AN ADDITIVE VERSION OF HIGHER CHOW GROUPS
UNE VERSION ADDITIVE DES GROUPES DE CHOW SUPÉRIEURS

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Abstract. The cosimplicial scheme

$$\Delta^\bullet = \Delta^0 \rightrightarrows \Delta^1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \ldots; \quad \Delta^n := \mathrm{Spec}\Big(k[t_0, \ldots, t_n]/(\sum t_i - t)\Big)$$

was used in[3] to define higher Chow groups. In this note, we let t tend to 0 and replace $\Delta^\bullet$ by a degenerate version

$$Q^\bullet = Q^0 \rightrightarrows Q^1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \ldots; \quad Q^n := \mathrm{Spec}\Big(k[t_0, \ldots, t_n]/(\sum t_i)\Big)$$

to define an additive version of the higher Chow groups. For a field k, we show the Chow group of 0-cycles on Q^n in this theory is isomorphic to the group of absolute $(n-1)$-Kähler forms Ω_k^{n-1}.

An analogous degeneration on the level of de Rham cohomology associated to "constant modulus" degenerations of varieties in various contexts is discussed.

Résumé en français: Le schéma cosimplicial

$$\Delta^\bullet = \Delta^0 \rightrightarrows \Delta^1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \ldots; \quad \Delta^n := \mathrm{Spec}\Big(k[t_0, \ldots, t_n]/(\sum t_i - t)\Big)$$

a été utilisé dans [3] afin de définir des groupes de Chow supérieurs. Dans cette note, nous définissons une version additive des groupes de Chow supérieurs en faisant tendre t vers 0 et en remplaçant $\Delta^\bullet$ par une version dégénérée

$$Q^\bullet = Q^0 \rightrightarrows Q^1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \ldots; \quad Q^n := \mathrm{Spec}\Big(k[t_0, \ldots, t_n]/(\sum t_i)\Big).$$

Nous montrons que sur un corps k, le groupe de Chow des 0-cycles dans cette théorie est isomorphe aux formes de Kähler absolues de degré $(n-1)$.

Nous discutons une dégénerescence analogue en cohomologie de de Rham apparaissant dans diverses situations pour des familles à module constant.

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1. Introduction

The purpose of this note is to study a common sort of limiting phenomenon which occurs in the study of motives. Here is a simple example. Let k be a field. Let $S = \mathbb{A}^1_t = \mathrm{Spec}\,(k[t])$ and let $T = \mathrm{Spec}\,(k[x,t]/(x(x-t)) \hookrightarrow \mathbb{A}^2_{t,x}$. Over $S[1/t]$ the Picard scheme $\mathrm{Pic}(\mathbb{A}^2_{x,t}, T)/S$ is represented by $\mathbb{G}_{m,S[1/t]}$. On the other hand, when $t = 0$ one gets $\mathrm{Pic}(\mathbb{A}^1_x, \{x^2 = 0\}) \cong \mathbb{G}_{a,k}$. In some sense, $\mathbb{G}_m$ has "jumped" to $\mathbb{G}_a$.

In higher dimension, for $t \neq 0$, the homology of the group of algebraic cycles associated to the cosimplicial scheme

$$(1.1) \quad \Delta^\bullet = \Delta^0 \rightrightarrows \Delta^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots; \quad \Delta^n := \mathrm{Spec}\Big(k[t_0,\ldots,t_n]/(\sum t_i - t)\Big)$$

is known to give motivic cohomology ([9]). What can one say about the algebraic cycle groups (1.3) of the degenerate cosimplicial complex:

$$(1.2) \qquad Q^\bullet = Q^0 \rightrightarrows Q^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots; \quad Q^n := \mathrm{Spec}\Big(k[t_0,\ldots,t_n]/(\sum t_i)\Big)?$$

Our main result is a calculation of the Chow groups of 0-cycles on $Q^\bullet$. Let $\mathcal{Z}^n(Q^r)$ be the free abelian group on codimension n algebraic cycles on Q^r satisfying a suitable general position condition with respect to the face maps. Let $SH^n(k,r)$ be the cohomology groups of the complex

$$(1.3) \qquad \ldots \mathcal{Z}^n(Q^{r+1}) \to \mathcal{Z}^n(Q^r) \to \mathcal{Z}^n(Q^{r-1}) \to \ldots$$

where the boundary maps are alternating sums of pullbacks along face maps. Write $\Omega^\bullet_k$ for the absolute Kähler differentials.

Theorem 1.1. $SH^n(k,n) \cong \Omega^{n-1}_k$.

Note for $n = 1$, this is the above $\mathbb{G}_a$.

We explain another example of this limiting phenomenon. We consider $\mathbb{P}^{n+1}$ with homogeneous coordinates $U_0, \ldots, U_{n+1}$ over a field k. Let

$$(1.4) \qquad X : f(U_0, \ldots, U_{n+1}) = 0$$

be a hypersurface X defined by a homogeneous polynomial f of degree $n+2$ (e.g. an elliptic curve in $\mathbb{P}^2$).

Write $u_i = U_i/U_0$, and define a log $(n+1)$-form on $\mathbb{P}^{n+1} \setminus X$

$$(1.5) \qquad \omega_f := \frac{du_1 \wedge \ldots \wedge du_{n+1}}{f(1, u_1, \ldots, u_{n+1})}.$$

As well known, this form generates $\omega_{\mathbb{P}^{n+1}}(X) \cong \mathcal{O}_{\mathbb{P}^{n+1}}$.

Let $r_1, \ldots, r_{n+1} \geq 0$ be integers. We consider the action of $\mathbb{G}_m$ given by the substitutions $u_i = t^{-r_i} v_i$. Let N be minimal such that

$t^N f(1, t^{-r_1}u_1, \ldots, t^{-r_{n+1}}u_{n+1})$ is integral in t. Assume $s := N - \sum r_i > 0$ and s is invertible in k. One checks easily that in the coordinates v_i one has

$$\omega_f = t^s \nu_f(t, v_1, \ldots, v_{n+1}) + st^{s-1} dt \wedge \gamma_f(t, v_1, \ldots, v_{n+1}) \tag{1.6}$$

with

$$\nu_f(t, v_1, \ldots, v_{n+1}) = \frac{dv_1 \wedge \ldots \wedge dv_{n+1}}{t^N f(1, t^{-r_1}v_1, \ldots, t^{-r_{n+1}}v_{n+1})}$$

$$\gamma_f(t, v_1, \ldots, v_{n+1}) = \frac{1}{s}\sum_i (-1)^i r_i v_i \frac{dv_1 \wedge \ldots \wedge \widehat{dv_i} \wedge \ldots \wedge dv_{n+1}}{t^N f(1, t^{-r_1}v_1, \ldots, t^{-r_{n+1}}v_{n+1})}.$$

Thus ν_f and γ_f are forms in dv_i of degrees $n+1$ and n respectively which are integral in t. As ω_f is an $(n+1)$-form on $\mathbb{P}^{n+1}$ in the u-coordinates, it is closed, so

$$\begin{aligned} st^{s-1} dt \wedge \big(\nu_f(t, v_1, \ldots, v_{n+1}) - d\gamma_f(t, v_1, \ldots, v_{n+1})\big) \\ + st^{s-1} \cdot \frac{t}{s} d\nu_f(t, v_1, \ldots, v_{n+1}) = 0. \end{aligned}$$

Dividing by t^{s-1} and restricting to $t = 0$ yields

$$\nu_f|_{t=0} = d\gamma_f|_{t=0}. \tag{1.7}$$

In other words, under the action of the 1-parameter subgroup, ν_f degenerates to the exact form $d\gamma_f$.

To see the relationship with the 0-cycles, take

$$\Delta_{n+1} : f(1, u_1, \ldots, u_{n+1}) = u_1 u_2 \cdots u_{n+1}(1 - u_1 - \ldots - u_{n+1}) = 0.$$

Substitute $v_i = tu_i,\ 1 \le i \le n+1$. Then $s = 1$, and a calculation yields

(1.8)

$$\gamma_{\Delta_{n+1}}(:= \gamma_f) = \frac{\sum_{i=1}^{n+1} (-1)^i d\log(v_1) \wedge \ldots \wedge \widehat{d\log(v_i)} \wedge \ldots \wedge d\log(v_{n+1})}{t - v_1 - \ldots - v_{n+1}};$$

$$\nu_{\Delta_{n+1}}(:= \nu_f) = \frac{dv_1 \wedge \ldots \wedge dv_{n+1}}{v_1 v_2 \cdots v_{n+1}(t - v_1 - \ldots - v_{n+1})}.$$

Note the limiting configuration as $t \to 0$ is (compare (1.2))

$$v_1 v_2 \cdots v_{n+1}(v_1 + \ldots + v_{n+1}) = 0.$$

For convenience we write

$$\gamma_n := \gamma_{\Delta_{n+1}}|_{t=0} \tag{1.9}$$

We view $\nu_{\Delta_{n+1}}|_{t=1}$ (resp. γ_n) as a map

$$\mathcal{Z}_0\Big(\mathbb{A}_k^{n+1} \setminus \{(1 - u_1 + \cdots + u_{n+1})u_1 \cdots u_{n+1} = 0\}\Big) \to \Omega_k^{n+1}$$

(resp.

$$\mathcal{Z}_0\Big(\mathbb{A}_k^{n+1} \setminus \{v_1 v_2 \cdots v_{n+1}(v_1 + \cdots + v_{n+1}) = 0\}\Big) \to \Omega_k^n).$$

Here $\mathcal{Z}_0$ denotes the free abelian group on closed points (0-cycles) and the maps are respectively

$$x \mapsto \mathrm{Tr}_{k(x)/k}\nu|_{\{x\}}; \quad x \mapsto \mathrm{Tr}_{k(x)/k}\gamma|_{\{x\}}. \tag{1.10}$$

In the first case, the Nesterenko-Suslin-Totaro theorem ([7], [8]) identifies the zero cycles modulo relations coming from curves in $\mathbb{A}^{n+2}$ with the Milnor K-group $K^M_{n+1}(k)$. The evaluation map (1.10) passes to the quotient, and the resulting map $K^M_{n+1}(k) \to \Omega_k^{n+1}$ is given on symbols by the $d\log$-map

$$\{x_1, \ldots, x_{n+1}\} \mapsto d\log(x_1) \wedge \cdots \wedge d\log(x_{n+1}). \tag{1.11}$$

In the second case, factoring out by the relations coming from curves on $\mathbb{A}^{n+2}$ as in (1.3) yields the Chow group of 0-cycles $SH^{n+1}(k, n+1)$, and our main result is that evaluation on γ_n gives an isomorphism

$$SH^{n+1}(k, n+1) \cong \Omega_k^n. \tag{1.12}$$

Sections 2-5 contain the proof of theorem 1.1. Section 6 contains some brief remarks on specialization of forms as it relates to Aomoto's theory of configurations, to 0-cycles on hypersurfaces, and to Goncharov's theory of hyperbolic motives. Finally, section 7 computes $SH^1(k, n)$, responding to a question of S. Lichtenbaum.

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2. The additive Chow groups

In this section, we consider a field k, and a k-scheme X of finite type. We will throughout use the following notations.

Notations 2.1. We set $Q^n = \mathrm{Spec}\ k[t_0, \ldots, t_n]/(\sum_{i=0}^n t_i)$, together with the faces

$$\partial_j : Q^{n-1} \to Q^n; \quad \partial_j^*(t_i) = \begin{cases} t_i & i < j \\ 0 & i = j \\ t_{i-1} & i > j \end{cases}.$$

One also has degeneracies

$$\pi_j : Q^n \to Q^{n-1}; \quad \pi_j^*(t_i) = \begin{cases} t_i & i < j \\ t_i + t_{i+1} & i = j \\ t_{i+1} & i > j \end{cases}.$$

We denote by $\{0\} \in Q^n$ the vertex defined by $t_i = 0$. We write $Q^n_X = Q^n \times_{\mathrm{Spec}\,(k)} X$. The above face and degeneracy maps make $Q^\bullet_X$ a cosimplicial scheme.

Definition 2.2. Let $\mathcal{SZ}_q(X,n)$ be the free abelian group on irreducible, dimension q subvarieties in Q^n_X with the property:

(i) They don't meet $\{0\} \times X$.
(ii) They meet all the faces properly, that is in dimension $\leq q$.

Thus the face maps induce restriction maps

$$\partial_i : \mathcal{SZ}_q(X,n) \to \mathcal{SZ}_{q-1}(X,n-1); \quad i = 0, \dots n; \quad \partial = \sum_{i=0}^{n} (-1)^i \partial_i;$$

yielding complexes $\mathcal{SZ}_{q-\bullet}(X,\bullet)$:

$$\dots \xrightarrow{\partial} \mathcal{SZ}_{q+1}(X,n+1) \xrightarrow{\partial} \mathcal{SZ}_q(X,n) \xrightarrow{\partial} \mathcal{SZ}_{q-1}(X,n-1) \xrightarrow{\partial} \dots$$

Definition 2.3. The additive higher Chow groups are given for $n \geq 1$ by

$$SH_q(X,n) = H_n(\mathcal{SZ}_{q-\bullet}(X,\bullet))$$

and (for X equidimensional)

$$SH^p(X,n) = SH_{\mathrm{dimX}-p}(X,n).$$

The groups are not defined for $n = 0$.

Remarks 2.4. (i) The above should be compared with the higher Chow groups $CH^p(X,n)$ defined as above with $Q^\bullet$ replaced by $\Delta^\bullet$, where $\Delta^n := \mathrm{Spec}\,(k[t_0,\dots,t_n]/(\sum t_i - 1))$.

(ii) The cosimplicial scheme $Q^\bullet$ admits an action of $\mathbb{G}_{m,k}$, which we define by

$$x \star (t_0,\dots,t_n) := (t_0/x,\dots,t_n/x).$$

(The reason for the inverse will be clear below.) By functoriality, we obtain a $k^\times$-action on the $SH^p(X,n)$.

(iii) Let $f : X' \to X$ be a proper map with $n = \dim X' - \dim X$. Then one has a push-forward map

$$f_* : \mathcal{SZ}^p(X',\bullet) \to \mathcal{SZ}^{p-n}(X,\bullet).$$

On homology this yields $SH^p(X',n) \to SH^{p-n}(X,n)$. We will be particularly interested in the case $X = \mathrm{Spec}\,(k)$, $X' = \mathrm{Spec}\,(k')$ with $[k' : k] < \infty$. We write $\mathrm{tr}_{k'/k} : SH^p(k',n) \to SH^p(k,n)$ for the resulting map. This trace map is compatible with the action of $k^\times$ from (ii) in the sense that for $x \in k^\times$ and $a' \in SH^p(k',n)$ we have $x \star \mathrm{tr}_{k'/k}(a') = \mathrm{tr}_{k'/k}(x \star a')$.

Lemma 2.5. *The $\star$ action of $k^\times$ on $SH^n(k,n)$ extends to an action of the multiplicative monoid k by setting $0 \star x = 0$. This action comes from a k-vector space structure on $SH^n(k,n)$.*

Proof. We have to show that for a closed point $x = (u_0, \ldots, u_n) \in Q^n \setminus \cup_{i=0}^n \partial_i(Q^{n-1})$, and $a, b \in k$, one has $(a+b) \star x = a \star x + b \star x$. For either a or $b = 0$, this is trivial. Thus we assume $ab \neq 0$. Let $k' = k(x)$. Then the class in $SH^n(k,n)$ of x is the trace from k' to k of a k'-rational point $x' \in SH^n(k', n)$. Using the compatibility of $\star$ and trace from Remarks 2.4(iii) above, we reduce to the case x k-rational.

Write $(t_0 - u_0, \ldots, t_n - u_n)$ for the ideal of x. Define $\ell(t) = -\frac{ab}{u_0}t + a + b$. Consider the curve $W \subset Q^{n+1}$ defined parametrically by

$$W = \{(t, -t + \frac{u_0}{\ell(t)}, \frac{u_1}{\ell(t)}, \ldots, \frac{u_n}{\ell(t)})\}.$$

To check that this parametrized locus is Zariski-closed, we consider the ideal:

$$I_W = \Big((t_1 + t_0)\ell(t_0) - u_0, t_2\ell(t_0) - u_1, \ldots, t_n\ell(t_0) - u_n\Big).$$

If $y = (y_0, \ldots, y_{n+1})$ is a geometric point in the zero locus of I_W, then since the $u_i \neq 0$ we see that $\ell(y_0) \neq 0$. Substituting $t = y_0$, we see that y lies on the parametrized locus W.

The equation $-t + \frac{u_0}{\ell(t)} = 0$ leads to a quadratic equation in t with solutions $t = \frac{u_0}{a}$, $t = \frac{u_0}{b}$. If $a + b \neq 0$ we have

$$\begin{aligned}
\partial_0(W) &= (\frac{u_0}{a+b}, \ldots, \frac{u_n}{a+b}) = (a+b) \star (u_0, \ldots, u_n);\\
\partial_1(W) &= (\frac{u_0}{a}, \ldots, \frac{u_n}{a}) + (\frac{u_0}{b}, \ldots, \frac{u_n}{b})\\
&= a \star (u_0, \ldots, u_n) + b \star (u_0, \ldots, u_n)\\
\partial_i(W) &= 0; \quad i \geq 2,
\end{aligned}$$

so the lemma follows in this case. If $a + b = 0$, then $\partial_0 W = 0$ as well, and again the assertion is clear. □

3. Additive Chow groups and Milnor K-theory

We consider the map (compatible with faces) $\iota : \Delta^\bullet \to Q^{\bullet+1}$ defined on k-rational points by $(u_0, \ldots, u_n) \mapsto (-1, u_0, \ldots, u_n)$. It induces a map of complexes $\mathcal{Z}^p(k, \bullet) \to \mathcal{SZ}^{p+1}(k, \bullet + 1)$, which in turn induces a map

$$\iota : CH^p(k,n) \to SH^{p+1}(k, n+1). \tag{3.1}$$

By [7] and [8], one has an isomorphism

$$K^M_n(k) \cong CH^n(k,n) \tag{3.2}$$

of the higher Chow groups of 0-cycles with Milnor K-theory. It is defined by:

$$(u_0,\dots,u_n) \mapsto \{-\frac{u_0}{u_n},\dots,-\frac{u_{n-1}}{u_n}\} \tag{3.3}$$

$$\{b_1,\dots,b_n\} \mapsto (\frac{b_1}{c},\dots,\frac{b_n}{c},-\frac{1}{c}); \quad c = -1+\sum_{i=1}^n b_i. \tag{3.4}$$

Note that if $\sum_{i=1}^n b_i = 1$, then the symbol $\beta := \{b_1,\dots,b_n\}$ is trivial in Milnor K-theory, and one maps β to 0.

In this way, one obtains a map

$$\begin{aligned} K^M_{n-1}(k) \to SH^n(k,n); \quad & \{x_1,\dots,x_{n-1}\} \mapsto \\ & \Big(-1, \frac{x_1}{-1+\sum_{i=1}^{n-1} x_i},\dots,\frac{x_{n-1}}{-1+\sum_{i=1}^{n-1} x_i},\frac{-1}{-1+\sum_{i=1}^{n-1} x_i}\Big). \end{aligned} \tag{3.5}$$

4. Differential forms

In this section we construct a k-linear map $\Omega^{n-1}_k \to SH^n(k,n)$. (Here Ω^i_k are the absolute Kähler differential i-forms.)

The following lemma is closely related to calculations in [5].

Lemma 4.1. *As a k-vector space, the differential forms Ω^{n-1}_k are isomorphic to $(k\otimes_{\mathbb{Z}} \wedge^{n-1}k^\times)/\mathcal{R}$. The k-structure on $k\otimes_{\mathbb{Z}} \wedge^{n-1}k^\times$ is via multiplication on the first argument. The relations $\mathcal{R}$, for $n\ge 2$, are the k-subspace spanned by*

$$a\otimes(a\wedge b_1\wedge\ldots\wedge b_{n-2}) + (1-a)\otimes((1-a)\wedge b_1\wedge\ldots\wedge b_{n-2}),$$

for $b_i \in k^\times$, $a\in k$. The map $(k\otimes_{\mathbb{Z}} \wedge^{n-1}k^\times)/\mathcal{R} \to \Omega^{n-1}_k$ is then defined by $(a,b_1,\dots,b_{n-1}) \mapsto a\,d\log b_1\wedge\ldots\wedge d\log b_{n-1}$.

Proof. Write $'\Omega^* := (k\otimes_{\mathbb{Z}} \wedge^* k^\times)/\mathcal{R}$. This is a quotient of the graded k-algebra $k\otimes_{\mathbb{Z}}\wedge^* k^\times$ by the graded ideal $\mathcal{R}$ and hence has a graded k-algebra structure, generated in degree 1. There is an evident surjection of graded algebras $'\Omega^* \twoheadrightarrow \Omega^*$, so, by the universal mapping property of the exterior algebra Ω^*, it suffices to check $'\Omega^1 \cong \Omega^1$.

Define $D: k\to (k\otimes_{\mathbb{Z}} k^\times)/\mathcal{R}$ by $D(a) = a\otimes a$ for $a\in k^\times$, else $D(0)=0$. To define the required inverse, it suffices to show D is a derivation. Clearly, $D(ab) = aD(b)+bD(a)$. Also $1\otimes -1$ is trivial in

$k \otimes k^\times$, so $D(-a) = -D(a)$. Given $a, b \in k^\times$, write $b = -ac$. We have

$$\begin{aligned} D(a+b) = D(a-ac) = aD(1-c) + (1-c)D(a) = \\ -aD(c) + D(a) - cD(a) = D(a) + D(-ac) = D(a) + D(b). \end{aligned} \tag{4.1}$$

Hence D is a derivation so the inverse map $\Omega^1 \to' \Omega^1$ is defined. $\square$

Remark 4.2. We will frequently use the relations in the equivalent form $a \otimes a \wedge (\cdots) - (1+a) \otimes (1+a) \wedge (\cdots) \sim 0$.

Proposition 4.3. *One has a well-defined k-linear map*

$$\phi : \Omega_k^{n-1} \to SH^n(k,n)$$
$$\alpha := a d\log b_1 \wedge \ldots \wedge d\log b_{n-1} \mapsto a \star (-1, \frac{b_1}{\gamma}, \ldots, \frac{b_{n-1}}{\gamma}, -\frac{1}{\gamma})$$

where $\gamma = -1 + \sum_{i=1}^{n-1} b_i$. (Define $\alpha = 0$ when $\gamma = 0$.) The diagram

$$\begin{array}{ccc} K_{n-1}^M(k) & \xrightarrow{d\log} & \Omega_k^{n-1} \\ \iota \downarrow (3.5) & & \phi \downarrow \\ SH^n(k,n) & = & SH^n(k,n) \end{array}$$

is commutative.

Proof. We write $c = -1 + a + \sum_{i=2}^{n-1} b_i$. By Lemma 4.1, we have to show

$$0 = \rho :=$$
$$a \star (-1, \frac{a}{c}, \ldots, \frac{b_{n-1}}{c}, -\frac{1}{c}) - (a+1) \star (-1, \frac{a+1}{c+1}, \ldots, \frac{b_{n-1}}{c+1}, -\frac{1}{c+1}).$$

If $a = 0$, then one has

$$\rho = -(-1, \frac{1}{c}, \ldots, \frac{b_{n-1}}{c}, -\frac{1}{c}) = -\iota\{1, b_2, \ldots, b_{n-1}\} = 0.$$

Similarly, $\rho = 0$ if $a = -1$. Assume now $a \neq 0, -1$. Set $b = -a \in k \setminus \{0, 1\}$. One defines, for $n \geq 3$ and $(u_1, \ldots, u_{n-1}) \in (\Delta^{n-2} \setminus \cup_{i=1}^{n-1} \Delta^{n-3})(k)$, the parametrized curve

$$\Gamma(b,u) := \{(\frac{-1}{b} + t, \frac{1}{b-1}, -t, \frac{-u_1}{b(b-1)}, \ldots, \frac{-u_{n-1}}{b(b-1)})\} \subset Q^{n+1}$$

and for $n = 2$

$$\Gamma(b) := \{(\frac{-1}{b} + t, \frac{1}{b-1}, -t, \frac{-1}{b(b-1)})\} \subset Q^3. \tag{4.2}$$

(See [8] for the origin of this definition). This curve is indeed in good position, so it lies in $\mathcal{SZ}^n(Q^{n+1})$. One computes

$$\begin{aligned}\partial\Gamma(b,u) &= (1-b)\star(-1,1-\frac{1}{b},\frac{u_1}{b},\dots,\frac{u_{n-1}}{b}) \\ &+b\star(-1,\frac{b}{b-1},\frac{-u_1}{b-1},\dots,\frac{-u_{n-1}}{b-1}).\end{aligned}\tag{4.3}$$

(Resp. in the case $n=2$

$$\partial\Gamma(b) = (1-b)\star(-1,1-\frac{1}{b},\frac{1}{b}) + b\star(-1,\frac{b}{b-1},\frac{-1}{b-1}).)$$

Now one has

$$\begin{aligned}(-1,1-\frac{1}{b},\frac{u_1}{b},\dots,\frac{u_{n-1}}{b}) &= \iota\{\frac{1-b}{u_{n-1}},-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\} \\ = \iota[\{1-b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\} &- \{u_{n-1},u_1,\dots,u_{n-2}\}],\end{aligned}$$

as the rest of the multilinear expansion contains only symbols of the shape $\{\dots,u_{n-1},\dots,-u_{n-1},\dots\}$. On the other hand, since $\sum_{i=1}^{n-1}u_i = 1$, one has $\{u_{n-1},u_1,\dots,u_{n-2}\}=0$. Similarly, one has

$$(-1,\frac{b}{b-1},\frac{-u_1}{b-1},\dots,\frac{-u_{n-1}}{b-1}) = \iota\{\frac{b}{u_{n-1}},-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\}.$$

The same argument yields that this is

$$\iota\{b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\}.$$

It follows now from (4.3) that for $n\geq 3$ we have the relation in $SH^n(k,n)$

$$\begin{aligned}(1-b)\star\iota\{1-b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\}+ \\ b\star\iota\{b,-\frac{u_1}{u_{n-1}},\dots,-\frac{u_{n-2}}{u_{n-1}}\} = 0\end{aligned}\tag{4.4}$$

(The analogous relation for $n=2$ is similar.) □

Proposition 4.4. *With notation as above, the map*

$$\phi:\Omega_k^{n-1}\to SH^n(k,n)$$

is surjective. In particular, $SH^n(k,n)$ is generated by the classes of k-rational points in Q^n.

Proof. It is easy to check that the image of ϕ coincides with the subgroup of $SH^n(k,n)$ generated by k-points. Clearly, $SH^n(k,n)$ is generated by closed points, and any closed point is the trace of a k'-rational point for some finite extension k'/k. We first reduce to the case k'/k

separable. If $x \in Q^n_k$ is a closed point in good position (i.e. not lying on any face) such that $k(x)/k$ is not separable, then a simple Bertini argument shows there exists a curve C in good position on Q^{n+1} such that $\partial C = x + y$ where y is a zero cycle supported on points with separable residue field extensions over k. Indeed, let $W \subset Q^{n+1}$ be the union of the faces. View $x \in W$. Since x is in good position, it is a smooth point of W. Bertini will say that a non-empty open set in the parameter space of n-fold intersections of hypersurfaces of large degree containing x will meet W in x plus a smooth residual scheme. Since k is necessarily infinite, there will be such an n-fold intersection defined over k. Since the residual scheme is smooth, it cannot contain inseparable points. Then $x \equiv -y$ which is supported on separable points.

We assume now k'/k finite separable, and we must show that the trace of a k'-point is equivalent to a zero cycle supported on k-points. Since the image of ϕ is precisely the subgroup generated by k-rational points, it suffices to check that the diagram

$$\begin{array}{ccc} \Omega^{n-1}_{k'} & \xrightarrow{\phi} & SH^n(k',n) \\ \downarrow{\scriptstyle \mathrm{Tr}_{k'/k}} & & \downarrow{\scriptstyle \mathrm{Tr}_{k'/k}} \\ \Omega^{n-1}_{k} & \xrightarrow{\phi} & SH^n(k,n) \end{array} \tag{4.5}$$

commutes. Because k'/k is separable, one has $\Omega^{n-1}_k \hookrightarrow \Omega^{n-1}_{k'}$, and $\Omega^{n-1}_{k'} = k' \cdot \Omega^{n-1}_k$. One reduces to showing, for $\alpha = (-1, \alpha_1, \ldots, \alpha_n) \in Q^n(k)$ and $t \in k'$, that $\mathrm{Tr}(t \star \alpha) = (\mathrm{Tr}(t)) \star \alpha$ in $SH^n(k,n)$.

Let $P(V) = V^N + a_{N-1}V^{N-1} + \ldots + a_1 V + a_0 \in k[V]$ be the minimal polynomial of $-\frac{1}{t}$. We set $b_N = \frac{-1}{\alpha_n}$, $b_i = \frac{-a_i}{\alpha_n}, i = N-1, \ldots, 2$ and $b_i = a_i, i = 1, 0$. We define the polynomial $Q(V,u) = b_N V^{N-1}u + \ldots b_2 V u + b_1 V + b_0 \in k[V,u]$, which by definition fulfills $Q(V, -\alpha_n V) = P(V)$. We define the ideal

$$\begin{aligned} \mathcal{I} = (Q(V_0,u), V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0) \\ \subset k[V_0, \ldots, V_{n-1}, u]. \end{aligned}$$

It defines a curve $W \subset \mathbb{A}^{n+1}$. We think of $\mathbb{A}^{n+1}$ as being Q^{n+1} with the faces $V_0 = 0, \ldots, V_{n-1} = 0, u = 0, u + \sum_{i=0}^{n-1} V_i = 0$. Then this curve is in general position and defines a cycle in $\mathcal{SZ}^1(k, n+1)$.

Since $b_0 \neq 0$, and $\alpha_i \neq 0$, one has

$$\partial_i W = 0, i = 0, 1, \ldots, n-1. \tag{4.6}$$

One has

$$\partial_u W \text{ defined by } (a_1V_0 + a_0, V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0).$$

To compute the last face, we observe that the ideal

$$(u + \sum_{i=0}^{n-1} V_i, V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0),$$

contains $u + \alpha_n V_0$. Consequently $\partial_{u+\sum_{i=0}^{n-1} V_i} W$ is defined by

$$\Big(Q(V_0, -\alpha_n V_0), V_1 + \alpha_1 V_0, \ldots, V_{n-1} + \alpha_{n-1}V_0\Big),$$

with $Q(V_0, -\alpha_n V_0) = P(V_0)$. Thus one obtains

$$0 \equiv (-1)^n \partial W = \frac{a_1}{a_0} \star (-1, \alpha) - t \star (-1, \alpha).$$

Since P is the minimal polynomial of $-\frac{1}{t}$, $\frac{a_1}{a_0}$ is the trace of t. $\square$

5. The main theorem

Recall (1.8) we have a logarithmic $(n-1)$-form γ_{n-1} on $Q^n = \operatorname{Spec}(k[v_0, \ldots, v_n]/(\sum_{i=0}^n v_i))$

$$\gamma_{n-1} = \frac{1}{v_0} \sum_1^n (-1)^i d\log(v_1) \wedge \ldots \wedge \widehat{d\log(v_i)} \wedge \ldots \wedge d\log(v_n) \tag{5.1}$$

$$d\gamma_{n-1} = \nu_n = \frac{dv_1 \wedge \ldots \wedge dv_n}{v_0 v_1 \cdots v_n}$$

Writing $v_i = V_i/V_{n+1}$, we can view γ_{n-1} as a meromorphic form on $\mathbb{P}^n = \operatorname{Proj}(k[V_0, \ldots, V_{n+1}]/(\sum_0^n V_i))$. Let $\mathcal{A} : V_0 \cdots V_n = 0$; $\infty : V_{n+1} = 0$. The fact that $d\gamma_{n-1}$ has log poles on the divisors $V_i = 0,\ 0 \le i \le n$ implies that

$$\gamma_{n-1} \in \Gamma\Big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A} + \infty))(-\infty)\Big) \tag{5.2}$$

In particular, γ_{n-1} has log poles, so we can take the residue along components of $\mathcal{A}$. (The configuration $\mathcal{A}$ does not have normal crossings. The sheaf $\Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A} + \infty))$ is defined to be the subsheaf of $j_*\Omega^{n-1}_{\mathbb{P}^n - \mathcal{A} - \infty}$, where $j : \mathbb{P}^n \setminus (\mathcal{A} \cup \infty) \to \mathbb{P}^n$ is the open embedding, generated by forms without poles and the evident log forms with residue 1 along one hyperplane and (-1) along another one. According to [1], the global sections of this naturally defined log sheaf compute de Rham cohomology.)

In the following, we adopt the sign convention $\operatorname{Res}_{t=0} \frac{dt}{t} \wedge \omega = \omega$. This yields $\operatorname{Res}_{t=0} d(\frac{dt}{t} \wedge \omega) = -d(\operatorname{Res}_{t=0} \frac{dt}{t} \wedge \omega)$.

Lemma 5.1. *We have the following residue formulae*

$$\mathrm{Res}_{v_i=0}\gamma_n = (-1)^i\gamma_{n-1}; \quad 0 \le i \le n+1.$$

Proof. One can either compute directly or argue indirectly as follows:

$$-\,d\mathrm{Res}_{v_i=0}\gamma_n = \mathrm{Res}_{v_i=0}d\gamma_n = \mathrm{Res}_{v_i=0}\nu_{n+1} = (-1)^{i+1}\nu_n = -d((-1)^i\gamma_{n-1}) \neq 0. \tag{5.3}$$

To conclude now, it suffices to show that the space of global sections (5.2) has dimension 1. To verify the dimension 1 property, let $\mathcal{A}' \subset \mathcal{A}$ be defined by $V_1 \cdots V_n = 0$. Then $\mathcal{A}'+\infty$ consists of $n+1$ hyperplanes in general position in $\mathbb{P}^n$, so $\Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A}'+\infty)) = \wedge^{n-1}\Omega^1_{\mathbb{P}^n}(\log(\mathcal{A}'+\infty)) \cong \mathcal{O}^{\oplus n}_{\mathbb{P}^n}$. One looks at the evident residue

$$\Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A}+\infty))(-\infty) \to \Omega^{n-2}_{\mathbb{P}^{n-1}}(\log(\mathcal{A}+\infty))(-\infty).$$

along $V_0 = 0$.) □

Remark 5.2. The computation of the lemma shows that γ_{n-1} is the *unique* $(n-1)$-differential form in $\Gamma\big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log(\mathcal{A}+\infty))(-\infty)\big)$ with $d\gamma_{n-1} = \nu_n$. The configuration $\mathcal{A}$ being given, fixing ∞ fixes γ_{n-1}.

Theorem 5.3. *The assignment* $x \mapsto \mathrm{Tr}_{k(x)/k}(\gamma(x))$ *gives an isomorphism for* $n \ge 1$

$$SH^n(k,n) \cong \Omega^{n-1}_k.$$

Proof. Let $X \subset Q^{n+1}_k$ be a curve in good position. For a zero-cycle c in good position on Q^n we write $\mathrm{Tr}\gamma_{n-1}(c) \in \Omega^{n-1}_k$ (absolute differentials) for the evident linear combination of traces from residue fields of closed points. We must show $\mathrm{Tr}\gamma_{n-1}(\partial X) = 0$. Let $\overline{X}$ denote the closure of X in $\mathbb{P}^n$. We consider $\gamma_{n-1}|_{\overline{X}} \in \Gamma(\overline{X}, \Omega^{n-1}_{\overline{X}/\mathbb{Z}}(*D))$, where D is the pole set of $\gamma_{n-1}|_{\overline{X}}$. The form γ_{n-1} dies when restricted to ∞. Thus $D = \cup_{j=0}^n D_j \subset X$, with $D_j := \partial_j(Q^n) \cap \overline{X}$. We define the residue along D_j as follows. One has an exact sequence

$$0 \to \Omega^n_k \otimes \mathcal{O}_{\overline{X}} \to \Omega^n_{\overline{X}}(\log D) \to \Omega^{n-1}_k \otimes \omega_{\overline{X}}(\log D) \to 0. \tag{5.4}$$

The residue map followed by the trace

$$\omega_{\overline{X}}(\log D) \xrightarrow{\mathrm{Tr}\circ\sum \mathrm{Res}_{D_j}} k \tag{5.5}$$

yields, by the reciprocity formula, a vanishing residue map on global n-forms

$$H^0(\overline{X}, \Omega^n_{\overline{X}/\mathbb{Z}}(\log D)) \to \Omega^{n-1}_k \otimes_k H^0(\overline{X}, \omega_{\overline{X}}(\log D)) \xrightarrow{0} \Omega^{n-1}_k. \tag{5.6}$$

On the other hand, the residue decomposes as

$$H^0(\overline{X}, \Omega^n_{\overline{X}/\mathbb{Z}}(\log D)) \xrightarrow{\sum_j \mathrm{Res}_{D_j}} \oplus_{j=0}^n \Omega^{n-1}_{D_j} \xrightarrow{\sum_j \mathrm{Tr}} \Omega^{n-1}_k. \tag{5.7}$$

By Lemma 5.1, $\gamma_{n-1}(D_j) = (\mathrm{Res}_{v_j=0}\gamma_n)(D_j)$. The desired vanishing follows.

We now have

$$\Omega^{n-1}_k \overset{\phi}{\twoheadrightarrow} SH^n(k,n) \xrightarrow{\gamma_{n-1}} \Omega^{n-1}_k.$$

It suffices to check the composition is multiplication by $(-1)^{n+1}$. Given $b_1, \dots, b_{n-1} \in k$ with $c := \sum b_i - 1 \neq 0$, the composition is computed to be (use (5.1) and Proposition 4.3)

$$a \cdot d\log(b_1) \wedge \dots \wedge d\log(b_{n-1}) \mapsto$$
$$-a \sum (-1)^i d\log(\frac{b_1}{ac}) \wedge \dots \wedge \widehat{d\log(\frac{b_i}{ac})} \wedge \dots \wedge d\log(\frac{-1}{ac})$$

Expanding the term on the right yields

$$-a \sum (-1)^i d\log(\frac{b_1}{c}) \wedge \dots \wedge \widehat{d\log(\frac{b_i}{c})} \wedge \dots \wedge d\log(\frac{-1}{c}) + \\ - a \cdot d\log(a) \wedge \Big(\dots\Big),$$

and it is easy to check that the terms involving $d\log(a)$ cancel. In this way, one reduces to the case $a = 1$. Here

$$-\sum (-1)^i d\log(\frac{b_1}{c}) \wedge \dots \wedge \widehat{d\log(\frac{b_i}{c})} \wedge \dots \wedge d\log(\frac{-1}{c}) = \\ (-1)^{n+1}\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_{n-1}}{b_{n-1}} + \frac{dc}{c} \wedge \Big(\dots\Big). \tag{5.8}$$

Again the terms involving $d\log(c)$ cancel formally, completing the proof. $\square$

Challenge 5.4. Finally, as a challenge we remark that the Kähler differentials have operations (exterior derivative, wedge product,...) which are not evident on the cycles $\mathcal{SZ}$. For example, one can show that the map

$$\nabla(x_0, \dots, x_n) = (x_0, -\frac{x_1 x_0}{1-x_0}, \dots, -\frac{x_n x_0}{1-x_0}, -\frac{x_0}{1-x_0}) \tag{5.9}$$

satisfies

$$\gamma_n(\nabla(x)) = (-1)^n d\gamma_{n-1}(x) \tag{5.10}$$

and hence induces the exterior derivative on the 0-cycles. The map is not uniquely determined by this property, and this particular map does not preserve good position for cycles of dimension > 0. Can one find a geometric correspondence on the complex $\mathcal{SZ}^\bullet$ which induces d on the 0-cycles? What about the pairings $(a,b) \mapsto a \wedge b$ or $(a,b) \mapsto a \wedge db$?

6. Specialization of Forms

In this section we consider again the specialization of differential forms as in §1. Recall $f(U_0, \ldots, U_{n+1})$ is homogeneous of degree $n+2$, $u_i = U_i/U_0$, and $v_i = t^{r_i}u_i$. N is minimal such that $t^N f(1, t^{-r_1}v_1, \ldots, t^{-r_{n+1}}v_{n+1})$ is integral in t, and we assume $s = N - \sum r_i > 0$. We have forms

$$\omega_f := \frac{d(t^{-r_1}v_1) \wedge \ldots \wedge d(t^{-r_{n+1}}v_{n+1})}{f(1, t^{-r_1}v_1, \ldots, t^{-r_{n+1}}v_{n+1})} = t^s \nu_f + st^{s-1}dt \wedge \gamma_f \tag{6.1}$$

and $\nu_f|_{t=0} = d\gamma_f|_{t=0}$.

We have already mentioned the case

$$f = \Delta_{n+1} = (t^{-1}v_1) \cdots (t^{-1}v_{n+1})(1 - t^{-1}v_1 - \ldots - t^{-1}v_{n+1}).$$

The forms $\nu_{\Delta_{n+1}}$ and $\gamma_{\Delta_{n+1}}$ are given in (1.8). As before, we write $\nu_{n+1} := \nu_{\Delta_{n+1}}|_{t=0}$; $\gamma_n := \gamma_{\Delta_{n+1}}|_{t=0}$. The differential form ν_{n+1} plays an interesting rôle in the computation of de Rham cohomology of the complement of hyperplane configurations. Let $\mathcal{A}_t$ be the configuration in $\mathbb{P}^n$ of $(n+1)$ hyperplanes in general position with with affine equation $u_0u_1 \cdots u_n = 0, u_0 + u_1 + \ldots + u_n = t \neq 0$. Then $H^n(\mathbb{P}^n \setminus \mathcal{A}_t) = H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(\log \mathcal{A}_t))$ is a pure Tate structure generated by $\nu_{\Delta n+1}$. Now make t tend to 0 and consider the degenerate configuration $\mathcal{A}_0$ with affine equation $u_0u_1 \cdots u_n = 0, u_0 + u_1 + \ldots + u_n = 0$. Exactness of $\nu_{n+1} = d\gamma_n$ in the case (1.7) follows from Aomoto's theory ([1], [4]). Due to the special shape of the configurations $\mathcal{A}_t$ considered, however, more is true. We have fixed ∞, and the differential form γ_n lies in $H^0(\mathbb{P}^{n+1}, \Omega^n(\log(\mathcal{A}_0 + \infty)(-\infty))$. That is, it has log poles along all components of $\mathcal{A}_0$, and it vanishes as a differential form at ∞ (see Lemma 5.1). This k-vector space of differential forms is 1-dimensional. In other words, the differential form γ_n is uniquely defined by the vanishing condition at ∞ and the secondary data $d\gamma_n = \nu_{n+1}$ (see remark 5.2). Do such primitives for more general degenerating configurations admit an interpretation identifying 0-cycles with spaces of differential forms?

Here is another example of specialization and 0-cycles. Define

$$f(1, u_1, u_2) = u_1^2 - u_2^3 - au_2 - b; \quad v_1 = t^3u_1,\ v_2 = t^2u_2 \tag{6.2}$$

One computes

$$\nu = \frac{dv_1 \wedge dv_2}{v_1^2 - v_2^3 - at^4 v_2 - bt^6}; \quad \gamma = \frac{2v_2 dv_1 - 3v_1 dv_2}{v_1^2 - v_2^3 - at^4 v_2 - bt^6}. \tag{6.3}$$

One checks that $\mathrm{Res}(\gamma|_{t=0}) = v_2/v_1$ and

$$d(v_2/v_1) = -dv_2/2 = d\mathrm{Res}(\nu|_{t=0}) \text{ on } v_1^2 - v_2^3 = 0 \tag{6.4}$$

The assignment $x \mapsto \mathrm{Tr}_{k(x)/k}(v_2/v_1)(x)$ identifies the jacobian of the special fibre $v_1^2 - v_2^3 = 0$ with $\mathbb{G}_a(k) = k$.

A final example of specialization, which we understand less well, though it was an inspiration for this article, concerns the hyperbolic motives of Goncharov [6]. The *matrix coefficients* of his theory (in the sense of [2]) are the objects $H^{2n-1}(\mathbb{P}^{2n-1} \setminus Q, M \setminus Q \cap M)$. Here $Q \subset \mathbb{P}^{2n-1}$ is a smooth quadric and M is a simplex (union of $2n$ hyperplanes in general position). The subschemes Q and M are taken in general position with respect to each other. The notation is intended to suggest a sort of abstract relative cohomology group. In de Rham cohomology, the non-trivial class in $H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus Q)$ is represented by

$$\omega = \frac{du_1 \wedge \ldots \wedge du_{2n-1}}{(u_1^2 + \ldots + u_{2n-1}^2 - 1)^n} \tag{6.5}$$

Substituting $u_i = v_i t$, we get

$$\omega = \frac{t^{2n-1} dv_1 \wedge \ldots \wedge dv_{2n-1} + t^{2n-2} dt \wedge \sum (-1)^i v_i dv_1 \wedge \ldots \widehat{dv_i} \ldots \wedge dv_{2n-1}}{\Big(t^2(v_1^2 + \ldots + v_{2n-1}^2) - 1\Big)^n} \tag{6.6}$$

We get $\nu|_{t=0} = \pm dv_1 \wedge \ldots \wedge dv_{2n-1}$ and $\gamma|_{t=0} = \frac{1}{2n-1} \sum (-1)^i v_i dv_1 \wedge \ldots \widehat{dv_i} \ldots \wedge dv_{2n-1}$. Let $\Delta_M \in H_{2n-1}(\mathbb{P}^{2n-1}, M; \mathbb{Z})$ be a generator. The hyperbolic volume

$$\int_{\Delta_M} \omega \tag{6.7}$$

is the *real period* ([6] section 4.1) of the Hodge structure associated to $H^{2n-1}(\mathbb{P}^{2n-1} \setminus Q)$. Goncharov remarks (op. cit., Question 6.4 and Theorem 6.5) that this volume degenerates to the euclidean volume as $t \to 0$. (More precisely, from (6.7), we see that as a relative form, $dv_1 \wedge \ldots \wedge dv_{2n-1} = \lim_{t \to 0} t^{1-2n} \omega$.) He asks for an interpretation of the degenerated volume in terms of some sort of motive over $k[t]/(t^2)$.

In the Goncharov picture we can view Q as fixed and degenerate M. Suppose $M : L_0 L_1 \cdots L_{2n-1} = 0$ where

$$L_i = L_i(v_1, \ldots, v_{2n-1}) = L_i(u_1/t, \ldots, u_{2n-1}/t)$$

Assuming the L_i are general, clearing denominators and passing to the limit $t \to 0$ yields a degenerate simplex M_0 consisting of $2n$ hyperplanes meeting at the point $v = 0$. This limiting configuration leads to the Chow groups $SH^*(k, 2n-1)$, but we do not see how to relate $\gamma|_{t=0}$ to cycles.

7. Cycle Groups of Divisors

We compute $SH^1(k, n)$ for $n \geq 1$.

Let $\mathcal{O}_{Q^n,0}$ be the local ring at $(0, \dots, 0) \in Q^n$. Let

$$I^n \subset J^n \subset \mathcal{O}_{Q^n,0} \tag{7.1}$$

be the ideals of all (resp. all but the last) face.

Lemma 7.1. $SH^1(k, n) \cong (1 + I^n)^\times / \partial_{n+1}(1 + J^{n+1})^\times$.

Proof. Let $D \subset Q^n$ be an effective divisor. Let

$$f \in k_n := k[t_0, \dots, t_n]/(\sum t_i)$$

be a defining equation for D. Then D meets faces properly in our sense if and only if $f \in \mathcal{O}^\times_{Q^n,0}$. Conversely, any $f \in \mathcal{O}^\times_{Q^n,0}$ defines a (not necessarily effective) divisor on Q^n meeting faces properly. Further

$$(f) \cdot Q_i^{n-1} = 0 \Leftrightarrow f \in k^\times \cdot (1 + t_i \mathcal{O}_{Q^n,0}). \tag{7.2}$$

By general simplicial considerations, the $SH^1(k, n)$ is given by divisors meeting all faces trivially, modulo the restriction to the last face of divisors on Q^{n+1} meeting all but the last face trivially. Scaling the f above to remove the factor $k^\times$, this is precisely the assertion of the lemma. □

Proposition 7.2. *We have* $SH^1(k, 1) = k$, *and* $SH^1(k, n) = (0)$ *for* $n \geq 2$.

Proof. The first assertion is part of theorem 5.3. The exact sequence

$$0 \to I^{n+1} \to J^{n+1} \xrightarrow{\partial_{n+1}} I^n \to 0 \tag{7.3}$$

yields the surjectivity $1 + J^{n+1} \twoheadrightarrow 1 + I^n$. We observe that when $n = 1$, we have $I^1 = (t_0)$ but $\partial_2(J^2) = (t_0^2)$. □

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