# THE ADDITIVE DILOGARITHM

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*To Kazuya Kato, with fondness and profound respect, on the occasion of his fiftieth birthday*

Abstract. A notion of additive dilogarithm for a field $k$ is introduced, based on the $K$-theory and higher Chow groups of the affine line relative to 2(0). Analogues of the $K_2$-regulator, the polylogarithm Lie algebra, and the $\ell$-adic realization of the dilogarithm motive are discussed. The higher Chow groups of 0-cycles in this theory are identified with the Kähler differential forms $\Omega^*_k$. It is hoped that these results will serve as a guide in developing a theory of contravariant motivic cohomology with modulus, modelled on the generalized Jacobians of Rosenlicht and Serre.

## 1. Introduction

In [18] Déf. (5.1.1), Laumon introduces the category of generalized 1-motives over a field $k$ of characteristic 0. Objects in this category are arrows $f : \mathcal{G} \to G$ where $\mathcal{G}$ and $G$ are commutative algebraic groups, with $\mathcal{G}$ assumed formal, torsion free, and $G$ assumed connected. These, of course, generalize the more restricted category of 1-motives introduced by Deligne [8] as a model for the category of mixed Hodge structures of types $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$. Of particular interest for us are motives of the form $\mathbb{Z} \to \mathbb{V}$ which arise in the study of algebraic cycles relative to a "modulus". Here $\mathbb{V} \cong \mathbb{G}_a^n$ is a vector group. The simplest example is

$$\mathrm{Pic}(\mathbb{A}^1, 2\{0\}) \cong \mathbb{G}_a. \tag{1.1}$$

which may be viewed as a degenerate version of the identification $\mathrm{Pic}(\mathbb{A}^1, \{0,\infty\}) \cong \mathbb{G}_m$ obtained by associating to a unit the corresponding Kummer extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$. (For more details, cf. [6],[5],[13],[14],[19].) We expect such generalized motives to play an important role in the (as yet undefined) contravariant theory of motivic sheaves and motivic cohomology for (possibly singular) varieties.

The polylog mixed motives of Beilinson and Deligne are generalizations to higher weight of Kummer extensions, so it seems natural to look for degenerate, or $\mathbb{G}_a$ versions of these. The purpose of this article is to begin to study an additive version of the dilogarithm motive. We assume throughout that $k$ is a field which for the most part will be taken to be of characteristic 0. Though our results are limited to the dilogarithm, the basic result from cyclic homology

$$Gr^n_\gamma \ker\Big(K_{2n-1}(k[t]/(t^2)) \to K_{2n-1}(k)\Big) \cong k \tag{1.2}$$

suggests that higher polylogarithms exist as well.

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In the first part of the article we introduce an additive "Bloch group" $TB_2(k)$ for an algebraically closed field $k$ of characteristic $\neq 2$. In lieu of the 4-term sequence in motivic cohomology associated to the usual Bloch group

$$\begin{aligned}(1.3)\quad 0 \to H^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to B_2(k) \to k^\times \otimes k^\times \otimes \mathbb{Q} \\ \to H^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to 0\end{aligned}$$

(with $H^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \cong K_3(k)_{ind} \otimes \mathbb{Q}$ and $H^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \cong K_2(k) \otimes \mathbb{Q}$), we find an additive 4-term sequence

$$\begin{aligned}(1.4)\quad 0 \to TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to TB_2(k) \to k \otimes k^\times \\ \xrightarrow{d\log} TH^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \to 0\end{aligned}$$

where

$$\begin{aligned}(1.5)\quad TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) := K_2(\mathbb{A}^1_t, (t^2)) \cong (t^3)/(t^4) \cong k; \\ TH^2_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) := K_1(\mathbb{A}^1, (t^2)) \cong \Omega^1_k = \text{absolute Kähler 1-forms}; \\ d\log(a \otimes b) = a\frac{db}{b}.\end{aligned}$$

Our construction should be compared and contrasted with the results of [7]. Cathelineau's group $\beta_2(k)$ is simply the kernel

$$(1.6)\qquad 0 \to \beta_2(k) \to k \otimes k^\times \to \Omega^1_k \to 0,$$

so there is an exact sequence

$$(1.7)\qquad \begin{array}{ccccccc} 0 & \to TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) & \to TB_2(k) & \to \beta_2(k) \to 0 \\ & \downarrow \cong & & \\ & k & & \end{array}$$

For $a \in k$ we define $\langle a \rangle \in TB_2(k)$ lifting similar elements defined by Cathelineau and satisfying his 4-term infinitesimal version

$$(1.8)\qquad \langle a \rangle - \langle b \rangle + a\langle b/a \rangle + (1-a)\langle (1-b)/(1-a) \rangle = 0; \quad a \neq 0, 1.$$

of the classical 5-term dilogarithm relation. Here, the notation $x\langle y \rangle$ refers to an action of $k^\times$ on $TB_2(k)$. Unlike $\beta_2(k)$, this action does *not* extend to a $k$-vector space structure on $TB_2(k)$. Thus (1.7) is an exact sequence of $k^\times$-modules, where the kernel and cokernel have $k$-vector space structures but the middle group does not.

Finally in this section we show the assignment $\langle a \rangle \mapsto a(1-a)$ defines a regulator map $\rho : TB_2(k) \to k$ and the composition

$$(1.9)\qquad TH^1_M(\mathrm{Spec}\,(k), \mathbb{Q}(2)) \hookrightarrow TB_2(k) \xrightarrow{\rho} k$$

is an isomorphism.

It seems plausible that $TB_2(k)$ can be interpreted as a Euclidean scissors-congruence group, with $\partial : TB_2(k) \to k \otimes k^\times$ the Dehn invariant and $\rho : TB_2(k) \to k$ the volume. Note the scaling for the $k^\times$-action is appropriate, with $\partial(x\langle y \rangle) = x\partial(\langle y \rangle)$ and $\rho(x\langle y \rangle) = x^3\rho(\langle y \rangle)$. For a careful discussion of Euclidean scissors-congruence and its relation with the dual numbers, the reader is referred to [17] and the references cited there.

In §4 we introduce an extended polylogarithm Lie algebra. The dual co-Lie algebra has generators $\{x\}_n$ and $\langle x\rangle_n$ for $x \in k - \{0,1\}$. The dual of the bracket satisfies $\partial\{x\}_n = \{x\}_{n-1}\cdot\{1-x\}_1$ and $\partial\langle x\rangle_n = \langle x\rangle_{n-1}\cdot\{1-x\}_1 + \langle 1-x\rangle_1\cdot\{x\}_{n-1}$ with $\langle x\rangle_1 = x \in k$. For example, $\partial\langle x\rangle_2 = x\otimes x + (1-x)\otimes(1-x) \in k\otimes k^\times$ is the Cathelineau relation [7]. It seems likely that there exists a representation of this Lie algebra, extending the polylog representation of the sub Lie algebra generated by the $\{x\}_n$, and related to variations of Hodge structure over the dual numbers lifting the polylog Hodge structure.

§5 was inspired by Deligne's interpretation of symbols [9] in terms of line bundles with connections. We indicate how this viewpoint is related to the additive dilogarithm. In characteristic 0, one finds affine bundles with connection, and the regulator map on $K_2$ linearizes to the evident map $H^0(X,\Omega^1)\to\mathbb{H}^1(X,\mathcal{O}\to\Omega^1)$. In characteristic $p$, Artin-Schreier yields an exotic flat realization of the additive dilogarithm motive. For simplicity we limit ourselves to calculations mod $p$. The result is a flat covering $T$ of $\mathbb{A}^1-\{0,1\}$ which is a torsor under a flat Heisenberg groupscheme $\mathcal{H}_{AS}$. This groupscheme has a natural representation on the abelian groupscheme $\mathbb{V} := \mathbb{Z}/p\mathbb{Z}\oplus\mu_p\oplus\mu_p$. The contraction

$$T \overset{\mathcal{H}_{AS}}{\times} \mathbb{V} \tag{1.10}$$

should, we think, be considered as analogous to the mod $\ell$ étale sheaf on $\mathbb{A}^1-\{0,1\}$ with fibre $\mathbb{Z}/\ell\mathbb{Z}\oplus\mu_\ell\oplus\mu_\ell^{\otimes 2}$ associated to the $\ell$-adic dilogarithm.

The polylogarithms can be interpreted in terms of algebraic cycles on products of copies of $\mathbb{P}^1-\{1\}$ ([3], (3.3)), so it seems natural to consider algebraic cycles on

$$(\mathbb{A}^1, 2\{0\})\times(\mathbb{P}^1-\{1\},\{0,\infty\})^n. \tag{1.11}$$

In the final section of this paper, we calculate the Chow groups of 0-cycles on these spaces. Our result:

$$CH_0\Big((\mathbb{A}^1,2\{0\})\times(\mathbb{P}^1-\{1\},\{0,\infty\})^n\Big)\cong\Omega^n_k,\quad n\ge 0, \tag{1.12}$$

is a "degeneration" of the result of Totaro [23] and Nesterenko-Suslin [21]

$$CH_0\Big((\mathbb{P}^1-\{1\},\{0,\infty\})^n\Big)\cong K_n^M(k) = n\text{-th Milnor } K\text{-group}, \tag{1.13}$$

and a cubical version of the simplicial result $SH^n(k,n)\cong\Omega_k^{n-1}$ (see [6]).

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## 2. Additive Bloch groups

Let $k$ be a field with $1/2\in k$. In this section, we mimic the construction in [2] §5, replacing the semi-local ring of functions on $\mathbb{P}^1$, regular at 0 and $\infty$ by the local ring of functions on $\mathbb{A}^1$, regular at 0, and the relative condition on $K$-theory at 0 and $\infty$ by the one at $2\cdot\{0\}$. In particular, as we fix only 0 and $\infty$ in this theory, we have a $k^\times$-action on the parameter $t$ on $\mathbb{A}^1$ so our groups will be $k^\times$-modules.

Thus let $R$ be the local ring at 0 on $\mathbb{A}^1_k$. One has an exact sequence of relative $K$-groups

$$K_2(\mathbb{A}^1_k)\to K_2(k[t]/(t^2))\to K_1(\mathbb{A}^1_k,(t^2))\to K_1(\mathbb{A}^1_k)\to K_1(k[t]/(t^2)). \tag{2.1}$$

Using Van der Kallen's calculation of $K_2(k[t]/(t^2))$ [24], [25], and the homotopy property $K_*(k)\cong K_*(\mathbb{A}^1_k)$, we conclude

$$K_1(\mathbb{A}^1_k,(t^2))\cong\Omega^1_k. \tag{2.2}$$

Now we localize on $\mathbb{A}^1$ away from 0. Assuming for simplicity that $k$ is algebraically closed, we get

$$\coprod_{k-\{0\}} K_2(k) \to K_2(\mathbb{A}^1_k, (t^2)) \to K_2(R, (t^2)) \to \coprod_{k-\{0\}} k^\times \to \Omega^1_k \to 0 \tag{2.3}$$

To $a \in (t^2)$ and $b \in R$ we associate the pointy-bracket symbol [20] $\langle a, b\rangle \in K_2(R, (t^2))$ which corresponds to the Milnor symbol $\{1 - ab, b\}$ if $b \neq 0$. These symbols generate $K_2(R, (t^2))$. If the divisors of $a$ and $b$ are disjoint, we get

$$\mathrm{tame}\langle a, b\rangle = a|_{\text{poles of } b} + b|_{ab=1} + b^{-1}|_{\text{poles of } a} \tag{2.4}$$

We continue to assume $k$ algebraically closed. Let $\mathcal{C} \subset K_2(R, (t^2))$ be the subgroup generated by pointy-bracket symbols with $b \in k$. For $a \in (t^2)$ write

$$a(t) = \frac{a_0 t^n + \ldots + a_{n-2}t^2}{t^m + b_1 t^{m-1} + \ldots + b_{m-1}t + b_m}; \quad b_m \neq 0. \tag{2.5}$$

We assume numerator and denominator have no common factors. If $\alpha_i$ are the solutions to the equation $a(t) = \kappa \in k^\times \cup \infty$, then $\sum \alpha_i^{-1} = -b_{m-1}/b_m$. In particular, this is independent of $\kappa$. It follows that one has an isomorphism

$$\coprod_{k-\{0\}} k^\times \Big/ \mathrm{tame}(\mathcal{C}) \cong k \otimes_{\mathbb{Z}} k^\times; \quad u|_v \mapsto v^{-1} \otimes u. \tag{2.6}$$

Define

$$TB_2(k) := K_2(R, (t^2))/\mathcal{C} \tag{2.7}$$
$$TH^1_M(k, 2) := \mathrm{image}\Big(K_2(\mathbb{A}^1, (t^2)) \to TB_2(k)\Big)$$
$$TH^2_M(k, 2) := \Omega^1_k = K_1(R, (t^2))$$

A basic result of Goodwillie [16] yields $K_2(\mathbb{A}^1, (t^2)) \cong k$, so $TH^1_M(k, 2)$ is a quotient of $k$. We will see (remark 2.6) that in fact $TH^1_M(k, 2) \cong k$. The above discussion yields

**Proposition 2.1.** *Let $k$ be an algebraically closed field of characteristic $\neq 2$. With notations as above, we have an exact sequence*

$$0 \to TH^1_M(k, 2) \to TB_2(k) \xrightarrow{\partial} k \otimes k^\times \xrightarrow{\pi} \Omega^1_k \to 0. \tag{2.8}$$

*Here $\pi(a \otimes b) = a\frac{db}{b}$ and $\partial$ is defined via the tame symbol.*

**Remark 2.2.** There is an evident action of the group $k^\times$ on $\mathbb{A}^1$ (multiplying the parameter) and hence on the sequence (2.8). This action extends to a $k$-vector space structure on all the terms except $TB_2(k)$.

Let $\mathfrak{m} = tR \subset R$. One has the following purely algebraic description of $K_2(R, \mathfrak{m}^2)$ ([22], formula (1.4), and the references cited there).
generators:

$$\langle a, b\rangle; \qquad (a, b) \in (R \times \mathfrak{m}^2) \cup (\mathfrak{m}^2 \times R) \tag{2.9}$$

Relations:

$$\langle a, b\rangle = -\langle b, a\rangle; \quad a \in \mathfrak{m}^2 \tag{2.10}$$

$$\langle a, b\rangle + \langle a, c\rangle = \langle a, b + c - abc\rangle; \quad a \in \mathfrak{m}^2 \text{ or } b, c \in \mathfrak{m}^2 \tag{2.11}$$

$$\langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle; \quad a \in \mathfrak{m}^2 \tag{2.12}$$

**Proposition 2.3.** *There is a well-defined and nonzero map*

$$\rho : K_2(R, \mathfrak{m}^2) \to \mathfrak{m}^3/\mathfrak{m}^4 \tag{2.13}$$

*defined by*

$$\rho\langle a, b\rangle := \begin{cases} -adb & a \in \mathfrak{m}^2 \\ bda & b \in \mathfrak{m}^2. \end{cases} \tag{2.14}$$

*Proof.* Note first if $a, b \in \mathfrak{m}^2$ then $adb \equiv bda \equiv 0 \mod \mathfrak{m}^4$ so the definition (2.14) is consistent. For $a \in \mathfrak{m}^2$

$$\langle a, b\rangle + \langle b, a\rangle \mapsto -adb + adb = 0, \tag{2.15}$$

so (2.10) holds. For $a \in \mathfrak{m}^2$

$$\langle a, b\rangle + \langle a, c\rangle \mapsto -ad(b + c) \equiv -ad(b + c - abc) \mod \mathfrak{m}^4 \tag{2.16}$$

for $b, c \in \mathfrak{m}^2$

$$\langle a, b\rangle + \langle a, c\rangle \mapsto (b + c)da \equiv (b + c - abc)da \mod \mathfrak{m}^4 \tag{2.17}$$

For $a \in \mathfrak{m}^2$,

$$\langle a, bc\rangle \mapsto -ad(bc) = -abdc - acdb = \rho(\langle ab, c\rangle + \langle ac, b\rangle) \tag{2.18}$$

□

**Remark 2.4.** Note that

$$-adb \equiv \log(1 - ab)db/b \in \mathfrak{m}^2\Omega^1_R/d\log(1 + \mathfrak{m}^4) \cong \mathfrak{m}^3/\mathfrak{m}^4. \tag{2.19}$$

The group $\mathfrak{m}^2\Omega^1_R/d\log(1 + \mathfrak{m}^4)$ is the group of isomorphism classes of rank 1 line bundles, trivialized at the order 4 at $\{0\}$, with a connection vanishing at the order 2 at $\{0\}$. Thus the regulator map $\rho$ assigns such a connection to a pointy symbol. Over the field of complex numbers $\mathbb{C}$, one can think of it in terms of "Deligne cohomology" $\mathbb{H}^2(\mathbb{A}^1, j_!\mathbb{Z}(2) \to t^4\mathcal{O} \to t^2\omega)$, and one can, as in [12], write down explicitely an analytic Cech cocycle for this regulator as a Loday symbol.

One has $\rho\langle t^2, x\rangle = -t^2dt \neq 0$, thus $\rho$ is not trivial. Note also, the appearance of $\mathfrak{m}^3$ is consistent with A. Goncharov's idea [15] that the regulator in this context should correspond to the volume of a simplex in hyperbolic 3 space in the scissors-congruence interpretation [17]. In particular, it should scale as the third power of the coordinate.

Proposition 2.3 yields

**Corollary 2.5.** *Let $\mathfrak{m} \subset R$ be the maximal ideal. One has a well-defined map*

$$\rho : TB_2(k) \to \mathfrak{m}^3/\mathfrak{m}^4 \tag{2.20}$$

*given on pointy-bracket symbols by*

$$\rho\langle a, b\rangle = -a \cdot db; \quad a \in \mathfrak{m}^2,\ b \in R. \tag{2.21}$$

*For $x \in TB_2(k)$ and $c \in k^\times$, write $c \star x$ for the image of $x$ under the mapping $t \mapsto c \cdot t$ on polynomials. Then $\rho(c \star x) = c^3 \cdot \rho(x)$.*

*Proof.* The first assertion follows because if $b \in k$, then $db = 0$. The second assertion is clear. □

**Remark 2.6.** The map $\rho$ is non-trivial on $TH^1_M(k,2)$ because $\rho\langle t^2, t\rangle = -t^2dt \neq 0$. Since this group is a $k^\times$-module (remark 2.2) and is a quotient of $k$ by the result of Goodwillie cited above, it follows that

$$TH^1_M(k,2) \cong (t^3)/(t^4) \cong k \tag{2.22}$$

## 3. Cathelineau elements and the entropy functional equation

We continue to assume $k$ is an algebraically closed field of characteristic $\neq 2$. Define for $a \in k - \{0,1\}$

$$\langle a\rangle := \langle t^2, \frac{a(1-a)}{t-1}\rangle \in TB_2(k) \tag{3.1}$$
$$\epsilon(a) := a\otimes a + (1-a)\otimes(1-a) \in k^\times \otimes k$$

**Lemma 3.1.** *Writing $\partial$ for the tame symbol as in proposition 2.1, we have $\partial(\langle a\rangle) = 2\epsilon(a)$.*

*Proof.*

$$\partial(\langle a\rangle) = \mathrm{tame}\Big\{\frac{\frac{1-t}{a(1-a)}+t^2}{\frac{1-t}{a(1-a)}}, \frac{a(1-a)}{t-1}\Big\} = \tag{3.2}$$

$$\frac{a(1-a)}{t-1}\Big|_{t=\frac{1}{a}} + \frac{a(1-a)}{t-1}\Big|_{t=\frac{1}{1-a}} \mapsto a^2\otimes a + (1-a)^2\otimes(1-a) = 2\epsilon(a) \in k^\times\otimes k.$$

□

**Lemma 3.2.** *We have $\rho(\langle a\rangle) = a(1-a)t^2dt \in (t^3)/(t^4)$.*

*Proof.* Straightforward from corollary 2.5. □

**Lemma 3.3.** *Let notations be as in corollary 2.5, Assume $k$ is algebraically closed, and $\mathrm{char}(k) \neq 2, 3$. Then every element in $TB_2(k)$ can be written as a sum $\sum c_i \star \langle a_i\rangle$. In other words, $TB_2(k)$ is generated as a $k^\times$-module by the $\langle a\rangle$.*

*Proof.* Define

$$\mathfrak{b} := \mathrm{Image}(\partial : TB_2(k) \to k^\times \otimes k) = \ker(k^\times\otimes k \to \Omega^1_k). \tag{3.3}$$

The $k$-vector space structure $c\cdot(a\otimes b)$ on $k^\times\otimes k$ is defined by $a\otimes cb$. By (2.6) and (2.4), the map $TB_2(k) \to k^\times\otimes k$ is $k^\times$-equivariant.

Let $A \subset TB_2(k)$ be the subgroup generated by the $c\star\langle a\rangle$. $\mathfrak{b}$ is a $k$-vector space which is generated [7] by the $\epsilon(a)$ so the composition $A \subset TB_2(k) \to \mathfrak{b}$ is surjective. For $c_1, c_2 \in k^\times$ with $c_1 + c_2 \neq 0$ we have $(c_1+c_2)\star\langle a\rangle - c_1\star\langle a\rangle - c_2\star\langle a\rangle \mapsto 0 \in \mathfrak{b}$, so this element lies in $A \cap H^1_M(k,2)$. It is not trivial because

$$\begin{aligned}\rho((c_1+c_2)\star\langle a\rangle - c_1\star\langle a\rangle - c_2\star\langle a\rangle) &= \\ \Big((c_1+c_2)^3 - c_1^3 - c_2^3\Big)a(1-a)t^2dt &= \\ 3\Big(c_1c_2(c_1+c_2)\Big)a(1-a)t^2dt.\end{aligned} \tag{3.4}$$

Since the equation $\lambda = 3\Big(c_1c_2(c_1+c_2)\Big)a(1-a)$ can be solved in $k$, one has $A \supset H^1_M(k,2)$. This finishes the proof. □

**Theorem 3.4.** *Under the assumptions of lemma 3.3, the group $TB_2(k)$ is generated as a $k^\times$-module by the $\langle a\rangle$. These satisfy relations*

$$\langle a\rangle - \langle b\rangle + a \star \langle b/a\rangle + (1-a) \star \langle (1-b)/(1-a)\rangle = 0. \tag{3.5}$$

*Proof.* The generation statement is lemma 3.3. Because we factor out by symbols with one entry constant, we get

$$x \star \langle a\rangle = \langle x^2t^2, \frac{a(1-a)}{xt-1}\rangle = \langle t^2, \frac{x^2a(1-a)}{xt-1}\rangle. \tag{3.6}$$

The identity to be established then reads

$$0 = \langle t^2, \frac{a(1-a)}{t-1}\rangle - \langle t^2, \frac{b(1-b)}{t-1}\rangle + \langle t^2, \frac{b(a-b)}{at-1}\rangle + \langle t^2, \frac{(1-b)(b-a)}{(1-a)t-1}\rangle. \tag{3.7}$$

The pointy bracket identity $\langle a, b\rangle + \langle a, c\rangle = \langle a, b + c - abc\rangle$ means we can compute the above sum using "faux" symbols

$$\{t^2, 1 - \frac{a(1-a)t^2}{t-1}\}\{t^2, 1 - \frac{b(1-b)t^2}{t-1}\}^{-1}\{t^2, 1 - \frac{b(a-b)t^2}{at-1}\}\times \\ \{t^2, 1 - \frac{(1-b)(b-a)t^2}{(1-a)t-1}\} = \{t^2, X\} \tag{3.8}$$

with

$$\begin{aligned} X &= \frac{(1-t+a(1-a)t^2)(1-at+b(a-b)t^2)(1-(1-a)t+(1-b)(b-a)t^2)}{(1-t+b(1-b)t^2)(1-at)(1-(1-a)t)} \\ &= \frac{(1-at)(1-(1-a)t)(1-bt)(1-(a-b)t)(1-(1-b)t)(1-(b-a)t)}{(1-bt)(1-(1-b)t)(1-at)(1-(1-a)t)} \\ &= (1-(a-b)t)(1-(b-a)t) = 1-(a-b)^2t^2 \end{aligned} \tag{3.9}$$

Reverting to pointy brackets, the Cathelineau relation equals

$$\{t^2, X\} = \{1-(a-b)^2t^2, (a-b)^2\} = \langle t^2, (a-b)^2\rangle = 0 \tag{3.10}$$

since we have killed symbols with one entry constant. □

**Remark 3.5.** In this remark, we assume $k$ to be algebraically closed of characteristic 0, in order to simplify the discussion by using [7], proposition 3, section 4.2. There is a presentation for $TB_2(k)$ as follows. Let $\mathcal{I} \subset \mathbb{Z}[k^\times]$ be the kernel of the group ring homomorphism induced by the group homomorphism $k^\times \to k^\times \times k^\times, x \mapsto (x, x^3)$. Then $TB_2(k)$ is the free $\mathbb{Z}[k^\times]$-module with generators $\langle a\rangle, a \in k^\times$ and with the relations spanned by $z \star \langle a\rangle$ for $z \in \mathcal{I}$ and by the submodule $\mathbb{Z}[k^\times] \star ((3.5))$. Indeed, $\mathbb{Z}[k^\times]/\mathcal{I} \star \{\langle a\rangle\} \cong (k \times k) \star \{\langle a\rangle\}$. We consider separately the two projections $k \times k \to k$. On the first, $k^\times$ acts linearly. By [7], proposition 6, section 4.2, one has $k \star \{\langle a\rangle\}/k \star ((3.5)) \cong \ker(k \otimes k^\times \to \Omega^1_k)$. On the second factor, $k^\times$ acts by $[z] \star \langle a\rangle = z^3\langle a\rangle$. The fact that $\langle a\rangle \mapsto a(1-a)$ (lemma 3.2) identifies this quotient with $k$ follows easily from uniqueness of solutions for the entropy equation [10], section 2, theorem 2.

We remark that $\mathcal{I}$ contains the ideal generated by

$$\Big((x+y+z+w)-(x+y+z)-(x+y+w)-\ldots-(x)-(y)-(z)-(w)\Big)\star\langle a\rangle=0 \tag{3.11}$$

$$(-1)\star\langle a\rangle=-\langle a\rangle. \tag{3.12}$$

Here $x\in k^\times$ corresponds to $(x)\in\mathbb{Z}[k^\times]$, and the first relation is imposed whenever it makes sense, i.e. whenever all the partial sums are non-zero. We haven't checked under what circumstances if ever (3.11) computes exactly $\mathcal{I}$.

**Remark 3.6.** It is remarkable that a functional equation equivalent to (3.5),

$$\langle a\rangle+(1-a)\star\langle\frac{b}{1-a}\rangle=\langle b\rangle+(1-b)\star\langle\frac{a}{1-b}\rangle \tag{3.13}$$

occurs in information theory, where it is known to have a unique continuous functional solution (up to scale) given by $y\star\langle x\rangle\mapsto -yx\log(x)-y(1-x)\log(1-x)$. If on the other hand, we interpret the torus action $y\star$ as multiplication by $y^p, p\neq 1$, then the unique solution is $\langle x\rangle\mapsto x^p+(1-x)^p-1$ [10]. We used explicitely in remark 3.5 that the regulator map $\rho(y\star\langle x\rangle)=y^3x(1-x)$, so $\rho$ is a solution for $p=3$. Indeed, $x(1-x)=\frac{1}{3}(x^3+(1-x)^3-1)$. (Again, one uses char $(k)\neq 3$.)

One can check that the functional equation (3.13) is equivalent to (3.5). To see this, one needs the following property of the elements $\langle a\rangle$.

**Lemma 3.7.** *$\langle a\rangle=-a\star\langle a^{-1}\rangle$.*

*Proof.* We remark again that $TB_2(k)\xrightarrow{\rho\oplus\partial}k\oplus\mathfrak{b}$ is an isomorphism, so it suffices to check the relations on $\epsilon(a)$ and on $\rho(a)=a(1-a)t^2dt$. These become respectively

$$a\otimes a+(1-a)\otimes(1-a)=a^{-1}\otimes -1+(1-a^{-1})\otimes(1-a)\in k^\times\otimes k \tag{3.14}$$

$$-a^3(a^{-1}(1-a^{-1}))=a(1-a). \tag{3.15}$$

The second relation is trivial. For the first one, one writes

$$\begin{aligned} a\otimes a+(1-a)\otimes(1-a)&=a\otimes a+(-a)\otimes(1-a)+(1-a^{-1})\otimes(1-a)\\ &=a^{-1}\otimes(-a+a-1)+(-1)\otimes(1-a).\end{aligned} \tag{3.16}$$

Since $k$ is 2-divisible, one has $(-1)\otimes b=0$. $\square$

## 4. A conjectural Lie algebra of cycles

The purpose of this section is to sketch a conjectural algebraic cycle based theory of additive polylogarithms. The basic reference is [4], where a candidate for the Tannakian Lie algebra of the category of mixed Tate motives over a field $k$ is constructed. The basic tool is a differential graded algebra (DGA) $\mathcal{N}$ with a supplementary grading (Adams grading)

$$\mathcal{N}^\bullet=\oplus_{j\geq 0}\mathcal{N}(j)^\bullet \tag{4.1}$$

$$\mathcal{N}(j)^i\subset\text{Codim. } j \text{ algebraic cycles on } (\mathbb{P}^1-\{1\})^{2j-i}$$

where $\mathcal{N}(j)^i$ consists of cycles which meet the faces (defined by setting coordinates $=0,\infty$) properly and which are alternating with respect to the action of the symmetric group on the factors and with respect to inverting the coordinates. The product structure is the external product $(\mathbb{P}^1-\{1\})^{2j_1-i_1}\times(\mathbb{P}^1-\{1\})^{2j_2-i_2}=$

$(\mathbb{P}^1 - \{1\})^{2j_1-i_1+2j_2-i_2}$ followed by alternating projection, and the boundary map is an alternating sum of restrictions to faces. For full details, cf. op. cit.

We consider an enlarged DGA

$$\widetilde{\mathcal{N}}^\bullet = \oplus_{j\geq 0}\widetilde{\mathcal{N}}(j)^\bullet \tag{4.2}$$
$$\widetilde{\mathcal{N}}(j)^i := \mathcal{N}(j)^i \oplus T\mathcal{N}(j)^i$$
$$T\mathcal{N}(j)^i \subset \text{Codim. } j \text{ algebraic cycles on } \mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2j-i-1}.$$

The same sort of alternation and good position requirements are imposed for the factors $\mathbb{P}^1 - \{1\}$. In addition, we impose a "modulus" condition at the point $0 \in \mathbb{A}^1$. The following definition is tentative, and is motivated by example 4.2 below.

**Definition 4.1.** Let $D$ be the effective divisor $\mathbb{A}^1 \times \Big((\mathbb{P}^1)^n - \mathbb{G}_m^n\Big)$ on $\mathbb{A}^1 \times (\mathbb{P}^1)^n$, where $\mathbb{G}_m = \mathbb{P}^1 - \{0,\infty\}$. Let $Z \subset \mathbb{A}^1 \times (\mathbb{P}^1)^n$ be an effective algebraic cycle. We assume no component of $Z$ lies on $D$. Let $m \geq 1$ be an integer. Let $F_i : y_i = 1$. We assume no component of $Z$ lies in an $F_i$. (Components lying in an $F_i$ can be ignored when computing motivic cohomology). Write $F_i \cdot Z = \sum r_{W,i}W$, and define $r_W = \max_i(r_{W,i})$. We say that $Z$ weakly satisfies the modulus $m$ ($Z \equiv 0 \mod m\{0\}\times(\mathbb{P}^1)^n$) if the intersection $Z\cdot(0)\times(\mathbb{P}^1)^n = \sum m_V \cdot V$ is defined, and for each $V$ with $m_V \neq 0$ we have

$$m \cdot m_V \leq \begin{cases} r_V & V \not\subset D \\ r_V - \epsilon_V & \text{else} \end{cases} \tag{4.3}$$

Here $\epsilon_V$ is the multiplicity with which $V$ occurs in $Z \cdot D$.

We say $Z$ satisfies modulus $m$ if $Z^0 := Z|_{\mathbb{A}^1\times(\mathbb{P}^1-\{1\})^n}$ is in good position with respect to all face maps, and if $Z$ and the closures of all faces of $Z^0$ weakly satisfy modulus $m$.

If $Z$ satisfies modulus $m$ and $X$ is any subvariety of $(\mathbb{P}^1)^r$ which is not contained in a face, then $Z \times X$ satisfies modulus $m$ on $\mathbb{A}^1 \times (\mathbb{P}^1)^{n+r}$.

**Example 4.2.** Milnor $K$-theory of a field can be interpreted in terms of 0-cycles [21], [23]. More generally, a Milnor symbol $\{f_1,\ldots,f_p\}$ over a ring $R$ corresponds to the cycle on $\mathrm{Spec}\,(R) \times (\mathbb{P}^1)^p$ which is just the graph

$$\{(x, f_1(x), \ldots f_p(x)) | x \in \mathrm{Spec}\,(R)\}$$

One would like cycles with modulus to relate to relative $K$-theory. Assume $R$ is semilocal, and let $J \subset R$ be an ideal. Then we have already used (2.9) that $K_2(R,J)$ has a presentation with generators given by pointy-bracket symbols $\langle a,b\rangle$ with $a \in R$ and $b \in J$ or vice-versa. The pointy-bracket symbol $\langle a,b\rangle$ corresponds to the Milnor symbol $\{1-ab, b\}$ when the latter is defined. Suppose $R$ is the local ring on $\mathbb{A}^1_k$ at the origin, with $k$ a field, and take $J = (s^m)$, where $s$ is the standard parameter. For $a \in J$ and $b \in (s^p)$ for some $p \geq 0$, we see that our definition of cycle with modulus is designed so the cycle $\{(x, 1-a(x)b(x), b(x))\}$ has modulus at least $m$.

The modulus condition is compatible with pullback to the faces $t_i = 0, \infty$.

**Definition 4.3.** $T\mathcal{N}(j)^i$, $-\infty \leq i \leq 2j-1$, is the $\mathbb{Q}$-vector space of codimension $j$ algebraic cycles on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2j-i-1}$ which are in good position for the face maps $t_i = 0,\infty$ and have modulus $2\{0\} \times (\mathbb{P}^1)^{2j-i-1}$. Here, in order to calculate the modulus, we close up the cycle to a cycle on $\mathbb{A}^1 \times (\mathbb{P}^1)^{2j-i-1}$.

Note that a cycle $Z$ of modulus $m \geq 1$ doesn't meet $\{0\} \times (\mathbb{P}^1)^{2j-i-1}$ on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2j-i-1}$.

We have a split-exact sequence of $DGA$'s,

$$0 \to T\mathcal{N}^\bullet \to \widetilde{\mathcal{N}}^\bullet \leftrightarrows \mathcal{N}^\bullet \to 0 \tag{4.4}$$

with multiplication defined so $T\mathcal{N}^\bullet$ is a square-zero ideal. Denote the cohomology groups by

$$\widetilde{H}^i_M(k,j) := H^i(\widetilde{\mathcal{N}}^\bullet(j)); \quad TH^i_M(k,j) := H^i(T\mathcal{N}^\bullet(j)). \tag{4.5}$$

As an example, we will see in section 6 that the Chow groups of 0-cycles in this context compute the Kähler differential forms:

$$TH^j_M(k,j) \cong \Omega^{j-1}_k; j \geq 0. \tag{4.6}$$

(Here $\Omega^0_k = k$.)

One may apply the bar construction to the DGA $\widetilde{\mathcal{N}}^\bullet$ as in [4]. Taking $H^0$ yields an augmented Hopf algebra (defining $TH^0$ as the augmentation ideal)

$$0 \to TH^0(B(\widetilde{\mathcal{N}}^\bullet)) \to H^0(B(\widetilde{\mathcal{N}}^\bullet)) \leftrightarrows H^0(\mathcal{N}^\bullet) \to 0. \tag{4.7}$$

The hope would be that the corepresentations of the co-Lie algebra of indecomposables (here $H^{0,+} := \ker(H^0 \to \mathbb{Q})$ denotes the elements of bar degree $> 0$, cf. op. cit. §2)

$$\widetilde{\mathcal{M}} := H^0(B(\widetilde{\mathcal{N}}^\bullet))^+/(H^0(B(\widetilde{\mathcal{N}}^\bullet))^+)^2 = \mathcal{M} \oplus T\mathcal{M} \tag{4.8}$$

correspond to contravariant motives over $k[t]/(t^2)$. In particular, the work of Cathelineau [7] suggests a possible additive polylogarithm Lie algebra. In the remainder of this section, we will speculate a bit on how this might work.

For a general DGA $A^\bullet$ which is not bounded above, the total grading on the double complex $B(A^\bullet)$ has infinitely many summands (cf. [4], (2.15)). For example the diagonal line corresponding to $H^0(B(A^\bullet))$ has terms ($A^+ := \ker(A^\bullet \to \mathbb{Q})$)

$$A^1,\ (A^+ \otimes A^+)^2,\ (A^+ \otimes A^+ \otimes A^+)^3, \ldots \tag{4.9}$$

When, however, $A^\bullet$ has a graded structure

$$A^i = \oplus_{j \geq 0} A^i(j); \quad dA(j) \subset A(j); \quad A^+ = \oplus_{j>0} A^+(j), \tag{4.10}$$

for each fixed $j$, only finitely many tensors can occur. For example $H^0(B(\widetilde{\mathcal{N}}^\bullet(1))) = H^1(\widetilde{\mathcal{N}}^\bullet(1)) = k \oplus k^\times$, and $H^0(B(\widetilde{\mathcal{N}}^\bullet(2)))$ is the cohomology along the indicated degree 0 diagonal in the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{N}}^1(1) \otimes \widetilde{\mathcal{N}}^1(1) & \xrightarrow{\delta} & \widetilde{\mathcal{N}}^2(2) \\ \uparrow \partial & \ddots\ deg.\ 0 & \uparrow \partial \\ (\widetilde{\mathcal{N}}^1(1) \otimes \widetilde{\mathcal{N}}^0(1)) \oplus (\widetilde{\mathcal{N}}^0(1) \otimes \widetilde{\mathcal{N}}^1(1)) & \xrightarrow{\delta} & \widetilde{\mathcal{N}}^1(2) \\ & & \uparrow \\ & & \widetilde{\mathcal{N}}^0(2). \end{array} \tag{4.11}$$

In the absence of more information about the DGA $\widetilde{\mathcal{N}}^\bullet$, it is difficult to be precise about the indecomposable space $\widetilde{\mathcal{M}}$. As an approximation, we have

**Proposition 4.4.** *Let $db(\widetilde{\mathcal{N}}) \subset \widetilde{\mathcal{N}}^1$ be the subspace of elements $x$ with decomposable boundary, i.e. such that there exists $y \in (\widetilde{\mathcal{N}} \otimes \widetilde{\mathcal{N}})^2$ with $\delta(y) = \partial(x) \in \widetilde{\mathcal{N}}^2$. Define*

$$q\widetilde{\mathcal{N}} := db(\widetilde{\mathcal{N}})/(\partial\widetilde{\mathcal{N}}^0 + \delta(\widetilde{\mathcal{N}}^+ \otimes \widetilde{\mathcal{N}}^+)^1) \tag{4.12}$$

*Then there exists a natural map, compatible with the grading by codimension of cycles (Adams grading)*

$$\phi : \widetilde{\mathcal{M}} \to q\widetilde{\mathcal{N}}. \tag{4.13}$$

*Proof.* Straightforward. □

As above, we can decompose

$$q\widetilde{\mathcal{N}} = q\mathcal{N} \oplus Tq\mathcal{N}, \tag{4.14}$$

where $q\mathcal{N}(p)$ is a subquotient of the space of codimension $p$ cycles on $(\mathbb{P}^1 - \{1\})^{2p-1}$, and $Tq\mathcal{N}$ is a subquotient of the cycles on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2p-2}$

**Example 4.5.** The polylogarithm cycle $\{a\}_p$ for $a \in \mathbb{C} - \{0, 1\}$ is defined to be the image under the alternating projection of $(-1)^{p(p-1)/2}$ times the locus in $(\mathbb{P}^1 - \{1\})^{2p-1}$ parametrized in nonhomogeneous coordinates by

$$(x_1, \ldots, x_{p-1}, 1 - x_1, 1 - x_2/x_1, \ldots, 1 - x_{p-1}/x_{p-2}, 1 - a/x_{p-1}) \tag{4.15}$$

(We take $\{a\}_1 = 1 - a \in \mathbb{P}^1 - \{1\}$.) To build a class in $H^0(B(\mathcal{N}^\bullet))^+$ and hence in $\mathcal{M}$ one uses that $\partial\{a\}_n = \{a\}_{n-1} \cdot \{1 - a\}_1$.

The following should be compared with [11], where a similar formula is proposed. The key new point here is that algebraic cycles make it possible to envision this formula in the context of Lie algebras.

**Conjecture 4.6.** There exist elements $\langle a\rangle_n \in T\mathcal{M}(n)$ (4.8) represented by cycles $Z_n(a)$ of codimension $n$ on $\mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^{2n-2}$ with $\langle a\rangle_1 = a \in \mathbb{A}^1 - \{0\}$. These cycles should satisfy the boundary condition

$$\partial\langle a\rangle_n = \langle a\rangle_{n-1} \cdot \{1 - a\}_1 + \langle 1 - a\rangle_1 \cdot \{a\}_{n-1} \in \Big(\bigwedge^2 \widetilde{\mathcal{M}}\Big)(n). \tag{4.16}$$

For example, for $n = 2$,

$$\partial\langle a\rangle_2 = a \otimes a + (1 - a) \otimes (1 - a) \tag{4.17}$$
$$\in k \otimes k^\times \cong T\mathcal{M}(1) \otimes \mathcal{M}(1) \subset \Big(\bigwedge^2 \widetilde{\mathcal{M}}\Big)(2)$$

gives Cathelineau's relation [7].

**Proposition 4.7.** *Assume given elements $\langle a\rangle_n$ satisfying* (4.16)*. Let $\widetilde{\mathcal{P}} = \bigoplus_{n=1}^\infty \mathbb{Q}\langle a\rangle_n \oplus \mathbb{Q}\{a\}_n$ be the constant graded sheaf over $\mathbb{A}^1 - \{0, 1\}$. Then $\widetilde{\mathcal{P}}, \partial : \widetilde{\mathcal{P}} \to \bigwedge^2 \widetilde{\mathcal{P}}$ is a sheaf of co-lie algebras.*

*Proof.* It suffices to show that $\partial^2 = 0$. Using the derivation property of the boundary,

$$\partial\partial\langle a\rangle_n = (\partial\langle a\rangle_{n-1}) \cdot \{1 - a\}_1 - \langle 1 - a\rangle_1 \cdot \partial\{a\}_{n-1} = \tag{4.18}$$
$$\Big(\langle a\rangle_{n-2} \cdot \{1 - a\}_1 + \langle 1 - a\rangle_1 \cdot \{a\}_{n-2}\Big) \cdot \{1 - a\}_1 - \langle 1 - a\rangle_1 \cdot \{a\}_{n-2} \cdot \{1 - a\}_1 =$$
$$0 \in \bigwedge^3 \widetilde{\mathcal{M}}.$$

□

We can make the definition (independent of any conjecture)

**Definition 4.8.** The additive polylogarithm sheaf of Lie algebras over $\mathbb{A}^1 - \{0,1\}$ is the graded sheaf of Lie algebras with graded dual the sheaf $\widetilde{\mathcal{P}}, \partial : \widetilde{\mathcal{P}} \to \bigwedge^2 \widetilde{\mathcal{P}}$ satisfying (4.16) above.

Of course, $\langle a\rangle_2$ should be closely related to the element $\langle a\rangle \in K_2(R,(t^2))$ (3.1). The cycle

$$\{(t, 1 - \frac{t^2 a(1-a)}{t-1}, \frac{a(1-a)}{t-1}) \mid t \in \mathbb{A}^1\} \subset \mathbb{A}^1 \times (\mathbb{P}^1 - \{1\})^2 \tag{4.19}$$

associated to the pointy bracket symbol in (3.1) satisfies the modulus 2 condition but is not in good position with respect to the faces. (It contains $(1,\infty,\infty)$.) It is possible to give symbols equivalent to this one whose corresponding cycle is in good position, but we do not have a canonical candidate for such a cycle, or a candidate whose construction would generalize in some obvious way to give all the $\langle a\rangle_n$.

## 5. The Artin-Schreier dilogarithm

The purpose of this section is to present a definition of what one might call an Artin-Schreier dilogarithm in characteristic $p$. To begin with, however, we take $X$ to be a complex-analytic manifold and sketch certain analogies between the multiplicative and additive theory. We write $\mathcal{O}$ (resp. $\mathcal{O}^\times$, $\Omega^1$) for the sheaf of analytic functions (resp. invertible analytic functions, analytic 1-forms). The reader is urged to compare with [9].

$$\begin{array}{rcl}
\text{MULTIPLICATIVE} & & \text{ADDITIVE} \\
\mathcal{O}^\times \overset{\mathbb{L}}{\otimes} \mathcal{O}^\times & \longleftrightarrow & \mathcal{O} \otimes \mathcal{O}^\times \\
\mathcal{O}^\times \overset{\mathbb{L}}{\otimes} \mathcal{O}^\times \to \Big(\mathcal{O}^\times(1) \to \Omega^1\Big)[1] & \longleftrightarrow & \mathcal{O} \otimes \mathcal{O}^\times \to \Big(\mathcal{O}(1) \to \Omega^1\Big)[1] \\
K_2 & \longleftrightarrow & \Omega^1 \\
\text{Steinberg rel'n } = & \longleftrightarrow & \text{Cathelineau rel'n } = \\
a \otimes (1-a) & & a \otimes a + (1-a) \otimes (1-a) \\
\text{exponential of dilogarithm } = & \longleftrightarrow & \text{Shannon entropy function } = \\
\exp\Big(\int_0^a \log(1-t)dt/t\Big) & & \int^a \log(\frac{t}{1-t})dt = \\
 & & a\log a + (1-a)\log(1-a).
\end{array} \tag{5.1}$$

In the multiplicative (resp. additive) theory, one applies $\mathcal{O}^\times \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} \bullet$ (resp. $\mathcal{O} \otimes_{\mathbb{Z}} \bullet$) to the exponential sequence (here $\mathbb{Z}(1) := \mathbb{Z} \cdot 2\pi i$)

$$0 \to \mathbb{Z}(1) \to \mathcal{O} \to \mathcal{O}^\times \to 0. \tag{5.2}$$

The regulator maps (5.1), line 2, come from liftings of these tensor products to

$$\begin{array}{ccccc}
\mathcal{O}^\times(1) & \to & \mathcal{O}^\times \overset{\mathbb{L}}{\otimes} \mathcal{O} & \to & \mathcal{O}^\times \otimes \mathcal{O}^\times \\
\| & & \downarrow & & \| \\
\mathcal{O}^\times(1) & \to & \mathcal{O}^\times \otimes \mathcal{O} & \to & \mathcal{O}^\times \otimes \mathcal{O}^\times \\
\downarrow & & \downarrow & & \\
\Omega^1 & = & \Omega^1. & &
\end{array} \tag{5.3}$$

$$\begin{tikzcd}\mathcal{O}(1) \arrow[r] \arrow[d] & \mathcal{O}\otimes\mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}\otimes\mathcal{O}^{\times} \\ \Omega^1 \arrow[r, phantom, "="] & \Omega^1 & \end{tikzcd} \tag{5.4}$$

In the multiplicative theory, the regulator map can be viewed as associating to two invertible analytic functions $f,\ g$ on $X$ a line bundle with connection $\mathcal{L}(f,g)$ on $X$, [9]. The exponential of the dilogarithm

$$\exp(\frac{1}{2\pi i}\int_0^f \log(1-t)\frac{dt}{t}) \tag{5.5}$$

determines a flat section trivializing $\mathcal{L}(1-g,g)$. Let $\{U_i\}$ be an analytic cover of $X$, and let $\log_i f$ be an analytic branch of the logarithm on $U_i$. Then $\mathcal{L}(f,g)$ is represented by the Cech cocycle

$$\Big(g^{\frac{1}{2\pi i}(\log_i f-\log_j f)}, \frac{1}{2\pi i}\log_i f\frac{dg}{g}\Big) \tag{5.6}$$

The trivialization comes from the 0-cochain

$$i\mapsto \exp(\frac{1}{2\pi i}\int_0^f \log_i(1-t)\frac{dt}{t}). \tag{5.7}$$

The additive theory associates to $a\otimes f\in\mathcal{O}\otimes\mathcal{O}^{\times}$ the class in $\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$ represented by the cocycle for $\mathcal{O}(1)\to\Omega^1$

$$\Big(a\otimes(\log_i f-\log_j f), \log_i f\cdot da\Big). \tag{5.8}$$

This can be thought of as defining a connection on the affine bundle $\mathcal{A}(a,f)$ associated to the coboundary of $a\otimes f$ in $H^1(X,\mathcal{O}(1))$. The affine bundle itself is canonically trivialized because in the diagram

$$\begin{tikzcd}0 \arrow[r] & \mathcal{O}(1) \arrow[r] \arrow[d] & \mathcal{O}\otimes\mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}\otimes\mathcal{O}^{\times} \arrow[r] & 0 \\ & \Omega^1 \arrow[r, phantom, "="] & \Omega^1 & & \end{tikzcd} \tag{5.9}$$

the top sequence is split (by multiplication $\mathcal{O}\otimes\mathcal{O}\to\mathcal{O}(1)$). The splitting is not compatible with the vertical arrows, so it does not trivialize the connection. More concretely, $a\otimes f\in\mathcal{O}\otimes\mathcal{O}^{\times}$ gives the 1-cocycle $(a\otimes(\log_i f-\log_j f),\ \log_i f\cdot da)\in\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$. Subtracting the coboundary of the 0-cochain $\frac{1}{2\pi i}a\log_i f\otimes 2\pi i$ leaves the cocycle $(0,a\frac{df}{f})$. We have proved:

**Proposition 5.1.** *The map $\partial: H^0(X,\mathcal{O}\otimes\mathcal{O}^{\times})\to\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$ factors*

$$H^0(X,\mathcal{O}\otimes\mathcal{O}^{\times})\xrightarrow{a\otimes f\mapsto adf/f} H^0(X,\Omega^1)\to\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1).$$

*In particular, for $a\in\mathcal{O}$ such that $a$ and $1-a$ are both units, the Cathelineau elements $\epsilon(a)=a\otimes a+(1-a)\otimes(1-a)$ (3.1) lift to*

$$a\otimes\log a+(1-a)\otimes\log(1-a)-\frac{1}{2\pi i}\int^a \log(\frac{t}{1-t})dt\otimes 2\pi i\in H^0(X,\mathcal{O}\otimes\mathcal{O}).$$

**Remark 5.2.** The element $a\otimes a\in H^0(X,\mathcal{O}\otimes\mathcal{O}^{\times})$ maps to $da=0\in\mathbb{H}^1(X,\mathcal{O}(1)\to\Omega^1)$, but the above construction does not give a canonical trivializing 0- cocycle.

We now suppose $X$ is a smooth variety in characteristic $p > 0$, and we consider an Artin-Schreier analog of the above construction. In place of the exponential sequence (5.2) we use the Artin-Schreier sequence of étale sheaves

$$0 \to \mathbb{Z}/p \to \mathbb{G}_a \xrightarrow{1-F} \mathbb{G}_a \to 0. \tag{5.10}$$

Here $F$ is the Frobenius map. We replace the twist by $\mathcal{O}_{\mathrm{an}}^{\times}$ over $\mathbb{Z}$ by the twist over $\mathbb{Z}/p$ by $\mathbb{G}_m/\mathbb{G}_m^p$ to build a diagram (compare (5.3). Here $Z^1 \subset \Omega^1$ is the subsheaf of closed forms.)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_m/\mathbb{G}_m^p \arrow[r] \arrow[d, "d\log"'] & \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p \arrow[r, "(1-F)\otimes 1"] \arrow[d, "f\otimes g \mapsto f^p dg/g"] & \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p \arrow[r] & 0 \\
& Z^1 \arrow[r, phantom, "="] & Z^1 & &
\end{tikzcd} \tag{5.11}$$

The group $\mathbb{H}^1(\mathbb{G}_m \to Z^1)$ is the group of isomorphism classes of line bundles with integrable connections as usual, and $H^1$(the subcomplex $\mathbb{G}_m^p \to 0$) is the subgroup of connections corresponding to a Frobenius descent. We get an exact sequence

$$\begin{aligned} 0 \to \{&\text{line bundle + integrable connection}\}/\{\text{lb + Frobenius descent}\} \\ &\to \mathbb{H}^1(X, \mathbb{G}_m/\mathbb{G}_m^p \to Z^1) \to {}_pH^2(X, \mathbb{G}_m). \end{aligned} \tag{5.12}$$

**Proposition 5.3.** *Let $\iota, C : Z^1 \to \Omega^1$ be the natural inclusion and the Cartier operator, respectively. One has a quasi-isomorphism $(\mathbb{G}_m/\mathbb{G}_m^p \to Z^1) \overset{\iota - C}{\sim} \Omega^1[-1]$. Then the diagram*

$$\begin{tikzcd}
H^0(X, \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p) \arrow[r, "\partial\ (5.11)"] \arrow[d, "a\otimes b \mapsto a db/b"'] & \mathbb{H}^1(X, \mathbb{G}_m/\mathbb{G}_m^p \to Z^1) \arrow[d, "\iota - C", "\cong"'] \\
H^0(X, \Omega^1) \arrow[r, phantom, "="] & H^0(X, \Omega^1)
\end{tikzcd}$$

*is commutative.*

*Proof.* Straightforward from the commutative diagram (with $\mathfrak{b}$ defined to make the columns exact and $\phi(a \otimes b) = a \cdot db/b$).

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & \mathfrak{b} \arrow[r, phantom, "="] \arrow[d] & \mathfrak{b} \arrow[d] & \\
0 \arrow[r] & \mathbb{G}_m/\mathbb{G}_m^p \arrow[r] \arrow[d, equal] & \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p \arrow[r, "F\otimes 1 - 1"] \arrow[d, "\phi\circ(F\otimes 1)"] & \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \mathbb{G}_m/\mathbb{G}_m^p \arrow[r] & Z^1 \arrow[r, "1-C"] \arrow[d] & \Omega^1 \arrow[r] \arrow[d] & 0 \\
& & 0 & 0. &
\end{tikzcd} \tag{5.13}$$

∎

Next we want to see what plays the role of the exponential of the dilogarithm or the Shannon entropy function in this Artin-Schreier context. Let $X = \mathrm{Spec}\,(\mathbb{F}_{p^2}[x])$. Begin with Cathelineau's element

$$\epsilon(x) := x \otimes x + (1-x) \otimes (1-x) \in \mathfrak{b}. \tag{5.14}$$

Choose an Artin-Schreier roots $y^p - y = x$ and $\beta^p - \beta = 1$. To simplify we view $\beta \in \mathbb{F}_{p^2}$ as fixed, and we write $\mathbb{F} = \mathbb{F}_{p^2}$. A local lifting of $\epsilon(x)$ on the étale cover $\mathrm{Spec}\,\mathbb{F}(y) \to \mathrm{Spec}\,\mathbb{F}(x)$ is given by

$$\rho(y) := y \otimes x + (\beta - y) \otimes (1-x) \in \Gamma(\mathrm{Spec}\,\mathbb{F}(y), \mathbb{G}_a \otimes \mathbb{G}_m/\mathbb{G}_m^p). \tag{5.15}$$

From diagram (5.13) there should exist a canonical global lifting, i.e. a lifting defined over $\mathrm{Spec}\,\mathbb{F}(x)$. This lifting, call it $\theta(x)$ has the form $\theta(x) = \rho(y) \cdot \delta(y)^{-1}$ for some $\delta(y) \in \Gamma(\mathrm{Spec}\,\mathbb{F}(y), \mathbb{G}_m/\mathbb{G}_m^p)$. We want to calculate $\delta(y)$.

To do this calculation, note

$$\begin{aligned}\phi \circ (F \otimes 1)(\rho(y)) = y^p dx/x - (\beta - y)^p dx/(1-x) = \\ \frac{(-y^p(1-y^p+y) + (\beta+1-y^p)(y^p-y))dy}{(y^p-y)(1-y^p+y)} = \\ \frac{(\beta(y^p-y)-y)dy}{(y^p-y)(1-y^p+y)} =: \eta(y) \in Z^1.\end{aligned} \tag{5.16}$$

Viewed as a meromorphic form on $\mathbb{P}^1_y$, $\eta$ has simple poles at the points $a$ and $\beta - a$ for $a \in \mathbb{F}_p$. The residue of a form $P/Qdy$ at a point $a$ where $Q$ has a simple zero is given by $P(a)/Q'(a)$. Using this, the residue of $\eta$ at $a \in \mathbb{F}_p$ is $a$. The residue at $\beta - a$ is $\frac{\beta \cdot 1 - (\beta - a)}{1} = a$. Necessarily, therefore, since $\eta$ is regular at $y = \infty$ we must have

$$\eta = d\log\Big(\prod_{a=1}^{p-1} \frac{(\beta - (y+a))^a}{(y+a)^a}\Big) \tag{5.17}$$

We conclude

$$\delta(y) = \prod_{a=1}^{p-1} \frac{(\beta - (y+a))^a}{(y+a)^a}. \tag{5.18}$$

Everything is invariant under the automorphism $y \mapsto \beta - y$. Indeed, the equation can be rewritten (of course $\bmod \mathbb{F}(y)^{\times p}$)

$$\delta(y) = \prod_{a=1}^{(p-1)/2} \Big[\frac{(\beta - (y+a))(y-a)}{(y+a)(\beta - (y-a))}\Big]^a \tag{5.19}$$

Note $\delta(y)$ depends on $y$, not just on $x$. Indeed the product (2.6) can be taken for $0 \le a \le p-1$, i.e. for $a \in \mathbb{F}_p$. One gets then

$$\frac{\delta(y+1)}{\delta(y)} \equiv \prod_{a=0}^{p-1} \frac{y+a}{\beta - (y+a)} = \frac{y^p - y}{1 - y^p - y} = \frac{x}{1-x} \mod \mathbb{F}(y)^{\times p}. \tag{5.20}$$

The fact that $\rho(y)\delta(y)^{-1}$ is defined over $\mathbb{F}(x)$ says that the Cech boundaries of $\rho(y)$ and $\delta(y)$ coincide. Since the latter is, by definition, the coboundary in $\mathbb{G}_m/\mathbb{G}_m^p$ of

$\epsilon(x) = x \otimes x + (1-x) \otimes (1-x)$, it follows that $\delta(y)$ is a 0-cochain for the Galois cohomology

$$\mathbb{H}^*(\mathbb{F}(y)/\mathbb{F}(x), \mathbb{G}_m/\mathbb{G}_m^p \to Z^1)$$

which trivializes the coboundary of $\rho(\epsilon(x))$.

Finally, in this section, we discuss a flat realization of the Artin-Schreier dilogarithm. To see the point, consider the $\ell$-adic realization of the usual dilogarithm mixed Tate motivic sheaf over $\mathbb{A}^1 - \{0,1\}$. Reducing mod $\ell$ yields a sheaf with fibre an $\mathbb{F}_\ell$-vector space of dimension 3. The sheaf has a filtration with successive quotients having fibres $\mathbb{Z}/\ell\mathbb{Z}, \mu_\ell, \mu_\ell^{\otimes 2}$. The geometric fundamental group acts on the fibre via a Heisenberg type group. We visualize this action as follows:

$$\begin{pmatrix} 1 & \mu_\ell & \mu_\ell^{\otimes 2} \\ 0 & 1 & \mu_\ell \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_\ell^{\otimes 2} \\ \mu_\ell \\ \mathbb{Z}/\ell\mathbb{Z} \end{pmatrix} \tag{5.21}$$

Here the notation means that for $g \in \pi_1^{geo}$ the corresponding matrix

$$\begin{pmatrix} 1 & a_{12}(g) & a_{13}(g) \\ 0 & 1 & a_{23}(g) \\ 0 & 0 & 1 \end{pmatrix}$$

has $a_{ij} \in \mathrm{Hom}(\mu_\ell^{\otimes i-1}, \mu_\ell^{\otimes j-1}) = \mu_\ell^{j-i}$.

The essential ingredients here are first the Heisenberg group $\mathcal{H}_\ell$, second the $\mathcal{H}_\ell$-torsor over $\mathbb{A}^1 - \{0,1\}$ corresponding to the kernel of the representation, and third the (standard) representation of $\mathcal{H}_\ell$ on $\mathbb{Z}/\ell\mathbb{Z} \oplus \mu_\ell \oplus \mu_\ell^{\otimes 2}$.

We define an Artin-Schreier Heisenberg group as the non-commutative flat group-scheme $\mathcal{H}_{AS}$ which we could suggestively write

$$\mathcal{H}_{AS} := \begin{pmatrix} 1 & \mathbb{Z}/p\mathbb{Z} & \mu_p \\ 0 & 1 & \mu_p \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.22}$$

More precisely, $\mathcal{H}_{AS}$ is a central extension

$$0 \to \mu_p \to \mathcal{H}_{AS} \to \mu_p \times \mathbb{Z}/p\mathbb{Z} \to 0. \tag{5.23}$$

Let

$$b : (\mu_p \times \mathbb{Z}/p\mathbb{Z}) \times (\mu_p \times \mathbb{Z}/p\mathbb{Z}) \to \mu_p \tag{5.24}$$
$$b((\zeta_1, a_1), (\zeta_2, a_2)) = \zeta_1^{-a_2}\zeta_2^{a_1}.$$

Define $\mathcal{H}_{AS} = \mu_p \times (\mu_p \times \mathbb{Z}/p\mathbb{Z})$ as a scheme, with group structure given by

$$(\zeta_1, \theta_1, a_1) \cdot (\zeta_2, \theta_2, a_2) := (\zeta_1\zeta_2\theta_2^{a_1}, \theta_1\theta_2, a_1 + a_2). \tag{5.25}$$

The commutator pairing on $\mathcal{H}_{AS}$ is given by

$$\Big[(\zeta_1, \theta_1, a_1), (\zeta_2, \theta_2, a_2)\Big] = \Big(b((\theta_1, a_1), (\theta_2, a_2)), 1, 0\Big) = b((\theta_1, a_1), (\theta_2, a_2)) \in \mu_p. \tag{5.26}$$

We fix a solution $\beta^p - \beta = 1$. We define a flat $\mathcal{H}_{AS}$-torsor $T = T_\beta$ over $\mathbb{A}^1_{\mathbb{F}_{p^2}}$ as follows. A local (for the flat topology) section $t$ is determined by

1. A $p$-th root of $\frac{x}{1-x}$: $w^p \equiv \frac{x}{1-x} \mod \mathbb{F}_{p^2}(x)^{\times p}$.
2. A $y$ satisfying $y^p - y = x$.
3. A $p$-th root $z$ of $\delta(y) : z^p \equiv \delta(y) \mod \mathbb{F}_{p^2}(x)^{\times p}$ (where $\delta(y)$ is as in (5.18).)

The action of $\mathcal{H}_{AS}$ is given by

$$(\zeta,\theta,a)\star(z,w,y)=(\zeta zw^a,\theta w,y+a). \tag{5.27}$$

Note $(\zeta zw^a)^p=\delta(y)(\frac{x}{1-x})^a=\delta(y+a)$ by (5.20), so the triple on the right lies in $T$. This is an action because

$$\begin{aligned}(\zeta',\theta',a')\star\Big((\zeta,\theta,a)\star(z,w,y)\Big)&=\\ (\zeta',\theta',a')\star\Big((\zeta zw^a,\theta w,y+a)\Big)&=\\ \Big(\zeta'\zeta zw^a(\theta w)^{a'},\theta'\theta w,y+a+a'\Big)=(\zeta'\zeta\theta^{a'},\theta'\theta,a+a')\star(z,w,y)&=\\ \Big((\zeta',\theta',a')\star(\zeta,\theta,a)\Big)\star(z,w,y).\end{aligned} \tag{5.28}$$

Define $\mathbb{V}:=\mu_p\times\mu_p\times\mathbb{Z}/p\mathbb{Z}$. There is an evident action of $\mathcal{H}_{AS}$ on $\mathbb{V}$, viewed as column vectors. We suggest that the contraction $T\overset{\mathcal{H}_{AS}}{\times}\mathbb{V}$ should be thought of as analogous to the mod $\ell$ étale sheaf on $\mathbb{A}^1-\{0,1\}$ with fibre $\mathbb{Z}/\ell\mathbb{Z}\oplus\mu_\ell\oplus\mu_\ell^{\otimes 2}$ associated to the $\ell$-adic dilogarithm.

## 6. The additive cubical (higher) Chow groups

In this section, we show that the modulus condition we introduced in definition (4.1) yields additive Chow groups which we can compute in weights $(n,n)$. We assume throughout that $k$ is a field and $\frac{1}{6}\in k$.

One sets

$$\begin{aligned}A&=(\mathbb{A}^1,2\{0\})\\ B&=(\mathbb{P}^1\setminus\{1\},\{0,\infty\}).\end{aligned} \tag{6.1}$$

The coordinates will be $x$ on $A$ and $(y_1,\ldots,y_n)$ on $B$. One considers

$$X_n=A\times B^n. \tag{6.2}$$

The boundary maps $X_{n-1}\hookrightarrow X_n$ defined by $y_i=0,\infty$ are denoted by $\partial_i^j, i=1,\ldots,n, j=0,\infty$. One denotes by $Y_n\subset X_n$ the union of the faces $\partial_i^j(X_{n-1})$. One defines

$$\begin{aligned}\mathcal{Z}_0(X_n)=\oplus\mathbb{Z}\xi,\ \xi\in X_n\setminus Y_n,\\ \xi \text{ closed point.}\end{aligned} \tag{6.3}$$

For any 1-cycle $C$ in $X_n$, one denotes by $\nu:\bar{C}\to\mathbb{P}^1\times(\mathbb{P}^1)^n$ the normalisation of its compactification. One defines

$$\begin{aligned}\mathcal{Z}_1(X_n)=\oplus\mathbb{Z}C,\ C\subset X_n \text{ with}\\ \partial_i^j(C)\in\mathcal{Z}_0(X_{n-1}) \text{ and (cf. definition 4.1)}\\ 2\nu^{-1}(\{0\}\times(\mathbb{P}^1)^n)+\nu^{-1}(Y_n)\subset \max_{i=1}^n\nu^{-1}(\mathbb{P}^1\times(\mathbb{P}^1)^{i-1}\times\{1\}\times(\mathbb{P}^1)^{n-i})\end{aligned} \tag{6.4}$$

One defines

$$\partial:=\sum_{i=1}^n(-1)^i(\partial_i^0-\partial_i^\infty):\mathcal{Z}_1(X_n)\to\mathcal{Z}_0(X_{n-1}) \text{ for all } i,j. \tag{6.5}$$

Further one defines the differential form

$$\psi_n = \frac{1}{x}\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_n}{y_n} \in \Gamma(\mathbb{P}^1 \times (\mathbb{P}^1)^n, \Omega^1_{(\mathbb{P}^1\times(\mathbb{P}^1)^n)/\mathbb{Z}}(\log Y_n)(\{x=0\})). \tag{6.6}$$

We motivate the choice of this differential form as follows. One considers

$$V_n(t) = (\mathbb{P}^1 \setminus \{0,t\}, \infty) \times (\mathbb{P}^1 \setminus \{0,\infty\}, 1)^n. \tag{6.7}$$

Its cohomology

$$H^{n+1}(V_n(t)) = H^1 \otimes (H^1)^n = F^1 \otimes (F^1)^n \tag{6.8}$$

is Hodge-Tate for $t \neq 0$. The generator is given by

$$\omega_{n+1}(t) = (\frac{dx}{x} - \frac{d(x-t)}{(x-t)}) \wedge \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_n}{y_n}. \tag{6.9}$$

Thus

$$\frac{\omega_{n+1}(t)}{t}|_{t=0} = d(\psi_n). \tag{6.10}$$

**Definition 6.1.** We define the additive cubical (higher) Chow groups

$$TH^n_M(k,n) = \mathcal{Z}_0(X_{n-1})/\partial\mathcal{Z}_1(X_n)$$

One has the following reciprocity law

**Proposition 6.2.** *The map* $\mathcal{Z}_0(X_{n-1}) \to \Omega^{n-1}_k$ *which associates to a closed point* $\xi \in X_{n-1} \setminus Y_{n-1}$ *the value* $\mathrm{Trace}(\kappa(\xi)/k)(\psi_{n-1}(\xi))$ *factors through*

$$TH^n_M(k,n) := \mathcal{Z}_0(X_{n-1})/\partial\mathcal{Z}_1(X_n).$$

*Proof.* Let $C$ be in $\mathcal{Z}_1(X_n)$. Let $\Sigma \subset \bar{C}$ be the locus of poles of $\nu^*\psi_n$. One has the functoriality map

$$\nu^* : \Omega^n_{\mathbb{P}^1\times(\mathbb{P}^1)^{n-1}}(\log Y_{n-1})(\{x=0\}) \to \Omega^{n-1}_{\bar{C}/\mathbb{Z}}(*\Sigma). \tag{6.11}$$

Thus reciprocity says

$$\sum_{\sigma\in\Sigma} \mathrm{res}_\sigma \nu^*(\psi_n) = 0. \tag{6.12}$$

Recall that here res means the following. One has a surjection

$$\Omega^n_{\bar{C}/\mathbb{Z}}(*\Sigma) \to \Omega^{n-1}_{k/\mathbb{Z}} \otimes \omega_{\bar{C}/k}(*\Sigma) \tag{6.13}$$

which yields

$$\Gamma(\bar{C}, \Omega^n_{\bar{C}/\mathbb{Z}}(*\Sigma)) \to \Gamma(\bar{C}, \Omega^{n-1}_{k/\mathbb{Z}} \otimes \omega_{\bar{C}/k}(*\Sigma)) = \Omega^{n-1}_{k/\mathbb{Z}} \otimes \Gamma(\bar{C}, \omega_{\bar{C}/k}(*\Sigma)). \tag{6.14}$$

By definition, res on $\Omega^{n-1}_{k/\mathbb{Z}} \otimes \omega_{\bar{C}/k}(*\Sigma)$ is $1 \otimes \mathrm{res}$. This explains the reciprocity.

Now we analyze $\Sigma \subset \sigma^{-1}(Y_n \cup \{x=0\})$. Let $t$ be a local parameter on $\bar{C}$ in a point $\sigma$ of $\nu^{-1}(\{x=0\})$.

We write $x = t^m \cdot u$, where $u \in \mathcal{O}^\times_{\bar{C},\sigma}, m \geq 0$. If $m \geq 1$, the assumption we have on $\mathcal{Z}_1$ says that there is at least one $i$ such that $\{t=0\}$ lies in $\nu^{-1}(\{y_i = 1\})$. Let us order $i = 1, \ldots, n$ such that $\{t=0\}$ lies in $\nu^{-1}(\{y_i=1\})$ for $i = 1, 2, \ldots, r$. Thus we write

$$\begin{aligned} y_i - 1 = t^{m_i} \cdot u_i, m_1 \geq m_2 \geq \ldots \geq 1, u_i \in \mathcal{O}^\times, i = 1, \\ y_i = t^{p_i} u_i, p_i \geq 0, u_i \in \mathcal{O}^\times, i = r+1, \ldots, n. \end{aligned} \tag{6.15}$$

The assumption we have says

$$2m \leq m_1. \tag{6.16}$$

One has around the point $\sigma$

$$\nu^{-1}(\psi_n)|_\sigma = \frac{u^{-1}}{t^m} \cdot \frac{d(t^{m_1} \cdot u_1)}{1 + t^{m_1} \cdot u_1} \wedge \ldots \wedge \frac{d(t^{m_r} \cdot u_r)}{1 + t^{m_r} \cdot u_r} \wedge_{i=r+1}^{n} \frac{d(t^{p_i} \cdot u_i)}{t^{p_i} \cdot u_i}. \tag{6.17}$$

We analyze the poles of the right hand side. The numerator of this expression is divisible by $t^{(m_1+\ldots+m_r)-1}$. Thus the condition for $\nu^{-1}\psi_n$ to be smooth in $\sigma$ is

$$m + 1 \leq (m_1 + \ldots + m_r). \tag{6.18}$$

This is always fulfilled for $2m \leq m_1$. We have

$$\nu^{-1}\psi_n \text{ smooth in } \nu^{-1}(\{x = 0\}). \tag{6.19}$$

On the other hand, one obviously has

$$\mathrm{res}_{y_i=0}\psi_n = -\mathrm{res}_{y_i=\infty}\psi_n = (-1)^i \psi_{n-1}. \tag{6.20}$$

Thus one concludes

$$\sum_{\sigma \in \Sigma} \mathrm{res}_\sigma \nu^{-1}\psi_n = \sum_{i=1}^{n} (-1)^i \psi_{n-1}(\partial_i^0 - \partial_i^\infty)(C) = 0. \tag{6.21}$$

□

We want to see that the reciprocity map in proposition 6.2 is an isomorphism. Define

$$\begin{aligned} k \otimes_{\mathbb{Z}} \wedge_{i=1}^{n-1} k^\times &\to TH^n_M(k, n) \\ a \otimes (b_1 \wedge \ldots \wedge b_{n-1}) &\mapsto (\frac{1}{a}, b_1, \ldots, b_{n-1}) \text{ for } a \neq 0 \\ &\mapsto 0 \text{ for } a = 0. \end{aligned} \tag{6.22}$$

**Proposition 6.3.** *Assume $\frac{1}{6} \in k$. Then (6.22) factors through*

$$\Omega_k^{n-1} \to TH^n_M(k, n).$$

*Proof.* One has the following relations, where $\equiv$ means equivalence modulo $\partial \mathcal{Z}_1(X_{n+1})$:

$$(\frac{1}{x + x'}, y_1, \ldots, y_n) \equiv (\frac{1}{x}, y_1, \ldots, y_n) + (\frac{1}{x'}, y_1, \ldots, y_n) \tag{6.23}$$

$$(x, y_1 z_1, y_2, \ldots, y_n) \equiv (x, y_1, y_2, \ldots, y_n) + (x, z_1, y_2, \ldots, y_n) \tag{6.24}$$

$$(x, -1, y_2, \ldots, y_n) \equiv 0 \in TH^n_M(k, n). \tag{6.25}$$

Note, the last is obviously a consequence of the first two:

$$(2x, -1, y_2, \ldots, y_n) = 2(x, -1, y_2, \ldots, y_n) = (x, 1, y_2, \ldots, y_n) = 0, \tag{6.26}$$

so we need only consider (6.23) and (6.24). Assume first $xx'(x+x') \neq 0$, and define

$$C = (t, y_1 = \frac{(1 - xt)(1 - x't)}{1 - (x + x')t}, y_2, \ldots, y_n) \in \mathcal{Z}_1(X_n). \tag{6.27}$$

Indeed, the expansion of $y_1$ in $t = 0$ reads $1 + t^2 c_2 +$ (higher order terms), so our modulus condition is fulfilled. Also we have taken $y_i \in k^\times$ so $C$ meets the faces properly. Then one has

$$\partial(C) = (\frac{1}{x}, y_2, \ldots, y_n) + (\frac{1}{x'}, y_2, \ldots, y_n) - (\frac{1}{x + x'}, y_2, \ldots, y_n). \tag{6.28}$$

Similarly, if $x + x' = 0$, then one sets

$$C = (t, y_1 = (1 - \frac{t^2}{x^2}), y_i) \in \mathcal{Z}_1(X_n). \tag{6.29}$$

One has

$$\partial(C) = (x, y_i) + (-x, y_i), \tag{6.30}$$

proving (6.23). Note the proposition for $n = 1$ is a consequence of this identity.

To show multiplicativity in the $y$ variables, one uses Totaro's curve [23]. There is a $\mathcal{C} \in \mathcal{Z}_1(B^{n+1})$ with $\partial(\mathcal{C}) = (y_1 z_1, y_2, \ldots, y_n) - (y_1, y_2, \ldots, y_n) - (z_1, y_2, \ldots, y_n)$. One sets $C = (x, \mathcal{C}) \in \mathcal{Z}_1(X_{n+1})$. Here $x$ is fixed and nonzero, so the modulus condition is automatic, and one has $\partial(C) = (x, \partial(\mathcal{C}))$. This proves (6.24).

It remains to verify the Cathelineau relation (cf. [6], [7])

$$(\frac{1}{a}, a, b_2, \ldots, b_n) + (\frac{1}{1-a}, (1-a), b_2, \ldots, b_n) \equiv 0. \tag{6.31}$$

In fact, the $b_2, \ldots b_n \in k^\times$ play no role, so we will drop them. One considers the 1-cycle which is given by its parametrization

$$\begin{aligned} Z(a) &= -Z_1(a) + Z_2 \\ Z_1(a) &= (t, 1 + \frac{t}{2}, 1 - \frac{a^2 t^2}{4}) \\ Z_2 &= (\frac{t}{4}, 1 + \frac{t}{6}, 1 - \frac{t^2}{4}), \end{aligned} \tag{6.32}$$

We see immediately that $Z \in \mathcal{Z}_1(X_2)$. One has

$$\begin{gathered} \partial(Z_1(a)) \\ = (-2, 1 - a^2) - (\frac{2}{a}, 1 + \frac{1}{a}) - (-\frac{2}{a}, 1 - \frac{1}{a}) = \\ (-2, 1-a) + (-2, 1+a) + (\frac{2}{a}, \frac{a-1}{a+1}) = \\ (-2, 1-a) + (\frac{2}{a}, a-1) + (-2, 1+a) - (\frac{2}{a}, a+1) = \\ (-2, a-1) + (\frac{2}{a}, a-1) + (-2, 1+a) + (-\frac{2}{a}, a+1) = \\ (\frac{2}{a-1}, a-1) - (\frac{2}{a+1}, a+1). \end{gathered} \tag{6.33}$$

Setting $a = 1 - 2b$, one obtains

$$\begin{gathered} \partial(Z_1(a)) = (-\frac{1}{b}, -2b) - (\frac{1}{1-b}, 2(1-b)) = \\ -(\frac{1}{b}, b) - (\frac{1}{1-b}, 1-b) - (\frac{1}{b}, 2) - (\frac{1}{1-b}, 2) = \\ -(\frac{1}{b}, b) - (\frac{1}{1-b}, 1-b) - (1, 2). \end{gathered} \tag{6.34}$$

One has

$$\partial(Z_2) = (-\frac{3}{2}, -8) - (\frac{1}{2}, \frac{4}{3}) - (-\frac{1}{2}, \frac{2}{3}) = \\ 3(-\frac{3}{2}, 2) - (\frac{1}{2}, 2) = \\ (-\frac{3}{2}, 2) + (-\frac{3}{4}, 2) - (\frac{1}{2}, 2) = \\ (-\frac{1}{2}, 2) - (\frac{1}{2}, 2) = -(1, 2). \tag{6.35}$$

In conclusion

$$\partial(Z(1-2a)) = (\frac{1}{a}, a, b_i) + (\frac{1}{1-a}, 1-a, b_i). \tag{6.36}$$

□

We now have well defined maps $\phi_n, \psi_n$

$$\Omega_k^{n-1} \xrightarrow{\phi_n} TH_M^n(k, n) \xrightarrow{\psi_n} \Omega_k^{n-1} \tag{6.37}$$

which split $TH_M^n(k, n)$. The image of the differential forms consists of all 0-cycles which are equivalent to 0-cycles $\sum m_i p_i$ with $p_i \in X_n(k)$.

**Theorem 6.4.** . *Assume $\frac{1}{6} \in k$. The above maps identify $TH_M^n(k, n)$ with $\Omega_k^{n-1}$.*

*Proof.* It suffices to show that the class of a give closed point $p \in (\mathbb{A}^1 - \{0\}) \times (\mathbb{P}^1 - \{0, 1, \infty\})^{n-1}$ lies in the image of $\phi_n$. Write $\kappa = \kappa(p)$ for the residue field at $p$. One first applies a Bertini type argument as in [6], Proposition 4.5, to reduce to the case where $\kappa/k$ is separable. Then we follow the argument in loc.cit. The degree $[\kappa : k] < \infty$, so standard cycle constructions yield a norm map $N : TH_M^n(\kappa, n) \to TH_M^n(k, n)$. We claim the diagram

$$\begin{array}{ccc} \Omega_\kappa^{n-1} & \xrightarrow{\phi_{n,\kappa}} & TH_M^n(\kappa, n) \\ \mathrm{Tr}\big\downarrow & & \big\downarrow N \\ \Omega_k^{n-1} & \xrightarrow{\phi_{n,k}} & TH_M^n(k, n) \end{array} \tag{6.38}$$

is commutative, where Tr is the trace on differential forms. Indeed, $\Omega_\kappa^n = \kappa \otimes \Omega_k^n$, so it suffices to check on forms $a d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1})$ with $a \in \kappa$ and $b_i \in k$. But in this situation, we have projection formulas, both for 0-cycles and for differential forms, and it is straightforward to check (ignore the $b_i$ and reduce to $n = 1$)

$$\mathrm{Tr}(a d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1})) = \mathrm{Tr}_{\kappa/k}(a) d\log(b_1) \wedge \ldots \wedge d\log(b_{n-1}) \tag{6.39}$$

$$N(\frac{1}{a}, b_1, \ldots, b_{n-1}) = \begin{cases} (\frac{1}{\mathrm{Tr}_{\kappa/k} a}, b_1, \ldots, b_{n-1}) & \mathrm{Tr}(a) \neq 0 \\ 0 & \mathrm{Tr}(a) = 0 \end{cases}$$

Write $[p]_\kappa$ (resp. $[p]_k$) for the class of $p \in TH_M^n(\kappa, n)$ (resp. $TH_M^n(k, n)$.) One has $[p]_k = N([p]_\kappa)$. Since $p$ is $\kappa$-rational, $[p]_\kappa \in \mathrm{Image}(\phi_\kappa)$. Commutativity of (6.38) implies $[p]_k \in \mathrm{Image}(\phi_k)$. It follows that $\phi_k$ is surjective, proving the theorem. □

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