CHAPTER I

BASIC DEFINITIONS AND PROPERTIES

1. Introduction

This chapter contains the basic definitions and properties of the higher chow groups. The focus will be entirely on cycles; any mention of K-theory will be purely (co)incidental. We fix a ground field, usually denoted k, and *scheme* will usually mean quasi-projective algebraic k-scheme. A *variety* is a reduced, irreducible algebraic k-scheme. Occasionally, we may work with schemes like Spec(R), where R is the local ring at a point on an algebraic k-scheme.

1.1. An *algebraic cycle* on a scheme X is a finite, formal linear combination $\sum n_V[V]$ of subvarieties $V \subset X$ with integer coefficients. When X is an equidimension algebraic k-scheme, we write $\mathcal{Z}^r(X)$ for the group of all codimension r algebraic cycles on X. Our reference for foundational questions involving cycles is [FULTON]. Let $D : f = 0$ be a principal divisor on X, and let $V \subset X$ be an irreducible subvariety. Assume $V \not\subset D$. We can then define an intersection cycle $D \cdot V$ as in [FULTON, 1.3].We extend by linearity to define $D \cdot \big(\sum n_V[V]\big)$. More generally, if $W \subset X$ is a complete intersection, $W : f_1 = \ldots = f_n = 0$, one says that W and V *meet properly* if $W \cap V$ has codimension $\geq n$ in V. When W and V meet properly, the above procedure can be iterated to define a cycle $W \cdot V$. Again, $W \cdot \big(\sum n_V[V]\big)$ is defined by linearity. Because it is done one divisor at a time, this intersection is clearly associative. It is commutative, i.e. $W_1{\cdot}W_2{\cdot}V = W_2{\cdot}W_1{\cdot}V$ by [FULTON, 2.4].

1.2. We write Δ^n for an affine simplex of the form

$$\Delta^n = \text{Spec}\left(k[t_0, \ldots, t_n]/(\sum t_i - 1)\right)$$

Let $\Delta^{\bullet}$ be the cosimplicial variety associated to the Δ^n. In other words, given $\rho : \{0, \ldots, m\} \to \{0, \ldots, n\}$ a (weakly) increasing map, define $\tilde{\rho} : \Delta^m \to \Delta^n$ by

$$\tilde{\rho}^*(t_i) = \begin{cases} 0 & \text{if } \rho^{-1}(i) = \emptyset \\ \sum_{\rho(j)=i} t_j & \text{otherwise} \end{cases}$$

If ρ is injective, $\tilde{\rho}$ is called a *face map* and $\tilde{\rho}(\Delta^m)$ is a *face*. Maps $\tilde{\rho}$ for ρ surjective are *degeneracies*. Face maps are closed immersions, and degeneracies are flat. The notions of faces, face maps, and degeneracies extend in an obvious way to schemes $X \times \Delta^n$.

1.3. Let X be an equidimension algebraic k-scheme, $n, r \geq 0$ integers.

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DEFINITION 1.3.1. *$\mathcal{Z}^r(X,n) \subset \mathcal{Z}^r(X \times \Delta^n)$ is the free abelian group on irreducible subvarieties $V \subset X \times \Delta^n$ of codimension r such that V meets all faces of $X \times \Delta^n$ properly.*

PROPOSITION 1.3.2. *$\mathcal{Z}^r(X,\cdot)$ is a simplicial abelian group.*

PROOF. The assertion is that one can functorially associate to a weakly increasing map $\rho : \{0,\ldots,m\} \to \{0,\ldots,n\}$ a pullback map

$$\rho^* : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,m).$$

Suppose first that ρ is surjective. Then $\tilde{\rho} : X \times \Delta^m \to X \times \Delta^n$ is flat, so by [FULTON, 1.7], one has defined a pullback $\rho^* : \mathcal{Z}^r(X \times \Delta^n) \to \mathcal{Z}^r(X \times \Delta^m)$. Note that $\rho^*(V)$ has support in $\tilde{\rho}^{-1}(V)$. For ρ injective, $\tilde{\rho}(X \times \Delta^m) \subset X \times \Delta^n$ is a face, and, for V meeting faces properly, we define $\rho^*(V) = \tilde{\rho}(X \times \Delta^m)\cdot V$. A general ρ can be written uniquely as $\rho = \rho_1 \circ \rho_2$ with ρ_1 injective and ρ_2 surjective. We define $\rho^*(V) = \rho_2^* \circ \rho_1^*(V)$. Another way to understand this is to notice that, given $T \subset X \times \Delta^m$ a closed subvariety of codimension r and $V \subset X \times \Delta^n$ a closed subvariety of codimension r meeting faces properly, the multiplicity of T in $\rho^*(V)$ coincides with the length of $\tilde{\rho}^{-1}(V)$ at the generic point of T. This description of multiplicities also makes functoriality of pullback clear, providing that at each stage the cycle meets faces properly.

Finally, for $V \subset X \times \Delta^n$ meeting faces properly, we must show that $\rho^*(V)$ meets faces properly in $X \times \Delta^m$. We argue by induction on m. The assertion is immediate for ρ injective. In general, one sees easily that $\tilde{\rho}^{-1}(V)$ has pure codimension r in $X \times \Delta^m$, i.e. $\rho^*(V)$ meets the codimension 0 face properly. Let $\sigma \subset \Delta^m$ be a proper face. Then $\tilde{\rho}|\sigma$ corresponds to a map $\tau : \{0,\ldots,p\} \to \{0,\ldots,n\}$ for some $p < m$. By induction, $\tau^*(V)$ meets faces properly on $X \times \tau$. We must check that $\tau^*(V) = \rho^*(V)\cdot(X \times \sigma)$, but this follows from the above description of the multiplicities of components of $\tau^*(V)$.

We will write

$$\partial_i : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,n-1)$$

for the pullback map on cycles associated to the inclusion $\Delta^{n-1} \subset \Delta^n$ defined by $t_i = 0$.

1.4 Higher Chow groups.

DEFINITION 1.4.1. *Let X be an equidimensional algebraic k-scheme as above. The higher Chow groups are defined to be the homotopy groups of the simplicial abelian group $\mathcal{Z}^*(X,\cdot)$:*

$$CH^r(X,n) := \pi_n(\mathcal{Z}^r(X,\cdot)).$$

Let $\delta = \sum(-1)^i\partial_i : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,n-1)$. One checks easily that $\delta \circ \delta = 0$, so $\big(\mathcal{Z}^r(X,\cdot),\delta\big)$ is a chain complex. On the other hand, one defines the *normalized complex*

$$\mathcal{Z}^r(X,n)_{norm} = \bigcap_{i=0}^{i=n-1} \ker(\partial_i) \subset \mathcal{Z}^r(X,n)$$

with boundary maps $\partial_n : \mathcal{Z}^r(X,n)_{norm} \to \mathcal{Z}^r(X,n-1)_{norm}$.

PROPOSITION 1.4.2. *We have*

$$CH^r(X,n) = \pi_n(\mathcal{Z}^r(X,\cdot)) \cong H_n(Z^r(X,\cdot),\delta) \cong H_n(Z^r(X,\cdot)_{norm}).$$

PROOF. This is standard from the structure of simplicial abelian groups, cf. [MAY, §22].

PROPOSITION 1.4.3. $CH^r(X,0) \cong CH^r(X)$, *the usual Chow group of algebraic cycles on X modulo rational equivalence.*

PROOF. By definition, the Chow group coincides with the cokernel of $\mathcal{Z}(X,1) \rightrightarrows \mathcal{Z}(X,0)$. QED

2. Functoriality

PROPOSITION 2.1. *The complex $\mathcal{Z}(X,.)$ is covariant functorial for proper maps and contravariant functorial for flat maps.*

PROOF. Recall $\mathcal{Z}(X,n) \subset \mathcal{Z}(X \times \Delta^n)$ is generated by subvarieties meeting all the faces $X \times \Delta^m \subset X \times \Delta^n$ properly. Let $f : X \to Y$ be proper. Then $f \times 1 : X \times \Delta^n \to Y \times \Delta^n$ is also proper, so there is a map on cycles $f_* : \mathcal{Z}(X \times \Delta^n) \to \mathcal{Z}(Y \times \Delta^n)$. For $Z \subset X \times \Delta^n$ and $\Delta^m \subset \Delta^n$ one has

$$f(Z) \cap (Y \times \Delta^m) = f(Z \cap (X \times \Delta^m))$$
$$\dim(f(Z) \cap Y \times \Delta^m) \leq \dim(Z \cap (X \times \Delta^m)).$$

It follows that the arrows in the square

$$\begin{array}{ccc} \mathcal{Z}(X,n) & \xrightarrow{\text{pullback}} & \mathcal{Z}(X,m) \\ \downarrow f_* & & \downarrow f_* \\ \mathcal{Z}(Y,n) & \xrightarrow{\text{pullback}} & \mathcal{Z}(Y,m) \end{array}$$

are defined. The fact that the square commutes is e.g. ([**], Th. 6.2(a)). Actually, in the reference, Fulton works with cycles mod rational equivalence, but we can replace $X \times \Delta^n$, $Y \times \Delta^n$, $X \times \Delta^m$, $Y \times \Delta^m$ with Z, $f(Z)$, $Z \cap (X \times \Delta^m)$, $f(Z \cap (X \times \Delta^m))$ and assume we are working with cycles of codimension 0 where rational equivalence changes nothing.

The assertion of contravariant functoriality for flat maps is ([**], 1.7). QED

COROLLARY 2.1.1. *Let $\pi : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ be a finite extension of fields. Then one has*

$$\mathcal{Z}(X_k,.) \xrightarrow{\pi^*} \mathcal{Z}(X_{k'},.) \xrightarrow{\pi_*} \mathcal{Z}(X_k,.).$$

If π has degree n, the composition

$$\pi_*\pi^* = \textit{multiplication by } n.$$

PROOF. Immediate. QED

2.2. The following lemma is the prototype for an argument which will be used repeatedly in the sequel.

LEMMA 2.2.1. *Let $k \subset K$ be a purely transcendental extension. Then the pullback $\pi^* : \mathcal{Z}(X_k, .) \to \mathcal{Z}(X_K, .)$ is injective on homology.*

PROOF. Let $Z \in \mathcal{Z}(X_k, n)$ with $\partial Z = 0$ in $\mathcal{Z}(X_k, n-1)$. Let $W \in \mathcal{Z}(X_K, n+1)$ satisfy $\partial W = \pi^* Z$. Clearly, W is defined over a finitely generated purely transcendental extension $k(t_1, \dots, t_p) \subset K$. Standard "spreading out" results in algebraic geometry imply there exists U an open subvariety of $\mathbf{A}^p$ and a cycle $\mathcal{W}$ in $\mathcal{Z}(X \times_k U, n+1)$ such that

$$\partial \mathcal{W} = Z \times_k U. \tag{2.2.1.1}$$

Again by standard results in algebraic geometry, there exists a closed subvariety $V \subsetneq U$ such that if $u \in U - V$ is a k-point, then the intersection $\mathcal{W}_u = \mathcal{W} \cdot X \times \{u\}$ is defined and $\partial \mathcal{W}_u = Z \times \{u\}$. If k is infinite, then such a k-point u exists, and the class of Z in $H_n(\mathcal{Z}(X_k, .))$ is zero.

If k is a finite field, for any prime ℓ there exists an infinite pro-ℓ extension k_ℓ of k. By (2.1.1), the kernel of the map $CH(X_k, n) \to CH(X_{k_\ell}, n)$ is ℓ-torsion. By the above argument, since k_ℓ is infinite, the kernel of $CH(X_k, n) \to CH(X_K, n)$ is ℓ-torsion. Since ℓ is arbitrary, this map is injective. QED

Here is a variant on (2.2.1) needed in the next section. Let Y be a smooth k-variety, $W \subset Y$ a closed algebraic subset. Let

$$\mathcal{Z}_W(Y, .) \subset \mathcal{Z}(Y, .) \tag{2.2.2}$$

be generated by irreducible subvarieties of $X \times \Delta^n$ meeting $W \times \Delta^m$ properly for all faces $\Delta^m \subset \Delta^n$. More generally, if $\mathcal{W} = \{W_1, \dots, W_q\}$ is a collection of closed algebraic subsets of Y, we define $\mathcal{Z}_{\mathcal{W}}(Y, .) \subset \mathcal{Z}(Y, .)$ to be generated by irreducible cycles meeting all $W_i \times \Delta^m$ properly. Returning to the case of a single closed set, write $\mathcal{C}_*$ for the quotient complex $\mathcal{Z}(Y, .)/\mathcal{Z}_W(Y, .)$. Let $K \supset k$ be a purely transcendental extension, and write $\mathcal{C}_{K,*}$ for the corresponding quotient for $W_K \subset Y_K$.

LEMMA 2.2.3. *The pullback map $\iota : \mathcal{C}_* \to \mathcal{C}_{K,*}$ induces an injection on homology.*

PROOF. The proof is precisely analogous to (2.2.1) and is left for the reader.

The interest in the complex $\mathcal{Z}_W(Y, .)$ arises from the following lemma:

LEMMA 2.2.4. *With notation as above, assume either that Y is smooth or $W \subset Y$ is a local complete intersection. Then pullback of cycles induces a map of complexes*

$$\mathcal{Z}_W(Y, .) \to \mathcal{Z}(W, .).$$

More generally, if $\mathcal{W} = \{W_1, \dots, W_q\}$ with $W = W_1 \subset Y$ as above and $W_i \subset W_1$ for $i \geq 2$ then pullback induces a map of complexes

$$\mathcal{Z}_{\mathcal{W}}(Y, .) \to \mathcal{Z}_{\{W_2, \dots, W_q\}}(W_1, .).$$

PROOF. This follows from commutativity of the intersection product [FULTON, 6.4]. QED

2.3 Cones. The purpose of this section and the next is to prove what we are tempted to call the "easy" moving lemma:

PROPOSITION 2.3.1. *Let Y be a smooth k-variety, $\mathcal{W} = \{W_1, \dots, W_r\}$ a collection of closed algebraic subsets of Y. Assume Y is either affine or projective. Then with notation as in (2.2.2), the inclusion $\mathcal{Z}_{\mathcal{W}}(Y, \bullet) \subset \mathcal{Z}(Y, \bullet)$ is a quasi-isomorphism.*

PROOF. In this section, we will assume (2.3.1) for $Y = \mathbb{A}^n$ and $Y = \mathbb{P}^n$ and prove the general result. The result for $\mathbb{A}^n$ and $\mathbb{P}^n$ will be proved in the following section by a different method.

To fix ideas, suppose Y is projective. Let K be a large, purely transcendental extension of k as above, and define $\mathcal{C}_*$ and $\mathcal{C}_{K,*}$ as in (2.2). By (2.2.3) it will suffice to prove

LEMMA 2.3.1.1. *The map $\iota : \mathcal{C}_* \to \mathcal{C}_{K,*}$ induces the zero map on homology.*

PROOF OF 2.3.1.1. Using (2.2.3), we may enlarge K if we want. Let $\dim(Y) = d$. Fix an embedding $Y \hookrightarrow \mathbb{P}^N$ for $N >> 0$. Let $L \subset \mathbb{P}^N$ be a linear space of codimension $d+1$ which is k-generic and defined over K. We have an inclusion $j : Y \hookrightarrow \mathbb{P}^N - L$. Projecting from L defines a map

$$\pi_L : \mathbb{P}^N - L \to \mathbb{P}^d. \tag{2.3.1.2}$$

The composition $\rho = \pi \circ j : Y \to \mathbb{P}^d$ is finite so there is a map of complexes

$$\rho_* : \mathcal{Z}^*(Y, \bullet) \to \mathcal{Z}^*(\mathbb{P}^d, \bullet).$$

Let Q be the blowup of $\mathbb{P}^N$ along L. There is an evident diagram

$$\begin{array}{ccc} & Q & \\ f \swarrow & & \searrow g \\ \mathbb{P}^d & & \mathbb{P}^N \end{array}$$

with f flat and g proper. We define the cone map

$$\mathrm{cone}_L : \mathcal{Z}^r(Y, \bullet) \to \mathcal{Z}^r(\mathbb{P}^N_K, \bullet) \tag{2.3.1.3}$$

to be the composition $\mathrm{cone}_L = g_* \circ f^* \circ \rho_*$. In fact, the cones in question meet faces on Y properly. More generally, for $W \subset Y$ closed, if a cycle Z in $\mathcal{Z}^r(Y, n)$ meets $W \times \Delta^m$ properly for a face $\Delta^m \subset \Delta^n$ it is easy to check that $\mathrm{cone}_L(Z)$ does as well. We get maps of complexes

$$\mathrm{cone}_L : \mathcal{Z}^*(Y, \bullet) \to \mathcal{Z}^*_Y(\mathbb{P}^N_K, \bullet)$$
$$\mathrm{cone}_L : \mathcal{Z}^*_{\mathcal{W}}(Y, \bullet) \to \mathcal{Z}^*_{Y, W_1, \dots, W_r}(\mathbb{P}^N_K, \bullet)$$

Using (2.2.4), we have a composed map

$$\mathcal{C}_* \xrightarrow{\mathrm{cone}_L} \mathcal{Z}^*_Y(\mathbb{P}^N_K, \bullet)/\mathcal{Z}^*_{Y, W_1, \dots, W_r}(\mathbb{P}^N_K, \bullet) \xrightarrow{j^*} \mathcal{C}_{K,*}.$$

We will prove in the next section that the quotient complex in the middle here is acyclic, so $j^* \circ \mathrm{cone}_L$ is zero on homology. On the other hand, given $z \in H_n(\mathcal{C}_*)$, the standard construction for moving cycles ([**]) implies that for $N, p >> 0$ (depending on n), there exists a chain of purely transcendental extensions $k \subset K_1 \subset K_2 \ldots \subset K_p$ and a chain of codimension $d+1$ linear spaces L_i defined over K_i and independent generic over K_{i-1} such that the composition (with obvious notation)

$$(\iota_p - j^* \circ \mathrm{cone}_{L_p}) \circ \ldots \circ (\iota_1 - j^* \circ \mathrm{cone}_{L_1})$$

is zero on z. (We remark that the usual moving lemma is for cycles on Y, while our application is to cycles on $Y \times \Delta^n$. To deal with this we must apply the classical argument separately to $Y \times \mathrm{Spec}(k(t))$ as t runs through the generic points of all faces of Δ^n. This explains the possible dependence of N and p on n.)

It follows that ι is zero on homology. This completes the proof of (2.3.1.1) and (2.3.1) for Y projective, modulo the proof of the corresponding assertions for $Y = \mathbb{P}^N$. QED

For the case Y affine, we consider $\mathbb{A}^N_k \subset \mathbb{P}^N_k$ and we fix a linear space L of codimension d in the hyperplane at ∞, $\mathbb{P}^N - \mathbb{A}^N$. Let G be the Euclidean group (semi-direct product of $\mathbb{A}^N$ and $GL(N)$.) We take a k-embedding $Y \hookrightarrow \mathbb{A}^N$ which we compose with $g \in G$, where we think of K as containing the function field $k(G)$ and take g to be the k-generic K-point of G. In this way we obtain a K-embedding of Y in $\mathbb{A}^N_K$. Projection from L preserves cycles "at ∞" and so induces a coning map $\mathcal{Z}^*(Y, \bullet) \to \mathcal{Z}^*(\mathbb{A}^N_K, \bullet)$. The rest of the argument parallels the projective case. QED

2.4 Contravariant Functoriality for Projective and Affine Space.

In this section, we verify (2.3.1) for the cases $Y = \mathbb{A}^N$ and $Y = \mathbb{P}^N$.

LEMMA 2.4.1. *Let X be an algebraic k-scheme and G a connected algebraic k-group acting on X. Let A, $B \subset X$ be closed subsets, and assume the fibres of the map*

$$G \times A \to X \qquad (g, a) \mapsto g \cdot a$$

all have the same dimension, and that this map is dominant. (Note both conditions hold when G acts transitively on X.) Then there exists a non-empty open set $U \subset G$ such that for $g \in U$ the intersection $g(A) \cap B$ is proper.

PROOF. Consider the diagram

$$\begin{array}{ccccc} G & \leftarrow & G \times A & \rightarrow & X \\ & \nwarrow & \cup & \square & \cup \\ & & C & \rightarrow & B \end{array}$$

where C is the fibre product. Our hypotheses imply

$$\dim G + \dim A + \dim B - \dim X = \dim C.$$

We may take for U the open set in G where the fibres of $C \to G$ have smallest dimension. QED

LEMMA 2.4.2. *Let hypotheses be as in (2.4.1). Assume, moreover, we are given an overfield $K \supset k$ and a K-morphism $\psi : X_K \to G_K$. Let $V \subset X$ be non-empty open such that for every $x \in V_K$ a scheme point, we have*

$$\mathrm{tr.\,deg}_k\, k(\varphi \circ \psi(x), \pi(x)) \geq \dim G,$$

where $\pi : X_K \to X_k$ and $\varphi : G_K \to G_k$. (In other words, we assume that the subfield of $K(x)$ generated over k by the residue fields at x and $\psi(x)$ has transcendence degree over k at least equal to the dimension of G.) Define

$$\phi : X_K \to X_K \qquad \phi(x) = \psi(x) \cdot x.$$

Assume ϕ is an isomorphism. Then the intersection $\phi(A \cap V) \cap B$ is proper.

PROOF. the map $A_K \to A_k$. We have the diagram

$$\begin{array}{ccccc} (V \cap A)_K & \xrightarrow{(\psi,\pi)} & G \times_k A & \to & X \\ \cup & & \cup & & \cup \\ (V \cap A)_K \cap C & \to & C & \to & B \end{array}$$

As in (2.4.1), $\dim C = \dim G + \dim A + \dim B - \dim X$. Also $(V \cap A)_K \cap C$ coincides under ϕ with $\phi(V \cap A) \cap B$. It therefore suffices to show the intersection $C \cap (V \cap A)$ is proper on $G \times A$. We can replace A by $V \cap A$ and ignore V.

We regard this as an intersection problem on $G \times A$ for C arbitrary. Replacing C by a hyperplane section, we may assume $C \cap A$ is zero-dimensional. For the intersection to be improper we must have then $\dim C < \dim G$. Let $a \in A$ be a scheme point such that $(\psi(a), \pi(a)) \in C$. We must have $\mathrm{tr.\,deg}_k\, k(\psi(a), a) < \dim G$, contradicting the hypotheses. QED

REMARK 2.4.2.1. *In the application, $X = Y \times \mathbb{A}^1$, with G acting trivially on $\mathbb{A}^1$. We are given $\psi_0 : \mathbb{A}^1 \to G_K$ and an open $U_k \subset \mathbb{A}^1_k$ such that for all scheme points $w \in U_K$, $\mathrm{tr.\,deg}_k\, \psi_0(w) \geq \dim G - 1$ and $\mathrm{tr.\,deg}_k\, \psi_0(w) = \dim G$ if w is algebraic over k. Then the map $\psi = \psi_0 \circ \mathrm{projection} : X \to G$ satisfies the hypotheses of (2.4.2). Indeed, the map $\phi : X = Y \times \mathbb{A}^1 \to X$ is $\phi(x, a) = (\psi_0(a)x, a)$ and is thus an isomorphism.*

For the case $Y = \mathbb{A}^N$ (resp. $Y = \mathbb{P}^N$) we take $G = \mathbb{A}^N$ acting by translation (resp. $G = SL(N+1)$ acting in the usual way). In both cases, $k(G) \subset K$ with K purely transcendental over k, and we can find a map $\psi_0 : \mathbb{A}^1_K \to G_K$ with $\psi_0(0) =$ identity and $\psi_0(1)$ k-generic in G. (Use the fact that $SL(N+1)$ is generated by transvections.)

PROPOSITION 2.4.3. *With notation as above, let $\mathcal{W}$ be a finite collection of closed sets in Y. Then $\mathcal{C}_* = \mathcal{Z}(Y, \bullet)/\mathcal{Z}_{\mathcal{W}}(Y, \bullet)$ is acyclic.*

PROOF. We begin by fixing triangulations for $\triangle^n \times \triangle^1$ for $n = 0, 1, 2, \ldots$. Vertices in $\triangle^n \times \triangle^1$ will be pairs (v, w) with v a vertex of $\triangle^n$ and w a vertex of $\triangle^1$. For $S = ((v_0, w_0), \ldots, (v_r, w_r))$ a collection of $r+1$ vertices, we define

$$\theta_S : \triangle^r \to \triangle^n \times \triangle^1 \qquad (t_0, \ldots, t_r) \mapsto \sum t_i(v_i, w_i)$$

. We can consider formal linear combinations of maps between products of simplices. There is an obvious notion of composition of such linear combinations. For example

$$\partial_{\triangle^r} = \sum(-1)^i\partial_i : \triangle^{r-1} \to \triangle^r.$$

A triangulation T is a collection $\mathcal{S}_{n n=0,1,\ldots}$, where each $\mathcal{S}_n$ is itself a collection of sets S of $n+1$ vertices in $\triangle^{n-1} \times \triangle^1$ together with signs $\sigma(S)$ such that writing

$$T_n = \sum_{S \in \mathcal{S}_n} \sigma(S)\theta_S : \triangle^n \to \triangle^{n-1} \times \triangle^1$$

we have as formal linear combinations of maps $\triangle^n \to \triangle^n \times \triangle^1$

$$T_{n+1} \circ \partial_{\triangle^{n+1}} - ((\partial_{\triangle^{n-1}}) \times \mathrm{id}_{\triangle^1}) \circ T_n = \mathrm{id}_{\triangle^n} \times \iota_1 - \mathrm{id} \times \iota_0 : \triangle^n \to \triangle^n \times \triangle^1.$$

We fix such a triangulation $T = T_n$. For example, T_1 might look like

fig. 1

with signs arranged so the diagonal faces on $T_1 \circ \partial$ cancel. We would like to think that pulling back along T. would define a homotopy

$$T^* : \mathcal{Z}(Y \times \mathbb{A}^1, .) \to \mathcal{Z}(Y, . + 1).$$

Unfortunately, for example, with reference to fig. 1 above, taking $t \neq 0, 1$ the cycle $Y \times \{t\} \times \{t\} \in \mathcal{Z}(Y \times \mathbb{A}^1, 1)$ does not meet the diagonal faces properly.

Lemma 2.4.3.1. *Consider the composition*

$$Y \times \mathbb{A}^1_K \xrightarrow{\phi} Y \times \mathbb{A}^1_K \xrightarrow{\mathrm{pr}_1} Y_K \xrightarrow{\pi} Y_k.$$

The mapping

$$\rho^* = T^* \circ \phi^* \circ \mathrm{pr}_1^* \circ \pi^* : \mathcal{Z}(Y_k, .) \to \mathcal{Z}(Y_K, . + 1) \tag{2.4.3.2}$$

is well defined. This same map carries $\mathcal{Z}_{\mathcal{W}}(Y_k, .) \to \mathcal{Z}_{\mathcal{W}_K}(Y_K, . + 1)$.

Proof of 2.4.3.1. We prove the last assertion. Let Z represent a cycle in $\mathcal{Z}_{\mathcal{W}}(Y_k, n)$ and let $F \subset \triangle^n \times \triangle^1$ be a face of some $\triangle^{n+1}$ entering into the triangularization. We must show the intersection

$$\phi^* \, \mathrm{pr}_1^* \, \pi^*(Z) \cap (W \times F)$$

is proper for $W \in \mathcal{W}$. Replacing ψ_0 by its image under composition with the inverse map in G_K, we might as well consider

$$pr_1^* \pi^*(Z) \cap \phi^*(W \times F).$$

If $F \subset \triangle^n \times \{0\}$ this intersection is proper since $\psi_0(0) = \mathrm{id}$ and $Z \in \mathcal{Z}_{\mathcal{W}}(Y, n)$. Otherwise, we apply (2.4.2) with $B = W \times F$, $A = \mathrm{pr}_1^*(Z) \cap (Y \times F)$. (Note the intersection in the definition of A is proper.) This proves (2.4.3.1). QED

END OF PROOF OF 2.4.3. Note the homotopy ρ^* in (2.4.3.2) satisfies

$$\partial \circ \rho_n^* - \rho_{n-1}^* \circ \partial = \psi_0(1)^* - \psi_0(0)^* : \mathcal{Z}(Y_k, n) \to \mathcal{Z}(Y_K, \text{.}).$$

Next we claim

$$\psi_0(1)^* : \mathcal{Z}(Y_k, \text{.}) \to \mathcal{Z}_{\mathcal{W}}(Y_k, \text{.}).$$

We apply (2.4.2) with ψ the constant map with image $\psi_0(1)$. For ambiant space we take $Y \times \Delta^m$ for some face $\Delta^m \subset \Delta^n$. We take $A = W \times \Delta^m$, and $B = z \cap (Y \times \Delta^m)$ with $z \in \mathcal{Z}(Y_k, n)$.

In conclusion, we see the map $\mathcal{C}_* \to \mathcal{C}_{K,*}$ induces 0 on homology. Again by the argument in (2.2.3), $\mathcal{C}_*$ is acyclic, and (2.4.3) follows. QED

REMARK 2.4.3.3. *Let Y and $\mathcal{W} = \{W_1, \ldots, W_n\}$ be as in (2.4.3) and let X be a k-scheme. Write $X \times \mathcal{W} = \{X \times W_1, \ldots, X \times W_n\}$. The same argument as in the proof of (2.4.3) shows that the inclusion $\mathcal{Z}_{X \times \mathcal{W}}(X \times Y_k, \text{.}) \subset \mathcal{Z}(X \times Y_k, \text{.})$ is a quasi-isomorphism.*

2.5 Contravariant Functoriality; Applications.

PROPOSITION 2.5.1. *Let $f : X \to Y$ be a morphism of algebraic k-schemes. Assume Y is smooth and either affine or projective. Then there is defined a pullback map*

(2.5.1.1) $$f^* : CH^*(Y, n) \to CH^*(X, n).$$

Given a composition $W \xrightarrow{f} X \xrightarrow{g} Y$ with X and Y smooth and either affine or projective, we have

$$(g \circ f)^* = f^* \circ g^* : CH^*(Y, n) \to CH^*(W, n).$$

PROOF. There exists a collection $\mathcal{T} = \{T_1, \ldots, T_p\}$ of closed subsets of Y such that for any n and any cycle $Z \in \mathcal{Z}_{\mathcal{T}}(Y, n)$, the pullback $f^{-1}(Z)$ is defined in $\mathcal{Z}(X, n)$. Indeed, the diagonal $\Delta_Y \subset Y \times_k Y$ is a local complete intersection since Y is smooth. Hence, the graph $\Gamma_f = (f \times 1_Y)^{-1} \Delta_Y \subset X \times Y$ is also a local complete intersection. Let T_i be the set over which the fibres of f have dimension $\geq \dim X - \dim Y + i$. If Z meets $T_i \times \Delta^n$ properly for all i, then $X \times Z \subset X \times Y \times \Delta^n$ meets $\Gamma_f \times \Delta^n$ properly, so the intersection is defined. Under the identification $\Gamma_f \cong X$, this is the pullback. Since $H_*(\mathcal{Z}_{\mathcal{T}}(Y, \text{.})) \cong H_*(\mathcal{Z}(Y, \text{.}))$, the pullback (2.5.1.1) is defined. The remaining argument is similar and is left for the reader. QED

The hypothesis that Y be either projective or affine in (2.3.1) and (2.5.1) can be removed using the "strong" moving lemma to be proved in (**).

PROPOSITION 2.5.2. *Let Y be smooth quasi-projective, and let $\mathcal{W} = \{W_1, \ldots, W_p\}$ be closed subsets of Y. Assume the strong moving lemma (**) holds. Then the inclusion $\mathcal{Z}_{\mathcal{W}}(Y, \text{.}) \subset \mathcal{Z}(Y, \text{.})$ is a quasi-isomorphism.*

PROOF. Induction on $d = \dim Y$. For $d \leq 1$, Y is affine or projective and the assertion is (2.3.1). In general, let $\bar{Y}$ be a projective closure of Y with $Z = \bar{Y} - Y$.

Let $\bar{X}$ be a hypersurface section of large degree on $\bar{Y}$ which contains Z but which is otherwise generic. Let $X = \bar{X} - Z$. Then X is smooth quasi-projective of dimension $d-1$, and $Y - X$ is affine. Write $\mathcal{W} \cap X = \{W_1 \cap X, \dots, W_p \cap X\}$. We have a diagram

$$\begin{array}{ccccc} \mathcal{Z}_{\mathcal{W}\cap X}(X,\cdot) & \longrightarrow & \mathcal{Z}_{\mathcal{W}}(Y,\cdot) & \longrightarrow & \mathcal{Z}_{\mathcal{W}}(Y-X,\cdot) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}(X,\cdot) & \longrightarrow & \mathcal{Z}(Y,\cdot) & \longrightarrow & \mathcal{Z}(Y-X,\cdot) \end{array}$$

By induction, the outside vertical arrows are quasi-isomorphisms. The strong moving lemma (**) and (**) implies the two horizontal sequences give distinguished triangles in the derived category. It follows from the five-lemma that the middle vertical arrow is a quasi-isomorphism. QED

3. The Homotopy Theorem

THEOREM 3.1. *Let X be a quasi-projective k-variety. The pullback map p^* : $\mathcal{Z}(X,\cdot) \to \mathcal{Z}(X \times \mathbb{A}^n,\cdot)$ is a quasi-isomorphism.*

PROOF. One reduces by induction to the case $n = 1$. Let $K = k(\mathbb{A}^1)$ be the function field of $\mathbb{A}^1$. Let

$$(1_X \times \rho)^* : \mathcal{Z}(X \times_k \mathbb{A}^1,\cdot) \to \mathcal{Z}((X \times_k \mathbb{A}^1)_K, \cdot + 1)$$

be defined as in (2.4.3.2). The first point is that this map is defined, and that

$$(1_X \times \rho)^*(\mathcal{Z}_{X\times\{0,1\}}(X \times_k \mathbb{A}^1,\cdot)) \subset \mathcal{Z}_{X\times\{0,1\}}((X \times_k \mathbb{A}^1)_K, \cdot + 1).$$

The arguments here are the same as in (2.4) (with $Y = \mathbb{A}^1$), and are left to the reader. In particular, the inclusion $\mathcal{Z}_{X\times\{0,1\}}(X \times \mathbb{A}^1,\cdot) \subset \mathcal{Z}(X \times \mathbb{A}^1,\cdot)$ is a quasi-isomorphism, and, writing θ^* for the pullback on cycles along the map translation by the k-generic point t, we have that θ^* is homotopic to the natural pullback π^* mapping $\mathcal{Z}(X \times \mathbb{A}^1_k,\cdot) \to \mathcal{Z}(X \times \mathbb{A}^1_K,\cdot)$ with $K = k(t)$. Also,

$$\theta^*(\mathcal{Z}(X \times \mathbb{A}^1_k,\cdot)) \subset \mathcal{Z}_{X\times\{0,1\}}(X \times \mathbb{A}^1_K,\cdot).$$

Writing T for a triangulation, the composition

$$\mathcal{Z}_{X\times\{0,1\}}(X \times \mathbb{A}^1_k,\cdot) \xrightarrow{1_X\times\theta^*} \mathcal{Z}_{X\times\{0,1\}}(X \times \mathbb{A}^1_K,\cdot) \xrightarrow{T^*} \mathcal{Z}(X_K, \cdot + 1)$$

is defined. Since T is a triangulation, we have

$$\partial\circ T^*\circ\theta^* - T^*\circ\theta^*\circ\partial = (\partial\circ T^* - T^*\circ\partial)\circ\theta^* = (\iota_1^* - \iota_0^*)\circ\theta^*.$$

If we view these as maps $\mathcal{Z}_{X\times\{0,1\}}(X \times \mathbb{A}^1_k,\cdot) \to \mathcal{Z}(X_K,\cdot)$ and consider the effect on homology we have

$$0 = H_*((\iota_1^* - \iota_0^*)\circ\theta^*) = H_*((\iota_1^* - \iota_0^*)\circ\pi^*) = H_*(\pi^*\circ(\iota_1^* - \iota_0^*)).$$

Since π^* is injective on homology, we conclude

$$H_*(\iota_0^*) = H_*(\iota_1^*) : H_*(\mathcal{Z}_{X\times\{0,1\}}(X \times \mathbb{A}^1_k, \bullet)) \to H_*(\mathcal{Z}(X, \bullet)).$$

Let

$$\tau : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \qquad \tau(a,b) = ab.$$

The map τ is flat, so we have a pullback (2.1) $1_X \times \tau^* : \mathcal{Z}(X \times \mathbb{A}^1, \bullet) \to \mathcal{Z}(X \times \mathbb{A}^1 \times \mathbb{A}^1, \bullet)$. For $Z \subset X \times \mathbb{A}^1 \times \Delta^n$ one has

$$(1_X \times \tau)^{-1}(Z) \cap (X \times \mathbb{A}^1 \times \{0\} \times \Delta^n) \subset (1_X \times \tau)^{-1}(Z \cap (X \times \{0\} \times \Delta^n)).$$

It follows easily, again using flatness, that

$$1_X \times \tau^*(\mathcal{Z}_{X\times\{0\}}(X \times \mathbb{A}^1, \bullet)) \subset \mathcal{Z}_{X\times\mathbb{A}^1\times\{0,1\}}(X \times \mathbb{A}^1 \times \mathbb{A}^1, \bullet)).$$

The compositions

$$\iota_j^* \circ \tau^* : \mathcal{Z}_{X\times\{0\}}(X \times \mathbb{A}^1, \bullet) \to \mathcal{Z}(X \times \mathbb{A}^1, \bullet)$$

are defined for $j = 0, 1$ and agree on homology. But $\iota_1^* \circ \tau^* =$ identity while $\iota_0^* \circ \tau^* = (\tau \circ \iota_0)^* = (\iota_0 \circ p)^*$. It follows that

$$p^* : H_*(\mathcal{Z}(X, \bullet)) \to H_*(\mathcal{Z}(X \times \mathbb{A}^1, \bullet))$$

is surjective. Since $p^*(\mathcal{Z}(X, \bullet)) \subset \mathcal{Z}_{X\times\{0\}}(X \times \mathbb{A}^1, \bullet)$, we have identity $= \iota_0^* \circ p^*$ defined so p^* is injective. this completes the proof of (3.1). QED

4. Cubical Chow Groups

In this section we show how the higher Chow groups can be computed using cubes rather than simplices. The use of cubes simplifies both the product structure (**) and the proof of the strong moving lemma (**).

4.1. Let $\square^n = \mathbb{A}^n_X$. (The variety X will be fixed throughout the discussion and will frequently be omitted from the notation.)

(4.1.1)
$$\delta_i^j : \square^n \hookrightarrow \square^{n+1}$$
$$\delta_i^j(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, j, x_i, \dots, x_n) \qquad 1 \le i \le n, \quad j = 0, 1$$

and

(4.1.2)
$$p_i : \square^n \to \square^{n-1}$$
$$p_i(x_1, \dots, x_n) = (x_1, \dots, \hat{x}_i, \dots, x_n)$$

Codimension 1 faces of $\square^n$ are defined by setting a coordinate $= 0, 1$. Codimension r faces are intersections of r codimension 1 faces. Let

(4.1.3)
$$\mathcal{Z}_c(n) = \mathcal{Z}_c(X, n)$$

be the free abelian group on irreducible subvarieties of $\square^n$ meeting all faces properly (**). We have pullback maps

$$d_i^j = (\delta_i^j)^* : \mathcal{Z}_c(n) \to \mathcal{Z}_c(n-1) \tag{4.1.4}$$

$$\pi_i = p_i^* : \mathcal{Z}_c(n-1) \to \mathcal{Z}_c(n) \tag{4.1.5}$$

These satisfy the following identities for $\nu < \mu$ and $j, k = 0, 1$

$$d_{\mu-1}^k \circ d_\nu^j = d_\nu^j \circ d_\mu^k \tag{4.1.5}$$

$$\pi_\nu \circ \pi_{\mu-1} = \pi_\mu \circ \pi_\nu \tag{4.1.6}$$

$$\pi_\nu \circ d_{\mu-1}^j = d_\mu^j \circ \pi_\nu \tag{4.1.7}$$

$$\pi_{\mu-1} \circ d_\nu^j = d_\nu^j \circ \pi_\mu \tag{4.1.8}$$

Finally, we define

$$d = \sum_{i=1}^n (-1)^i (d_i^0 - d_i^1) : \mathcal{Z}_c(n) \to \mathcal{Z}_c(n-1). \tag{4.1.8}$$

It follows from 4.1.5 that $d^2 = 0$, so $\mathcal{Z}_c(*)$ is a complex.

4.2. Define the *degenerate cubical cycles*

$$\mathcal{D}(n) := \sum_{i=1}^n \pi_i(\mathcal{Z}_c(n-1)) \subset \mathcal{Z}_c(n). \tag{4.2.1}$$

It follows from 4.1.7 and 4.1.8 that $\mathcal{D}(*)$ is a subcomplex of $\mathcal{Z}_c(*)$.

The *reduced cubical cycles* are given by

$$\mathcal{Z}_c(n)_0 := \bigcap_{i=1}^n \ker(d_i^0) \subset \mathcal{Z}_c(n). \tag{4.2.2}$$

It follows from 4.1.5 that $\mathcal{Z}_c(*)_0$ is a subcomplex of $\mathcal{Z}_c(*)$. The differential is

$$d = \sum (-1)^{i-1} d_i^1 : \mathcal{Z}_c(n)_0 \to \mathcal{Z}_c(n-1)_0.$$

LEMMA 4.2.3. $\mathcal{Z}_c(n) = \mathcal{D}(n) \bigoplus \mathcal{Z}_c(n)_0$.

PROOF. Let $z \in \mathcal{Z}_c(n)$ and assume $d_{r+1}^0(z) = \ldots = d_n^0(z) = 0$. It follows from 4.1.5 that $z' = z - \pi_r \circ d_r^0(z)$ satisfies $d_r^0(z') = \ldots = d_n^0(z') = 0$. By induction we conclude that $\mathcal{Z}_c(n) = \mathcal{D}(n) + \mathcal{Z}_c(n)_0$. To see the sum is direct, let r be minimal such that there exists $0 \neq z \in \mathcal{Z}_c(n)_0$ with $z = \sum_{i=1}^r \pi_i w_i$. Using 4.1.7 and 4.1.8 togher with $d_r^0 z = 0$, we get

$$w_r = -\sum_{i=1}^{r-1} d_r^0 \pi_i w_i = -\sum_{i=1}^{r-1} \pi_i d_{r-1}^0 w_i$$

so by 4.1.6

$$z = \sum_{i=1}^{r-1} \pi_i (w_i - \pi_{r-1} d_{r-1}^0 w_i),$$

contradicting the minimality of r. QED

DEFINITION 4.2.4. *The cubical cycle groups,* $CH_c^*(X,p) := H_p(\mathcal{Z}_c(*)_0)$.

REMARK 4.2.5. *Note the full cubical complex* $\mathcal{Z}_c(*)$ *does not give the correct groups. For example, one checks easily that* $CH_c^0(X,p) = \mathbb{Z}$ *for all* $p \geq 0$.

THEOREM 4.3. $CH_c^*(X,p) \cong CH^*(X,p)$

LEMMA 4.3.1. *Let* $\mathcal{W} = \{W_1, \dots, W_q\}$ *be closed subsets of* $\mathbb{A}^n$. *Then the inclusion* $\mathcal{Z}_{c,X\times\mathcal{W}}(X\times\mathbb{A}^n,*)_0 \subset \mathcal{Z}_c(X\times\mathbb{A}^n,*)_0$ *is a quasi-isomorphism.*

PROOF OF 4.3.1. The proof is analogous to the proof of 2.4.3 and 2.4.3.3 and is left for the reader. QED

LEMMA 4.3.2. $CH_c^*(X,p) \cong CH_c^*(X\times\mathbb{A}^1,p)$

PROOF OF LEMMA 4.3.2. The proof is analogous to, but easier than, the proof of (**). One simply notes that the identification $X\times\mathbb{A}^1\times\square^n = X\times\square^{n+1}$ gives a homotopy $T^* : \mathcal{Z}_{c,X\times\{0,1\}}(X\times\mathbb{A}^1,n)_0 \to \mathcal{Z}_c(X,n+1)_0$. Indeed, for Z on $X\times\mathbb{A}^1\times\square^n$, write $\bar{Z}$ for the corresponding cycle on $X\times\square^{n+1}$ under the obvious identification. Define $T^*Z = \bar{Z} - \pi_1\circ\iota_0^*(Z)$, where $\iota_i^* : \mathcal{Z}_{c,X\times\{0,1\}}(X\times\mathbb{A}^1,*) \to \mathcal{Z}_c(X,*)$ is the pullback, $i = 0,1$.. One has $dT^* - T^*d = \iota_1^* - \iota_0^*$. The rest of the proof is just as in (**), using 4.3.1 in place of 2.4.3.3. QED

PROOF OF THEOREM 4.3. Define $\mathcal{Z}_c(p,q)_0$ to be cycles on $X\times\Delta^p\times\square^q$ meeting all (intersections of simplicial and cubical) faces properly and trivial on faces defined by setting cubical coordinates $= 0$. One has boundary maps $d' : \mathcal{Z}_c(p,q)_0 \to \mathcal{Z}_c(p-1,q)_0$ (simplicial) and $d'' : \mathcal{Z}_c(p,q)_0 \to \mathcal{Z}_c(p,q-1)_0$ (cubical). It is immediate from 4.3.1 and 4.3.2 that

$$''H_q(\mathcal{Z}_c(p,*)_0) \cong CH_c(X\times\Delta^p,q) \cong CH_c(X,q).$$

In particular the boundary map $''H_q(\mathcal{Z}_c(p,*)_0) \to {}''H_q(\mathcal{Z}_c(p-1,*)_0)$ is zero for p odd and an isomorphism for p even. We have

$$E^2_{q,p} = {}'H_p(\,''H_q(\mathcal{Z}_c(*,*)_0)) = \begin{cases} 0, & p > 0 \\ CH_c(X,q), & p = 0 \end{cases}$$

so $H_p(\mathcal{Z}_c(*,*)_0) \cong CH_c(X,p)$.

On the other hand, if we drop the condition that cycles be trivial on faces defined by setting cubical coordinates $= 0$, we get a double complex $\mathcal{Z}_c(*,*) \supset \mathcal{Z}_c(*,*)_0$, and by the homotopy theorem 3.1 for the simplicial groups:

$$'H_p(\mathcal{Z}_c(*,q)) \cong CH(X\times\square^q,p) \cong CH(X\times\square^0,p) \cong {}'H_p(\mathcal{Z}_c(*,0)).$$

It follows that the map restriction to the point with cubical coordinates $(0,\dots,0)$ induces a quasi-isomorphism $\mathcal{Z}_c(*,q) \to \mathcal{Z}_c(*,0)$. Since $\mathcal{Z}_c(*,q)_0 \subset \mathcal{Z}_c(*,q)$ is a direct summand (mapping to 0 in $\mathcal{Z}_c(*,0)$ if $q > 0$) we conclude

$$'H_p(\mathcal{Z}_c(*,q)_0) = \begin{cases} 0, & q > 0 \\ CH(X,p), & q = 0. \end{cases}$$

The spectral sequence $E^2_{p,q} = {}''H_q('H_p(\mathcal{Z}_c(*,*)_0))$ therefore degenerates, so $H_p(\mathcal{Z}_c(*,*)_0) \cong CH(X,p)$. We conclude $CH(X,p) \cong CH_c(X,p)$. QED

4.4. Finally, for the hard moving lemma, it will be useful to work with a complex of cubical cycles which are trivial on all but one face.

DEFINITION 4.4.1. $\mathcal{Z}_c(*)_{0i} = \{x \in \mathcal{Z}_c(*)_0 \mid d_j^1 x = 0$ *for* $j \geq i+2.\}$.

For example, d_1^1 is the only non-trivial face map on $\mathcal{Z}_c(*)_{00}$. It follows from 4.1.5 that $\mathcal{Z}_c(*)_{0i} \subset \mathcal{Z}_c(*)_0$ is a subcomplex.

THEOREM 4.4.2. *The inclusion* $\mathcal{Z}_c(*)_{00} \subset \mathcal{Z}_c(*)_0$ *is a homotopy equivalence.*

PROOF. Given integers $\ell \leq n-1$ define $h^\ell : \square^{n+1} \to \square^n$ by

$$h^\ell(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n-\ell-1}, x_{n-\ell} \cdot x_{n-\ell+1}, x_{n-\ell+2}, \dots, x_{n+1}).$$

For Z a cycle on $\square^n$, define $H^\ell(Z) = (-1)^{n-\ell}(h^\ell)^{-1}(Z)$ on $\square^{n+1}$. For $\ell \geq n$ define $H^\ell(Z) = 0$. It is easy to check that $H^\ell : \mathcal{Z}_c(*)_0 \to \mathcal{Z}_c(*+1)_0$. Consider the composition of endomorphisms of $\mathcal{Z}_c(*)_0$:

$$\varphi := \cdots (\mathrm{id} - (d \circ H^\ell + H^\ell \circ d)) \circ (\mathrm{id} - (d \circ H^{\ell-1} + H^{\ell-1} \circ d)) \circ \cdots \circ (\mathrm{id} - (d \circ H^0 + H^0 \circ d)).$$

Although this is an infinite composition, it stabilizes in any degree and so defines an endomorphsim $\varphi : \mathcal{Z}_c(*)_0 \to \mathcal{Z}_c(*)_0$ which is homotopic to the identity.

It will be convenient to write an element $z \in \mathcal{Z}_c(n)_0$ as $z(x_1, \dots, x_n)$. We have, for example, $d_i^1 z(x_1, \dots, x_n) = z(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})$. Thus,

$$\begin{aligned} dH^\ell z = {} & (-1)^{n-\ell} z(1, x_1, \dots, x_{n-\ell-2}, x_{n-\ell-1} \cdot x_{n-\ell}, x_{n-\ell+1}, \dots, x_n) \\ & - \dots + z(x_1, \dots, x_{n-\ell-2}, 1, x_{n-\ell-1} \cdot x_{n-\ell}, x_{n-\ell+1}, \dots, x_n) \\ & - z(x_1, \dots, x_{n-\ell-1}, x_{n-\ell} \cdot x_{n-\ell+1}, 1, x_{n-\ell+2}, \dots, x_n) \\ & + \dots + (-1)^\ell z(x_1, \dots, x_{n-\ell-1}, x_{n-\ell} \cdot x_{n-\ell+1}, x_{n-\ell+2}, \dots, x_n, 1). \end{aligned}$$

Also

$$\begin{aligned} H^\ell dz = {} & (-1)^{n-\ell-1} z(1, x_1, \dots, x_{n-\ell-2}, x_{n-\ell-1} \cdot x_{n-\ell}, x_{n-\ell+1}, \dots, x_n) \\ & - \dots + (-1)^\ell z(x_1, \dots, x_{n-\ell-2}, x_{n-\ell-1} \cdot x_{n-\ell}, x_{n-\ell+1}, \dots, x_n, 1) \end{aligned}$$

It follows that

$$\begin{aligned} dH^\ell z + H^\ell dz = {} & - z(x_1, \dots, x_{n-\ell-1}, x_{n-\ell} \cdot x_{n-\ell+1}, 1, x_{n-\ell+2}, \dots, x_n) \\ & + \dots + (-1)^\ell z(x_1, \dots, x_{n-\ell-1}, x_{n-\ell} \cdot x_{n-\ell+1}, x_{n-\ell+2}, \dots, x_n, 1) \\ & + z(x_1, \dots, x_{n-\ell-2}, x_{n-\ell-1} \cdot x_{n-\ell}, 1, x_{n-\ell+1}, \dots, x_n) \\ & - \dots + (-1)^\ell z(x_1, \dots, x_{n-\ell-2}, x_{n-\ell-1} \cdot x_{n-\ell}, x_{n-\ell+1}, \dots, x_n, 1) \end{aligned}$$

Suppose now $z \in \mathcal{Z}_c(n)_{0i}$ for some $i \leq n-1$. If $\ell \leq n-i-2$, we see from the above that $dH^\ell z + H^\ell dz = 0$. Thus the first few factors of φ may act as the identity on z, but eventually we get to $\ell = n-i-1$. At this point,

$$\begin{aligned} z' &= z - (dH^\ell z + H^\ell dz) \\ &= z(x_1, \dots, x_n) - z(x_1, \dots, x_{n-\ell-2}, x_{n-\ell-1} \cdot x_{n-\ell}, 1, x_{n-\ell+1}, \dots, x_n). \end{aligned}$$

Note that $d_j^1 z' = 0$ for $j \geq i+1$, i.e. $z' \in \mathcal{Z}_c(n)_{0,i-1}$. The next factor of φ is $\ell = n-i = n-(i-1)-1$, so z' is mapped to some $z'' \in \mathcal{Z}_c(n)_{0,i-2}$, etc. Once $z \in \mathcal{Z}_c(n)_{00}$, it follows again from the above that $dH^\ell z + H^\ell dz = 0$ so factors of φ are the identity. Thus, $\varphi : \mathcal{Z}_c(*)_0 \to \mathcal{Z}_c(*)_{00}$, φ is the identity on $\mathcal{Z}_c(*)_{00}$, and φ composed with the inclusion is homotopic to the identity on $\mathcal{Z}_c(*)_0$. QED