A FEYNMAN INTEGRAL VIA HIGHER NORMAL FUNCTIONS

by

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Abstract. —
We study the Feynman integral for the three-banana graph defined as the scalar two-point self-energy at three-loop order. The Feynman integral is evaluated for all identical internal masses in two space-time dimensions. Two calculations are given for the Feynman integral; one based on an interpretation of the integral as an inhomogeneous solution of a classical Picard-Fuchs differential equation, and the other using arithmetic algebraic geometry, motivic cohomology, and Eisenstein series. Both methods use the rather special fact that the Feynman integral is a family of regulator periods associated to a family of $K3$ surfaces. We show that the integral is given by a sum of elliptic trilogarithms evaluated at sixth roots of unity. This elliptic trilogarithm value is related to the regulator of a class in the motivic cohomology of the $K3$ family. We prove a conjecture by David Broadhurst that at a special kinematical point the Feynman integral is given by a critical value of the Hasse-Weil L-function of the $K3$ surface. This result is shown to be a particular case of Deligne's conjectures relating values of L-functions inside the critical strip to periods.

IPHT-T/14/015, IHES/P/14/06.

Contents

1. Introduction

The computation of scattering amplitudes in quantum field theory requires the evaluation of Feynman integrals. This is a non-trivial task for which many techniques have been developed by physicists over the years (cf. the reviews [**BDK, Bri, EKMZ, EH**].) Feynman integrals are multivalued functions of the physical parameters, given by the external momenta and internal masses. Differentiating with respect to the physical parameters leads to a first order system of differential equations as in e.g. [**H, CHH**] or to higher order differential equations as in e.g. [**LR, MSWZ, MSWZ2, Va, ABW, ABW2**].

The Feynman integral associated to a graph Γ with n edges (propagators) is an integral over the positive simplex $\Delta_n := \{[x_1 : \cdots : x_n] \in \mathbb{P}^{n-1}(\mathbb{R}) \,|\, x_i \geq 0\}$ in projective $(n-1)$-space of a meromorphic differential $(n-1)$-form:

$$I_\Gamma = \int_{\Delta_n} \Omega_\Gamma \,. \tag{1.1}$$

The form Ω_Γ depends on the physical parameters – that is, the external momenta and internal masses attached to the graph – and is expressed in terms of the first and second Symanzik polynomial [**IZ**]. The variables x_i are the Schwinger proper times indexed by edges (propagators).

For the algebro-geometric approach of [**BEK**], the Feynman integral I_Γ is a period of the mixed Hodge structure on the relative cohomology group $H^{n-1}(\mathbb{P}^{n-1}\backslash X_\Gamma, B\backslash(B\cap X_\Gamma))$, where X_Γ is the graph hypersurface defined by the poles of Ω_Γ and B is a blow-up of the simplex Δ_n. Varying the physical parameters leads to a variation of the Hodge structure. As a result, the Feynman integral satisfies a set of first order differential equations under the action of the the Gauss-Manin connection [**G**], leading to an inhomogeneous Picard-Fuchs equation. The inhomogeneous term has its origin in the extension of mixed Hodge structure associated with Feynman graphs. The dependence on external momenta means that we have a family of extensions, also known as a *normal function* from the work of Poincaré [**P**] and Griffiths [**G2**].

This point of view enables us to bring to bear a number of techniques including Picard-Fuchs differential equations, motivic cohomology and regulators, Eisenstein series, and Hodge structures, for the analysis of the properties of Feynman integrals.

The main topic of this paper is the evaluation of the Feynman integral for the three-banana graph

$$I_{\ominus}(t) := \int_{x_1,x_2,x_3\geq 0} \frac{1}{(1+\sum_{i=1}^3 x_i)(1+\sum_{i=1}^3 x_i^{-1}) - t} \prod_{i=1}^{3} \frac{dx_i}{x_i}\,. \tag{1.2}$$

The associated graph hypersurface $X_{\ominus}(t) := \{(1+\sum_{i=1}^3 x_i)(1+\sum_{i=1}^3 x_i^{-1}) - t = 0\}$ leads to a family of $K3$ surfaces with (generic) Picard number 19, over the modular curve $\mathbb{P}^1 \setminus \{0,4,16,\infty\} \cong Y_1(6)^{+3}$. It is closely related to the family of elliptic curves over $Y_1(6)$, which was studied in [**BV**] in connection with the Feynman integral arising from the sunset (two-loop banana) graph.

We prove in theorems 2.3.2 and 5.3.1 that the Feynman integral evaluates to the product of a period $\varpi_1(\tau)$ of the $K3$ surface and an Eichler integral of an Eisenstein series. Explicitly, we have

$$I_{\ominus}(t) = \varpi_1(\tau)\left(\sum_{n\geq 1} \frac{\psi(n)}{n^3}\frac{q^n}{1-q^n} - 4(\log q)^3 + 16\zeta(3)\right), \tag{1.3}$$

where $q = \exp(2\pi i\tau)$, $\psi(n)$ is a mod-6 character given in eq. (2.3.24), and t is related to τ by the Hauptmodul (2.3.11) for $\Gamma_1(6)^{+3}$.

Remarkably, the Eichler integral factor can be expressed as a combination of the Beilinson-Levin elliptic trilogarithms [**BL, L, Z**]

$$I_{\ominus}(t) = \varpi_1(\tau)\Big(40\pi^2 \log q + 24\mathcal{L}i_3(\tau,\zeta_6) + 21\,\mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1)\Big) \tag{1.4}$$

where $\zeta_6 := \exp(i\pi/3)$ is the same sixth root of unity that enters the expression of the sunset integral studied in [**BV**].

It turns out that the three-banana integral is associated to a *generalized normal function* arising from a family of "higher" algebraic cycles or motivic cohomology classes [**KL, DK**]. The passage from classical normal functions associated to families of cycles to normal functions associated to motivic classes suggests interesting new links between mathematics and physics (op.cit.). Actually motivic normal functions can, in many cases, be associated with multiple-valued holomorphic functions which arise as amplitudes as in this work or in the context of open mirror symmetry as in [**MW**] for instance.

The plan of the paper is the following. In section 2 we derive the inhomogeneous Picard-Fuchs equation satisfied by the three-banana integral. The solution of the differential equation in terms of the elliptic trilogarithm is given in theorem 2.3.2. In section 3 we give a construction of the family of $K3$ surfaces associated with the three-banana graph.

In section 4 we show that the three-banana integral $I_{\ominus}(t)$ is an higher normal function, originating from a family of elements in $K_3(K3's)$ (a charming sort of mathematical eponym). Specifically, we show that the Milnor symbols $\{-x_1, -x_2, -x_3\} \in K_3^M\left(\mathbb{C}(X_{\ominus}(t))\right)$ extend to classes $\Xi_t \in H^3_M(X_{\ominus}(t), \mathbb{Q}(3))$. We construct a family of closed 2-currents $\tilde{R}_t$ representing the Abel-Jacobi classes $AJ(\Xi_t) \in H^2(X_{\ominus}(t), \mathbb{C}/\mathbb{Q}(3))$, and a family of holomorphic forms $\tilde{\omega}_t \in \Omega^2(X_{\ominus}(t))$, such that

$$I_{\ominus}(t) = \int_{X_{\ominus}(t)} \tilde{R}_t \wedge \tilde{\omega}_t$$

(Theorem 4.3.2). This has immediate consequences, including a conceptual proof of the inhomogeneous Picard-Fuchs equation for $I_{\ominus}(t)$ (Corollary 4.3.3).

In section 5 we pull the higher cycle Ξ_t back from the family of $K3$ surfaces to a modular Kuga 3-fold, where we are able to recognize it as an *Eisenstein symbol* in the sense of Beilinson. Applying a general computation (Theorem 5.1.1ff) of higher normal functions associated to Beilinson's cycles, gives a "motivic" proof (Theorem 5.3.1) that the three-banana integral $I_{\ominus}(t)$ takes the form claimed in (1.3)-(1.4). In section 6 we give the abstract Hodge-theoretic formulation of the Feynman integral in our case.

Finally, in sections 2.4 and theorem 7.2.1 we show that the integral at $t = 0$ takes the value $I_{\ominus}(0) = 7\zeta(3)$ recovering at result of [**BBDG, Broad1, Broad2**]. And in sections 2.5 and 7.1.1 we evaluate the three-banana at the special value $t = 1$. (The results in section 7 again make crucial use of Theorem 4.3.2.) We show the regulator to be trivial, which means that the Feynman integral is actually a classical rational period of the $K3$ up to a factor of $12\pi i/\sqrt{-15}$. A conjecture of Deligne then relates the Feynman integral to the critical value of the Hasse-Weil L-function of the $K3$ at $s = 2$. This proves a result first obtained numerically by Broadhurst in [**Broad1, Broad2**].

Acknowledgements

We thank A. Clingher and C. Doran for helpful discussions. MK thanks the IHÉS for support and good working conditions while part of this paper was written. We acknowledge support from the ANR grant reference QFT ANR 12 BS05 003 01, and the PICS 6076, and partial support from NSF Grant DMS-1068974.

2. The three-banana Feynman integral

2.1. The integral. — We look at the three-loop banana graph in two space-time dimensions associated with the Feynman graph in figure 2.1.1

$$I_{\ominus}(m_1, m_2, m_3, m_4; K) := \int_{\mathbb{R}^8} \frac{\delta(\sum_{i=1}^4 \ell_i + K) \prod_{i=1}^4 d^2\ell_i}{\prod_{i=1}^4(\ell_i^2 + m_i^2)} \,. \tag{2.1.1}$$

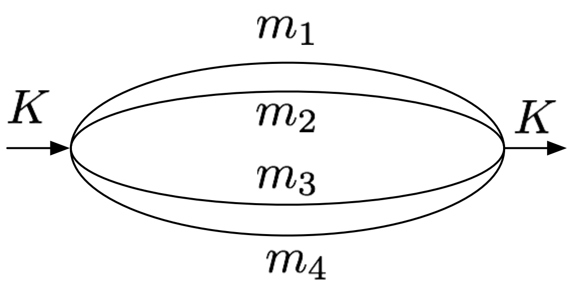


Figure 2.1.1. The three-loop three-banana Feynman graph. K is the external momentum in $\mathbb{R}^2$ and $m_i \geq 0$ with $i = 1, \dots, 4$ are internal masses.

Setting $t = K^2$, this integral can be equivalently represented as (see for instance [**Va**, section 8])

$$I_{\ominus}(m_i; t) = \int_{x_i \geq 0} \frac{1}{(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) - t} \prod_{i=1}^3 \frac{dx_i}{x_i} \tag{2.1.2}$$

Theorem 2.1.1. — *The integral $I_{\ominus}(m_i; t)$ defined in eq* (2.1.2) *has the following integral representation for $t < (\sum_{i=1}^4 m_i)^2$*

$$I_{\ominus}(m_i; t) = 2^3 \int_0^\infty x\, I_0(\sqrt{t}x) \prod_{i=1}^4 K_0(m_i x)\, dx\,. \tag{2.1.3}$$

The Bessel functions K_0, I_0 are defined by

$$K_0(2\sqrt{ab}) := \frac{1}{2} \int_0^\infty e^{-ax - \frac{b}{x}} \frac{dx}{x}; \qquad \text{for } a, b > 0\,, \tag{2.1.4}$$

and

$$I_0(x) := \sum_{k \geq 0} \left(\frac{x}{2}\right)^{2k} \frac{1}{\Gamma(k+1)^2}\,. \tag{2.1.5}$$

For the all equal mass case this Bessel representation has already been given in [**BBDG, Broad2**].

Proof. — For $t < (\sum_{i=1}^4 m_i)^3$ we can perform the series expansion

$$I_{\ominus}(m_i; t) = \sum_{k \geq 0} t^k I_k \tag{2.1.6}$$

with

$$I_k := \int_{x_i \geq 0} \frac{1}{(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)^{k+1}(1 + \sum_{i=1}^3 x_i^{-1})^{k+1}} \prod_{i=1}^3 \frac{dx_i}{x_i} \tag{2.1.7}$$

Exponentiating the denominators using $\int_0^\infty dx x^k \exp(-ax) = \Gamma(k+1)/a^{k+1}$ for $a > 0$ we have

(2.1.8)

$$I_k = \frac{1}{\Gamma(k+1)^2} \int_{x_i \geq 0} \int_{u,v \geq 0} e^{-u(1+\sum_{i=1}^3 x_i^{-1}) - v(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)} \frac{du dv}{(uv)^{-k}} \prod_{i=1}^3 \frac{dx_i}{x_i} \,.$$

Using the definition in (2.1.4) the integral over each x_i leads to a $K_0(x)$ Bessel function, therefore

$$I_k = \frac{2^3}{\Gamma(k+1)^2} \int_{u,v \geq 0} e^{-u - v m_4^2} \prod_{i=1}^3 K_0(2\sqrt{uv} m_i) \frac{du dv}{(uv)^{-k}} \,. \tag{2.1.9}$$

Changing variables $(u, v) \to (x = 2\sqrt{uv}, v)$ then

$$\begin{aligned} I_k &= \frac{2^4}{\Gamma(k+1)^2} \int_{v,x \geq 0} e^{-\frac{x^2}{4v} - v m_4^2} \prod_{i=1}^3 K_0(2\sqrt{uv} m_i) \left(\frac{x}{2}\right)^{2k+2} \frac{dx dv}{xv} \\ &= \frac{2^5}{\Gamma(k+1)^2} \int_0^{+\infty} \prod_{i=1}^4 K_0(m_i x) \left(\frac{x}{2}\right)^{2k+2} \frac{dx}{x} \end{aligned} \tag{2.1.10}$$

Now we can perform the summation over k using the series expansion of the Bessel function $I_0(\sqrt{t}\, x)$ given in (2.1.5) to conclude the proof. □

For the all equal masses case $m_1 = m_2 = m_3 = m_4 = 1$ we have

$$\begin{aligned} I_{\ominus}(t) &:= \int_{x_i \geq 0} \frac{1}{(1 + \sum_{i=1}^3 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) - t} \prod_{i=1}^3 \frac{dx_i}{x_i} \\ &= 2^3 \int_0^\infty x I_0(\sqrt{t}\, x) K_0(x)^4 \, dx \,. \end{aligned} \tag{2.1.11}$$

2.2. The Picard-Fuchs equation. — In this section we show the three-loop banana integral $I_{\ominus}(t)$ satisfies an inhomogeneous Picard-Fuchs equation given in [**MSWZ2, Va**], following the derivation given in [**Va**] for the equal masses banana graphs at all loop orders.

Theorem 2.2.1. — *The three-loop banana integral*

$$I_{\ominus}(t) = \int_{x_i \geq 0} \frac{1}{(1 + \sum_{i=1}^{3} x_i)(1 + \sum_{i=1}^{3} x_i^{-1}) - t} \prod_{i=1}^{3} \frac{dx_i}{x_i} \tag{2.2.1}$$

satisfies the inhomogeneous Picard-Fuchs equation $\mathcal{L}_t^3 I_{\ominus}(t) = -24$ *with the Picard-Fuchs operator* $\mathcal{L}_t^3$ *given by*

(2.2.2)

$$\mathcal{L}_t^3 := t^2(t-4)(t-16)\frac{d^3}{dt^3} + 6t(t^2-15t+32)\frac{d^2}{dt^2} + (7t^2-68t+64)\frac{d}{dt} + t - 4\,.$$

This Picard-Fuchs operator already appeared in the work by Verrill in [**Ve**] and [**MSWZ**]. We will comment on the relation to this work in §3.2.

Proof. — We consider the Bessel integral representation of the previous section

$$I_{\ominus}(t) = \sum_{k \geq 0} t^k I_k \tag{2.2.3}$$

where I_k is given by (2.1.10) with $m_1 = m_2 = m_3 = m_4 = 1$

$$I_k = \frac{2^4}{\Gamma(k+1)^2} \int_0^{+\infty} \left(\frac{x}{2}\right)^{2k+1} K_0(x)^4\, dx\,. \tag{2.2.4}$$

Then the action of the Picard-Fuchs operators on this series expansion gives

$$\mathcal{L}_t^3 I_{\ominus}(t) = \sum_{k \geq 0} \left(t\alpha_k + \beta_k + \frac{\gamma_k}{t}\right) t^k I_k \tag{2.2.5}$$

therefore

$$\mathcal{L}_t^3 I_{\ominus}(t) = \frac{\gamma_0 I_0}{t} + \gamma_1 I_1 + \beta_0 I_0 + \sum_{k \geq 1} (\alpha_k I_k + \beta_{k+1} I_{k+1} + \gamma_{k+2} I_{k+2})\, t^k \tag{2.2.6}$$

Using the result of the lemma 2.2.2 below, we have $\mathcal{L}_t^3 I_{\ominus}(t) = \gamma_1 I_1 + \beta_0 I_0$. Evaluating the integrals gives that $\gamma_1 I_1 + \beta_0 I_0 = -24$, which proves the theorem.

□

Lemma 2.2.2. — *The Bessel moment integrals*

$$I_k = \frac{2^4}{\Gamma(k+1)^2} \int_0^{+\infty} \left(\frac{x}{2}\right)^{2k+1} K_0(x)^4 \, dx \tag{2.2.7}$$

satisfy the recursion relation

$$\alpha_k I_k + \beta_{k+1} I_{k+1} + \gamma_{k+2} I_{k+2} = 0, \qquad k \geq 0 \tag{2.2.8}$$

with for $k \geq 0$

$$\begin{aligned} \alpha_k &:= (k+1)^3 \\ \beta_k &:= -2(2k+1)\,(5k^2+5k+2) \\ \gamma_k &:= 64k^3 \,. \end{aligned} \tag{2.2.9}$$

Proof. — The proof has been given in [**BS**, Example 6] (see [**O**] for related considerations). Following this reference we introduce the Bessel moment integrals $c_{4,2k+1} = 2^{2k-3}\Gamma(k+1)^2\, I_k$. One notices that $K_0(x)^4$ satisfies the differential equation $L_5 K_0(x)^4 = 0$ where
(2.2.10)

$$L_5 := \left(x\frac{d}{dx}\right)^5 - 20x^2 \left(x\frac{d}{dx}\right)^3 - 60x^2 \left(x\frac{d}{dx}\right)^2 + 8x^2(8x^2-9)\left(x\frac{d}{dx}\right) + 32x^2(4x^2-1)\,.$$

And finally one notices the identities

$$\int_0^{+\infty} x^{k+j} \left(x\frac{d}{dx}\right)^m \left(K_0(x)^4\right) \, dx = (-1-k-j)^m \, c_{4,k+j}\,. \tag{2.2.11}$$

Therefore integrating term by term the expression

$$\int_0^{+\infty} t^{2k+1}\, L_5 K_0(x)^4 \, dx = 0 \tag{2.2.12}$$

leads to the recursion (2.2.8). □

2.3. Solution of the inhomogeneous Picard-Fuchs equation. — We need an intermediate result expressing the solution of the third order differential equation using the Wronskian method. Recall the Wronskian of a linear differential equation

$$f_n(x)y(x)^{(n)} + \ldots + f_1(x)y' + f_0(x)y = 0 \tag{2.3.1}$$

is the determinant $W(x) := \det(y_j^{(i)})$ where $y_1, \ldots, y_n$ are independent solutions. Viewing the equation (2.3.1) as a system of n first order equations, the Wronskian is the solution of the first order equation given by the determinant of the system. This yields the formula

$$W(t) = \exp(-\int^t f_{n-1}(x)/f_n(x)\,dx). \tag{2.3.2}$$

Consider the inhomogeneous differential equation

$$f_3(x)y'''(x) + f_2(x)y''(x) + f_1(x)y'(x) + f_0(x)y(x) = S(x) \tag{2.3.3}$$

Let $y_i(x)$ with $i = 1, 2, 3$ be three independent solutions of the homogeneous equation. Let

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{vmatrix} \tag{2.3.4}$$

be the Wronskian of these solutions, and introduce the modified Wronskian

$$\widetilde{W}(t,x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1(t) & y_2(t) & y_3(t) \end{vmatrix}. \tag{2.3.5}$$

We have the following identities

$$\widetilde{W}(t,t) = 0; \quad \frac{\partial \widetilde{W}(t,x)}{\partial t}|_{x=t} = 0; \quad \frac{\partial^2 \widetilde{W}(t,x)}{\partial t^2}|_{x=t} = W(t) \tag{2.3.6}$$

$$\sum_{i=0}^{3} f_i(t) \frac{\partial^i}{\partial t^i} \widetilde{W}(t,x) = 0 \tag{2.3.7}$$

A simple computation now yields the general solution for the inhomogeneous equation (2.3.3)

$$y(t) = \sum_{i=1}^{3} \alpha_i \, y_i(t) + \int_0^t \widetilde{W}(t,x) \, \frac{S(x) \, dx}{W(x) f_3(x)} \, . \tag{2.3.8}$$

For the three-banana graph, the Picard-Fuchs operators in (2.2.2) has $f_3(x) = x^2(x-4)(x-16)$ and $f_2(x) = 6x(x^2 - 15x + 32) = \frac{3}{2}\frac{df_3(x)}{dx}$, therefore the Wronskian is given by

$$W(t) = \exp\left(-\int^t \frac{f_2(x)}{f_3(x)} \, dx\right) = \frac{W_0}{(t^2(t-4)(t-16))^{\frac{3}{2}}} \, . \tag{2.3.9}$$

The arbitrary normalization W_0 of the Wronskian is be determined in (2.3.18). We now use the fact showed in [**Ve**, theorem 3], and reviewed in §3.2, that Picard-Fuchs operator is a symmetric square of the sunset Picard-Fuch operator. For $t \in \mathbb{P}^1 \backslash \{0, 4, 16, \infty\}$ the solutions of the homogenous equations are given by

$$(y_1(t), y_2(t), y_3(t)) = \varpi_1(\tau) \, (1, 2\pi i \tau, (2\pi i \tau)^2) \, . \tag{2.3.10}$$

In this expression $\varpi_1(\tau)$ is a period and τ is the period ratio. The parameter t is the Hauptmodul given by [**Ve**]

$$t(\tau) = H_{\ominus}([\tau]) = -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 \, . \tag{2.3.11}$$

We recall that the Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = \exp(\pi i \tau/12) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n \tau)) \tag{2.3.12}$$

The special values of the Hauptmodul $t = \{0, 4, 16, +\infty\}$ are obtained for the values of $\tau = \{0, \frac{-3+i\sqrt{3}}{12}, \frac{3+i\sqrt{3}}{6}, +i\infty\}$. The nature of the fibers for these values of the Hauptmodul are discussed in §3.2. The value $t = 4$ is the pseudo-threshold of the Feynman integral and the value $t = 16$ is the normal threshold of the Feynman integral.

In the neighborhood $|t| > 16$ of $t = \infty$ the holomorphic period is given by

$$\begin{aligned} \varpi_1(\tau) &= \frac{1}{(2\pi i)^3} \int_{|x_1|=|x_2|=|x_3|=1} \frac{1}{(1+\sum_{i=1}^3 x_i)(1+\sum_{i=1}^3 x_i^{-1}) - t} \prod_{i=1}^3 \frac{dx_i}{x_i} \\ &= -\sum_{n\geq 0} t^{-n-1} \frac{1}{(2\pi i)^3} \int_{|x_1|=|x_2|=|x_3|=1} (1+\sum_{i=1}^3 x_i)^n (1+\sum_{i=1}^3 x_i^{-1})^n \prod_{i=1}^3 \frac{dx_i}{x_i} \\ &= -\sum_{n\geq 0} t^{-n-1} \sum_{p+q+r+s=n} \left(\frac{n!}{p!q!r!s!}\right)^2 . \end{aligned} \tag{2.3.13}$$

Using the above expression for the Hauptmodul t, the period is expressed as

$$\varpi_1(\tau) := \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2} . \tag{2.3.14}$$

Expanding the modified Wronskian

$$\begin{aligned} \widetilde{W}(t,x) &= y_1(t)\, W_{23}(x) - y_2(t)\, W_{13}(x) + y_3(t)\, W_{12}(x) \\ &= \varpi_1 \left(W_{23}(x) - \tau(t)\, W_{13}(x) + \tau(t)^2\, W_{12}(x) \right) . \end{aligned} \tag{2.3.15}$$

and then evaluating yields

$$W_{12}(t) = 2\pi i \varpi_1^2 \frac{d\tau}{dt}, \quad W_{13}(t) = (2\pi i)^2 \varpi_1^2\, 2\tau \frac{d\tau}{dt}, \quad W_{23}(t) = (2\pi i)^3 \varpi_1^2\, \tau^2 \frac{d\tau}{dt} . \tag{2.3.16}$$

Thus

$$\widetilde{W}(t,x) = (2\pi i)^3 \varpi_1(\tau) \varpi_1(x)^2 \left(\tau(x) - \tau(t)\right)^2 \frac{d\tau}{dx} . \tag{2.3.17}$$

The condition

$$\partial_t^2 \widetilde{W}(t,x)\Big|_{x=t} = W(t) \tag{2.3.18}$$

determines the normalization $W_0 = 2$ of the Wronskian.

Therefore the tree-loop banana integral is given by

(2.3.19)
$$I_{\circleddash}(t) = I^{\rm period} - 12(2\pi i)^3 \varpi_1(t) \int_0^t (\tau(x)-\tau(t))^2 \, (x^2(x-4)(x-16))^{\frac12} \, \frac{d\tau(x)}{dx}\, dx\,.$$

where $I^{\rm period}$ is an homogeneous solution belonging to $\varpi_1(\tau)(\mathbb{C}+\tau\mathbb{C}+\tau^2\mathbb{C})$.

Lemma 2.3.1. — *Using the expressions for the Hauptmodul t and the period ϖ_1 then the function $\sigma(\tau) := -24\varpi_1(\tau)^2\,(t(\tau)^2(t(\tau)-4)(t(\tau)-16))^{\frac12}$ has the following representation*

(2.3.20)
$$\sigma(\tau) = \frac{1}{5}\,(-E_4(\tau)+16E_4(2\tau)+9E_4(3\tau)-144E_4(6\tau))$$

where $E_4(\tau)$ is the Eisenstein series

(2.3.21)
$$E_4(\tau) = \frac{1}{2\zeta(4)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^4} = 1+240\sum_{n\geq1} n^3 \frac{q^n}{1-q^n}$$

With $q := \exp(2\pi i\tau)$ the coefficients σ_n of the q-expansion

(2.3.22)
$$\sigma(\tau) = \sum_{n\geq 0} \sigma_n\, q^n$$

are given by $\sigma_0 = -24$ and

(2.3.23)
$$\sigma_n = n^3 \sum_{m|n} \frac{1}{m^3}\,\psi(m)$$

where $\psi(n+6)=\psi(n)$ is an even mod 6 character taking the values

(2.3.24)
$$\begin{aligned}\psi(1) &= -48, \quad \psi(2)=720, \quad \psi(3)=384,\\ \psi(4) &= 720, \quad \psi(5)=48, \quad \psi(6)=-5760\,.\end{aligned}$$

Proof. — The expression in (2.3.20) is obtained by performing a q expansion and verifying that the coefficients are the same to very high-order in the q-expansion using [**Sage**].

The expression for the Fourier coefficients in (2.3.23) are easily obtained by using that

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n \tag{2.3.25}$$

where $\sigma_3(n) = \sum_{m|n} m^3$ is the divisor sum, and a reorganization of the q-expansion mod 6. □

Recall the polylogarithm functions $Li_r(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^r}$.

Theorem 2.3.2. — *The integral $I_{\ominus}(t)$ in* (2.1.11) *with t given in* (2.3.11), *is given by the following function of q*

$$I_{\ominus}(t(\tau)) = \varpi_1(\tau) \left(16\zeta(3) + \sum_{n\geq 1} \frac{\psi(n)}{n^3} \frac{q^n}{1-q^n} - 4(\log q)^3 \right) . \tag{2.3.26}$$

with $\varpi_1(\tau)$ the period in (2.3.14) *and ψ the even mod 6 character with the values given in* (2.3.24). *This integral can be expressed as linear combination of the elliptic trilogarithms introduced by Beilinson and Levin* [**BL, L, Z**].

$$I_{\ominus}(t(\tau)) = \varpi_1(\tau)(40\pi^2 \log q - 48\mathcal{H}_{\ominus}(\tau)) \tag{2.3.27}$$

where

$$\mathcal{H}_{\ominus}(\tau) := 24\mathcal{L}i_3(\tau,\zeta_6) + 21\, \mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1) \tag{2.3.28}$$

with $\mathcal{L}i_3(\tau,z)$ defined by

$$\begin{aligned}\mathcal{L}i_3(\tau,z) := {}& Li_3\left(z\right) + \sum_{n\geq 1} (Li_3\left(q^n z\right) + Li_3\left(q^n z^{-1}\right)) \\ & - \left(-\frac{1}{12}(\log z)^3 + \frac{1}{24} \log q\, (\log z)^2 - \frac{1}{720} (\log q)^3 \right) .\end{aligned} \tag{2.3.29}$$

Proof. — In order to prove the theorem we just evaluate the integral in (2.3.19). We perform the change of variables $2\pi i\tau(t) = \log q$ and

$2\pi i\tau(x) = \log\hat{q}$ to get

$$I_{\ominus}(t) = I^{\text{period}} + \frac{1}{2}\varpi_1(t)\int_1^q \left(\log\frac{\hat{q}}{q}\right)^2 \sigma(\hat{q})\, d\log\hat{q}\,. \tag{2.3.30}$$

(Here we used that $t = 0$ for $\tau = 0$, and I^{period} is a solution of the homogenous Picard-Fuchs equation in $\varpi_1(\tau)(\mathbb{C} + \tau\,\mathbb{C} + \tau^2\,\mathbb{C})$.) The form of the homogenous solution is determined in (2.3.39).

Using the q-expansion for $\sigma(\tau)$ and the following integrals

$$\begin{aligned}\int_1^q \left(\log\frac{\hat{q}}{q}\right)^2 \hat{q}^n d\log\hat{q} &= \frac{2(q^n-1) - 2n\log q - n^2(\log q)^2}{n^3}\\ \int_1^q \log\left(\frac{\hat{q}}{q}\right)^2 d\log\hat{q} &= \frac{(\log q)^3}{3}\,.\end{aligned} \tag{2.3.31}$$

Summing all the terms we find that

$$\begin{aligned} I_{\ominus}(t(\tau)) = I^{\text{period}} & \\ &+ \varpi_1(\tau)\left(\frac{\sigma_0}{6}(\log q)^3 + \sum_{n\geq 1}\frac{\sigma_n}{n^3}\left(q^n - \frac{1}{2}\left(1+\log(q^n)\right)^2\right)\right)\end{aligned} \tag{2.3.32}$$

This leads to

$$I_{\ominus}(t(\tau)) \;=\; I^{\text{period}} \;+\; \frac{\sigma_0}{6}\varpi_1(\tau)(\log q)^3 \;+\; \varpi_1(t)\sum_{n\geq 1}\frac{\sigma_n}{n^3}\,q^n\,. \tag{2.3.33}$$

We remark that the expression for the coefficient σ_n in (2.3.23) can be expressed in term of the sixth root of unity $\zeta_6 = \exp(i\pi/3)$

$$\sigma_n = -48n^3\left(\sum_{r=1}^{6} c_r \sum_{m|n}\frac{1}{m^3}\zeta_6^{rm}\right) \qquad n \geq 1 \tag{2.3.34}$$

with $c_r = \{24, 21, 16, 21, 24, 14\}$. This allows to express the q-expansion

$$\frac{\sigma_0}{6}(\log q)^3 + \sum_{n\geq 1}\frac{\sigma_n}{n^3}\,q^n = -48\mathcal{H}_{\ominus}(\tau) \tag{2.3.35}$$

where

$$\mathcal{H}_{\ominus}(\tau) := 24\mathcal{L}i_3(\tau,\zeta_6) + 21\,\mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1) \tag{2.3.36}$$

is expressed in terms of the elliptic trilogarithms $\mathcal{L}i_3(\tau, z)$ of Beilinson and Levin [**BL, L**] defined by

$$\mathcal{L}i_3(\tau,z) := \mathrm{Li}_3(z) + \sum_{n\geq 1}\left(\mathrm{Li}_3(q^n z) + \mathrm{Li}_3\left(q^n z^{-1}\right)\right) - \left(-\frac{1}{12}(\log z)^3 + \frac{1}{24}\log q\,(\log z)^2 - \frac{1}{720}(\log q)^3\right). \tag{2.3.37}$$

Therefore the three-loop banana integral is given by a sum of elliptic trilogarithms modulo periods solutions of the homogeneous Picard-Fuchs equation

$$I_{\ominus}(t(\tau)) = \varpi_1(\tau)(\alpha_1 + \alpha_2\tau + \alpha_3\tau^2) - 48\mathcal{H}_{\ominus}(\tau) \tag{2.3.38}$$

where we have expressed the homogeneous solution I^{period} as $\varpi_1(\tau)(\alpha_1 + \alpha_2\tau + \alpha_3\tau^2)$ with α_1, α_2 and α_3 arbitrary complex numbers. Using [**Sage**] we have numerically evaluated the integral and the elliptic trilogarithms at the particular values given in table 1. This leads to the determinations of the constants

$$\alpha_1 = \alpha_3 = 0; \qquad \alpha_2 = 40\pi^2\,. \tag{2.3.39}$$

This proves the representation in (2.3.27) for the three-loop banana integral.

Using the series expansion for the trilogarithms we have

$$\sum_{n\geq 1}\frac{\sigma_n}{n^3}\,q^n = \sum_{n\geq 1}\frac{\psi(n)}{n^3}\,\frac{q^n}{1-q^n} \tag{2.3.40}$$

with $\psi(n)$ given in (2.3.24). One can rewrite the sum over all non zero n

$$I_{\ominus}(t(\tau)) = \varpi_1(\tau)\left(\sum_{n\geq 1}\frac{\psi(n)}{n^3}\,\frac{q^n}{1-q^n} - 4(\log q)^3 + 16\zeta(3)\right) \tag{2.3.41}$$

This proves the expression in (2.3.26) and the theorem. □

τ	$\frac{-3+i\sqrt{3}}{12}$
$I_{\ominus}(t)$	9.109181165853514
$-48\mathcal{H}_{\ominus}(\tau)$	$347.868145888636 + 637.725764198092i$
$\varpi_1(\tau)$	$-0.224110197194 - 0.388170248035i$
τ	$\frac{-3+i\sqrt{15}}{24}$
$t(\tau)$	1
$I_{\ominus}(t)$	8.570280443360948
$-48\mathcal{H}_{\ominus}(\tau)$	$404.292203809358 + 325.565905143148i$
$\varpi_1(\tau)$	$0.133813847482 - 0.518258802791i$
τ	$-(3 + 1.80224199747123i)^{-1}$
$t(\tau)$	$\frac{319}{80}$
$I_{\ominus}(t)$	9.106670607198028
$-48\mathcal{H}_{\ominus}(\tau)$	$355.272552751915 + 625.839953492151i$
$\varpi_1(\tau)$	$-0.206610686713 - 0.388422174005i$

TABLE 1. Numerical evaluations of the Hauptmodul $t(\tau)$ the three-loop banana integral $I_{\ominus}(t)$, the elliptic trilogarithm sum $-48\mathcal{H}_{\ominus}(\tau)$ and the period $\varpi_1(\tau)$.

Remark 2.3.3. — The integral expression in (2.3.19)

(2.3.42)

$$I_{\ominus}(t(\tau)) = (2\pi i)^3\varpi_1(\tau)\int_0^t (\tau(x)-\tau(t))^2\,\sigma(\tau(x))\,d\tau+\varpi_1\,(\mathbb{C}+\tau\,\mathbb{C}+\tau^2\mathbb{C})$$

shows that $I_{\ominus}(t(\tau))/\varpi_1(\tau)$ is an Eichler integral of the modular form $\sigma(\tau)$. Another proof of this will be given in §5 and in theorem 5.3.1.

Remark 2.3.4. — Using the Kronecker-regularization for the sum [**W**]

$$\sum_{m\in\mathbb{Z}} {}_e\frac{1}{m+n\tau} = -i\pi\frac{1+q^n}{1-q^n} \tag{2.3.43}$$

and that

$$16\zeta(3) + \sum_{n\geq 1}\frac{\psi(n)}{n^3}\frac{q^n}{1-q^n} = \frac{1}{2}\sum_{n\geq 1}\frac{\psi(n)}{n^3}\frac{1+q^n}{1-q^n} \tag{2.3.44}$$

we conclude that

$$\frac{I_{\ominus}(t)}{\varpi_1(\tau)} + 4(\log q)^3 = -\frac{1}{2\pi i}\sum_{n\geq 1}\sum_{m\in\mathbb{Z}} {}_e\frac{\psi(n)}{n^3}\frac{1}{m+n\tau}\,, \tag{2.3.45}$$

which can be rewritten as a converging sum

$$\frac{\tau}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}}\frac{\psi(n)}{n^2}\frac{1}{(m+n\tau)(m-n\tau)}\,. \tag{2.3.46}$$

It is immediate from this that $I_{\ominus}(t)/\varpi_1(\tau)$ is antisymmetric under the transformation $\tau\to-\tau$.

2.4. Value of the integral at $t=0$. — We evaluate the integral at $t=0$ which corresponds to $\tau=0$. We have

$$I_{\ominus}(0) = \lim_{\tau\to 0}\varpi_1(\tau)\left(-4(\log q)^3 + \frac{\tau}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}}\frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau)^2}\right)\,. \tag{2.4.1}$$

since $\lim_{\tau\to 0}\varpi_1(\tau)\sim(48\tau^2)^{-1}$, we therefore have to evaluate the following limit

$$I_{\ominus}(0) = \frac{1}{48}\lim_{\tau\to 0}\tau^{-2}\,\frac{\tau}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}}\frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau)^2}\,. \tag{2.4.2}$$

For that we use the series representation and perform a Poisson summation on the integer n. We start by rewriting the sum as

$$\frac{\tau}{2\pi i}\sum_{\substack{n\neq 0\\ m\geq 1}}\frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau)^2} = \frac{\tau^3}{2\pi i}\sum_{\substack{n\neq 0\\ m\geq 1}}\psi(n)\left(\frac{1}{n^4m^2\tau^2}+\frac{1}{m^2(m^2-(n\tau)^2)}\right)$$

$$= \frac{\tau^3}{2\pi i} \sum_{\substack{n\in\mathbb{Z}, n\neq 0\\ m\geq 1}} \frac{\psi(n)}{m^2} \frac{1}{m^2-(n\tau)^2}\,. \tag{2.4.3}$$

where we used $\sum_{n\geq 1}\psi(n)/n^4 = 0$. Therefore

(2.4.4)
$$\frac{\tau}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}} \frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau)^2} = \frac{\tau^3}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ m\geq 1}} \frac{\psi(n)}{m^2}\frac{1}{m^2-(n\tau)^2} + \frac{5760\,\tau^3}{2\pi i}\zeta(4)\,.$$

We perform a Poisson summation on n to get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{1}{m^2+((r+6n)\tau)^2} &= \sum_{\hat n\in\mathbb{Z}} \int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\hat n}}{m^2+((r+6x)\tau)^2}\,dx\\ &= \frac{\pi}{6m\tau}\sum_{\hat n\in\mathbb{Z}} e^{-\pi\frac{m|\hat n|}{3\tau}+i\pi\frac{\hat n r}{3}}\,.\end{aligned} \tag{2.4.5}$$

Therefore

(2.4.6)
$$\frac{\tau}{2\pi}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}} \frac{\psi(n)}{n^2}\frac{1}{m^2+(n\tau)^2} = -\frac{\tau^2}{12}\sum_{r=1}^{6}\sum_{\substack{\hat n\in\mathbb{Z}\\ m\geq 1}} \frac{\psi(r)}{m^3}\, e^{-\pi\frac{m|\hat n|}{3\tau}+i\pi\frac{\hat n r}{3}} - \frac{63\pi^3}{2}\,\tau^3$$

which has the limit for $\tau\to i0^+$

(2.4.7)
$$\lim_{\tau\to i0^+} \tau^{-2}\,\frac{\tau}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}} \frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau)^2} = -\frac{\zeta(3)}{12}\sum_{r=1}^{6}\psi(r) = 336\zeta(3)\,.$$

In the expression (2.4.2) this yields

$$I_{\ominus}(0) = 7\zeta(3)\,. \tag{2.4.8}$$

Recovering the result of [**BBDG, Broad1, Broad2**].

This result we will obtained using the higher normal function analysis with the theorem 7.1.2.

2.5. Value of the integral at $t = 1$. — It is numerically obtained in [**Broad1, Broad2**] that the value at $t = 1$ of the banana graph is given by a L-function value

$$I_{\ominus}(1) \stackrel{?}{=} \frac{12\pi}{\sqrt{15}} L(f^+, 2)\,, \tag{2.5.1}$$

with $L(f^+,s) = \sum_{n\geq 1} a_n/n^s$ the L-function associated to the weight three modular form $f^+(q) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+4n^2} = \sum_{n\geq 0} a_n q^n$ constructed in [**PTV**]. Because the functional equation equation is $\Gamma(s)\,(\sqrt{15}/(2\pi))^s\, L(s) = \Gamma(3-s)\,(\sqrt{15}/(2\pi))^{3-s}\, L(3-s)$, the value $s=2$ is inside the critical band. We show in §7.1.1 that for $t=1$ the mixed Hodge structure (motive) associated to the Feynman integral has rank two.

Anticipating on the relation between the three-banana and sunset geometry described in §3.2, we use the relation $t(-1/(6\tau)) = 10-9/t_{\ominus}(\tau) - t_{\ominus}(\tau)$ between the three-banana Hauptmodul t and the sunset Hauptmodul $t_{\ominus}(\tau) = 9+72\,\eta(\tau)^5\eta(2\tau)\eta(3\tau)^{-1}\eta(6\tau)^5)$, one finds that the value $t=1$ is reached[(1)] for $t_{\ominus}(\tau_{\ominus}) = \frac{3}{2}(1-\sqrt{5})$ with $\tau_{\ominus} = (3+i\sqrt{15})/6$ and the sunset elliptic curve is defined over $\mathbb{Q}[\sqrt{5}]$

$$\mathcal{E}_{\ominus}: \qquad y^2 = x^3 + \frac{3}{8}\left(1-3\sqrt{5}\right)x^2 + \frac{3}{2}\left(3-\sqrt{5}\right)x\,. \tag{2.5.2}$$

This curve has complex multiplication (CM) with discriminant -15 as can be seen by fact that $(1+i\sqrt{15})(\mathbb{Z}+\tau_{\ominus}\mathbb{Z}) = (\mathbb{Z}+\tau_{\ominus}\mathbb{Z})$.

Getting back to the banana period ratio by $\tau_{\ominus} = -1/(6\tau_{\ominus}) = (-3+i\sqrt{15})/24$, yields

$$I_{\ominus}(1) = \varpi_1(\tau_{\ominus})\left(-4(2\pi i\tau_{\ominus})^3 + \frac{\tau_{\ominus}}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq 1}} \frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau_{\ominus})^2}\right)\,. \tag{2.5.3}$$

(1) There is of course another solution obtained for $t'_{\ominus}(\tau'_{\ominus}) = \frac{3}{2}(3+\sqrt{5})$ and $\mathcal{E}'_{\ominus}: y^2 = x^3 + \frac{3}{8}\left(1+3\sqrt{5}\right)x^2 + \frac{3}{2}\left(3+\sqrt{5}\right)x$. These two elliptic curves are isogeneous. We refer to §3.2 for a review of the relation between the three-banana and the sunset geometry.

We remark that $\varpi_1(\tau_{\ominus}) = -\frac{3}{4}\,\tau_{\ominus}^2\,\varpi_r$ with

$$\varpi_r = \frac{(\eta(\tau_{\ominus})\eta(3\tau_{\ominus}))^4}{(\eta(2\tau_{\ominus})\eta(6\tau_{\ominus}))^2} = (\theta_4(e^{4i\pi\tau_{\ominus}})\theta_4(e^{12i\pi\tau_{\ominus}}))^2 \tag{2.5.4}$$

which has the following sum expression

$$\varpi_r = \left(1+2\sum_{n\geq1}(-1)^n e^{-n^2\pi\sqrt{\frac{5}{3}}}\right)^2\left(1+2\sum_{n\geq1}(-1)^n e^{-n^2\pi\sqrt{15}}\right)^2 . \tag{2.5.5}$$

showing that $\varpi_r \in \mathbb{R}$. Since the integral is real we conclude that

$$\Im\mathrm{m}\left[\tau_{\ominus}^2\left(-4(2\pi i\tau_{\ominus})^3 + \frac{\tau_{\ominus}}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq1}}\frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau_{\ominus})^2}\right)\right] = 0\,, \tag{2.5.6}$$

that implies

(2.5.7)

$$\Im\mathrm{m}\left(\frac{\tau_{\ominus}}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq1}}\frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau_{\ominus})^2}\right) = \sqrt{15}\,\Re\mathrm{e}\left(\frac{\tau_{\ominus}}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq1}}\frac{\psi(n)}{n^2}\frac{1}{m^2-(n\tau_{\ominus})^2}\right) - \frac{2\pi^3}{3} .$$

To evaluate the real part of the series we use

$$\begin{aligned}
&\Re\mathrm{e}\left(\frac{\tau_{\ominus}}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ n\geq1}}\frac{\psi(n)}{n^2}\,\frac{1}{m^2-(n\tau_{\ominus})^2}\right) \\
&= \frac{\sqrt{15}}{2\pi}\sum_{\substack{m\geq1\\ n\geq1}}\frac{\psi(n)}{n^2}\,\left(\frac{1}{24m^2-6mn+n^2}+\frac{1}{24m^2+6mn+n^2}\right) \\
&= \frac{\sqrt{15}}{2\pi}\,11\,\zeta(4) = \frac{11\pi^3}{12\sqrt{15}}
\end{aligned} \tag{2.5.8}$$

It then follows

$$I_{\ominus}(1) = \frac{(2\pi i)^3}{\sqrt{-15}}\,\frac{1+\sqrt{-15}}{16}\,\varpi_1(\tau_{\ominus}) = \frac{(2i\pi)^3}{\sqrt{-15}}\frac{\varpi_r}{8}\,, \tag{2.5.9}$$

and the conjecture in (7.1.1) amounts showing

$$L(f^+,2) \stackrel{?}{=} (2\pi i)^2\,\frac{\varpi_r}{48}\,. \tag{2.5.10}$$

This relation between the period ϖ_r and the critical value of the L-function is proven in section 7.1.

3. The family of $K3$ surfaces

Our analysis of the three-banana pencil is based on its presentation both as a family of anticanonical toric hypersurfaces and as a modular family of Picard-rank-19 $K3$ surfaces. Modern research in this area is influenced by the theory of toric varieties, and most particularly the toric variety associated to the Newton polytope of a Laurent polynomial. Briefly, to a Laurent polynomial ϕ in n variables $x_1, \ldots, x_n$ we associate firstly the set $\mathfrak{M}_\phi \subset \mathbb{Z}^n$ corresponding to exponents of monomials appearing with non-zero coefficient in ϕ and secondly the convex hull

$$\Delta_\phi := \{ \sum_{m\in\mathfrak{M}} a_m m \mid a_m \geq 0,\ \sum a_m = 1\} \subset \mathbb{R}^n \tag{3.0.1}$$

of these points. Let x_0 be another variable and define the graded ring (graded by powers of x_0)

$$R_\phi := \mathbb{C}[\{x_0^r x^m \mid r \in \mathbb{Z}^{\geq 0}, m \in r\Delta_\phi \cap \mathbb{Z}^n\}] \subset \mathbb{C}[x_0, x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \tag{3.0.2}$$

Notice that $x_0\phi \in R_\phi$. By definition

$$\mathbb{P}_{\Delta_\phi} = \text{Proj } R_\phi \supset \mathbb{G}_m^n = \text{Proj } \mathbb{C}[x_0, x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \tag{3.0.3}$$

where Proj R is the set of homogeneous prime ideals in a graded ring R with the "trivial" graded ideal consisting of all elements of graded degree > 0 omitted. (Alternatively, one may construct $\mathbb{P}_{\Delta_\phi}$ by taking the normal fan to Δ_ϕ.) Divisors at ∞, i.e. in the complement $\mathbb{P}_{\Delta_\phi} \setminus \mathbb{G}_m^n$, correspond to codimension 1 faces (facets) of Δ_ϕ. For a summary of other important properties of this construction, see [**Bat1**].

We begin by reviewing the simplest example of a family of anticanonical modular toric hypersurfaces, the sunset family of elliptic curves studied in [**BV**].

3.1. Sunset in a nutshell. — Consider the Laurent polynomial

$$\phi_{\ominus}(x,y) := (1+x+y)(1+x^{-1}+y^{-1})$$

and its associated (hexagonal) Newton polytope $\Delta_{\ominus} \subset \mathbb{R}^2$, which defines a toric Fano surface $\mathbb{P}_{\Delta_{\ominus}}$ ($\mathbb{P}^2$ blown up at three points). Compactifying the hypersurface defined by

$$t_{\ominus} - \phi_{\ominus}(x,y) = 0$$

in $\mathbb{P}_{\Delta_{\ominus}} \times \mathbb{P}^1 \backslash \mathcal{L}_{\ominus}$ ($\mathcal{L}_{\ominus} := \{0,1,9,\infty\}$) defines the sunset family

$$\mathcal{X}_{\ominus} \overset{\pi_{\ominus}}{\twoheadrightarrow} \mathbb{P}^1 \backslash \mathcal{L}_{\ominus}.$$

For its modular construction, recall that the congruence subgroup

$$\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \mod 6,\ c \equiv 0 \mod 6 \right\}$$

of $SL_2(\mathbb{Z})$ produces a universal family

$$\mathcal{E}_1(6) := (\mathbb{Z}^2 \rtimes \Gamma_1(6)) \Big\backslash (\mathbb{C} \times \mathfrak{H}) \overset{\pi_1}{\twoheadrightarrow} \Gamma_1(6) \backslash \mathfrak{H} =: Y_1(6)$$

of elliptic curves with six marked 6-torsion points (forming a copy of $\mathbb{Z}/6\mathbb{Z}$). Write τ for the parameter on $\mathfrak{H}$, and $q := e^{2\pi i \tau}$. Then we have an isomorphism

$$\begin{array}{ccc} \mathcal{E}_1(6) & \xrightarrow[\cong]{\mathcal{H}_{\ominus}} & \mathcal{X}_{\ominus} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_{\ominus}} \\ Y_1(6) & \xrightarrow[\cong]{H_{\ominus}} & \mathbb{P}^1 \setminus \mathcal{L}_{\ominus} \end{array}$$

of families, in which the Hauptmodul $H_{\ominus}$

$$t_{\ominus} = H_{\ominus}([\tau]) = 9 + 72 \frac{\eta(2\tau)}{\eta(3\tau)} \left(\frac{\eta(6\tau)}{\eta(\tau)} \right)^5, \tag{3.1.1}$$

and maps $[\tau] = [0], [i\infty], [\frac{1}{2}], [\frac{1}{3}]$ to $t_{\ominus} = \infty, 9, 1, 0$, respectively. In the semistable compactification of either family, these points support fibers of (respective) Kodaira types I_6, I_1, I_3, I_2. $\mathcal{H}_{\ominus}$ sends the marked points on

$\pi_1^{-1}([\tau])$ to the six points where $\pi_{\ominus}^{-1}(H_{\ominus}([\tau]))$ meets the toric boundary $\mathbb{P}_{\Delta_{\ominus}}\backslash(\mathbb{C}^*)^2$.

3.2. Verrill's family. — Turning to the three-banana, the relevant pencil

$$\mathcal{X}_{\circledequal} \overset{\pi_{\circledequal}}{\twoheadrightarrow} \mathbb{P}^1\backslash\mathcal{L}_{\circledequal}$$

($\mathcal{L}_{\circledequal} = \{0, 4, 16, \infty\}$) of $K3$ surfaces is defined in the same fashion: namely, we compactify the hypersurface

$$t - \phi_{\circledequal}(x, y, z) = 0$$

in $\mathbb{P}_{\Delta_{\circledequal}} \times \mathbb{P}^1\backslash\mathcal{L}_{\circledequal}$, where $\Delta_{\circledequal} \subset \mathbb{R}^3$ is the Newton polytope of

$$\phi_{\circledequal} = (1 - x - y - z)(1 - x^{-1} - y^{-1} - z^{-1}).$$

Here we are using the coordinate change $x_1 = -x$, $x_2 = -y$, $x_3 = -z$, which swaps $\mathbb{R}_{>0}^{\times 3}$ with $\mathbb{R}_{<0}^{\times 3}$, for reasons related to the completion of the Milnor symbol below.

Laurent polynomials with Newton polytope contained in $\Delta_{\circledequal}$ may be regarded as sections of an ample sheaf $\mathcal{O}(1)$ on $\mathbb{P}_{\Delta_{\circledequal}}$ [**Bat1**, Def. 2.4]. The polytope $\Delta_{\circledequal}$ has 12 vertices $\{\pm e_i\}_{i=1}^3 \cup \{\pm(e_i - e_j)\}_{1\leq i<j\leq 3}$, and a computation shows that its polar polytope

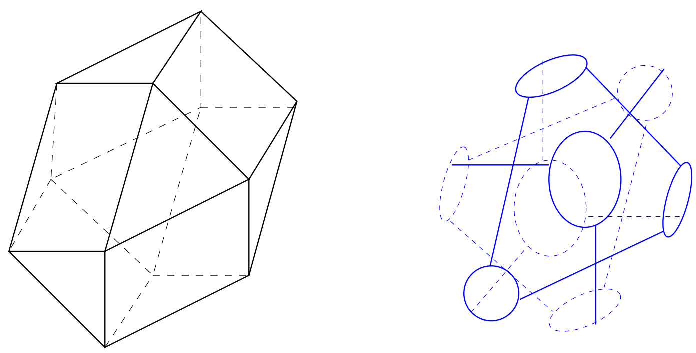
$$\Delta_{\circledequal}^{\circ} := \left\{ v \in \mathbb{R}^3 \,\middle|\, v \cdot w \geq -1 \,(\forall w \in \Delta_{\circledequal}) \right\}$$

has the 14 vertices $\{\pm e_i\}_{i=1}^3 \cup \{\pm(e_i + e_j)\}_{1\leq i<j\leq 3} \cup \{\pm(e_1 + e_2 + e_3)\}$. Since $\Delta_{\circledequal}^{\circ}$ is evidently integral, $\Delta_{\circledequal}$ is reflexive [**Bat2**, Def. 12.3], and so $\mathcal{O}(1)$ is the anticanonical sheaf [loc. cit, Thm. 12.2]. Moreover, as $\Delta_{\circledequal}^{\circ} \cap \mathbb{Z}^3$ consists only of vertices and $\underline{0}$, by [**Bat1**, Thm. 2.2.9(ii)], $\mathbb{P}_{\Delta_{\circledequal}}$ is smooth apart from 12 point singularities corresponding to vertices of $\Delta_{\circledequal}$. It follows that for any Laurent polynomial f which is $\Delta_{\circledequal}$-regular

in the sense of [**Bat1**, Defn. 3.1.1], the (anticanonical) hypersurface in $\mathbb{P}_{\Delta_{\circleddash}}$ defined by $f=0$ is a smooth $K3$ [**Bat1**, Thm. 4.2.2].(2)

We shall need to know the structure of "divisors at infinity" $\mathbb{D}_{\circleddash} := \mathbb{P}_{\Delta_{\circleddash}} \setminus (\mathbb{C}^*)^3$ and $D_{\circleddash} := \pi_{\circleddash}^{-1}(t) \cap \mathbb{D}_{\circleddash}$, the latter of which is the base locus of our pencil (and independent of t). This is understood by examining the facets of $\Delta_{\circleddash}$ and facet polynomials of $\phi_{\circleddash}$, as explained in [**DK**, §2]. Briefly, we draw a plane $\mathbb{R}_\sigma$ through each facet σ and (by choosing an origin) noncanonically identify $\mathbb{R}_\sigma \cap \mathbb{Z}^3 =: \mathbb{Z}_\sigma$ with $\mathbb{Z}^2$. The pair $(\sigma, \mathbb{Z}_\sigma)$ then yields a toric Fano surface $\mathbb{D}_\sigma$ in the usual manner; these are the components of $\mathbb{D}_{\circleddash}$. For $\Delta_{\circleddash}$, one may choose the identifications with $\mathbb{Z}^2$ so that the 8 triangular facets [resp. 6 quadrilateral facets] have vertices $(0,0), (1,0), (0,1)$ [resp. $(0,0), (1,0), (0,1), (1,1)$], whereupon the corresponding $\{\mathbb{D}_\sigma\}$ are evidently isomorphic to $\mathbb{P}^2$ [resp. $\mathbb{P}^1 \times \mathbb{P}^1$] (for instance by taking normal fans).

The components $D_\sigma := \pi_{\circleddash}^{-1}(t) \cap \mathbb{D}_\sigma$ of $D_{\circleddash}$ are obtained by retaining only the terms of the Laurent polynomial with exponent vectors in σ, and viewing this as a Laurent polynomial in two variables (in a manner made precise in §2.5 of [op. cit.]). One checks that $D_{\circleddash}$ is a union of 20 rational curves. The respective configurations of $\mathbb{D}_{\Delta_\phi}$ and $D_{\circleddash}$ are shown below.



(2)We need not carry out the MPCP-desingularization in [loc. cit.], as such a hypersurface avoids the 12 singular points (of $\mathbb{P}_{\Delta_{\circleddash}}$) which it resolves.

Note that $t-\phi_{\ominus}$ fails to be $\Delta_{\ominus}$-regular at the point in each boundary $\mathbb{P}^1\times\mathbb{P}^1$ where the two (rational curve) components of D_σ intersect. However, in local holomorphic coordinates at each such point, $t-\phi_{\ominus}$ takes the form $w=uv$; and it follows that for each $t\in\mathbb{P}^1\setminus\mathcal{L}_{\ominus}$, $\pi_{\ominus}^{-1}(t)$ is a smooth $K3$. Finally, as previously mentioned, $\mathbb{P}_{\Delta_{\ominus}}$ has 12 singular points; one way to construct it is by blowing up $\mathbb{P}^3$ at the 4 "vertices" then along the proper transforms of the 6 "edges", then blowing down 12 (-1)-curves. One choice of toric (MPCP-)desingularization (as in [**Bat1**]) in fact simply reverses this blow-down; note that this produces no additional components in $\mathbb{D}_{\ominus}$ and does not affect the $K3$ hypersurfaces. In subsequent sections, $\mathbb{P}_{\Delta_{\ominus}}$ will denote this smoothed toric 3-fold.

The family $\mathcal{X}_{\ominus}$ was studied by Verrill [**Ve**] (cf. also [**Ber, DK**]), who proved that the generic fiber $X_t=\pi_{\ominus}^{-1}(t)$ has Picard rank 19. More precisely the local system of $R^2(\pi_{\ominus})_*\mathbb{Z}$ contains a 19-dimensional subsystem spanned by divisors. We write $R^2_{var}(\pi_{\ominus})_*\mathbb{Z}$ for the quotient. The fibres $R^2_{var}(\pi_{\ominus})_*\mathbb{Z}=:H^2_{var}(X_t)$ have monodromy group isomorphic to $\Gamma_1(6)^{+3}$. The intersection form is $H\oplus\langle 6\rangle$ with discriminant 6. In particular, $\mathcal{X}_{\ominus}$ is a family of $M_6:=E_8(-1)^{\oplus 2}\oplus H\oplus\langle -6\rangle$-polarized $K3$ surfaces, and is thus of Shioda-Inose type (cf. [**Mo**]). There are countably many t for which the Picard rank is 20. For these fibres, the transcendental part $H^2_{tr}(X_t)$ is a quotient of H^2_{var} of rank 2. The motive $H^2_{tr}(X_t)$ for these fibres has complex multiplication, i.e. the rational endomorphism ring is an imaginary quadratic field.

We describe a modular construction of such a family, closely related to that of [**DK**, sec. 8.2.2]. Set

$$\alpha_3:=\begin{pmatrix}\sqrt{3} & \frac{2}{\sqrt{3}}\\ -2\sqrt{3} & -\sqrt{3}\end{pmatrix},\ \beta_3:=\begin{pmatrix}-\sqrt{3} & \frac{1}{\sqrt{3}}\\ -4\sqrt{3} & \sqrt{3}\end{pmatrix},\ \mu_6:=\begin{pmatrix}0 & \frac{-1}{\sqrt{6}}\\ \sqrt{6} & 0\end{pmatrix}$$

and note that

$$\text{(3.2.1)} \qquad \begin{cases} \beta_3\mu_6 = \mu_6\alpha_3 \\ \beta_3^{-1}\alpha_3 = \left(\begin{smallmatrix} 5 & 3 \\ 18 & 11 \end{smallmatrix}\right) \in \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_1(6). \end{cases}$$

We have $\alpha_3(\tau) = -\frac{\tau+\frac{2}{3}}{2\tau+1}$, $\mu_6(\tau) = \frac{-1}{6\tau}$. These induce involutions on $Y_1(6)$ since

$$\text{(3.2.2)} \qquad \begin{array}{ccccc} & \Gamma_1(6) & \lhd & \Gamma_1(6)^{+3} & := \langle \Gamma_1(6), \alpha_3 \rangle \\ & \triangle & & \triangle & \\ \langle \Gamma_1(6), \mu_6 \rangle =: & \Gamma_1(6)^{+6} & \lhd & \Gamma_1(6)^{+3+6} & := \langle \Gamma_1(6), \alpha_3, \mu_6 \rangle \end{array}$$

and $\alpha_3^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mu_6^2$. (The action on cusps is $[i\infty] \leftrightarrow [\frac{1}{2}]$, $[0] \leftrightarrow [\frac{1}{3}]$ for α_3 and $[i\infty] \leftrightarrow [0]$, $[\frac{1}{2}] \leftrightarrow [\frac{1}{3}]$ for μ_6.) From (3.2.1) one deduces that these involutions commute; and so μ_6 descends to $Y_1(6)^{+3} := \langle \alpha_3 \rangle \backslash Y_1(6)^{*\alpha_3}$ and α_3 to $Y_1(6)^{+6} := \langle \mu_6 \rangle \backslash Y_1(6)^{*\mu_6}$, where "$*$" means to delete fixed (elliptic) points.

Let $'\mathcal{E}_1(6) \overset{'\pi_1}{\twoheadrightarrow} Y_1(6)$ be the fiber-pullback of π_1 by α_3. (Note that α_3 and μ_6 do not lift to involutions of $\mathcal{E}_1(6)$, but do lift to $3:1$ resp. $6:1$ fiberwise isogenies.) Put $'\mathcal{E}_1^{[2]}(6) := \mathcal{E}_1(6) \underset{Y_1(6)}{\times} {'\mathcal{E}_1(6)}$, and let

$$I_3^{[2]} : {'\mathcal{E}_1^{[2]}(6)} \overset{\cong}{\to} {'\mathcal{E}_1^{[2]}(6)}$$

be the involution given by

$$(\tau; [z_1]_\tau, [z_2]_{\alpha_3(\tau)}) \mapsto (\alpha_3(\tau); [z_2]_{\alpha_3(\tau)}, [z_1]_\tau).$$

A first approximation to the three-banana family is then

$$\mathcal{E}_1^{[2]}(6)^{+3} := I_3^{[2]} \Big\backslash {'\mathcal{E}_1^{[2]}(6)^{*\alpha_3}} \overset{\pi_2}{\twoheadrightarrow} Y_1(6)^{+3}.$$

It has fibers of type $E_{[\tau]} \times E_{[\alpha_3(\tau)]}$, hence intersection form $H \oplus \langle 6 \rangle$ on H^2_{var}, and the same local system as $R^2_{var}(\pi_{\ominus})_* \mathbb{Z}_{\mathcal{X}_{\ominus}}$. By Schur's lemma and the Theorem of the Fixed Part [**Sc**], a $\mathbb{C}$-irreducible $\mathbb{Z}$-local system can underlie at most one polarized $\mathbb{Z}$-variation of Hodge structure, making the two variations isomorphic.

However, π_2 is not yet a family of $K3$ surfaces. Quotienting fibers by $(-\mathrm{id})^2$ and resolving singularities yields a family of Kummer $K3$ surfaces, with (incorrect) intersection form $(H \oplus \langle 6 \rangle)[2]$ on $H^2_{var,\mathbb{Z}}$. To correct this multiplication by 2, we require a fiberwise-birational $2:1$ cover of the Kummer family, which is the Shioda-Inose family [**Mo**] $\mathcal{X}_1(6)^{+3}$ over $Y_1(6)^{+3}$. Since this is a family of M_6-polarized $K3$ surfaces with integral H^2 isomorphic to $\pi_{\ominus}$, the relevant global Torelli theorem (cf. [**Do**, Cor. 3.2]) yields an isomorphism

$$\begin{array}{ccc} \mathcal{X}_1(6)^{+3} & \xrightarrow[\cong]{\mathcal{H}_{\ominus}} & \mathcal{X}_{\ominus} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\ominus}} \\ Y_1(6)^{+3} & \xrightarrow[\cong]{H_{\ominus}} & \mathbb{P}^1 \setminus \mathcal{L}_{\ominus}. \end{array}$$

Explicitly, the Hauptmodul (mapping $[i\infty] \mapsto \infty$, $[0] \mapsto 0$, elliptic points$\mapsto 4, 16$) is given by (2.3.11) and we have the relation

$$t = \frac{-64 t_{\ominus}}{(t_{\ominus} - 9)(t_{\ominus} - 1)} \,. \tag{3.2.3}$$

3.3. Miscellany. — Two observations about $\mathcal{H}_{\ominus}$ are in order. The first (used below in §5.2) is that we may construct a family $\tilde{\mathcal{X}} \to Y_1(6)^{+3}$ of smooth surfaces mapping onto $\mathcal{X}_1(6)^{+3}$ and $\mathcal{E}_1^{[2]}(6)^{+3}$ (over $Y_1(6)^{+3}$), with both projections generically $2:1$ on each fiber. We may then transfer generalized algebraic cycles from $\mathcal{E}_1^{[2]}(6)^{+3}$ to $\mathcal{X}_{\ominus}$ by composing this correspondence with $\mathcal{H}_{\ominus}$; and the Abel-Jacobi maps are then related by the action of this correspondence on cohomology (which is an *integral* isomorphism on H^2_{tr} after multiplication by $\frac{1}{2}$). To obtain the family $\tilde{\mathcal{X}}$, we take (a) the fiber product $\check{\mathcal{E}}_a$ of $\mathcal{E}_1^{[2]}(6)^{+3}$ and the Kummer family over $\mathcal{E}_1^{[2]}(6)^{+3}/\langle(-id)^{\times 2}\rangle$ and (b) the fiber product $\check{\mathcal{E}}_b$ of the Kummer family and $\mathcal{X}_1(6)^{+3}$ over the quotient of $\mathcal{X}_1(6)^{+3}$ by the Nikulin involution (cf.

[**Mo**]). Smoothing these families yields $\mathcal{E}_a$ and $\mathcal{E}_b$, whose fiber product over the Kummer family followed by resolution of singularities yields $\tilde{\mathcal{X}}$.

The second observation[(3)] is that we may use $\mathcal{H}_{\circleddash}$ to perform a rational involution on relative cohomology of the family over the automorphism $\mu : t \mapsto \frac{4^3}{t}$ induced by μ_6. First of all, $\mathcal{X}_{\circleddash}$ does not itself have a birational involution over μ, since $H^2_{var}(X_t, \mathbb{Z}) \cong H^2_{var}(E_\tau \times E_{\alpha_3(\tau)}, \mathbb{Z})$ and $H^2_{var}(X_{\frac{1}{4^3 t}}, \mathbb{Z}) \cong H^2_{var}(E_{\mu_6(\tau)} \times E_{\alpha_3(\mu_6(\tau))}, \mathbb{Z})$ are rationally but not integrally isomorphic. In particular, we only have a correspondence

$$\begin{array}{ccc} \mathcal{E}_1^{[2]}(6)^{+3} & \rightsquigarrow & \mathcal{E}_1^{[2]}(6)^{+3} \\ \downarrow & & \downarrow \\ Y_1(6)^{+3} & \xrightarrow[\cong]{\mu_6} & Y_1(6)^{+3} \end{array}$$

which is a $2:1$ isogeny in the first factor and $1:2$ multivalued map in the second factor, given by

$$\left(\tau; [z_1]_\tau, [z_2]_{\alpha_3(\tau)}\right) \mapsto \left(\mu_6(\tau); \left[\frac{(2\tau+1)z_2}{\tau}\right]_{\mu_6(\tau)}, \left[\frac{z_1}{2(-3\tau+1)}\right]_{\alpha_3(\mu_6(\tau))}\right).$$

However, the graph of this correspondence is a family of abelian surfaces, mapping fiberwise $2:1$ onto both $\mathcal{E}_1^{[2]}(6)^{+3}$ and its μ_6-pullback, which *does* have an involution over μ_6. This family, or its associated Shioda-Inose $K3$ family, can then be used as a correspondence (inducing isomorphisms of *rational* H^2_{tr}) between $\mathcal{X}_1(6)^{+3}$ and its μ_6-pullback over $Y_1(6)^{+3}$.

Finally, for future reference we shall write down a family of holomorphic 2-forms on the fibers of $\pi_{\circleddash}$. For any $t \in \mathbb{P}^1 \setminus \mathcal{L}_{\circleddash}$, let

$$\omega_t := Res_{X_t}\left(\frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{1 - t^{-1}\phi_{\circleddash}}\right) \tag{3.3.1}$$

(3) This is not used in the sequel, but illustrates an important difference between this family and the Apéry family of $K3$ surfaces (cf. [**DK**]), which *does* admit such an involution.

be the standard residue form. Remark that the holomorphic period in the neighborhood $|t| > 16$ of $t = \infty$ may be computed by integrating $\frac{1}{2\pi i}\frac{\frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z}}{1-t^{-1}\phi_{\ominus}}$ over the product $(S^1)^{\times 3}$ of unit circles. By the Cauchy residue theorem, this is

$$(2\pi i)^2 \sum_{k\geq 0} a_k t^{-k}, \tag{3.3.2}$$

where a_k, given in (2.3.13), is the constant term in $(\phi_{\ominus})^k$.

4. The three-banana integral as a higher normal function

In this section we shall explain the precise relationship between the integral $I_{\ominus}$ and the family $\mathcal{X}_{\ominus}$ of $K3$ surfaces defined by the denominator of the integrand. Properly understanding this, even without the modular description (done in §5), leads at once to the inhomogeneous equation (§4.3) and the special values at $t = 0$ and 1 (§7).

There are a number of general comments. The integral $I_{\ominus}$ (2.1.6) is a period, i.e. the integral of a rational differential form ω on a variety P over a chain c whose boundary ∂c is supported on a proper closed subvariety $\Sigma \subset P$. This theme goes back to Abel's theorem on Riemann surfaces. For Abel, P is a Riemann surface, $\Sigma = \{p, q\} \subset P$ is a set of two points, ω is a holomorphic 1-form on P and c is a path from p to q. In modern terms, this process associates to the 0-cycle $(p) - (q)$ an extension of Hodge structures

$$0 \to \mathbb{Q}(0) \to H \to H^1(P, \mathbb{Q}) \to 0.$$

The second point is that dependence on external momenta means that we have a family of integrals depending on a parameter t. The corresponding family of extensions is called a *normal function* and first appeared in the work of Poincaré [**P, G2**].

Finally, it turns out that the three-banana amplitude is associated to a *generalized normal function* arising from a family of "higher" algebraic cycles or motivic cohomology classes [**KL, DK**]. The passage from classical normal functions associated to families of cycles to normal functions associated to motivic classes suggests interesting new links between mathematics and physics (op.cit.). For one thing, motivic normal functions can, in many cases, be associated with multiple-valued holomorphic functions which arise as amplitudes. For a discussion of normal functions in physics, cf. [**MW**] for instance.

Briefly, the higher Chow groups $CH^p(X,q)$ of a variety X over a field k are the homology groups of a complex $\mathcal{Z}^p(X,\bullet)$. By definition $\mathcal{Z}^p(X,q)$ is the free abelian group on irreducible codimension p subvarieties $V \hookrightarrow X\times(\mathbb{P}^1\setminus\{1\})^q$ meeting faces properly, where faces are defined by setting various $\mathbb{P}^1$-coordinates to be 0 or ∞. Elements of $Z^p(X,q)$ are called (higher Chow) *precycles*. The face maps $\mathcal{Z}^p(X,q)\to\mathcal{Z}^p(X,q-1)$ are defined by restrictions to faces with alternating signs; elements of the kernel are called (higher Chow) *cycles*.

If $f_1,\dots,f_p$ are rational functions on X, the locus $\{x,f_1(x),\dots,f_p(x)\}$ will (assuming the zeroes and poles of the f_i are in general position) define a precycle in $\mathcal{Z}^p(X,p)$. The easiest way for its image under the face map to vanish, so that this precycle is a cycle and represents a class in $CH^p(X,p)$, is for the f_i to be units (invertible functions) on the complement of the the subvariety of X defined by $\prod_{j=1}^p(f_j(x)-1)=0$. A basic theorem of Suslin and Totaro identifies $CH^p(\mathrm{Spec}\ k,p)\cong K_p^M(k)$, the p-th Milnor K-group of the field k. These groups are linked to algebraic K-theory via the γ-filtration

$$CH^p(X,q)\otimes\mathbb{Q}\cong gr_\gamma^p K_q(X).$$

Finally, in keeping with modern usage, we will define *motivic cohomology* by

$$H^r_M(X,\mathbb{Z}(s)) := CH^s(X,2s-r)$$

when X is smooth. Notice that $H^r_M(X, \mathbb{Z}(r)) = CH^r(X, r)$ in this case. More generally, $H^r_M(X, \mathbb{Q}(s))$ may be constructed from higher Chow precycles as described in §1.3 of [**DK**], which leads to a long-exact sequence used only briefly at the end of §4.1 below.

4.1. $\mathbf{K_3}$ of a K3!— Let $X_t = \pi_{\circleddash}^{-1}(t)$ ($t \in \mathbb{P}^1 \backslash \mathcal{L}_{\circleddash}$) be as in §3.2, $X_t^* := X_t \cap (\mathbb{C}^*)^3 = X_t \backslash D_{\circleddash}$, $D_{\circleddash} = \cup_{j=1}^{20} D_j$ ($D_j \cong \mathbb{P}^1$). The Milnor symbol

$$\{x|_{X_t}, y|_{X_t}, z|_{X_t}\} \in K_3^M(\mathbb{C}(X_t)) \cong \varinjlim_{\substack{U \subset X_t \\ \text{Zar. op.}}} H^3_M(U, \mathbb{Z}(3))$$

extends to a (cubical) higher Chow cycle

$$[\xi_t] := \left[\Delta_{(\mathbb{C}^*)^3} \cap X_t^* \times \square^3\right] \in CH^3(X_t^*, 3) = H^3_M(X_t^*, \mathbb{Z}(3)),$$

where $\square := \mathbb{P}^1 \backslash \{1\}$ and $[\cdots]$ denotes cycle class. To (integrally) lift $[\xi_t]$ to a class

$$[\Xi_t] \in H^3_M(X_t, \mathbb{Z}(3))$$

in the exact sequence(4)

$$\oplus_j H^1_M(D_j, \mathbb{Z}(2)) \to H^3_M(X_t, \mathbb{Z}(3)) \to H^3_M(X_t^*, \mathbb{Z}(3)) \overset{Tame}{\to} \oplus_j H^2_M(D_j^*, \mathbb{Z}(2)),$$

we must check vanishing of the $Tame_{D_j^*}([\xi_t])$. Inspection of the edge polynomials [**DK**, sec. 2.5] shows that these are all of the form $\{\pm u, 1\}$, $\{1, \pm v\}$, and $\{\pm u, 1 - (\pm u)\}$ (in toric coordinates $\{u, v\}$ on $\mathbb{D}_j^* \cong (\mathbb{C}^*)^2$), which are trivial.

On the cycle level, the mechanism by which the lift takes place is given by the moving lemma for higher Chow groups [**Blo2**]. This yields a quasi-isomorphism

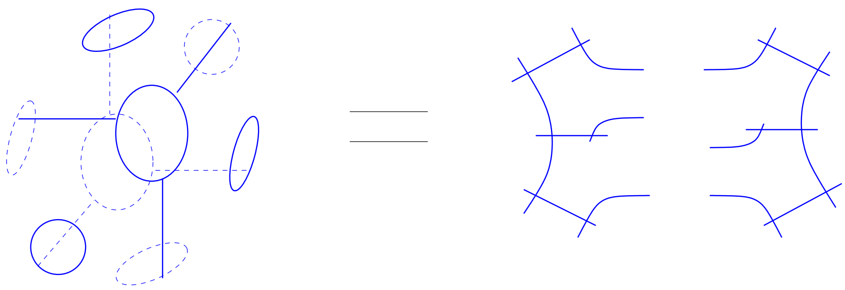
$$Z^3(X_t^*, \bullet) \underset{\jmath^*}{\overset{\simeq}{\leftarrow}} Z^3(X_t, \bullet)/\imath_*^D Z^2(D_{\circleddash}, \bullet)$$

(4)The ambiguities of this lift by the images of the $H^1_M(D_j, \mathbb{Z}(2))$ may for our purposes be ignored, as they have no bearing upon the transcendental part of its Abel-Jacobi image.

inducing the above exact sequence, and there exists $\mu_t \in Z^3(X_t^*, 4)$ such that[5]

$$\xi_t + \partial\mu_t = \jmath^* \Xi_t.$$

Moreover, there are 6 of the D_j (say, $j = 1, \ldots, 6$) on which x, y, or z is identically 1, so that we may replace in this argument X_t^* by $X_t^\sim := X_t \backslash \cup_{j=7}^{20} D_j$, ξ_t by its Zariski closure $\xi_t^\sim \in Z^3(X_t^\sim, 3)$, and μ_t by some $\mu_t^\sim$. The fact that the configuration $\mathcal{J} = \cup_{j=7}^{20} D_j$



has trivial H_1 will be crucial for the argument in §4.3 below.

Working modulo torsion, one can do somewhat better than a lift $[\Xi_t]$ for each $t \in \mathbb{P}^1 \backslash \mathcal{L}_{\circleddash}$ that is ambiguous by the image of $\oplus_j H^1_M(D_j, \mathbb{Q}(2))$. Let $\bar{\mathcal{X}}_{\circleddash} \xrightarrow{\bar{\pi}} \mathbb{P}^1 \backslash \{\infty\}$ be the Zariski closure of $\mathcal{X}_{\circleddash}$ in $\mathbb{P}_{\Delta_{\circleddash}} \times (\mathbb{P}^1 \backslash \{t = \infty\})$. One shows that $\phi_{\circleddash}$ is reflexive and tempered and the assumptions of [**DK**, Rem. 3.3(iv)] hold (with $K = \mathbb{Q}$). So by [**DK**, Thm. 3.1], there exists a motivic cohomology class $[\bar{\Xi}_{\circleddash}] \in H^3_M(\bar{\mathcal{X}}_{\circleddash}, \mathbb{Q}(3))$ defined over $\mathbb{Q}$ and restricting to $[\{x, y, z\}] \in H^3_M((\mathbb{C}^*)^3, \mathbb{Q}(3))$ under the inclusion $(\mathbb{C}^*)^3 \hookrightarrow \bar{\mathcal{X}}_{\circleddash}$ given by $(x, y, z) \mapsto (x, y, z, \phi_{\circleddash}(x, y, z)^{-1})$. Its fiberwise restrictions therefore produce rational lifts of ξ_t, and since $H^1_M\left((\mathbb{A}^1 \times D_j)_{/\mathbb{Q}}, \mathbb{Q}(2)\right) \cong H^1_M(Spec(\mathbb{Q}), \mathbb{Q}(2)) = \{0\}$ there is also no ambiguity. This guarantees that the processes described above can be carried out in a "continuous" fashion, and that the lift extends (as a motivic cohomology class) across the singular fibers over $t = 16, 4, 0$.

[5] Note: in this paper "∂" is used both to denote the boundary of a C^∞ cochain and the differential in the higher Chow complex.

In fact, the construction of $[\bar{\Xi}_{\ominus}]$ in this case is quite simple. The total space $\bar{\mathcal{X}}_{\ominus}$ has six singularities (of the local type $xy = zw$), situated over $t = 0$ in the base locus where the two D_j's in each $\mathbb{P}^1 \times \mathbb{P}^1$ component cross. Blowing these points $\{p_k\}_{k=1}^{6}$ up, we have exceptional divisors $E_k \cong \mathbb{P}^1 \times \mathbb{P}^1$ ($k = 1, \ldots, 6$) in $\tilde{\bar{\mathcal{X}}}_{\ominus}$, and the long exact sequence

$$\bigoplus_k H^2_M(E_k, \mathbb{Q}(3)) \to H^3_M(\bar{\mathcal{X}}_{\ominus}, \mathbb{Q}(3)) \to$$
$$H^3_M(\tilde{\bar{\mathcal{X}}}_{\ominus}, \mathbb{Q}(3)) \oplus \bigoplus_k H^3_M(\{p_k\}, \mathbb{Q}(3)) \xrightarrow{\alpha} \bigoplus_k H^3_M(E_k, \mathbb{Q}(3)).$$

One easily lifts $[\{x, y, z\}]$ to $[\tilde{\Xi}_{\ominus}] \in H^3_M(\tilde{\bar{\mathcal{X}}}_{\ominus}, \mathbb{Q}(3))$ (since the Tame symbols vanish), whereupon $\alpha([\tilde{\Xi}_{\ominus}], 0)$ vanishes since x, y, or z was 1 at each p_k.

In the sequel, the restriction of $[\bar{\Xi}_{\ominus}]$ to $H^3_M(\mathcal{X}_{\ominus}, \mathbb{Q}(3))$ will be denoted by $[\Xi_{\ominus}]$; we call this the *three-banana cycle*.

4.2. Review of Abel-Jacobi. — We shall need a few generalities on regulator currents for the arguments below. The presentation will be sketchy, as a more thorough exposition may be found in [**DK**, sec. 1].

Let X be a smooth projective variety with complexes of currents $\mathcal{D}^\bullet(X)$ and $(2\pi i)^p\mathbb{A}$-valued C^∞-cochains $C^\bullet_{top}(X; \mathbb{A}(p))$ ($\mathbb{A} \subset \mathbb{C}$ a subring). Given a cochain γ, we write δ_γ for the current of integration over it, and use this to define the Deligne complex

$$C^\bullet_{\mathscr{D}}(X, \mathbb{A}(p)) := C^\bullet_{top}(X; \mathbb{A}(p)) \oplus F^p\mathcal{D}^\bullet(X) \oplus \mathcal{D}^\bullet[-1]$$

with differential

$$D(T, \Omega, R) := (\partial T, -d[\Omega], d[R] - \Omega + \delta_T). \tag{4.2.1}$$

Its $(2p - n)^{\text{th}}$ cohomology sits in a short-exact sequence

$$0 \to J^{p,n}(X)_{\mathbb{A}} \to H^{2p-n}_{\mathscr{D}}(X, \mathbb{A}(p)) \to Hg^{p,n}(X)_{\mathbb{A}} \to 0,$$

where

$$\begin{cases} Hg^{p,n}(X)_{\mathbb{A}} := Hom_{\mathbb{A}\text{-MHS}}(\mathbb{A}(0), H^{2p-n}(X,\mathbb{A}(p))) \\ J^{p,n}(X)_{\mathbb{A}} := Ext^1_{\mathbb{A}\text{-MHS}}(\mathbb{A}(0), H^{2p-n-1}(X,\mathbb{A}(p))) \end{cases}.$$

Let $Z^p(X,\bullet)$ be the codimension-p higher Chow cycle complex with n^{th} homology $CH^p(X,n) = H_M^{2p-n}(X,\mathbb{Z}(p))$, and boundary map ∂; in particular, $Z^p(X,n)$ is a subgroup of the cycle group $Z^p(X\times\square^n)$. Denote by $Z^p_{\mathbb{R}}(X,\bullet) \subset Z^p(X,\bullet)_{\mathbb{Q}}$ the quasi-isomorphic subcomplex[(6)] described in [**KL**, sec. 8.2]. By [**KLM**, sec. 7], the cycle class map

$$c^{p,n}_{\mathscr{D}} : CH^p(X,n)_{\mathbb{Q}} = H_M^{2p-n}(X,\mathbb{Q}(p)) \to H_{\mathscr{D}}^{2p-n}(X,\mathbb{Q}(p))$$

defined in [**Blo1**] is computed by a map of complexes

$$Z^p_{\mathbb{R}}(X,\bullet) \to C_{\mathscr{D}}^{2p-\bullet}(X,\mathbb{Q}(p)).$$

Taking $\bullet = n$, it is defined on irreducible components by

$$\xi \longmapsto (2\pi i)^{p-n}\left((2\pi i)^n T_\xi, \Omega_\xi, R_\xi\right), \tag{4.2.2}$$

where (writing $\pi_X, \pi_\square$ for the projections from a desingularization $\tilde{\xi}$ to X, $\square^n$) R_ξ [resp. Ω_ξ, T_ξ] is defined by applying $(\pi_X)_*(\pi_\square)^*$ to[(7)]

$$R_n := \sum_{j=1}^n ((-1)^n 2\pi i)^{j-1} \log(z_j)\frac{dz_{j+1}}{z_{j+1}} \wedge \cdots \wedge \frac{dz_n}{z_n}\delta_{T_{z_1}\cap\cdots\cap T_{z_{j-1}}}$$

$$\left[\text{resp. } \Omega_n := \bigwedge_{j=1}^n \frac{dz_j}{z_j},\ T_n := \bigcap_{j=1}^n T_{z_j} := \bigcap_{j=1}^n z_j^{-1}(\mathbb{R}_{<0})\right].$$

Properties of T_n, Ω_n, R_n imply that

$$d[R_\xi] = \Omega_\xi - (2\pi i)^n \delta_{T_\xi} + 2\pi i R_{\partial\xi}, \tag{4.2.3}$$

(6) These are still precycles with $\mathbb{Q}$-coefficients; the "$\mathbb{R}$" refers to intersection conditions with real-analytic chains.

(7) Here $\log(z)$ is regarded as a 0-current on $\mathbb{P}^1$ with branch cut along $\mathbb{R}_{<0}$, so that $d[\log(z)] = \frac{dz}{z} - 2\pi i\delta_{T_z}$. Operations involving pullback are not in general defined on currents, but a convergence argument (when ξ is in the subcomplex) shows that R_ξ and Ω_ξ are in fact currents on X.

so that (by (4.2.1)) (4.2.2) gives a map of complexes.

Suppose $\partial\xi = 0$ (so that $[\xi] \in H_M^{2p-n}(X, \mathbb{Q}(p))$) and $n \geq 1$. Since $[T_\xi]$ and $[\Omega_\xi]$ define the map to $Hg^{p,n}(X)_{\mathbb{Q}}$(which is zero for $n \geq 1$), there exist $K \in F^p\mathcal{D}^{2p-n-1}(X)$ and $\Gamma \in C_{top}^{2p-n-1}(X; \mathbb{Q}(p))$ such that $\Omega_\xi = d[K]$ and $T_\xi = \partial\Gamma$, whereupon

$$\tilde{R}_\xi := R_\xi - K + (2\pi i)^n \delta_\Gamma$$

defines a closed current with class $[\tilde{R}_\xi] \in H^{2p-n-1}(X, \mathbb{C})$ projecting to

$$(c_{\mathscr{D}}^{p,n}(\xi) =)\, AJ_X^{p,n}(\xi) \in J^{p,n}(X)_{\mathbb{Q}} \cong \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Q}(p))}.$$

If X is a smooth algebraic $K3$ surface and $p = n = 3$, then

$$AJ_X^{3,3} : H_M^3(X, \mathbb{Z}(3)) \to H^2(X, \mathbb{C}/\mathbb{Z}(3)) = J^{3,3}(X) \tag{4.2.4}$$

is computed by

$$\tilde{R}_\xi := R_\xi + (2\pi i)^3 \delta_\Gamma, \tag{4.2.5}$$

since $\Omega_\xi \in F^3\mathcal{D}^3(X) = \{0\}$. Let $U \subset X$ be a Zariski open set. Any precycle $\xi \in Z^3_{\mathbb{R}}(U, 3)$ is a sum of components supported over divisors and components with generic support; the latter take the form

$$\{f_1, f_2, f_3\}_U := \overline{\{(x, f_1(x), f_2(x), f_3(x)) \,|\, x \in U \backslash \cup |(f_i)|\}},$$

where the bar denotes Zariski closure in $U \times \square^3$. One can show that

$$R_{\{f_1,f_2,f_3\}} = \log(f_1)\frac{df_2}{f_2} \wedge \frac{df_3}{f_3} + 2\pi i \log(f_2)\frac{df_3}{f_3}\delta_{T_{f_1}} + (2\pi i)^2 \log(f_3)\delta_{T_{f_1} \cap T_{f_2}}$$

extends to a 2-current on X (even if the closure of ξ over X is not a precycle).

This has the following application to the general situation of §4.1, where $\xi = \{f_1, f_2, f_3\}_U = \jmath^*\Xi - \partial\mu$ for $\Xi \in \ker(\partial) \subset Z^3_{\mathbb{R}}(X, 3)$ and $\mu \in Z^3_{\mathbb{R}}(U, 4)$, under the assumption that $\cap_j f_j^{-1}(\mathbb{R}_{<0}) \cap U = \emptyset$. Working modulo currents and chains supported on $D := X \backslash U$, formally applying

(4.2.2) (and noting that R_μ extends to X) gives

$$\left((2\pi i)^3 T_\xi, 0, R_\xi\right) + D\left((2\pi i)^3 T_\mu, 0, \frac{1}{2\pi i} R_\mu\right) \equiv \left((2\pi i)^3 T_\Xi, 0, R_\Xi\right), \tag{4.2.6}$$

while our assumption gives $T_\xi \equiv 0$. For the chains, this yields $T_\Xi = -\partial T_\mu + S_D$, where S_D is a (closed) 1-chain supported on D; since T_Ξ is exact, so is S_D (on X), and we write $S_D = \partial\gamma$. For the currents, (4.2.6) gives

$$R_\Xi = R_\xi + \frac{1}{2\pi i} d[R_\mu] + (2\pi i)^3 \delta_{T_\mu} + K_D$$

for some 2-current K_D supported on D, so that (taking $\Gamma = -T_\mu + \gamma$ in (4.2.5))

$$\tilde{R}_\Xi = R_\xi + \frac{1}{2\pi i} d[R_\mu] + (2\pi i)^3 \delta_\gamma + K_D$$

gives a lift of $AJ_X^{3,3}(\Xi)$.

The key point is now that *if* $H_1(D) = \{0\}$*, then we may take* γ *to be supported on* D, and up to exact currents on X and arbitrary currents supported on D,

$$\tilde{R}_\Xi \equiv R_{\{f_1, f_2, f_3\}}. \tag{4.2.7}$$

This is precisely what occurs in §4.1 with $X = X_t$, $\Xi = \Xi_t = \Xi_{\circleddash}|_{X_t}$, $U = X_t^\sim$, $D = \mathcal{J}$, and $\{f_1, f_2, f_3\} = \{x|_{X_t^\sim}, y|_{X_t^\sim}, z|_{X_t^\sim}\}$; the assumption $T_{\{x,y,z\}} \equiv 0$ holds for

$$t \notin \overline{\phi_{\circleddash}(\mathbb{R}_{<0}^{\times 3})} = [16, \infty].$$

Writing

$$\overline{AJ}_{X_t}^{3,3} := \pi_{var} \circ AJ_{X_t}^{3,3} : H_M^3(X_t, \mathbb{Q}(3)) \to H_{var}^2(X_t, \mathbb{C}/\mathbb{Q}(3)),$$

(4.2.7) provides a *well-defined* lift (for $t \notin [16, \infty] \cup \{0, 4\}$)

$$\mathcal{R}_t := \pi_{var}[\tilde{R}_{\Xi_t}] \in H_{var}^2(X_t, \mathbb{C})$$

of $\bar{\mathcal{R}}_t := \overline{AJ}_{X_t}^{3,3}(\Xi_t)$. As the extension of $R_{var}^2(\pi_{\circleddash})_*\mathbb{Q}$ across $t = 0$ has only rank 1, and $\bar{\mathcal{R}}_t$ must extend through $t = 0$, we conclude part (i) of the

Proposition 4.2.1. — *(i)* $\mathcal{R}_t$ *yields a holomorphic section of the sheaf* $\mathcal{O} \otimes R^2_{var}(\pi_{\circleddash})_*\mathbb{C}$ *over* $\mathbb{P}^1 \backslash [16, \infty] \cup \{0, 4\}$*, and is the unique such section lifting* $\bar{\mathcal{R}}_t$ *with no monodromy about* $t = 0$ *and* $t = 4$*.*

(ii) Writing $\delta_t := t\frac{d}{dt}$ *and* ∇ *for the Gauss-Manin connection, we have*

$$\nabla_{\delta_t} \mathcal{R}_t = -[\omega_t],$$

with $\omega_t := Res_{X_t}\left(\frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{1 - t^{-1}\phi_{\circleddash}}\right)$*.*

Proof. — (ii) follows at once from [**DK**, Cor. 4.1] (note $t_{DK} = t^{-1}$). □

4.3. Reinterpreting the Feynman integral. — The term "higher normal function" has been used in several different ways in the theory of algebraic cycles – for instance, to describe the section of $\cup_t J^{3,3}(X_t)$ (i.e. the family of extension classes (4.2.4)) associated to a family of higher cycles like Ξ_t. Here we shall pair this section with a specific family of holomorphic forms to get an actual function (Definition 4.3.1). We preface this with a brief discussion of the pairings used here and in later sections.

Let X be a smooth projective surface, $[X] \in H_4(X, \mathbb{Q})$ its fundamental class, and

$$\int_{[X]} : H^4(X, \mathbb{Q}) \to \mathbb{Q}(0)$$

the map (of Hodge type $(-2, -2)$) induced by pairing with $[X]$. We can define a Poincaré pairing in one of two ways:

$$\langle\ ,\ \rangle : H^2(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q}) \overset{\int_{[X]}}{\to} \mathbb{Q}(0)\,;$$

$$\langle\ ,\ \rangle' : H^2(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q}) = \mathbb{Q}(-2).$$

While the second has type $(0, 0)$, we prefer to work with the first bracket.

We now turn to the main content of this subsection.

Definition 4.3.1. — The (truncated) higher normal function associated to $\Xi_{\circleddash}$ is

$$V_{\circleddash}(t) := \langle \mathcal{R}_t, [\tilde{\omega}_t] \rangle \in \mathcal{O}(U_{\circleddash}),$$

where $\tilde{\omega}_t := \frac{-1}{(2\pi i)^2 t}\omega_t \in \Omega^2(X_t)$ and $U_{\ominus} \subset \mathbb{P}^1 \setminus \mathcal{L}_{\ominus} = \mathbb{P}^1 \setminus \{0, 4, 16, \infty\}$ is the complement of the real segment $16 < t < \infty$.

Note that $V_{\ominus}$ extends holomorphically across $t = 4$ and 0, since it pairs finite (in fact nonzero) homology resp. cohomology classes ($[\omega'_t]$ resp. $\mathcal{R}_t$) on those singular fibers.

Theorem 4.3.2. — *$I_{\ominus}(t) = V_{\ominus}(t)$ on $U_{\ominus}$.*

Proof. — Begin by noting that

$$\tilde{\omega}_t = \frac{-1}{(2\pi i)^2} Res_{X_t}\left(\frac{\frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z}}{t - \phi_{\ominus}(x,y,z)}\right) =: Res_{X_t}\left(\tilde{\Omega}_t\right)$$

so that (regarding $\tilde{\Omega}_t \in F^3 D^3(\mathbb{P}_{\Delta_{\ominus}})$ as a 3-current)

$$d\left[\tilde{\Omega}_t\right] = 2\pi i \imath_*^{X_t}\tilde{\omega}_t.$$

Furthermore, $R_3^* := R_{\{x,y,z\}}$ extends to a 2-current on $\mathbb{P}_{\Delta_{\ominus}}$, and writing $\Omega_3^* := \frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z}$, $T_3^* := \overline{T_x \cap T_y \cap T_z}$, we have on $\mathbb{P}_{\Delta_{\ominus}}$

$$d[R_3^*] = \Omega_3^* - (2\pi i)^3\delta_{T_3^*} + K_{\mathbb{D}}, \tag{4.3.1}$$

where $K_{\mathbb{D}}$ ($\in F^1 D^3(\mathbb{P}_{\Delta_{\ominus}})$) is supported on $\mathbb{D}_{\ominus}$.

Now[(8)] $I_{\ominus}(t) =$

$$= \int_{\mathbb{R}_{<0}^{\times 3}} \frac{\frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z}}{t - \phi_{\ominus}(x,y,z)} = -(2\pi i)^2\int_{T_3^*}\tilde{\Omega}_t$$

$$= -(2\pi i)^2 \int_{\mathbb{P}_{\Delta_{\ominus}}} \delta_{T_3^*}\wedge\tilde{\Omega}_t.$$

By (4.3.1), this

$$= \frac{1}{2\pi i}\int_{\mathbb{P}_{\Delta_{\ominus}}} (d[R_3^*] - \Omega_3^* - K_{\mathbb{D}})\wedge\tilde{\Omega}_t.$$

(8) The apparent sign change in the denominator (compare (2.2.1)) arises from the orientation of T_3^* and the change of variables.

Noting that $K_{\mathbb{D}} \wedge \tilde{\Omega}_t$ and $\Omega_3^* \wedge \tilde{\Omega}_t$ are zero by type, it becomes

$$= \frac{1}{2\pi i} \int_{\mathbb{P}_{\Delta_\circledcirc}} d[R_3^*] \wedge \tilde{\Omega}_t$$

$$= \frac{1}{2\pi i} \int_{\mathbb{P}_{\Delta_\circledcirc}} R_3^* \wedge d[\tilde{\Omega}_t]$$

$$= \int_{\mathbb{P}_{\Delta_\circledcirc}} R_3^* \wedge \imath_*^{X_t} \tilde{\omega}_t$$

$$= \int_{X_t} R_{\left\{x|_{X_t^\sim}, y|_{X_t^\sim}, z|_{X_t^\sim}\right\}} \wedge \tilde{\omega}_t. \tag{4.3.2}$$

Finally, the argument of (4.2.7)ff allows us to rewrite this as

$$= \int_{X_t} \tilde{R}_{\Xi_t} \wedge \tilde{\omega}_t = V_\circledcirc(t).$$

□

Without the last step, (4.3.2) would not pair two *closed* currents and would have no cohomological meaning. So the seemingly bizarre criterion that $H_1(\mathcal{J}) = \{0\}$, in the end, is absolutely essential.

To give an idea of the power of Theorem 4.3.2, we conclude this section with one of its basic consequences: namely, an alternate proof of Theorem 2.2.1. The characterization of $I_\circledcirc$ as a higher normal function can also be used to compute some special values, cf. §7.

For deriving the Picard-Fuchs equation, we shall modestly abuse notation and regard the family of forms as a section

$$\tilde{\omega}_t \in \Gamma\left(\mathbb{P}^1 \backslash \mathcal{L}_\circledcirc, \mathcal{O} \otimes R^2_{var}(\pi_\circledcirc)_* \mathbb{C}\right).$$

Let ∇_{PF} be the operator on cohomology obtained from $D_{PF} := \mathcal{L}_t^3 = \sum_{k=0}^3 f_k(t) \frac{d^k}{dt^k}$ by replacing $\frac{d}{dt}$ by $\nabla_t := \nabla_{\frac{d}{dt}}$, so that by [**Ve**, Prop. 8] $\nabla_{PF} \tilde{\omega}_t = 0$. Note that $f_3(t) = t^2(t-4)(t-16)$ and $f_2(t) = 6t(t^2 - 15t +$

$32) = \frac{3}{2} f_3'(t)$. Introduce the *Yukawa coupling*

$$\tilde{Y}(t) := \langle \tilde{\omega}_t, \nabla_t^2 \tilde{\omega}_t \rangle,$$

which may be computed as follows. Observe that by type $0 = \langle \tilde{\omega}_t, \nabla_t \tilde{\omega}_t \rangle$ implies

$$0 = \frac{d^2}{dt^2} \langle \tilde{\omega}_t, \nabla_t \tilde{\omega}_t \rangle = \langle \tilde{\omega}_t, \nabla_t^3 \tilde{\omega}_t \rangle + 3 \langle \nabla_t \tilde{\omega}_t, \nabla_t^2 \tilde{\omega}_t \rangle,$$

so that

$$\frac{d}{dt} \tilde{Y}(t) = \langle \tilde{\omega}_t, \nabla_t^3 \tilde{\omega}_t \rangle + \langle \nabla_t \tilde{\omega}_t, \nabla_t^2 \tilde{\omega}_t \rangle = \frac{2}{3} \langle \tilde{\omega}_t, \nabla_t^3 \tilde{\omega}_t \rangle$$

implies

$$\begin{aligned} f_3(t) \tilde{Y}'(t) &= \frac{2}{3} \langle \tilde{\omega}_t, -f_2(t) \nabla_t^2 \tilde{\omega}_t - f_1(t) \nabla_t \tilde{\omega}_t - f_0(t) \tilde{\omega}_t \rangle \\ &= -f_3'(t) \tilde{Y}(t) \end{aligned}$$

that implies

$$\tilde{Y}(t) = \frac{\kappa}{f_3(t)} \in \mathbb{C}(t).$$

We will see below in §5.2 that $\kappa = \frac{-24}{(2\pi i)^2}$. Assuming this, we conclude

Corollary 4.3.3. — *$D_{PF}\Big(I_{\circledcirc}(t)\Big) = -24$.*

Proof. — By Prop. 4.2.1(ii),

$$\nabla_t \mathcal{R}_t = (2\pi i)^2 \tilde{\omega}_t.$$

Now $I_{\circledcirc}(t) = V_{\circledcirc}(t) = \langle \mathcal{R}_t, \tilde{\omega}_t \rangle$, and

$$\begin{aligned} \frac{d}{dt} \langle \mathcal{R}_t, \tilde{\omega}_t \rangle &= (2\pi i)^2 \langle \tilde{\omega}_t, \tilde{\omega}_t \rangle + \langle \mathcal{R}_t, \nabla_t \tilde{\omega}_t \rangle = \langle \mathcal{R}_t, \nabla_t \tilde{\omega}_t \rangle \\ \frac{d^2}{dt^2} \langle \mathcal{R}_t, \tilde{\omega}_t \rangle &= (2\pi i)^2 \langle \tilde{\omega}_t, \nabla_t \tilde{\omega}_t \rangle + \langle \mathcal{R}_t, \nabla_t^2 \tilde{\omega}_t \rangle = \langle \mathcal{R}_t, \nabla_t^2 \omega_t \rangle \end{aligned}$$

by type (and Griffiths transversality [**G**]). Together with

$$\frac{d^3}{dt^3} \langle \mathcal{R}_t, \tilde{\omega}_t \rangle = (2\pi i)^2 \tilde{Y}(t) + \langle \mathcal{R}_t, \nabla_t^3 \tilde{\omega}_t \rangle,$$

these give

$$D_{PF}\langle \mathcal{R}_t, \tilde{\omega}_t\rangle = \langle \mathcal{R}_t, \nabla_{PF}\tilde{\omega}_t\rangle + (2\pi i)^2 f_3(t)Y(t)$$
$$= (2\pi i)^2 f_3(t)\tilde{Y}(t) = -24.$$

□

Remark 4.3.4. — For later reference we note that $Y(t) := \langle \omega_t, \nabla^2_{\delta_t}\omega_t\rangle = (2\pi i)^4 t^4 \tilde{Y}(t) \implies Y(\infty) = (2\pi i)^4\kappa$.

5. A second computation of the three-banana integral: the Eisenstein symbol

As an application of the results in sections 3 and 4, we will use $\mathcal{H}_{\circleddash}$ to pull back the toric three-banana cycle $\Xi_{\circleddash} \in H^3_M(\mathcal{X}_{\circleddash}, \mathbb{Q}(3))$ to $\mathcal{X}_1(6)^{+3}$. We then apply a correspondence to produce a higher Chow cycle on a Kuga variety $\mathcal{E}^{[2]}(6)$ (defined below), and recognize this as an Eisenstein symbol in the sense of Beilinson [**Beil, DS, DK**]. This will allow us to write the pullback $V \circ H_{\circleddash}$ of the higher normal function (i.e. Feynman integral) as an elliptic trilogarithm, giving another proof of Theorem 2.3.2.

5.1. Higher normal functions of Eisenstein symbols.

— For simplicity of exposition we shall restrict to the setting of Kuga 3-folds. We begin with an explanation of Beilinson's construction of higher cycles ("Eisenstein symbols") on these 3-folds and their relationship to Eisenstein series of weight 4. Each such cycle gives rise to a higher normal function over a modular curve (defined in (5.1.3)), which turns out to be an Eichler integral of the corresponding Eisenstein series. The main result of this subsection, Proposition 5.1.1, computes the q-expansion (5.1.5) of this normal function. In many cases it may be rewritten in terms of trilogarithms (cf. theorems 2.3.2 and 5.3.1). Everything in this subsection is general. In §§5.2-5.3 we shall apply this general computation to our special case, by pulling back the three-banana cycle from $\mathcal{X}_{\circledcirc}$ to the Kuga 3-fold and interpreting the result (up to Abel-Jacobi equivalence) as one of Beilinson's cycles.

To describe these cycles, consider the elliptic modular surface $\mathcal{E}(N) := (\mathbb{Z}^2 \rtimes \Gamma(N)) \backslash (\mathbb{C} \times \mathfrak{H})$ over $Y(N) = \Gamma(N) \backslash \mathfrak{H}$, where $\Gamma(N) = \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\}$ and $N > 3$. Its fibers are elliptic curves with 1-form dz and standard Betti 1-cycles $\alpha = [0,1]$, $\beta = [0,\tau]$. By duality we may regard α, β as defining H^1 classes, and write $[dz] = [\beta] - \tau[\alpha]$.

Let $\mathcal{E}^{[2]}(N) \overset{\pi^{[2]}(N)}{\longrightarrow} Y(N)$ be the self-fiber product of $\mathcal{E}(N)$. There exists a semistable compactification $\overline{\mathcal{E}^{[2]}(N)} \to \overline{Y(N)}$ due to Shokurov [**Sh**], with singular fibers $D^{[2]}(N) = \overline{\mathcal{E}^{[2]}(N)} \backslash \mathcal{E}^{[2]}(N)$. Choose for each cusp $\sigma = \left[\frac{r}{s}\right] \in \kappa(N) := \overline{Y(N)} \backslash Y(N)$ an element $M_\sigma := \begin{pmatrix} p & q \\ -s & r \end{pmatrix} \in SL_2(\mathbb{Z})$. Define modular forms of weight n for $\Gamma(N)$ by

$$M_k(N) := \left\{ F \in \mathcal{O}(\mathfrak{H}) \,\middle|\, \begin{array}{lll} (i) & F(\tau) = \frac{F(\gamma(\tau))}{(c\tau+d)^k} =: F|^k_\gamma & (\forall \gamma \in \Gamma(N)) \\ (ii) & r_\sigma(F) := \lim\limits_{\tau \to i\infty} F|^k_{M_\sigma{}^{-1}} < \infty & (\forall \sigma \in \kappa(N)) \end{array} \right\}.$$

There is an isomorphism ([**Sh**], or [**DK**, Prop. 7.1])

$$\begin{array}{rccl} \Psi: & M_4(N) & \overset{\cong}{\to} & \Omega^3(\overline{\mathcal{E}^{[2]}(N)})\langle \log D^{[2]}(N) \rangle \\ & F(\tau) & \mapsto & (2\pi i)^3 F(\tau) dz_1 \wedge dz_2 \wedge d\tau. \end{array}$$

Let $\Phi_2^K(N)$ denote the vector space of K-valued functions on $(\mathbb{Z}/N\mathbb{Z})^2$, with subspaces $\Phi_2^K(N)_\circ := \ker\{\text{evaluation at } (\bar{0},\bar{0})\}$ and

$$\Phi_2^K(N)^\circ := \ker\{\text{augmentation}\} = \{f : (\mathbb{Z}/N\mathbb{Z})^2 \to K \mid \sum_{0 \le m,n \le N-1} f(m,n) = 0\}$$

Assuming $K \supset \mathbb{Q}(\zeta_N)$ ($\zeta_N := e^{\frac{2\pi i}{N}}$), these are exchanged by the finite Fourier transform

$$\varphi(m,n) \mapsto \hat{\varphi}(\mu,\eta) := \sum_{(m,n)\in(\mathbb{Z}/N\mathbb{Z})^2} \varphi(m,n)\zeta_N^{\mu n-\eta m}.$$

This allows us to define the $\mathbb{Q}$-Eisenstein series $E_4^{\mathbb{Q}}(N)$ by the image of the map

$$\begin{array}{llll} E: & \Phi_2^{\mathbb{Q}} & \to & M_4(N) \\ & \varphi & \mapsto & E_\varphi(\tau) \quad := -\dfrac{3}{(2\pi i)^4} \displaystyle\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \dfrac{\hat{\varphi}(m,n)}{(m\tau+n)^4}. \end{array}$$

The horospherical maps

$$\begin{array}{llll} \mathscr{H}_\sigma: & \Phi_2^{\mathbb{Q}}(N)^\circ & \to & \mathbb{Q} \\ & \varphi & \mapsto & \mathscr{H}_\sigma(\varphi) \quad := \frac{1}{8}\sum_{a=0}^{N-1} B_4\left(\frac{a}{N}\right)\cdot((\pi_\sigma)_*\varphi)(a) \end{array}$$

record the "values" $\lim_{\tau\to i\infty} E_\varphi(\tau)|^4_{M_\sigma^{-1}}$ ($= \mathscr{H}_\sigma(\varphi)$) of the Eisenstein series E_φ at the cusps. Here $\pi_\sigma : (\mathbb{Z}/N\mathbb{Z})^2 \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$ sends $(m,n) = a(p,q) + b(-s,r) \mapsto a$, while $(\pi_\sigma)_*$ sums along fibers of π_σ, and $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$ is the fourth Bernoulli polynomial. Alternatively, one has

$$\mathscr{H}_\sigma(\varphi) = -\frac{6}{(2\pi i)^4} L(\iota_\sigma^*\hat{\varphi}, 4)$$

where $\iota_\sigma : \mathbb{Z}/N\mathbb{Z} \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^2$ sends $a \mapsto a(-s,r)$ and

$$L(\phi, n) := \sum_{k\geq 1} \frac{\phi(k)}{k^n}.$$

To construct the cycles, let $U \subset \mathcal{E}(N)$ [resp. $U^{[2]} \subset \mathcal{E}^{[2]}(N)$] be the complement of the N^2 [resp. N^4] N-torsion sections over $Y(N)$. Fix $\varphi \in \Phi_2^{\mathbb{Q}}(N)^\circ$, and (thinking of it as a $\mathbb{Q}$-divisor supported on $\mathcal{E}(N)\backslash U$) let $m_\alpha \in \mathbb{Q}$ and $f_{\alpha 1}, f_{\alpha 2}, f_{\alpha 3} \in \mathcal{O}^*(U)$ satisfy $\sum_\alpha m_\alpha (f_{\alpha 1}) * (f_{\alpha 2}) * (f_{\alpha 3}) = \varphi$ (Pontryagin product). (Here $(f_{\alpha i})$ is the divisor of $f_{\alpha i}$, the divisor being viewed as a function on $(\mathbb{Z}/N\mathbb{Z})^2$, and the Pontryagin product of two functions on a finite abelian group is defined by $(f *$

$g)(a) = \sum_{b+c=a} f(b)g(c)$. The group[(9)] $\mathcal{G} := D_4 \ltimes (\mathbb{Z}/N\mathbb{Z})^4$ acts on $H^3_M(U^{[2]}, \mathbb{Q}(3))$, and the $\mathcal{G}$-symmetrization of

$$\sum_\alpha m_\alpha \{f_{\alpha 1}(-z_1), f_{\alpha 2}(z_1 - z_2), f_{\alpha 3}(z_2)\}$$

extends to a cycle in $H^3_M(\mathcal{E}^{[2]}(N), \mathbb{Q}(3))$ (cf. [**DK**, sec. 7.3.4]). By abuse of notation we shall call it $\mathfrak{Z}_\varphi$, since its fiberwise $AJ^{3,3}$-classes

$$(5.1.1) \qquad \mathcal{R}_\varphi(y) \in H^2_{var}\left(\pi^{[2]}(N)^{-1}(y), \mathbb{C}/\mathbb{Q}(3)\right), \quad y \in Y(N),$$

depend only on φ – indeed, only on the $\{\mathscr{H}_\sigma(\varphi) | \sigma \in \kappa(N)\}$ – and not the choice of $\{f_{\alpha i}\}$ [**DK**, Cor 9.1].[(10)]

The connection between the cycle $\mathfrak{Z}_\varphi$ and the Eisenstein series E_φ comes about as follows. First, by using the moving lemma [**Blo2**] and log complexes of currents, it is possible to extend the (T, Ω, R) calculus of §4.2 to the quasi-projective setting ([**KLM**, §5.9],[**KL**, §3.1]). In particular, the fundamental class of $\mathfrak{Z}_\varphi$ (i.e. the image of $c^{3,3}_{\mathscr{D}}(\mathfrak{Z}_\varphi)$ in $Hg^{3,3}(\mathcal{E}^{[2]}(N))_\mathbb{Q}$) is computed by the holomorphic $(3,0)$-form $\Omega_{\mathfrak{Z}_\varphi}$. According to a result of Beilinson (in the form of [**DK**, Thm. 8.1]), we have

$$(5.1.2) \qquad \Omega_{\mathfrak{Z}_\varphi} = \Psi(E_\varphi) = \{(2\pi i)^3 E_\varphi(\tau) dz_1 \wedge dz_2\} \otimes d\tau.$$

It follows that $\mathcal{R}_\varphi(y)$ is given (up to an important "constant of integration") by the Gauss-Manin integral of (5.1.2); that is, (5.1.2) is $\nabla \mathcal{R}_\varphi$.

Define the associated higher normal function by[(11)]

$$(5.1.3) \qquad V_\varphi(\tau) := \langle \tilde{\mathcal{R}}_\varphi([\tau]), [dz_1 \wedge dz_2] \rangle,$$

(9) D_4 denotes the dihedral group of order 8.

(10) The reader is warned of the typo "surjective" for "injective" in the statement of [loc. cit., Lemma 9.1(ii)].

(11) Note: *a priori* this just uses the Poincaré pairing $H^2(E_\tau^{\times 2}, \mathbb{C})^{\otimes 2} \to \mathbb{C}$ on each fiber. However, it is better to think of $[dz_1 \wedge dz_2]$ as a class in $H_2(E_\tau^{\times 2}, \mathbb{C})$ by Poincaré duality and (5.1.3) as pairing $H^2 \times H_2 \to \mathbb{C}$, since this approach will extend across the singular fibers of $\overline{\mathcal{E}^{[2]}(N)}$ over cusps σ for which $\mathscr{H}_\sigma(\varphi) = 0$.

where for now $\tilde{\mathcal{R}}_\varphi$ is an indeterminate lift of $\mathcal{R}_\varphi$ to $\mathcal{O}\otimes R^2_{var}\pi^{[2]}(N)_*\mathbb{C}$. Arguing as in the proof of Corollary 4.3.3 above, and noting $\nabla^3_\tau[dz_1\wedge dz_2]=0$,

$$\begin{aligned}\frac{d^3}{d\tau^3}V_\varphi(\tau) &= \frac{d^2}{d\tau^2}\left\langle \mathcal{R}_\varphi, \nabla_\tau[dz_1\wedge dz_2]\right\rangle \\ &= \frac{d}{d\tau}\left\langle \mathcal{R}_\varphi, \nabla^2_\tau[dz_1\wedge dz_2]\right\rangle \\ &= \left\langle (2\pi i)^3 E_\varphi(\tau)[dz_1\wedge dz_2], 2[\alpha_1\times\alpha_2]\right\rangle \\ &= -2(2\pi i)^3 E_\varphi(\tau). \end{aligned}\tag{5.1.4}$$

That is, V_φ is an Eichler integral of E_φ. This leads to the following result, which is closely related to [**DK**, Prop. 9.2].

Proposition 5.1.1. — *Assume for simplicity that $\hat{\varphi}(m,n)=\hat{\varphi}(-m,-n)$. Then up to a $\mathbb{Q}(3)$-period $(2\pi i)^3Q_0+(2\pi i)^2Q_1\log q+(2\pi i)Q_2(\log q)^2$ ($Q_i\in\mathbb{Q}$),*

$$\begin{aligned}V_\varphi(q)\equiv\ & \tfrac{2}{(2\pi i)^4}L\left(\iota^*_{i\infty}\hat{\varphi},4\right)(\log q)^3+\tfrac{1}{N}L\left((\pi_{i\infty})_*\hat{\varphi},3\right)\\ &+\tfrac{2}{N}\textstyle\sum_{M\geq 1}q^{\frac{M}{N}}\sum_{d|M}\frac{1}{d^3}\sum_{a\in\mathbb{Z}/N\mathbb{Z}}\zeta_N^{\frac{aM}{d}}\hat{\varphi}(d,a).\end{aligned}\tag{5.1.5}$$

(Note that $\frac{1}{(2\pi i)^4}L(\iota^*_{i\infty}\hat{\varphi},4)\in\mathbb{Q}$.)

Proof. — By a classical result (cf. [**Gu**]), we have

$$E_\varphi(\tau)=\mathscr{H}_{[i\infty]}(\varphi)-\frac{1}{N^4}\sum_{M\geq 1}q^{\frac{M}{N}}\sum_{r|M}r^3\sum_{a\in\mathbb{Z}/N\mathbb{Z}}\zeta_N^{ar}\hat{\varphi}\left(\frac{M}{r},a\right).\tag{5.1.6}$$

In accordance with (5.1.4), we must take three indefinite integrals of $-2(2\pi i)^3E_\varphi(\tau)$ with respect to $d\tau=\frac{1}{2\pi i}d\log q$, i.e. of $-2E_\varphi(\tau)$ with respect to $d\log q$. Applying this to the second term of (5.1.6) gives

$$\frac{2}{N}\sum_{M\geq 1}\frac{q^{\frac{M}{N}}}{M^3}\sum_{r|M}r^3\sum_{a\in\mathbb{Z}/N\mathbb{Z}}\zeta_N^{ar}\hat{\varphi}\left(\frac{M}{r},a\right),\tag{5.1.7}$$

and replacing r by $d=\frac{M}{r}$ recovers the sum in (5.1.5). Doing the same to $\mathscr{H}_{[i\infty]}(\varphi)$ would give $-\frac{1}{3}\mathscr{H}_{[i\infty]}(\varphi)(\log q)^3$ plus an arbitrary quadratic polynomial in $\log q$. The more precise stated result follows at once from

[**DK**, (9.29)],[12] which is based on the delicate fiberwise $AJ^{3,3}$ computation for $\mathfrak{Z}_\varphi$ carried out in §9.2 of [op. cit.]. □

The connection of this formula to trilogarithms arises as follows. Define

$$\widehat{Li}_3(x) := \sum_{k\geq 1} Li_3(x^k) = \sum_{k\geq 1}\sum_{\delta\geq 1}\frac{x^{k\delta}}{\delta^3} = \sum_{m\geq 1} x^m \sum_{\delta|m}\frac{1}{\delta^3}, \tag{5.1.8}$$

and suppose that we can write

$$\hat{\varphi} = \sum_{\substack{\alpha|N\\ \beta|N}} \mu_{\alpha\beta}\psi_{\alpha,\beta}$$

where

$$\psi_{\alpha,\beta}(m,n) := \begin{cases} 1, \text{ if } \alpha|m \text{ and } \beta|n \\ 0, \text{ otherwise} \end{cases}. \tag{5.1.9}$$

In the $\sum_{M\geq 1}$ term

$$\frac{2}{N}\sum_{\alpha,\beta}\mu_{\alpha\beta}\sum_{M\geq 1} q^{\frac{M}{N}}\sum_{d|M}\frac{1}{d^3}\sum_{a\in\mathbb{Z}/N\mathbb{Z}}\zeta_N^{\frac{aM}{d}}\psi_{\alpha,\beta}(d,a)$$

of (5.1.5), the sum $\sum_{a\in\mathbb{Z}/N\mathbb{Z}}\zeta_N^{\frac{aM}{d}}\psi_{\alpha,\beta}(d,a)$ is zero unless $\alpha|d$ and $\frac{N}{\beta}|\frac{M}{d}$ (implies $\frac{\alpha N}{\beta}|M$), in which case it is $\frac{N}{\beta}$. So after putting $M=\frac{\alpha N}{\beta}m$ and $d=\alpha\delta$, the last displayed expression becomes

$$= \frac{2}{N}\frac{N}{\beta}\sum_{\alpha,\beta}\mu_{\alpha\beta}\sum_{m\geq 1}\left(q^{\frac{\alpha}{\beta}}\right)^m\sum_{\delta|m}\frac{1}{\delta^3\alpha^3},$$

which (upon putting $k=\frac{m}{\delta}$)

$$= 2\sum_{\alpha,\beta}\frac{\mu_{\alpha\beta}}{\beta\alpha^3}\widehat{Li}_3\left(q^{\frac{\alpha}{\beta}}\right). \tag{5.1.10}$$

[12] Note that while this formula is derived in [op. cit.] for φ of the form $\frac{1}{N}\pi^*_{i\infty}\varphi'$, any φ is of this form modulo $\ker(\mathscr{H}_{[i\infty]})\subset\Phi_2^{\mathbb{Q}}(N)^\circ$.

5.2. Modular pullback of the three-banana cycle. — In this subsection, we identify the pullback of $\Xi_{\circledcirc}$ to $\mathcal{E}^{[2]}(6)$ as an Eisenstein symbol. We begin with a general statement.

Let $\mathbb{X} \overset{\rho}{\to} \mathbb{P}^1$ be a 1-parameter family of anticanonical hypersurfaces in a toric Fano 3-fold $\mathbb{P}_\Delta$, with smooth total space obtained by a blow-up $\mathbb{X} \overset{\beta}{\twoheadrightarrow} \mathbb{P}_\Delta$, and $\beta(\mathrm{X}_0) := \beta(\rho^{-1}(0)) = \mathbb{D}_\Delta := \mathbb{P}_\Delta \backslash (\mathbb{C}^*)^3$. Suppose we have a higher cycle $\bar{\Xi} \in H^3_M(\mathbb{X}\backslash \mathrm{X}_0, \mathbb{Q}(3))$ with $\partial T_{\bar{\Xi}}$ (4.2.2) the integral generator of $H_2(\mathrm{X}_0, \mathbb{Z})$,[13] and a rational map (or even a correspondence)

$$\begin{array}{ccc} \overline{\mathcal{E}^{[2]}(N)} & \overset{\Theta}{\dashrightarrow} & \mathbb{X} \\ {\scriptstyle \overline{\pi^{[2]}(N)}}\downarrow & & \downarrow \\ \overline{Y(N)} & \overset{\mathsf{H}}{\twoheadrightarrow} & \mathbb{P}^1. \end{array}$$

Let $\Theta : \mathcal{E}^{[2]}(N) \dashrightarrow \mathcal{X}$ be the restriction to the complement of the singular fibers, and $\Xi \in H^3_M(\mathcal{X}, \mathbb{Q}(3))$ the restriction of $\bar{\Xi}$. Defining coefficients $r_\sigma(\Xi) \in \mathbb{Q}$ by

$$\bar{\Theta}^*(\mathrm{X}_0) = \sum_{\sigma \in \kappa(N)} r_\sigma(\Xi) \cdot \overline{\pi^{[2]}(N)}^{-1}(\sigma),$$

we have

Proposition 5.2.1. — *Modulo cycles with trivial fiberwise $AJ^{3,3}$, we have*

$$\Theta^* \Xi = \mathfrak{Z}_\varphi \in H^3_M(\mathcal{E}^{[2]}(N), \mathbb{Q}(3))$$

for any $\varphi \in \Phi_2^{\mathbb{Q}}(N)^\circ$ with $\mathscr{H}_\sigma(\varphi) = r_\sigma(\Xi)$ $(\forall \sigma \in \kappa(N))$.

Proof. — This is immediate from the fact that (5.1.1) depends only on the "residues" $\mathscr{H}_\sigma(\varphi)$. □

[13] Alternatively, $Res(\bar{\Xi}) \in H^4_{M,X_0}(\mathbb{X}, \mathbb{Q}(3))$ has cycle class in $H^4_{X_0}(\mathbb{X}, \mathbb{Q}(3))$, $\frac{1}{(2\pi i)^3}$ of which integrally generates $H_2(X_0, \mathbb{Z})$.

To apply this general statement to the three-banana cycle $\Xi_{\ominus}$ constructed in §4.1, we begin by analyzing the transformation of the family of holomorphic forms $\omega := \{\omega_t\} \in \Gamma(\mathbb{P}^1 \backslash \mathcal{L}_{\ominus}, (\pi_{\ominus})_* \Omega^2_{\pi_{\ominus}})$ (cf. (3.3.1)) under the correspondence

$$\begin{array}{ccccccccc} & & & & \overset{\theta}{\frown} & & & & \\ \mathcal{X}_{\ominus} & \underset{\cong}{\overset{\mathcal{H}_{\ominus}}{\longleftarrow}} & \mathcal{X}_1(6)^{+3} & \underset{2:1}{\overset{p_2}{\twoheadleftarrow}} & \tilde{\mathcal{X}} & \underset{2:1}{\overset{p_1}{\twoheadrightarrow}} & {}'\mathcal{E}_1^{[2]}(6)^{+3} & \overset{\bar{J}_3^{[2]}}{\twoheadleftarrow} & \mathcal{E}_1^{[2]}(6) \\ \downarrow{\scriptstyle \pi_{\ominus}} & & & \searrow & \downarrow & \swarrow & & & \downarrow \\ \mathbb{P}^1 \setminus \mathcal{L}_{\ominus} & & \underset{\cong}{\overset{H_{\ominus}}{\longleftarrow}} & & Y_1(6)^{+3} & & \underset{2:1}{\twoheadleftarrow} & & Y_1(6). \end{array}$$

between $\mathcal{E}_1^{[2]}(6)$ and $\mathcal{X}_{\ominus}$. Here $\tilde{\mathcal{X}}$ is described in §3.3, and

$$J_3^{[2]} : \mathcal{E}_1^{[2]}(6) := \mathcal{E}_1(6) \underset{Y_1(6)}{\times} \mathcal{E}_1(6) \overset{3:1}{\twoheadrightarrow} {}'\mathcal{E}_1^{[2]}(6)$$

is the map over $Y_1(6)$ defined by

$$(\tau; [z_1]_\tau, [z_2]_\tau) \longmapsto \left(\tau; [z_1]_\tau, \left[\frac{-z_2}{2\tau+1}\right]_{\alpha_3(\tau)}\right),$$

and $\bar{J}_3^{[2]}$ its composition with the quotient by $I_3^{[2]}$ (cf. §3.2).

By (2.3.13), the period of $\mathcal{H}_{\ominus}^*\omega$ over the minimal invariant cycle in $H_{2,\mathbb{Z}}^{tr}$ about $q = 0$ ($t = \infty$) limits to $(2\pi i)^2$. Applying p_2^*, $(p_1)_*$, $(\bar{J}_3^{[2]})^*$ multiplies this by 2, 2, resp. 3. Writing $\theta^* := (\bar{J}_3^{[2]})^*(p_1)_* p_2^*$, it follows that

$$\theta^*\omega \equiv 12(2\pi i)^2 dz_1 \wedge dz_2 \mod \mathcal{O}(q) \tag{5.2.1}$$

hence (noting $\delta_q = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$)

$$\begin{aligned} Y(\infty) &= \left\langle \omega, \nabla^2_{\delta_t}\omega \right\rangle |_{t=\infty} \\ &= \frac{1}{12} \left\langle \theta^*\omega, \nabla^2_{\delta_q}\theta^*\omega \right\rangle |_{q=0} \\ &= \frac{12^2(2\pi i)^4}{12(2\pi i)^2} \left\langle [dz_1 \wedge dz_2], \nabla^2_\tau [dz_1 \wedge dz_2] \right\rangle \end{aligned}$$

$$= -24(2\pi i)^2,$$

where $Y(t)$ was defined in Remark 4.3.4. In fact, by that remark we now have $\kappa = \frac{-24}{(2\pi i)^2}$ as claimed in the proof of Corollary 4.3.3.

Turning to the computation of the $\{r_\sigma(\Xi_\ominus)\}$, we take Θ to be the composition of θ with the base change over $Y(6) \twoheadrightarrow Y_1(6)$. We examine the pullback by Θ of the $(3,0)$ form $\Omega_{\Xi_\ominus}$ which computes the fundamental class of the cycle. By (4.2.3) and Proposition 4.2.1(ii), $\Omega_{\Xi_\ominus} = -\omega \wedge \frac{dt}{t} \in \Omega^3(\mathcal{X}_\ominus)$, and (5.2.1) now gives

$$\Omega_{\Theta^*\Xi_\ominus} = -\Theta^*\Omega_{\Xi_\ominus} = \Theta^*\omega \wedge \mathrm{dlog} H_\ominus(\tau)$$
$$\equiv 12(2\pi i)^3 dz_1 \wedge dz_2 \wedge d\tau \mod \mathcal{O}(q),$$

which implies at once that $r_{[i\infty]}(\Xi_\ominus) = 12$. (Note the consistency with (5.1.2) and (5.1.6).) Now the (partial) pullback of $\Xi_\ominus$ to $'\mathcal{E}_1^{[2]}(6)$ is invariant under $I_3^{[2]}$; a calculation as in [**DK**, sec. 8.2.2] shows that consequently $r_{[-\frac{1}{2}]}(\Xi_\ominus) = r_{[\alpha_3(i\infty)]}(\Xi_\ominus) = -\frac{r_{[i\infty]}(\Xi_\ominus)}{3^2} = -\frac{4}{3}$. In fact, writing $\Omega_{\Theta^*\Xi_\ominus} = (2\pi i)^3 E_\ominus(\tau) dz_1 \wedge dz_2 \wedge d\tau$, we have $E_\ominus(\tau) \in M_4(\Gamma_1(6)^{+3})$; and $r_\sigma(\Xi_\ominus) : \kappa(6) \to \mathbb{Q}$ is the pullback of the function on $\kappa_1(6) = \left\{[i\infty], [0], [\frac{1}{2}], [\frac{1}{3}]\right\}$ taking the respective values $12, 0, -\frac{4}{3}, 0$. (Under $\kappa(6) \twoheadrightarrow \kappa_1(6)$, the preimage of $[i\infty]$ resp. $[\frac{1}{2}]$ is $\{[i\infty]\}$ resp. $\left\{[\frac{1}{2}], [\frac{3}{2}], [-\frac{1}{2}]\right\}$.) Using the formula for $\mathscr{H}_\sigma$, one then finds that the function $\varphi_\ominus$ on $(\mathbb{Z}/N\mathbb{Z})^2$ with Fourier transform

$$(5.2.2) \qquad \hat{\varphi}_\ominus(m,n) := \begin{cases} -2^6 3^5/5, & (m,n) \equiv (0, \pm 1) \mod 6 \\ 2^6 3^3/5, & (m,n) \equiv (\pm 2, \pm 1 \text{ or } 3) \mod 6 \\ 0, & \text{otherwise} \end{cases}$$

satisfies $\mathscr{H}_\sigma(\varphi_\ominus) = r_\sigma(\Xi_\ominus)$.

Finally we determine the pullbacks of ω and $\tilde{\omega}$. In [**Ve**], it is shown that $\varpi_1(\tau) = (\eta(2\tau)\eta(6\tau))^4 (\eta(\tau)\eta(3\tau))^{-2}$ is the $H_\ominus$-pullback of a solution to D_{PF}; so $\Theta^*(\tilde{\omega}) = C \cdot \varpi_1(\tau) dz_1 \wedge dz_2$ for some constant C. But then $\Theta^*(\omega) = -(2\pi i)^2 C \varpi_1(\tau) H_\ominus(\tau) dz_1 \wedge dz_2$ and by (5.2.1) $C = 12$.

Remark 5.2.2. — One further immediate consequence is that $E_{\circleddash}(\tau) = 12\frac{\varpi_1(\tau)}{2\pi i}\frac{dH_{\circleddash}^{-1}(\tau)}{d\tau} = 12+24q-168q^2+\cdots$; but the equality $E_{\circleddash}(\tau) = E_{\varphi_{\circleddash}}(\tau)$ is more useful for us as it allows us to apply Proposition 5.1.1 and get the "constant of integration" right.

5.3. The main result. — Recall that $V_{\circleddash}(t) = \langle \mathcal{R}_t, [\tilde{\omega}_t]\rangle$. Putting everything together, we arrive at the

Theorem 5.3.1. — *Up to a* $\mathbb{Q}(3)$*-period* $(2\pi i)^3Q_0+(2\pi i)^2Q_1\tau+(2\pi i)Q_2\tau^2$ $(Q_i \in \mathbb{Q})$, *we have* $\frac{V_{\circleddash}(H_{\circleddash}(\tau))}{\varpi_1(\tau)} =$

$$-4(\log q)^3 + 16\zeta(3) - 16\left\{2\widehat{Li}_3(q^6) - \widehat{Li}_3(q^3) - 6\widehat{Li}_3(q^2) + 3\widehat{Li}_3(q)\right\},$$

where $\widehat{Li}_3(x) := \sum_{k\geq 1} Li_3(x^k)$.

Proof. — First notice that

$$\begin{aligned} V_{\circleddash} = \langle \mathcal{R}, \tilde{\omega}\rangle &= \frac{1}{12}\langle \Theta^*\mathcal{R}, \Theta^*\tilde{\omega}\rangle \\ &= \frac{1}{12}\langle \mathcal{R}_{\varphi_{\circleddash}}, 12\varpi(\tau)[dz_1\wedge dz_2]\rangle \\ &= \varpi_1(\tau)\langle \mathcal{R}_{\varphi_{\circleddash}}, [dz_1\wedge dz_2]\rangle \end{aligned}$$

so that $V_{\circleddash} = \varpi_1 V_{\varphi_{\circleddash}}$. The leading term in (5.1.5) is $-\frac{2}{3!}\mathscr{H}_{[i\infty]}(\varphi_{\circleddash})(\log q)^3 = -4(\log q)^3$. For the constant term we compute

$$\left((\pi_{[i\infty]})_*\hat{\varphi}_{\circleddash}\right)(n) = \begin{cases} -2^7 3^5/5, & n\equiv 0 \mod 6 \\ 2^6 3^4/5, & n \equiv \pm 2 \mod 6 \\ 0, & \text{otherwise}\end{cases}$$

$$\begin{aligned} \implies \frac{1}{6}L\left((\pi_{[i\infty]})_*\hat{\varphi}_{\circleddash}, 3\right) &= \frac{1}{6}\sum_{n\geq 1}\frac{(\pi_{[i\infty]})_*\hat{\varphi}_{\circleddash}(n)}{n^3} \\ &= \frac{1}{6}\cdot\frac{-2\cdot 6^5}{5}\left\{\frac{7}{3}\cdot\frac{1}{6^3}\zeta(3) - \frac{1}{3}\cdot\frac{1}{2^3}\zeta(3)\right\} \\ &= \frac{-2\cdot 6^4}{5}\cdot\frac{-20}{3\cdot 6^3}\zeta(3) = 16\zeta(3). \end{aligned}$$

Finally, we write using the character $\psi_{a,b}$ defined in (5.1.9)

$$\hat{\varphi}_{\circledcirc} = \frac{-3^3 2^6}{5}\left\{10\psi_{6,1} - 10\psi_{6,2} - 9\psi_{6,3} + 9\psi_{6,6} - \psi_{2,1} + \psi_{2,2}\right\}.$$

Substituting this into (5.1.10) gives the remaining terms in the result. □

6. Foundational Results via Hodge Theory

The methodology of sections 4 and 5 involving higher Chow cycles and currents is delicate. Care is needed to avoid bad position and ill-defined multiplication of currents. The purpose of this section is to give a general Hodge-theoretic context for proving basic results about periods in related situations. In the context of this paper, arguments using currents are required to lift the Milnor symbol regulator, defined a priori only on X_t^*, over all of X_t. Arguments in this section only give results upto periods over X_t^*. Because $X_t \backslash X_t^*$ in our case is a union of rational curves, it turns out that these extra periods associated to 2-chains on X_t relative to $X_t \backslash X_t^*$ are themselves of motivic interest. This point will be discussed briefly at the end of the section.

6.1. Some lemmas. — In this subsection, we give an elementary but useful application of Verdier duality (Lemma 6.1.4) – also known, thanks to R. MacPherson, as "red-green duality" (cf. Remark 6.1.5).We work throughout with sheaves for the complex topology.

Lemma 6.1.1. — *Let P be a smooth, quasi-projective variety over $\mathbb{C}$, and let $X, Y \subset P$ be closed subvarieties. Consider the diagram*

$$\begin{array}{ccccc} P\setminus(X\cup Y) & \xrightarrow{j'} & P\setminus X & & \\ \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k} & & \\ P\setminus Y & \xrightarrow{j} & P & \xleftarrow{i} & Y \end{array} \tag{6.1.1}$$

Assume that for every point $z \in X \cap Y$ there exists a ball B about z in P and a decomposition $B = B_X \times B_Y$ (where B_X, B_Y are smaller *dimensional balls). Assume further there exist analytic subvarieties $X' \subset B_X$ and $Y' \subset B_Y$ such that $X \cap B = X' \times B_Y$ and $Y \cap B = B_X \times Y'$. Then viewed as maps on the respective derived categories of sheaves for the complex topology (in keeping with modern usage we write e.g. j_* in place of Rj_*) we have*

$$j_! k'_* \mathbb{Q}_{P\setminus(X\cup Y)} = k_* j'_! \mathbb{Q}_{P\setminus(X\cup Y)}. \tag{6.1.2}$$

Proof. — We have

$$j^* k_* j'_! \mathbb{Q}_{P\setminus(X\cup Y)} = k'_* \mathbb{Q}_{P\setminus(X\cup Y)}. \tag{6.1.3}$$

Since $j_!$ is left adjoint to j^* we deduce the existence of a map (extending the identity map on $P \setminus (X \cup Y)$) from left to right in (6.1.2). To check that this map is a quasi-isomorphism is a local problem. The assertion is evident except at points of $X \cap Y \subset P$. By assumption, near such a point our diagram (6.1.1) looks like

(6.1.4)
$$\begin{array}{ccccc} (B_X \setminus X') \times (B_Y \setminus Y') & \longrightarrow & (B_X \setminus X') \times B_Y & & \\ \downarrow & & \downarrow & & \\ B_X \times (B_Y \setminus Y') & \longrightarrow & B_X \times B_Y & \longleftarrow & B_X \times Y'. \end{array}$$

The assertion is now clear by a variant of the Kunneth formula. Namely, both sides are identified with

$$(k_{B_X *} \mathbb{Q}_{B_X \setminus X'}) \otimes (j_{B_Y !} \mathbb{Q}_{B_Y \setminus Y'}). \tag{6.1.5}$$

□

Remark 6.1.2. — The hypotheses of the lemma are satisfied if $X \cup Y \subset P$ is a normal crossings divisor locally at points of $X \cap Y$.

Lemma 6.1.3. — *Let notation be as above and write $Z = X \cap Y$. We have*

$$H^*(P \setminus X, Y \setminus Z; \mathbb{Q}) \cong H^*(P, j_! k'_* \mathbb{Q}) \tag{6.1.6}$$

Proof. — We have

$$j_!k'_*\mathbb{Q}_{P\setminus(X\cup Y)} = j_!j^*k_*\mathbb{Q}_{P\setminus X}. \tag{6.1.7}$$

The functorial distinguished triangle of sheaves on P

$$j_!j^*\mathcal{S} \to \mathcal{S} \to i_*i^*\mathcal{S} \xrightarrow{+1} \dots$$

yields a distinguished triangle

$$j_!k'_*\mathbb{Q}_{P\setminus(X\cup Y)} \to k_*\mathbb{Q}_{P\setminus X} \to i_*i^*k_*\mathbb{Q}_{P\setminus X}. \tag{6.1.8}$$

Consider the diagram

$$\begin{array}{ccc} P\setminus X & \xleftarrow{\ \ell\ } & Y\setminus Z \\ \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle k''} \\ P & \xleftarrow{\ i\ } & Y \end{array} \tag{6.1.9}$$

The lemma will follow if we show $i^*k_*\mathbb{Q}_{P\setminus X} \xrightarrow{\cong} k''_*\ell^*\mathbb{Q}_{P\setminus X}$ in (6.1.9). Since i^* is left-adjoint to i_*, the existence of such a map is equivalent to the existence of a map

$$k_* \to i_*k''_*\ell^* = k_*\ell_*\ell^*. \tag{6.1.10}$$

It is enough to define a map from the identity functor to $\ell_*\ell^*$. But again by adjunction, this is the same as a map $\ell^* \to \ell^*$. Here we can take the identity.

Arguing as before, the problem is now local and we can work in a small ball $B = B_X \times B_Y$. The local picture with the notation of the previous lemma is

$$\begin{array}{ccc} (B_X\setminus X')\times B_Y & \longleftarrow & (B_X\setminus X')\times Y' \\ \Big\downarrow & & \Big\downarrow \\ B_X\times B_Y & \longleftarrow & B_X\times Y'. \end{array} \tag{6.1.11}$$

Again the assertion is clear by Kunneth. □

Lemma 6.1.4. — *Let notation and assumptions be as above, and write* $n = \dim P$. *Assume* P *is smooth and projective. Then we have a perfect pairing*

$$H^*(P \setminus Y, X \setminus Z; \mathbb{Q}(n)) \times H^{2n-*}(P \setminus X, Y \setminus Z; \mathbb{Q}) \to \mathbb{Q}. \tag{6.1.12}$$

Said another way, we have

$$H^*(P \setminus Y, X \setminus Z; \mathbb{Q}(n)) \cong H_{2n-*}(P \setminus X, Y \setminus Z; \mathbb{Q}). \tag{6.1.13}$$

Proof. — From the previous lemma applied twice we are reduced to showing

$$H^*(P, j_! k'_* \mathbb{Q})(-n) \cong H^{2n-*}(P, k_! j'_* \mathbb{Q})^\vee. \tag{6.1.14}$$

The Verdier duality functor $\mathbb{D}$ is a contravariant functor on the derived category of sheaves on P such that the sheaves $\mathcal{S}$ and $\mathbb{D}\mathcal{S}$ are Poincaré dual, i.e. there is a perfect pairing $H^i(P, \mathcal{S}) \times H^{-i}(P, \mathbb{D}\mathcal{S}) \to \mathbb{Q}$. We have $\mathbb{D}\mathbb{Q} = \mathbb{Q}[-2n](n)$, and $\mathbb{D}$ intertwines lower shriek and lower star. Thus

$$\begin{aligned} H^{2n-*}(P, k_! j'_* \mathbb{Q})^\vee &= H^{-*}(P, k_! j'_* \mathbb{D}\mathbb{Q})(-n) = \\ H^{-*}(P, \mathbb{D}(k_* j'_! \mathbb{Q}))(-n) &= H^*(P, k_* j'_! \mathbb{Q})(-n) = H^*(P, j_! k'_* \mathbb{Q})(-n). \end{aligned} \tag{6.1.15}$$

□

Remark 6.1.5. — In the analytic context, one way of representing the factors of (6.1.12) is in terms of topological cycles (using (6.1.13) and its analogue for the other factor). For the left-hand factor, these must avoid X (red) but are allowed to bound on Y (green); whereas for the right-hand factor, red and green are swapped.

6.2. Applications: CY periods. — Take $n \geq 2$ and assume (various generalizations are possible) that $\pi : P \to \mathbb{P}^n$ is a toric variety obtained by a sequence of blowups. Let $X \subset P$ be the strict transform of a hypersurface of degree $n+1, X_0 \subset \mathbb{P}^n$. Let $Y_0 \subset \mathbb{P}^n$ be the coordinate simplex $Y_0 : \prod_0^n T_i = 0$ where the T_i are homogeneous coordinates, and let $Y = \pi^{-1}Y_0$. We assume that X is smooth, and $Y \cup X$ is a normal crossings divisor. Let $Z = X \cap Y$. Note that $P \setminus Y \cong \mathbb{P}^n \setminus Y_0 \cong \mathbb{G}_m^n$. The exact sequence of relative cohomology yields

$$(6.2.1)\quad H^{n-1}(\mathbb{G}_m^n, \mathbb{Q}(n)) \to H^{n-1}(X \setminus Z, \mathbb{Q}(n)) \to \\ H^n(P \setminus Y, X \setminus Z; \mathbb{Q}(n)) \to H^n(\mathbb{G}_m^n, \mathbb{Q}(n)) \to 0.$$

This can be rewritten (the superscript $\tilde{}$ indicating we take the quotient modulo the image of $H^{n-1}(\mathbb{G}_m^n, \mathbb{Q}(n))$)

$$(6.2.2)\quad 0 \to H^{n-1}(X \setminus Z, \mathbb{Q}(n))^{\sim} \stackrel{\alpha}{\to} \\ H^n(P \setminus Y, X \setminus Z; \mathbb{Q}(n)) \to \mathbb{Q}(0) \to 0.$$

Assume further that the topological chain given by $T_i \geq 0, 0 \leq i \leq n$, lifts to a chain σ on P with $\partial\sigma \subset Y$ and $\sigma \cap X = \emptyset$.[(14)] Then σ represents a class in $H_n(P \backslash X, Y \backslash Z; \mathbb{Q})$ which maps to $1 \in \mathbb{Q}(0) = H_n(P \backslash Y, \mathbb{Q})$. Via equation (6.1.13) above, we can interpret $\sigma \in H^n(P \backslash Y, X \backslash Z; \mathbb{Q}(n))$ as a splitting of (6.2.2) as an exact sequence of $\mathbb{Q}$-vector spaces. The extension class of (6.2.2) in the ext group of mixed Hodge structures

$$(6.2.3)\quad Ext^1_{MHS}(\mathbb{Q}(0), H^{n-1}(X \backslash Z, \mathbb{Q}(n))^{\sim}) \cong H^{n-1}(X \backslash Z, \mathbb{C}(n)/\mathbb{Q}(n))^{\sim}$$

is computed as follows. By [**D**] corollaire 3.2.15(ii) it follows that

$$F^0 H^{n-1}(X \backslash Z, \mathbb{C}(n))^{\sim} = (0).$$

As a consequence, one has $F^0 H^n(P \backslash Y, X \backslash Z; \mathbb{C}(n)) \cong \mathbb{C}(0)$, so there is a unique $s_F \in F^0 H^n(P \backslash Y, X \backslash Z; \mathbb{C}(n))$ lifting 1. So the class of the extension (6.2.2) is given by

$$\varepsilon \in H^{n-1}(X \backslash Z, \mathbb{C}(n)/\mathbb{Q}(n))^{\sim},$$

where ε is the unique class with $\alpha(\varepsilon) = \sigma - s_F$.

---

(14) One can check for our family of $K3$-surfaces that blowing up the vertices and then the faces of dimension 1 in $\mathbb{P}^3$ suffices to achieve $\sigma \cap X = \emptyset$.

By assumption, X_0 is an anti-canonical hypersurface in $\mathbb{P}^n$. Let $\Omega_0 \neq 0$ be a global n-form on $\mathbb{P}^n$ with a pole of order 1 along X_0 and no other singularities. Assume further the pullback $\Omega := \pi^*\Omega_0$ has a pole along the strict transform X of X_0 and no other singularities, so Ω represents a class in $F^nH^n(P\backslash X, \mathbb{C})$. We have $H^n(Y\backslash Z, \mathbb{C}) = (0)$ by cohomological dimension, and $F^nH^{n-1}(Y\backslash Z, \mathbb{C}) = (0)$ by [**D**] corollaire 3.2.15(ii), so the exact sequence of relative cohomology yields an isomorphism $F^nH^n(P\backslash X, \mathbb{C}) \cong F^nH^n(P\backslash X, Y\backslash Z; \mathbb{C})$. Thus, Ω lifts canonically to $\Omega \in F^nH^n(P\backslash X, Y\backslash Z; \mathbb{C})$. We have a perfect pairing of mixed Hodge structures by lemma 6.1.4[15]

$$\langle\ ,\ \rangle' : H^n(P\backslash X, Y\backslash Z; \mathbb{Q}) \otimes H^n(P\backslash Y, X\backslash Z; \mathbb{Q}(n)) \to \mathbb{Q}(0) \tag{6.2.4}$$

In particular, the element $\langle \Omega, s_F\rangle' \in F^n\mathbb{C}(0) = (0)$. We have proven

Proposition 6.2.1. — *With notation as above, the pairing of* Ω *with the extension class* (6.2.2) *is given up to (relative) periods*

$$\left\{ \int_\Gamma \Omega \,\middle|\, \Gamma \in \mathit{image}\left\{ H_{n-1}(X, Z; \mathbb{Q}) \stackrel{Tube}{\to} H_n(P\backslash X, Y\backslash Z; \mathbb{Q}) \right\} \right\}$$

by the integral of Ω *over the chain* σ*:*

$$\langle \Omega, \sigma - s_F\rangle' = \int_\sigma \Omega. \tag{6.2.5}$$

Alternatively, with $\omega := Res_X(\Omega)$*, we have*

$$\langle \omega, \varepsilon\rangle' \equiv \frac{1}{2\pi i}\int_\sigma \Omega$$

modulo relative periods $\int_\gamma \omega$*,* $\gamma \in H_{n-1}(X, Z; \mathbb{Q})$*.*

To relate the above to the Abel-Jacobi viewpoint for Milnor symbols explained in section 4.2, one can use Deligne cohomology $H^p_{\mathcal{D}}(V, \mathbb{Z}(q))$ for any quasi-projective variety V over $\mathbb{C}$, [**EV**]. There is a functorial cycle class map $CH^a(V, b) \xrightarrow{[\ ]} H^{2a-b}_{\mathcal{D}}(V, \mathbb{Z}(a))$. One has the universal Milnor

[15] We refer to the beginning of section 4.3 for the definition of $\langle\ ,\ \rangle'$.

symbol in degree n which represents a class $sym_n \in CH^n(\mathbb{G}_m^n, n)$. In our situation, one has $X\backslash Z \hookrightarrow P\backslash Y = \mathbb{G}_m^n$. Consider the diagram

(6.2.6)
$$\begin{array}{ccccc} CH^n(\mathbb{G}_m^n, n) & \longrightarrow & CH^n(X\backslash Z, n) & \longrightarrow & CH^n(\mathbb{G}_m^n, X\backslash Z; n-1) \\ \downarrow & & \downarrow & & \downarrow \\ H^n_{\mathcal{D}}(\mathbb{G}_m^n, \mathbb{Z}(n)) & \longrightarrow & H^n_{\mathcal{D}}(X\backslash Z, \mathbb{Z}(n)) & \longrightarrow & H^{n+1}_{\mathcal{D}}(\mathbb{G}_m^n, X\backslash Z; \mathbb{Z}(n)) \end{array}$$

Deligne cohomology fits into an exact sequence

(6.2.7)
$$0 \to \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^{n-1}_{Betti}(V, \mathbb{Z}(r))) \to H^n_{\mathcal{D}}(V, \mathbb{Z}(r)) \to H^n_{Betti}(V, \mathbb{Z}(r))$$

By cohomological dimension, we have

$$H^{n+1}_{Betti}(\mathbb{G}_m^n, \mathbb{Z}) = (0) = H^n_{Betti}(X\backslash Z, \mathbb{Z}),$$

so the bottom line in (6.2.6) can be written

$$\begin{aligned} (6.2.8) \quad H^n_{\mathcal{D}}(\mathbb{G}_m^n, \mathbb{Z}(n)) &\xrightarrow{a} \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^{n-1}_{Betti}(X\backslash Z, \mathbb{Z}(n))) \\ &\to \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^n_{Betti}(\mathbb{G}_m^n, X\backslash Z; \mathbb{Z}(n))) \end{aligned}$$

Consider the diagram with top row the extension given by $a[sym_n]$ in (6.2.8).

(6.2.9)
$$\begin{array}{ccccccccc} 0 & \to & H^{n-1}(X\backslash Z, \mathbb{Q}(n)) & \to & M & \to & \mathbb{Q}(0) & \to & 0 \\ & & \downarrow & & \downarrow b & & \| & & \\ 0 & \to & H^{n-1}(X\backslash Z, \mathbb{Q}(n))^{\sim} & \to & H^n(\mathbb{G}_m^n, X\backslash Z; \mathbb{Q}(n)) & \to & \mathbb{Q}(0) & \to & 0 \end{array}$$

It follows from (6.2.8) that there exists an arrow b as indicated. This means that up to rational scale, the Milnor symbol extension coincides with the extension (6.2.2). Note that this does not recover Theorem 4.3.2. Indeed, quite generally, the ambiguity is given by periods the $\int_c \omega$ where c represents a class in $H_{n-1}(X, Z; \mathbb{Q})$. In our situation, where we have a family X_t of $K3$-surfaces, the resulting multi-valued function of t does not satisfy the inhomogeneous Picard-Fuchs equation because the local system with fibres $H^2(X_t\backslash Z_t)$ is larger than the local system $H^2(X_t)$.

For us, the "extra" periods have the form $\int_{c_t} \omega_t$ where c_t is a 2-disc on X_t with boundary on Z_t. Since Z_t is a union of rational curves, such periods are associated to motivic cohomology classes in $H^3_M(X_t, \mathbb{Q}(2))$. For more detail on these interesting periods, see [**K**] and the references cited there.

7. Special values of the integral

As promised in §4.3, we present some consequences for special values of the identification of the Feynman integral as a higher normal function (Theorem 4.3.2), by evaluating the three-banana integral at the special values $t = 1$ and $t = 0$.

7.1. Special value at $t = 1$**.** — It has been conjectured in [**BBDG, Broad1, Broad2**] that the value at $t = 1$ of the three-banana integral is given by an L-function value

$$I_{\ominus}(1) = \frac{12\pi i}{\sqrt{-15}} L(f^+, 2) \tag{7.1.1}$$

where $L(f^+, s) = \sum_{n\geq 1} a_n/n^s$ is the L-function associated to the weight-three conductor 15 modular form

$$f^+(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+4n^2} = \sum_{n\geq 1} a_n q^n \tag{7.1.2}$$

constructed in [**PTV**] .

We will prove 7.1.1 using a triviality result, theorem 7.1.2 below, for the trace of a certain $\mathbb{Z}/5\mathbb{Z}$-action on the Milnor symbol. The proof invokes Deligne's conjecture [**D2**] for critical values of L-functions. In this case, the L-function in question (7.1.2) is a Hecke L-series associated to an algebraic Hecke character, and Deligne's conjecture was proven by Blasius [**B**]. The specific application we will use of their work is the following

Proposition 7.1.1. — *Let* $\omega_1 \in \Gamma(X_1, \Omega^2)$ *be the algebraic differential form over* $\mathbb{Q}$*,* (3.3.1)*.*

(i) Let $0 \neq c \in H_2(X_1, \mathbb{Q})_{tr}$ *be a 2-cycle. Then* $L(f^+, 2) \in \mathbb{Q}(\sqrt{-15}) \cdot \int_c \omega_1$.
(ii) Let $0 \neq x \in H^2(X_1, \mathbb{Q}(2))_{tr}$ *be a Betti cohomology class. Then* $L(f^+, 2) \in \mathbb{Q}(\sqrt{-15}) \cdot \langle x, \omega_1 \rangle'$*. Here* $\langle x, \omega_1 \rangle'$ *is the Poincaré duality pairing.*

Proof. — Note that (i) and (ii) are equivalent because $H_2(X_1, \mathbb{Q}) \cong H^2(X_1, \mathbb{Q}(2))$, an isomorphism of Hodge structures which is compatible with the pairings with H^2. (To see that the L-function is critical at $s = 2$ the reader can consult [**HS**, §2].) The usual formulation of Deligne's conjecture would say that if x in (ii) is invariant under the real conjugation, then $L(f^+, 2) \in \mathbb{Q} \cdot \langle x, \omega_1 \rangle'$. However, in this case we have complex multiplication by $\mathbb{Q}(\sqrt{-15})$, i.e. $H^2(X_1, \mathbb{Q})_{tr}$ is a rank one $\mathbb{Q}(\sqrt{-15})$-vector space, so changing x multiplies the pairing by an element in the CM field. □

7.1.1. Special fiber at $t = 1$. — Recall from §3.2 that countably many fibers X_t in the $K3$ family have Picard number 20, and hence are of CM type. That X_1 is one of these CM fibers is shown in [**PTV**] (so that $H^2_{tr}(X_1)$ is a CM Hodge structure). What makes X_1 special amongst the CM fibers is an *additional* symmetry property which arises as follows.

Consider $\mathbb{P}^4$ with homogeneous coordinates $T_0, \ldots, T_4$, hyperplane $H = \left\{\sum_{i=0}^4 T_i = 0\right\}$, and hypersurface $Y = \left\{\sum_{i=0}^4 \prod_{j \neq i} T_j = 0\right\}$. Then X_1 is a resolution of singularities of $H \cap Y$, which can be seen by writing $U_i := T_i|_H$ $(i = 0, \ldots, 4)$ and $x_i := \frac{U_i}{U_0}$ $(i = 1, 2, 3)$. Since Y and H are stable under the permutation action of the symmetric group $\mathfrak{S}_5$ on the $\{T_i\}$, it is clear that $\mathfrak{S}_5$ acts on $H \cap Y$ hence birationally on X_1. Let $\omega_1 \in \Omega^2(X_1)$ be as in (3.3.1). Since we may express ω_1 as

$$Res_X Res_H \left(\frac{\sum_{i=0}^4 (-1)^i T_i dT_0 \wedge \cdots \wedge \widehat{dT_i} \wedge \cdots \wedge dT_4}{\left(\sum_i \prod_{j \neq i} T_j\right) \left(\sum T_i\right)} \right) \in \Omega^2(H \cap Y),$$

the action of $\mathfrak{S}_5$ on $\mathbb{C}\omega_1$ hence $H^2_{tr}(X_1)\,(\subsetneq H^2_{var}(X_1))$ is through the alternating representation.

7.1.2. The higher normal function analysis. —

Theorem 7.1.2. — *$I_{\circleddash}(1)$ is a $(2\pi i)^3$ times a period of*

$$\omega_1 := Res_{X_1}\left(\frac{\frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z}}{1-(1-x-y-z)(1-x^{-1}-y^{-1}-z^{-1})}\right).$$

Proof. — Let $\sigma : X_1 \to X_1$ be the automorphism induced by the cyclic permutation $T_0 \mapsto T_1 \mapsto \cdots \mapsto T_4 \mapsto T_0$ of the $\{T_i\}$. Write $\hat{\Xi}_1 := \sum_{j=0}^{4}(\sigma^j)^*\Xi_1 \in H^3_M(X_1, \mathbb{Q}(3))$. Since $\sigma_*\tilde{\omega}_1 = \tilde{\omega}_1$, we have

$$5V_{\circleddash}(1) = 5\langle\mathcal{R}_1, \tilde{\omega}_1\rangle$$
$$= \sum_{j=0}^{4}\langle\mathcal{R}_1, (\sigma^j)_*\tilde{\omega}_1\rangle = \langle\sum_{j=0}^{4}(\sigma^j)^*\mathcal{R}_1, \tilde{\omega}_1\rangle,$$

where the cohomology class $\sum_{j=0}^{4}(\sigma^j)^*\mathcal{R}_1 \in H^2_{var}(X_1, \mathbb{C})$ gives a lift of $\overline{AJ}^{3,3}_{X_1}(\hat{\Xi}_1) \in H^2_{var}(X_1, \mathbb{C}/\mathbb{Q}(3))$. To show that $V_{\circleddash}(1)$ is a $\mathbb{Q}(3)$-period, it will suffice to establish that the image of the latter in $H^2_{tr}(X_1, \mathbb{C}/\mathbb{Q}(3))$ is zero.

Let $U \subset X_1$ be any Zariski open set, $Y = X\backslash U$. In the commutative diagram[16]

$$\begin{array}{ccccc}
H^3_{M,Y}(X, \mathbb{Q}(3)) & \longrightarrow & H^3_M(X_1, \mathbb{Q}(3)) & \longrightarrow & H^3_M(U, \mathbb{Q}(3)) \\
\downarrow{\scriptstyle AJ_Y} & & \downarrow{\scriptstyle AJ_{X_1}} & & \downarrow{\scriptstyle AJ_U} \\
H^2_Y(X_1, \mathbb{C}/\mathbb{Q}(3)) & \longrightarrow & H^2(X_1, \mathbb{C}/\mathbb{Q}(3)) & \xrightarrow{\nu} & H^2(U, \mathbb{C}/\mathbb{Q}(3)),
\end{array}$$

the image of ν factors the projection from $H^2(X_1)$ to $H^2_{tr}(X_1)$. This reduces the problem to checking that the image $\hat{\Xi}_1|_{\eta_{X_1}}$ of $\hat{\Xi}_1$ in

$$\varinjlim_U H^3_M(U, \mathbb{Q}(3)) \cong K^M_3(\mathbb{C}(X_1)) \otimes \mathbb{Q}$$

is zero.

[16]Note: $H^3_{M,Y}(X, \mathbb{Q}(3)) \cong CH^2(Y, 3)_{\mathbb{Q}}$.

This is now a simple computation in Milnor K-theory (written additively). Working modulo (2-)torsion, we have

$$\begin{aligned}
\xi := \{x, y, z\} &= \left\{x, \frac{y}{x}, \frac{z}{x}\right\},\\
\sigma^*\xi = \left\{\frac{y}{x}, \frac{z}{x}, \frac{1+x+y+z}{x}\right\} &= \left\{\frac{1+x+y+z}{x}, \frac{y}{x}, \frac{z}{x}\right\},\\
(\sigma^2)^*\xi = \left\{\frac{z}{y}, \frac{1+x+y+z}{y}, \frac{1}{y}\right\} &= -\left\{1+x+y+z, y, z\right\},\\
(\sigma^3)^*\xi = \left\{\frac{1+x+y+z}{z}, \frac{1}{z}, \frac{x}{z}\right\} &= -\left\{1+x+y+z, \frac{1}{x}, z\right\},\\
(\sigma^4)^*\xi = \left\{\frac{1}{1+x+y+z}, \frac{x}{1+x+y+z}, \frac{y}{1+x+y+z}\right\}&\\
&= -\left\{1+x+y+z, \frac{y}{x}, \frac{1}{x}\right\}.
\end{aligned}$$

Now observe that

$$\xi + \sigma^*\xi = \left\{1+x+y+z, \frac{y}{x}, \frac{z}{x}\right\}$$

and

$$(\sigma^2)^*\xi + (\sigma^3)^*\xi + (\sigma^4)^*\xi = -\left\{1+x+y+z, \frac{y}{x}, \frac{z}{x}\right\},$$

so that $\hat{\Xi}_1|_{\eta_{X_1}} = \sum_{j=0}^{4}(\sigma^j)^*\xi = 0$. □

7.1.3. Value at $t = 1$. — The proof for Broadhurst's formula (7.1.1) is now straightforward. By theorem 7.1.2, the regulator class in $H^2(X_1, \mathbb{C}/\mathbb{Q}(3))_{tr}$ is trivial, which implies that the lifting $\mathcal{R}$ of this class to $H^2(X_1, \mathbb{C})_{tr}$ lies in $H^2(X_1, \mathbb{Q}(3))_{tr} = 2\pi i \cdot H^2(X_1, \mathbb{Q}(2))_{tr}$. Thus,

(7.1.3)

$$I_{\circleddash}(1) = \langle \mathcal{R}, \omega_1\rangle' \in 2\pi i \langle H^2(X_1, \mathbb{Q}(2)), \omega_1\rangle' = \mathbb{Q}(\sqrt{-15}) \cdot 2\pi i L(f^+, 2).$$

The identity on the right follows from proposition 7.1.1.

7.2. Special value at $t = 0$**.** — It has been showed in [**BBDG, Broad2**] that $I_{\circleddash}(0) = 7\zeta(3)$. We provide this section a derivation of this result form the point of view of higher normal functions.

Theorem 7.2.1. — $I_{\ominus}(0) = 7\zeta(3)$.

Proof. — The fiber X_0 (after semistable reduction) has the two components Y_1 resp. Y_2 arising from $1-x-y-z=0$ resp. $1-x^{-1}-y^{-1}-z^{-1}=0$, and six arising from the semistable reduction process which we may ignore since $R_{\{x,y,z\}}$ is zero there. The motivic cohomology formalism tells us to compute the pairing

$$V_{\ominus}(0) = \langle [R_{\{x,y,z\}}], [\tilde{\omega}_0] \rangle = \sum_{i=1}^{2} \int_{Y_i} R_{\{x,y,z\}} \wedge \tilde{\omega}_0$$

of a cohomology and homology class.

Observing that $Y_1 \cap Y_2$ is essentially the "triangle" $\{(x, y, 1-x-y) \mid (1-x)(1-y)(x+y) = 0\}$, let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ be a generator of $H_1(Y_1 \cap Y_2, \mathbb{Z})$. Also let $\beta = \beta_1 + \beta_2$ be a 2-cycle on X_0 with $\partial\beta_1 = \gamma = -\partial\beta_2$, and where $(x, y, z) \mapsto (x^{-1}, y^{-1}, z^{-1})$ sends $\beta_1 \mapsto \beta_2$. We have in $H_2(X_0, \mathbb{Q})$ (really in H^2_{var}, i.e. working modulo classes in the limit of the fixed part) that $[\tilde{\omega}_0] = \frac{1}{2}\beta$. The $\frac{1}{2}$ is obtained by computing

$$Res_{x=1} Res_{y=1} Res_{z=1-x-y} \frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{\phi_{\ominus}(x, y, z)}$$

$$= Res_{x=1} Res_{y=1} \frac{\frac{dx}{x} \wedge \frac{dy}{y}}{\left(1 - x^{-1} - y^{-1} - \frac{1}{1-x-y}\right)(x + y - 1)}$$

$$= Res_{x=1} Res_{y=1} \frac{dx \wedge dy}{(1-x)(1-y)(x+y)} = \frac{1}{2},$$

which is a period of $\tilde{\omega}_0$ over a vanishing cycle $\alpha \in H^2(X_0)$ with $\langle \alpha, \beta \rangle = 1$.

It remains to compute

$$\frac{1}{2} \sum_{i=1}^{2} \int_{\beta_i} R_{\{x,y,z\}} = \int_{\beta_1} R_{\{x,y,z\}}$$

$$= \int_{\beta_1} \log(x) \frac{dx}{x} \wedge \frac{dy}{y} = \int_{\beta_1} \frac{\log(x)}{y(1-x-y)} dx \wedge dy$$

$$= \int_{\beta_1} d\left\{\frac{\log\left(\frac{1-x-y}{y}\right)\log(x)}{1-x}dx\right\} = \int_{\gamma_1+\gamma_2+\gamma_3} \frac{\log\left(\frac{1-x-y}{y}\right)\log(x)}{1-x}dx$$
$$= 2\int_{-1}^{1} \frac{\log(-x)\log(x)}{1-x}dx\,.$$

This integral is readily evaluated as follows:

$$2\int_{-1}^{1}\frac{\log(-x)\log(x)}{1-x}dx \equiv 4\int_{-1}^{1}\log(1-x)\log(x)\frac{dx}{x} \mod \mathbb{Q}(3)$$
$$\equiv -4\sum_{k\geq 1}\frac{1}{k}\int_{-1}^{1}\log(x)x^{k-1}dx \mod \mathbb{Q}(3)$$
$$\equiv 8\sum_{\substack{k\geq 1\\ \text{odd}}}\frac{1}{k^3} \equiv 7\zeta(3) \mod \mathbb{Q}(3)\,.$$

Now $I_{\ominus}(0)$ is obviously real, so we can ignore the $\mathbb{Q}(3)$ ambiguity. □

Remark 7.2.2. — Alternatively we can give a very different proof of Theorem 7.2.1 using the Eisenstein analysis of §5. Referring to the proof of Theorem 5.3.1, we have

$$I_{\ominus} = V_{\ominus} = \varpi_1(\tau)\cdot V_{\varphi_{\ominus}}(\tau).$$

Applying Props. 9.2 and 9.4 of [**DK**] (the former suitably modified for the cusp [0]), we have that

$$V_{\varphi_{\ominus}}(\tau) \sim -\frac{\tau^2}{6}L((\pi_0)_*\hat{\varphi}_{\ominus},3) = 7\cdot 3\cdot 2^4\zeta(3)\tau^2$$

as $\tau \to 0$. For the other factor, the property $\eta(-1/\tau) = \sqrt{\tau}\eta(\tau)$ of Dedekind eta allows us to to pull back $\varpi_1(\tau) = \frac{(\eta(6\tau)\eta(2\tau))^4}{(\eta(3\tau)\eta(\tau))^2}$ under μ_6 : $\tau \mapsto -1/6\tau =: \tilde{\tau}$. Namely, we have

$$\varpi_1(\tau) = \varpi_1(-1/6\tilde{\tau}) = -\frac{3}{4}\tilde{\tau}^2 H_{\ominus}(\tilde{\tau})\varpi_1(\tilde{\tau}) \sim \frac{3}{4}\tilde{\tau}^2 = \frac{1}{2^4 3\tau^2}$$

as $\tau \to 0$. Taking the product (and noting the correspondence $\tau = 0 \leftrightarrow t = 0$) gives $I_{\ominus}(0) = 7\zeta(3)$.

Appendix A

Higher symmetric powers of the sunset motive

In this section we consider the higher symmetric powers for the sunset regulator. This leads immediately to generalization of the Eichler integral found for the two-loop sunset (cf. [**BV**] and §3.1) and three-banana (cf. §3.2) Feynman integrals. It remains to be seen whether this has any relevance for the higher loop banana integrals studied in [**Va**, §9].

Consider the series

$$\sum_{a\neq 0}{}^{e}\, \frac{\psi(a,b)}{a^{n-1}(a\tau+b)} \quad \text{Eisenstein summation, } n=3,4 \tag{A.1}$$

(Here $\psi : (\mathfrak{Z}/N\mathfrak{Z})^2 \to \mathbb{C}$ is some map.)

Let A be a finite dimensional $\mathbb{Q}$-vector space, and let $A^\vee := \mathrm{Hom}(A,\mathbb{Q})$ be the dual. There is a natural embedding $A^\vee \hookrightarrow \mathrm{Der}(\mathrm{Sym}(A))$ identifying $A^\vee$ with the translation invariant derivations of $\mathrm{Sym}(A)$, the symmetric algebra. (For example, if a_i is a basis of A, the dual basis elements $a_i^\vee$ are identified with $\frac{\partial}{\partial a_i}$.) This leads to perfect pairings

$$\langle , \rangle : \mathrm{Sym}^n(A^\vee)\otimes \mathrm{Sym}^n(A) \to \mathbb{Q}; \quad \langle D^I, a^J\rangle = D^I(a^J)|_0 \tag{A.2}$$

Notice, however, that because of factorials, this pairing is not perfect integrally. (The integral dual of Sym is the divided power algebra.)

Let $B := \mathfrak{Z}\varepsilon_1 \oplus \mathfrak{Z}\varepsilon_2$. Identify $B \cong B^\vee$ via the pairing $\langle \varepsilon_1, \varepsilon_2\rangle = -\langle \varepsilon_2,\varepsilon_1\rangle = 1$. With the above identification we find

$$\langle \varepsilon_1^{i_1}\varepsilon_2^{i_2}, \varepsilon_1^{j_1}\varepsilon_2^{j_2}\rangle = \begin{cases} (-1)^{i_2} i_1! i_2! & i_k = j_{1-k} \\ 0 & \text{else} \end{cases} \tag{A.3}$$

We now compute

$$\begin{aligned} &\langle (\tau\varepsilon_1+\varepsilon_2)^{n-2}, \int_\tau^{i\infty} \frac{(x\varepsilon_1+\varepsilon_2)^{n-2}d\tau}{(ax+b)^n}\rangle \\ = \;& (n-2)! \sum_{k=0}^{n-2} \binom{n}{k} (-\tau)^{n-2-k} \int_\tau^{i\infty} \frac{dx\, x^k}{(ax+b)^n} \end{aligned}$$

$$= (n-2)! \int_{\tau}^{i\infty} \frac{(x-\tau)^{n-2}}{(ax+b)^n}\, dx$$

$$= \frac{(n-2)!}{(n-1)a^{n-1}(ax+b)} \qquad \text{(A.4)}$$

Notice the left-hand-side is exactly the pairing we would expect to compute for $\mathrm{Sym}^{n-2}H^1(\mathcal{E}_t)$, where $\mathcal{E}_t$ is the sunset elliptic curve, while the right-hand-side when Eisenstein summed over a, b yields the corresponding function (A.1).

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June 11, 2014

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