Algebraic Cycles

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Outline

Introductory Remarks

- Sheaves
  - Sophisticated; powerful tools (Grothendieck's 6 functors, homological methods)
  - Deep conjectures (Geometric Langlands); viable programs of study.
- Algebraic cycles
  - Traditionally quite crude (moving lemmas and issues of functoriality)
  - Work of Voevodsky (h-topology, $\mathbb{A}^1$-homotopy and methods from algebraic topology)
  - Deep conjectures (cycles $\leftrightarrow$ morphisms in the category of motives)
    - ⋆ Beilinson conjectures; special values of L-functions
    - ⋆ Hodge conjectures
  - Cycle schizophrenia; tools used to study cycles (higher K-theory, $\mathbb{A}^1$-homotopy, cyclic homology) are often quite removed from the cycles themselves.

    Proving theorems about cycles $\neq$ understanding cycles.

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Affine and Projective Varieties.

- k a field; $\mathbb{A}^n_k$ is affine n-space over k.
- Affine variety V defined by polynomials $f_i(x_1, \ldots, x_n) = 0$; $V \subset \mathbb{A}^n_k$.
- Projective space $\mathbb{P}^n_k$ = Space of lines through 0 in $\mathbb{A}^{n+1}_k$
- Projective variety $X \subset \mathbb{P}^n_k$ defined by vanishing of homogeneous polynomials $F_i(T_0, \ldots, T_n)$.

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Vector Bundles and Coherent Sheaves

- $\mathcal{O}_X$ sheaf of functions on X for the Zariski topology
- Vector bundles $\mathcal{V}$ and coherent sheaves $\mathcal{F}$.
  - Example: Kähler differential forms $\Omega^r_{X/k} := \bigwedge^r \Omega^1_{X/k}$.
  - If X is *smooth* then $\Omega^r_{X/k}$ is a vector bundle.
- $K_0(X)$ Grothendieck group of vector bundles
  generators $[\mathcal{V}]$, $\mathcal{V}$ a vector bundle on X
  relations $[\mathcal{V}] = [\mathcal{V}'] + [\mathcal{V}'']$ if

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0.$$

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Algebraic cycles

- Algebraic cycles are finite formal linear combinations of closed subvarieties of X.

$$Z = \sum_i n_i Z_i; \quad Z_i \subset X.$$

- $\mathcal{Z}^r(X)$ cycles of codimension r (resp. $\mathcal{Z}_r(X)$ dimension r)

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Chow groups

- Intersection product of cycles $Z \cdot W$

$$\mathcal{Z}^r(X) \otimes \mathcal{Z}^s(X) \dashrightarrow \mathcal{Z}^{r+s}(X)$$

  - defined for X smooth and cycles Z, W in good position
  - $Z \cdot W =$
    sum over irreducible components of $Z_i \cap W_j$ with multiplicities

Chow groups II

- Functoriality; $f : X \to Y$
  - $f_* : \mathcal{Z}^r(X) \to \mathcal{Z}^{r-\dim X/Y}(Y)$, f *proper*
  - $f^* : \mathcal{Z}^r(Y) \to \mathcal{Z}^r(X)$, f *flat*
- Rational equivalence $\mathcal{Z}^r(X)^{\mathrm{rat}} \subset \mathcal{Z}^r(X)$
  -

$$X \xleftarrow{pr_1} X \times \mathbb{P}^1 \xrightarrow{pr_2} \mathbb{P}^1$$

$$pr_{1*}(Z \cdot pr_2^*((0) - (\infty))) \in \mathcal{Z}^r(X)^{\mathrm{rat}}$$

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Chow groups III

- Chow group

$$CH^r(X) := \mathcal{Z}^r(X)/\mathcal{Z}^r(X)^{\text{rat}}.$$

- General reference:
  Fulton, Intersection Theory (Springer Verlag).

Divisors and Line Bundles

- X smooth variety, $D \subset X$ effective divisor, $x \in X$ a point.
  $\exists\ x \in U \subset X$ open, $f \in \Gamma(U, \mathcal{O}_X)$, $D \cap U : f = 0$.
- $\{U_i, f_i\}$ defining D;
  $\{U_i \cap U_j, f_i/f_j\}$ 1-cocycle with values in $\mathcal{O}_X^\times$, sheaf of units.
- $CH^1(X) \cong H^1(X, \mathcal{O}_X^\times)$; group of Line Bundles (Picard Group)

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Function Theory for Riemann Surfaces

- X projective algebraic curve (Assume $X(k) \neq \emptyset$) (compact Riemann surface for $k = \mathbb{C}$)
- 0-cycles $\mathcal{Z}_0(X)$ (Note $\mathcal{Z}_0(X) = \mathcal{Z}^1(X)$; 0-cycles are divisors)
  - Degree map

$$\mathcal{Z}_0(X)^0 := \ker(\mathcal{Z}_0(X) \xrightarrow{\deg} \mathbb{Z}); \quad \sum n_i(x_i) \mapsto \sum n_i$$

  - Divisors of functions

$$\mathcal{Z}_0(X)^{\mathrm{rat}} = \{(f) | \; f \in k(X)^\times\} \subset \mathcal{Z}(X)^0$$

- $$CH_0(X)^0 := \mathcal{Z}_0(X)^0 / \mathcal{Z}_0(X)^{\mathrm{rat}} \subset CH_0(X) = CH^1(X)$$

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Abel's Theorem

- Abel's Theorem:

$$CH_0(X)^0 \cong J_X(k).$$

- The 0-cycles of degree 0 modulo divisors of functions are the k-points of an *Abelian Variety* J_X.
- Case $k = \mathbb{C}$.
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  $$J_X(\mathbb{C}) = \Gamma(X, \Omega^1_{X/\mathbb{C}})^\vee / H_1(X, \mathbb{Z})$$
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  $$\mathfrak{A} \in \mathcal{Z}_0(X)^0 \mapsto \int_c \in \Gamma(X, \Omega^1)^\vee; \quad \partial c = \mathfrak{A}$$

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- Define more general sorts of algebraic cycles (Motivic Cohomology)
- Interpret J_X and other similar abelian Lie groups J as Ext groups in the category of Hodge structures.
- Generalize Abel's construction to define more general cycle classes.

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Hodge Structures

- Free, f.g. $\mathbb{Z}$-module $H_{\mathbb{Z}}$ (or $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$, or $\mathbb{R}$-vector space $H_{\mathbb{R}}$)
- Decreasing filtration $F^*H_{\mathbb{C}}$; conjugate filtration $\overline{F}^*H_{\mathbb{C}}$.
- H pure of weight n if $F^p \cap \overline{F}^{n-p+1} = (0), \forall p$.

$$H_{\mathbb{C}} = \oplus H^{p,n-p}; \quad H^{p,n-p} := F^p \cap \overline{F}^{n-p}$$

- Mixed Hodge structure: $W_*H_{\mathbb{Q}}$ increasing weight filtration. F^* induces pure HS of weight n on $gr_n^W H$ for all n.
- Category of HS's is abelian. Weight and Hodge filtrations are exact functors.

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- Decreasing filtration $F^* H_{\mathbb{C}}$; conjugate filtration $\overline{F}^* H_{\mathbb{C}}$.
- H pure of weight n if $F^p \cap \overline{F}^{n-p+1} = (0), \forall p$.

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- Mixed Hodge structure: $W_* H_{\mathbb{Q}}$ increasing weight filtration. F^* induces pure HS of weight n on $gr_n^W H$ for all n.
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Ext Groups in the Category of Hodge Structures

- J Non-compact.
  - Suitable algebraic cycles define cycle classes $[Z] \in J$.
  - $J \cong \mathbb{R}^N/\Gamma$; $\Gamma \cong \mathbb{Z}^n$ discretely embedded in $\mathbb{R}^N$.
  - $0 \to$ compact torus $\to J \xrightarrow{\rho} \mathbb{R}^{N-n} \to 0$
- J compact (Height pairings).
  - $\mathbb{G}_m$ bundle (biextension) $\mathcal{B} \to J \times J^\vee$
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Beilinson Conjectures (Viewed from outer space)

- Smooth, projective variety X over number field $\leftrightarrow$ collection of Hasse-Weil L-functions $L(H^r(\overline{X}), s)$
- "If conjectures were horses, mathematicians would ride..."
  - $\Lambda(H^r(\overline{X}), s) = L(H^r(\overline{X}), s) \cdot \Gamma$ factor $\cdot$ exponential term
  - Given $n \geq \frac{r}{2} + 1$, $\exists$ motivic cohomology group (group of algebraic cycles) $H_M(X)_{\mathbb{Z}}$, Ext group of Hodge structures $J(X)$, and injective cycle map $H_M(X)_{\mathbb{Z}} \to J(X)$
  - $J(X)$ has a volume form which is well-defined upto $\mathbb{Q}^{\times}$.
  - $\mathrm{vol}(J/(H_M)_{\mathbb{Z}}) \in \mathbb{Q}^{\times} \cdot L(H^r, s = n)$.
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Chern classes

- $\mathcal{V}$ rank r vector bundle on X
- Chern classes $c_i(\mathcal{V}) \in CH^i(X)$ defined for $1 \leq i \leq r$.
- Given $s_1, \ldots, s_p \in \Gamma(X, \mathcal{V}), p \leq r$ sections in general position.
- Locally, $\mathcal{V} \cong \mathcal{O}_X^r$; the s_i yield $r \times p$ matrix of functions.
- $c_{r-p+1}(\mathcal{V}) \in CH^{r-p+1}(X)$ cycle defined by vanishing of all $p \times p$ minors.

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Chern Character

- Chern Character (X smooth)

$$(*) \quad ch : K_0(X)_{\mathbb{Q}} \cong \bigoplus_i CH^i(X)_{\mathbb{Q}}.$$

  Here ch is a power series in the c_i.
- Assume $f : X \to Y$ proper map, X, Y smooth.
- $f_! : K_0(X) \to K_0(Y)$
  - Example, $Y =$ point.
  - $f_![\mathcal{V}] = \sum(-1)^i[H^i(X,\mathcal{V})] = \chi(\mathcal{V}) \in K_0(\text{point}) = \mathbb{Z}$.

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Riemann Roch

- Todd class $Td(X) \in CH^*(X)$
- Riemann Roch

$$f_*(Td(X) \cdot ch([\mathcal{V}])) = Td(Y) \cdot ch(f_![\mathcal{V}]).$$

- On Tuesday, we will use (*) (but not RR) to study the Hodge conjecture.

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- Deep and important problems involving intersection numbers of cycles.
  - Schubert calculus (intersection theory on Grassmannians)
  - Enumerative problems arising in physics (intersection theory on orbifolds)
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Linear Series; Classification Problems

- L line bundle on X projective variety.
- Complete linear series $\Gamma(X, L)$.
- $x \in X$ defines $\theta(x) \in \mathrm{Hom}(\Gamma(X, L), k)$(!?)
- $\theta(x)(\ell) := \ell(x) \in L(x) \cong k$.
  - Problem: $L(x) \cong k$ not canonical: only get line in $\mathrm{Hom}(\Gamma(X, L), k)$
  - Possibly $\ell(x) = 0, \ \forall \ell, \ \theta(x) = 0$. Don't even get a line!
  - $X \xrightarrow{\text{rational map}} \mathbb{P}(\Gamma(X, L)^\vee)$
- Linear Series yield classification for $\dim X = 2$ (Algebraic Surfaces).

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- K-theory spectrum $K(X)$ (Quillen).
  - Higher K-groups $K_n(X)$, $n \geq 0$
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- Resolution of $\mathcal{K}_{n,X}$ for X smooth

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- Corollary: $H^n(X,\mathcal{K}_n) \cong CH^n(X)$.
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- References
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Motivic Cohomology and K-theory

- Beilinson definition

$$H^p_M(X, \mathbb{Q}(q)) := gr^q_\gamma K_{2q-p}(X)_\mathbb{Q}.$$

  - Example:

$$H^{2p}_M(X, \mathbb{Q}(p)) = gr^p_\gamma K_0(X) \cong CH^p(X)_\mathbb{Q}$$

Higher Chow Groups

- $\Delta^n_k := \mathrm{Spec}\, k[t_0, \ldots, t_n]/(\sum t_i - 1)$ algebraic n-simplex.
- $\iota_i : \Delta^{n-1} \hookrightarrow \Delta^n$ locus $t_i = 0$.
- $\mathcal{Z}^p(X \times \Delta^n)' \subset \mathcal{Z}^p(X \times \Delta^n)$ cycles in good position w.r.t. faces.
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Higher Chow Groups and Motivic Cohomology

- X smooth, $H^p_M(X, \mathbb{Z}(q)) \cong CH^q(X, 2q - p)$.
  - Variant: Cubical cycles: $\square := \mathbb{P}^1 - \{1\}$; Replace Δ^n with $\square^n$; factor out by degeneracies.
  - Face maps $\iota^j_i : \square^{n-1} \hookrightarrow \square^n, j = 0, \infty$

Examples

- Chow groups $CH^p(X) = H^{2p}_M(X, \mathbb{Z}(p))$.
- Milnor classes: $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X^\times)$. $\{f_1, \dots, f_n\} := [(x, f_1(x), \dots, f_n(x)) \cap (X \times \square^n)] \in CH^n(X, n) = H^n_M(X, \mathbb{Z}(n))$.
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Higher Chow DGA

- $X = \mathrm{Spec}\, k$ a point. Product

$$\mathcal{Z}^p(\square^n) \otimes \mathcal{Z}^q(\square^m) \to \mathcal{Z}^{p+q}(\square^{m+n}).$$

- $\mathfrak{N}^p(r) := \mathcal{Z}^r(\square_k^{2r-p})_{\mathbb{Q},Alt}$
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Cycles and the Tannakian Category of Mixed Tate Motives

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Example: Dilogarithm Motive

- $$\begin{array}{ccc} \mathfrak{N}^1(1) \otimes \mathfrak{N}^1(1) & \xrightarrow{\text{mult}} & \mathfrak{N}^2(2) \\ \uparrow \partial & & \uparrow \partial \\ (\mathfrak{N}^1(1) \otimes \mathfrak{N}^1(0)) \oplus (\mathfrak{N}^1(0) \otimes \mathfrak{N}^1(1)) & \longrightarrow & \mathfrak{N}^1(2)/\partial\mathfrak{N}^0(2) \end{array}$$
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Example: Dilogarithm Motive II

- $T_x = \{(t, 1-t, 1-xt^{-1}) \mid t \in \mathbb{P}^1\}$ parametrized curve in $\square^3$.
- $\partial T_x = (x, 1-x) \in \mathcal{Z}^2(\square^2) = \mathfrak{N}^2(2)$.
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The Hodge Conjecture

- $k = \mathbb{C}$, X smooth, projective variety.
- $Z \in \mathcal{Z}^r(X)$, $[Z]_{DR} \in F^r\mathbb{H}^{2r}(X, \Omega^*_X)$, $[Z]_{Betti} \in H^{2r}_{Betti}(X, \mathbb{Z}(r))$.
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Infinitesimal and Variational Hodge Conjecture

- Variational Hodge Conjecture (Grothendieck): X/S family, σ_s horizontal family of cohomology classes. If σ_0 is algebraic at one point $0 \in S$, then it is algebraic everywhere.
- Infinitesimal Hodge theorem: X/Λ formal family ($\Lambda = \overline{\mathbb{Q}}[[t]]$ or mixed characteristic $\Lambda = W(k)$). Then algebraic classes on the closed fibre lift to algebraic classes on all thickenings iff the horizontal lift (or crystalline lift in mixed char.) of the cohomology class is Hodge.

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