugate of $P \cap Q$ which lies in P is a conjugate in $N(P)$.*

As we remarked in Section 2, $P \cap Q$ is a maximal Sylow intersection since $P \cap Q$ is a maximal Sylow intersection with P so that this theorem is just the corollary given in [*1*]. We now supply the proof that $P \cap Q$ is a maximal Sylow intersection. We argue by contradiction and assume that R and S are distinct Sylow p-subgroups of G with $P \cap Q < R \cap S$. Let $D = P \cap Q$, $E = R \cap S$, $F = N_E(D)$, and $H = N_P(D)$, so that $D < F$ and $D < H$. By Sylow's theorem, applied to $N(D)$, there is x in $N(D)$ such that the subgroup K which is generated by H and F^x is a p-subgroup. We claim that K is a subgroup of P. Indeed if it is not and if T is a Sylow p-subgroup of G containing K, then $P \neq T$. However, $P \cap T \geqslant P \cap K \geqslant H > D$, which is a contradiction. Hence, as $K \leqslant P$, it follows that $F^x \leqslant P$ so that $P \cap R^x \geqslant F^x > D$ and $P \cap S^x \geqslant F^x > D$. Therefore, as either $P \neq R^x$ or $P \neq S^x$, we have a contradiction. This remark we now state as a

PROPOSITION. *It P and Q are distinct Sylow p-subgroups of the group G and $P \cap Q$ is a maximal Sylow intersection with P then $P \cap Q$ is a maximal Sylow intersection.*

We now prove Theorem 4.11.

Proof. Suppose that $(P \cap Q)^g \leqslant P$ for g in G. Choose elements $x_1, ..., x_n, y$ and Sylow p-subgroups $Q_1, ..., Q_n$ of G, such that $g = x_1 \cdots x_n y$, in accordance with the Main Theorem with $A = P \cap Q$ and $B = (P \cap Q)^g$. We shall prove, by induction on i, that $(P \cap Q)^{x_1 \cdots x_i}$ is conjugate to $P \cap Q$ in P. This will suffice for the proof since $y \in N(P)$.

First, suppose that $i = 1$. In this case, $P \cap Q \leqslant P \cap Q_1$ so that either $P \cap Q = P \cap Q_1$ or $P = Q_1$. In the first case $(P \cap Q)^{x_1} = P \cap Q$, as $x_1 \in N(P_1 \cap Q_1)$, and in the second case $x_1 \in P$ since x_1 is a p-element of $N(P)$.

Suppose that our assertion is true for $i = k$, $k < n$ and that $(P \cap Q)^{x_1 \cdots x_k} = (P \cap Q)^t$ where $t \in P$. We first establish that $P \cap Q^t = (P \cap Q)^t$ and $P \cap Q^t$ is a maximal intersection with P. The last part of this assertion is clear; indeed, since $P \cap Q$ is a maximal intersection with P it follows that $P^t \cap Q^t = P \cap Q^t$ is a maximal intersection with $P^t = P$ as $t \in P$. On the other hand,

$$(P \cap Q)^t = (P^{t^{-1}} \cap Q)^t = P \cap Q^t.$$

Thus, $P \cap Q^t = (P \cap Q)^t = (P \cap Q)^{x_1 \cdots x_k} \leqslant P \cap Q_{k+1}$ as $k < n$, by condition (4) of the Main Theorem. Hence, $P \cap Q_{k+1} = P \cap Q^t$ or $P \cap Q_{k+1} = P$. In the first case. $(P \cap Q^t)^{x_{k+1}} = P \cap Q^t$ as $x_{k+1} \in N(P \cap Q_{k+1})$ while in the second case x_{k+1} is a p-element of $N(P)$ so $x_{k+1} \in P$ and

$$(P \cap Q)^{x_1 \cdots x_{k+1}} = (P \cap Q)^{t x_{k+1}}$$

is conjugate to $P \cap Q$ in P. This completes the proof.

5. Conjugation Families

It is natural to ask whether there are sets of subgroups, other than the tame intersections, for which a result similar to the Main Theorem will hold. It is the purpose of this section to give a general criterion for the existence of such subgroups. Furthermore, we have several applications in mind. First, we shall show how Grun's second theorem is a consequence of these general considerations. Second, the work concerning fusion and transfer [2] depends on these results.

Given a class of subgroups of a Sylow p-subgroup it is rather clear how to define a relation "$\sim$" for these subgroups. If one then attempts to carry through the development of Section 2 then Lemma 2.4 is the only obstacle. This idea motivates much of what now follows.

Throughout this section G will denote a fixed finite group, p a prime divisor of $|G|$ and P a fixed Sylow p-subgroup of G.

Definition 5.1. *A family* is a collection of pairs (H, T) where H is a subgroup of P and T is a subset of $N(H)$.

Note that this concept, as well as several to follow, depend entirely on the special nature of P.

Definition 5.2. If F is a family and R and Q are Sylow p-subgroups of G then we write $R \sim Q$, *with respect to* F, if $Q = R$ *or* if there are elements

$(H_1, T_1), \ldots, (H_n, T_n)$ of F and elements $x_1, \ldots, x_n$ of G with $x_i \in T_i$, $1 \leqslant i \leqslant n$, and

(a) $R^x = Q$, where $x = x_1 \cdots x_n$

(b) $P \cap R \leqslant H_1$, $(P \cap R)^{x_1 \cdots x_i} \leqslant H_{i+1}$, $1 \leqslant i \leqslant n-1$.

We shall also say that $R \sim Q$ via x.

In section two we dealt with the family F_t of all pairs (H, T) where H is a tame intersection of the form $H = P \cap Q$ and T consists of the set of p-elements of $N(H)$. We shall now generalize the results we obtained to other families.

DEFINITION 5.3. A family F is called a *conjugation* family provided that whenever A and B are subsets of P and $B = A^g$ for g in G then there are elements $(H_1, T_1), \ldots, (H_n, T_n)$ of F and elements $x_1, \ldots, x_n, y$ of G such that

(a) $g = x_1 \cdots x_n y$,

(b) $x_i \in T_i$, $1 \leqslant i \leqslant n$ and $y \in N(P)$,

(c) $A \leqslant H_1$, $A^{x_1 \cdots x_i} \leqslant H_{i+1}$, $1 \leqslant i \leqslant n-1$.

Note that if $(P, N(P)) \in F$ then the definition may be stated in a simpler fashion. The Main Theorem states that the family F_t is a conjugation family. Another related concept is the following:

DEFINITION 5.4. A family F is called a *weak conjugation* family provided that whenever A and B are subsets of P and $B = A^g$ for g in G, then there are elements (H_i, T_i), $1 \leqslant i \leqslant n$ of F and elements $x_1, \ldots, x_n, y$ of G such that $B = A^{x_1 \cdots x_n y}$ and (b) and (c) of Definition 5.3 hold.

Since condition (a) of Definition 5.3 implies that $B = A^{x_1 \cdots x_n y}$, it follows that every conjugation family is a weak conjugation family. The last new definition is the following.

DEFINITION 5.5. A family F is called an *inductive* family provided that, whenever Q is a Sylow p-subgroup of G such that

(a) $P \cap Q$ is a tame intersection,

(b) $S \sim P$, with respect to F, for all Sylow p-subgroups, S of G for which $|P \cap S| > |P \cap Q|$,

then $Q \sim P$, with respect to F.

The key result of this part of the paper may now be stated.

MAIN THEOREM (second form). *Every inductive family is a conjugation family.*

The proof is merely a step-by-step modification of the proof of the Main

Theorem, as given in Sections 2 and 3. Therefore, we only sketch the proof, noting the changes that need to be made.

First, Lemmas 2.1, 2.2, and 2.3 hold as long as the symbol "$\sim$" is interpreted as "$\sim$, with respect to F." The proofs carry over with little modification. Each reference to a Sylow p-subgroup Q_i, tame intersection $P \cap Q_i$, and p-element x_i of $N(P \cap Q_i)$ must be replaced by a reference to an element (H_i, T_i) of F and element x_i of T_i, Furthermore, the proof of Lemma 2.1 must be divided into two cases: $S = R$ or $R = Q$ and $S \neq R$, $R \neq Q$. The first of these cases is trivial and the second proceeds as in the proof of Section 2. The reason for this change is that in Definition 5.2 we gave two conditions, either of which defines $Q \sim R$, while in Definition 2.2 there was only one possibility.

The next step is to redo Lemma 2.5. Again the proof proceeds as before with the above changes being made. However, the one reference to Lemma 2.4 must be replaced by mention of the definition of an inductive family. This is the only place where such a reference need be made. Finally, the proof of the Main Theorem, as given in Section 3, can be readily generalized to the present situation.

Similarly, one may derive many theorems as applications of the new form of our Main Theorem, in the same was as we proceeded in Section 4. The interested reader may easily state and prove these results. We shall limit ourselves to stating a result which is the analog, for general conjugation families, of Corollary 4.3. The proof is immediate from the suitably generalized form of Theorem 4.2.

COROLLARY. *Let P be a Sylow p-subgroup of the group G and let F be a conjugation family. If P^* is a subgroup of P and $[P, N(P)] \leqslant P^* < P \cap G'$ then there is an element (H, T) of F such that $[H, T] \nleqslant P^*$.*

We now proceed to define several families and discuss their properties. First, let F_c consist of all (H, T) where H is a subgroup of P for which there is a Sylow p-subgroup Q of G with $H = P \cap Q$ a tame intersection, and where $T = C(H)$ if $C_P(H) \nleqslant H$ and $T = N(H)$ if $C_P(H) \leqslant H$. We then have

THEOREM 5.1. *F_c is an inductive family.*

Proof. Throughout this proof, the symbol "$\sim$" shall mean "$\sim$, with respect to F_c." Let $H = P \cap Q$ be a tame intersection, where Q is a Sylow p-subgroup of G such that $S \sim P$ for all Sylow p-subgroups S of G for which $| P \cap S | > | P \cap Q |$. We need only show that $Q \sim P$. However, in view of Lemma 2.1, as generalized for the family F_c and as $P \sim P$ by definition, we may assume that $Q \neq P$ and need only show that $Q \sim R$ for some Sylow

p-subgroup R of G for which $|P \cap R| > |P \cap Q|$. Indeed, this being the case, it follows that $Q \sim R$ and $R \sim P$; so $Q \sim P$.

First, suppose that $C_P(H) \leqslant H$. We then proceed as in Lemma 2.4. Let Let $P_1 = N_P(H)$, $Q_1 = N_Q(H)$ and choose $x \in N(H)$ such that $Q_1^x = P_1$. We may do this because $P \cap Q$ is tame. Since $(H, N(H)) \cap F_c$, we have that $Q \sim Q^x$. However, $Q^x \geqslant Q_1^x = P_1$ so that $P \cap Q^x \geqslant P_1 > H$, and $H < P$, and the lemma is proved in this case.

Second, assume that $C_P(H) \nleqslant H$. Let $D = HC(H)$ so that D is a normal subgroup of $N(H)$. However, $N_P(H)$ and $N_Q(H)$ are Sylow p-subgroups of $N(H)$, as $P \cap Q$ is tame, so that $P_1 = N_P(H) \cap D$ and $Q_1 = N_Q(H) \cap D$ are Sylow p-subgroups of D. But $C_P(H) \leqslant P_1$; so $P_1 > H$. We now choose $x \in D$ such that $Q_1^x = P_1$. We may assume that $x \in C(H)$ as $D = HC(H)$ and $H \leqslant Q_1$. Therefore, $Q \sim Q^x$ since $(H, C(H)) \in F$ and $x \in C(H)$. Finally, again we have that

$$P \cap Q^x \geqslant P \cap Q_1^x = P \cap P_1 = P_1 > H,$$

and the proof is complete.

Let F_c' be the family of all pairs $(H, N(H))$ where H is a subgroup of P such that there exists a Sylow p-subgroup Q of G with $H = P \cap Q$ a tame intersection and with $C_P(H) \leqslant H$. We can then prove

THEOREM 5.2. *F_c' is a weak conjugation family.*

Proof. Let A and B be subsets of P with $B = A^g$ for g in G. By the second form of the Main Theorem and Theorem 5.1 we know that F_c is a conjugation family. Therefore, choose elements $(H_i, T_i) \in F_c$, $1 \leqslant i \leqslant n$, and elements x_i, y, $1 \leqslant i \leqslant n$ in accordance with the definition of conjugation family such that $g = x_1 \cdots x_n y$. Let $i_1, \ldots, i_m$ be the integers i, $1 \leqslant i \leqslant n$, in increasing order, such that $x_i \notin C(H_i)$. For such i, $(H_i, T_i) \in F_c'$ since $T_i \nleqslant C(H_i)$ implies that $C_P(H_i) \leqslant H_i$.

In order to complete the proof it will suffice to choose elements (K_j, U_j) of F_c' and elements w_j, v of G where $1 \leqslant j \leqslant m$ and m is as above, such that

(a) $A^{u_1 \cdots u_m v} = B$,

(b) $w_j \in U_j$, $1 \leqslant j \leqslant m$, $v \in N(P)$,

(c) $A \leqslant K_1$, $A^{w_1 \cdots w_j} \leqslant K_{j+1}$, $1 \leqslant j \leqslant m - 1$.

We define

$$K_j = H_{i_j}, \qquad U_j = T_{i_j}, \qquad w_j = x_{i_j}, \qquad v = y, \qquad 1 \leqslant j \leqslant m,$$

and claim that these satisfy the above conditions. First of all, we have just seen that $(K_j, U_j) \in F_c'$ and certainly (b) is fulfilled. As for (a) and (c), they both follow from the fact that

$$A^{w_1 \cdots w_j} = A^{x_1 \cdots x_{i_j}} \quad \text{for all} \quad j, \quad 1 \leqslant j \leqslant m.$$

Indeed,

$$A^{x_1 \cdots x_{i_1}} = A^{x_{i_1}} = A^{w_1}$$

as $x_1, \ldots, x_{i_1-1}$ all centralize A, and the rest follows by an easy induction and a similar argument.

Two consequences of this are as follows:

COROLLARY 5.3. *If $Z(P)$ is a weakly closed subgroup of the Sylow p-subgroup P of the group G then two elements of P are conjugate in G if and only if they are conjugate in $N(Z(P))$.*

Proof. Let a and b be elements of P such that $a^g = b$ for g in G. We apply the previous theorem and choose elements (H_i, T_i) of F_c', $1 \leqslant i \leqslant n$ and elements $x_1, \ldots, x_n, y$ of G fulfilling the definition of a weak conjugation family with $A = a$ and $B = b$ so that $a^{x_1 \cdots x_n y} = b$. We claim that each x_i and y are elements of $N(Z(P))$; this will suffice for the proof. Indeed, certainly $y \in N(Z(P))$ as $N(P) \leqslant N(Z(P))$, while $x_i \in N(H_i)$. But $C_P(H_i) \leqslant H_i$, so $Z(P) \leqslant H_i$. Thus $Z(P)^{x_i} \leqslant H_i \leqslant P$, so $Z(P)^{x_i} = Z(P)$ since $Z(P)$ is weakly closed in P.

COROLLARY 5.4 (Grun's second theorem). *If $Z(P)$ is a weakly closed subgroup of the Sylow p-subgroup P of the group G then $P \cap G' = P \cap N(Z(P))'$.*

Proof. We know, by the focal subgroup theorem, that $P \cap G'$ is generated by all the elements $a^{-1}b$ where $a, b \in P$ and a and b are conjugate in G. However, by the previous result, a and b are conjugate in $N(Z(P))$ so $a^{-1}b \in P \cap N(Z(P))'$. Thus $P \cap G' \leqslant P \cap N(Z(P))'$. Finally, the reverse inclusion is obvious and the corollary is proved.

We remark that the last two results are easy to prove directly; we have just illustrated our methods. The applications of section four, as generalized for F_c and F_c' are more interesting.

We shall conclude this section with one further example of a weak conjugation family. This result is motivated by Theorem 4.7, the theorem of Frobenius. Let F_p be the family of all $(H, N(H))$ where H is a subgroup of P for which $N(H)/C(H)$ is not a p-group or $H = P$. We then have the

THEOREM 5.5. *F_p is a weak conjugation family.*

Proof. In view of the Main Theorem in section three and the definition of a weak conjugation family it suffices to prove the following assertion: If A

and B are subsets of a subgroup H of P for which $N_P(H)$ is a Sylow p-subgroup of $N(H)$ and A and B are conjugate in $N(H)$, then there is a subgroup K of P such that $H \leqslant K$, $(K, N(K)) \in F_p$ and A and B are conjugate in $N(K)$. Indeed, this being the case, the sequence of conjugations performed in the Main Theorem can be replaced by a sequence of conjugations given by F_p. However, if $N(H)/C(H)$ is not a p-group then $(H, N(H)) \in F_p$ and we may take $K = H$. Suppose that $N(H)/C(H)$ is a p-group. Since $N_P(H)$ is a Sylow p-subgroup of $N(H)$ we then have that $N(H) = C(H) N_P(H)$. As $C(H)$ centralizes A, it follows that A and B are conjugate in $N_P(H)$ and so they are certainly conjugate in P. Thus, in this case we take $K = P$ and the proof is complete.

6. Concluding Remarks

It seems not to be clear whether there is a relation between the Hall-Wielandt theorem ([7], p. 211) and the methods developed here. Both results lead to many of the same applications. However, the Hall-Wielandt theorem is based upon a detailed analysis of transfer while we need only the relatively weak focal subgroup theorem. Perhaps some synthesis of these two approaches is possible; this would certainly be desirable.

The role of Sylow intersections seems to be crucial in the above results. These intersections appear again and again in group theory and it is natural to ask if there are connections between our work and these other results. For example, defect groups are always tame intersections [6]. However, it is not true that the family $(D, N(D))$ of all defect groups D contained in a Sylow p-subgroup P is a conjugation family or even a weak conjugation family, as the symmetric group on four letters shows for $p = 2$.

Nevertheless, it is reasonable to expect further relations to be established between Sylow intersections, fusion of p-elements, and transfer.

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