

Feynman Amplitudes in Mathematics and Physics

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ABSTRACT. These are notes of lectures given at the CMI conference in August, 2014 at ICMAT in Madrid. The focus is on some mathematical questions associated to Feynman amplitudes, including Hodge structures, relations with string theory, and monodromy (Cutkosky rules).

CONTENTS

1. Introduction	2
2. Configuration Polynomials	3
3. The First Symanzik Polynomial for Graphs	4
4. X_G and Λ_G	8
5. The Second Symanzik Polynomial for Graphs	11
6. Riemann Surfaces	14
7. Biextensions and Heights	17
8. The Poincaré Bundle	18
9. The main theorem; nilpotent orbit and passage to the limit	21
10. Cutkosky Rules	26
11. Cutkosky rules: Pham's vanishing cycles	27
12. Cutkosky's Theorem	31
References	33

1. Introduction

What follows are notes from some lectures I gave at a Clay Math conference at CMAT in Madrid in August, 2014. I am indebted to the organizers for inviting me, and most particularly to José Burgos Gil for giving me his notes.

The subject matter here is (a slice of) modern physics as viewed by a mathematician. To understand what this means, I told the audience about my grandson who, when he was 4 years old, was very much into trains. (He is 5 now, and more into dinosaurs.) For Christmas, I bought him an elaborate train set. At first, the little boy was at a complete loss. The train set was much too complicated. Amazingly, everything worked out! Even though the train itself was hopelessly technical and difficult, the box the train came in was fantastic; with wonderful, imaginative pictures of engines and freight cars. Christmas passed happily in fantasy play inspired by the images of the trains on the box. Modern physics is much too complicated for anyone but a “trained” and dedicated physicist to follow. However, “the box it comes in”, the superstructure of mathematical metaphor and analogy which surrounds it, can be a delightful inspiration for mathematical fantasy play.

The particular focus of these notes is the Feynman amplitude. Essentially, to a physical theory the physicist associates a lagrangian, and to the lagrangian a collection of graphs, and to each graph a function called the amplitude on a space of external momenta. Section 2 develops the basic algebra of the first and second Symanzik polynomials in the context of configurations, i.e. invariants associated to a finite dimensional based vector space \mathbb{Q}^E with a given subspace $H \subset \mathbb{Q}^E$. Section 3 considers the first Symanzik (aka Kirchoff polynomial) ψ_G in the case $E = \text{edges}(G)$ and $H = H_1(G)$ for a graph G . $\psi_G = \det(M_G)$ is a determinant, so the *graph hypersurface* $X_G : \psi_G = 0$ admits a birational cover $\pi : \Lambda_G \rightarrow X_G$ with fibre over $x \in X_G$ the projective space associated to $\ker M_G(x)$. The main result in this section is theorem 3.7. A classical theorem of Riemann considers singularities of the theta divisor $\Theta \subset J^{g-1}(C)$ where C is a Riemann surface of genus g . Here $J^{g-1}(C)$ is the space of rational equivalence classes of divisors of degree $g-1$ on C and $\Theta \subset J^{g-1}$ is the subspace of effective divisors. One shows that Θ is (locally) defined by the vanishing of a determinant, and the evident map $\text{Sym}^{g-1}(C) \rightarrow \Theta$ has fibres the projectivized kernels. Riemann’s result is that the dimension of the fibre over $x \in \Theta$ is one less than the multiplicity of x on Θ . Theorem 3.7 is the analogous result for X_G , viz. $\dim(\pi^{-1}(x)) = \text{Mult}_x(X_G) - 1$.

Section 4 considers in more detail the Hodge structure of Λ_G . It turns out that whereas the Hodge structure for X_G is subtle and complicated, the Hodge structure on Λ_G is fairly simple. In particular, Λ_G is mixed Tate.

Sections 5-9 focus on the amplitude A_G , viewed as a multi-valued function of external momenta. Various standard formulas for A_G are given in section 5. Sections 6-9 develop from a Hodge-theoretic perspective a result relating A_G to a limit in string theory when the string tension $\alpha' \rightarrow 0$. Associated to G is a family of stable rational curves which are unions of Riemann spheres with dual graph G . We view this family as lying on the boundary of a moduli space of pointed curves of genus $g = h_1(G)$. We show how external momenta are associated to limits at the boundary of marked points, and we explicit the second Symanzik as a limit of heights. The amplitude A_G then becomes an integral over the space of nilpotent orbits associated to the degeneration of the Hodge structure given by H_{Betti}^1 of the curves. This is joint work with José Burgos Gil, Omid Amini, and Javier Fresan.

We are greatly indebted to P. Vanhove and P. Tourkine for their insights, ([22] and references cited there.) I met my co-authors at the conference, and this work grew out of an attempt to make sense of vague and imprecise suggestions I had made in the lectures.

The last topic (sections 10 - 12) concerns joint work with Dirk Kreimer on Cutkosky Rules. The amplitude A_G is a multi-valued function of external momenta. Cutkosky rules give a formula for the *variation* as external momenta winds around the *threshold divisor* which is the discriminant locus in the space of external momenta for the quadric propagators. Over the years there has been some uncertainty in the physics community as to the precise conditions for Cutkosky rules to apply. Using a modified notion of vanishing cycle due to F. Pham, we prove that Cutkosky rules do apply in the case of what are called *physical singularities*.

Perhaps a word about what is *not* in these notes. In recent years, there has been enormous progress in calculating Feynman amplitudes. The work I should have liked to talk about is due to Francis Brown and Oliver Schnetz [2] which exhibits an algorithm for calculating many Feynman amplitudes and gives a beautiful example where the algorithm fails. In this case the amplitude is the period of a modular form. The whole subject of polylogarithms, which is intimately linked to amplitudes, is not addressed at all. I apologize; there simply wasn't time.

Finally, a shout-out to the organizers, most particularly Kurusch Ebrahimi-Fard who worked very very hard on both the practical and scientific aspects of the conference.

2. Configuration Polynomials

Let H be a finite dimensional vector space of dimension g over a field k , and suppose we are given a finite set E and an embedding $\iota : H \hookrightarrow k^E$, where $k^E := \{\sum_{e \in E} \kappa_e e \mid \kappa_e \in k\}$. Write $W = k^E/H$. Let $e^\vee : k^E \rightarrow k$ be the evident functional that simply takes the e -th coordinate of a vector, and write e^\vee as well for the composition $H \hookrightarrow k^E \rightarrow k$. The function $h \mapsto (e^\vee(h))^2$ defines a rank 1 quadratic form $e^{\vee,2}$ on H . If we fix a basis of H , we can identify the quadratic form with a rank 1 $g \times g$ symmetric matrix M_e in such a way that, thinking of elements of H as column vectors, we have

$$(2.1) \quad e^{\vee,2}(h) = h^t M_e h.$$

Let A_e , $e \in E$ be variables, and consider $M := \sum_{e \in E} A_e M_e$. We can interpret M as a $g \times g$ symmetric matrix with entries that are linear forms in the A_e . More canonically, $M : H \rightarrow H^\vee$ is independent of the choice of basis of H .

DEFINITION 2.1. *The first Symanzik polynomial $\psi(H, \{A_e\})$ associated to $H \hookrightarrow k^E$ is the determinant*

$$(2.2) \quad \psi(H, \{A_e\}) := \det(M).$$

REMARK 2.1. A different choice of basis multiplies $\psi(H, \{A_e\})$ by an element in $k^{\times,2}$. Indeed, M is replaced by $N^t M N$ where N is a $g \times g$ invertible matrix with entries in k .

For $w \in W$, define H_w to be the inverse image in k^E of $w \in W = k^E/H$. We have $H \subset H_w \subset k^E$, and we can calculate the first Symanzik polynomial of $\psi(H_w, \{A_e\})$.

DEFINITION 2.2. *The second Symanzik polynomial*

$$(2.3) \quad \phi(H, w, \{A_e\}) := \psi(H_w, \{A_e\}).$$

LEMMA 2.3. (i) *The first Symanzik $\psi(H, \{A_e\})$ is homogeneous of degree $g = \dim H$ in the A_e .*

(ii) *The second Symanzik $\phi(H, w, \{A_e\})$ is homogeneous of degree $g + 1$ in the A_e and is quadratic in w .*

PROOF. The assertions about homogeneity in A_e are clear. To check that ϕ is quadratic in w , it suffices to note that

$$(2.4) \quad \phi(H, w, \{A_e\}) = \det \left(\sum A_e \begin{pmatrix} M_e & W_e \\ {}^t W_e & Q(W_e) \end{pmatrix} \right)$$

is quadratic in W_e , where $W_e = (w_{e,1}, \dots, w_{e,g})$ is a column vector and $Q(W_e) = \sum_1^n w_{e,i}^2$. This is straightforward, expanding the determinant by the last row for example. \square

REMARK 2.2. In fact, one can do a bit better. Let V be a vector space and let q be a quadratic form on V . We can take the $W_e = (v_{e,1}, \dots, v_{e,g})$ to have entries in V and define $Q(W_e) := \sum_1^g q(v_{e,i})$. Using the quadratic form q , one can make sense of the determinant expression on the right in (2.4) and define $\phi(H, w, \{A_e\})$ for $w \in W \otimes V$. Typically, in physics $V = \mathbb{R}^D$ is space-time and q is the Minkowski metric.

REMARK 2.3. To relate the second Symanzik to the height (see sections 5-9), it will be convenient to rewrite the above determinant expression as a bilinear form. Suppose we are given a symmetric $(g+1) \times (g+1)$ matrix of the form

$$(2.5) \quad A := \begin{pmatrix} M & W \\ W^t & S \end{pmatrix}$$

where M is $g \times g$ and invertible, W is a column vector of length g , and S is a scalar. Recall the classical formula $(\det M)M^{-1} = \text{adj}(M)$ where $\text{adj}(M)$ is the matrix of minors. Then

$$(2.6) \quad \det A = -W^t \text{adj}(M)W + S \det M.$$

In the context of the previous remark, if W has entries in a vector space V with a quadratic form Q , the determinant in (2.4) can be written

$$(2.7) \quad \phi(H, w, \{A_e\})/\psi(H, \{A_e\}) = -W^t M^{-1}W + Q(W^t \cdot W)$$

where the product $W^t M^{-1}W$ is interpreted via the quadratic form.

3. The First Symanzik Polynomial for Graphs

Of primary importance for us will be configurations associated to graphs.

DEFINITION 3.1. *A graph G is determined by sets $E = E(G)$ (edges) and $V = V(G)$ (vertices) together with a set Λ which we can think of as the set of half edges of G . We are given a diagram of projections*

$$(3.1) \quad E \xleftarrow{p} \Lambda \xrightarrow{q} V.$$

We assume p is surjective and $p^{-1}(e)$ has 2 elements for all $e \in E$.

(This definition may seem a bit fussy, but half edges are very useful when one wants to talk about automorphisms of a graph. For example, there is a unique graph G with one edge and one vertex, and $\text{Aut}(G) = \mathbb{Z}/2\mathbb{Z}$.) An *orientation* of G is an ordering of $p^{-1}(e)$ for all edges e , i.e. a map $\Lambda \rightarrow E \times \{-1, 1\}$ satisfying obvious conditions. For an oriented graph G there is a boundary map $\partial : \mathbb{Z}^E \rightarrow \mathbb{Z}^V$, $e \mapsto \partial^+(e) - \partial^-(e)$. We define the homology of G via the exact sequence

$$(3.2) \quad 0 \rightarrow H_1(G, \mathbb{Z}) \rightarrow \mathbb{Z}^E \xrightarrow{\partial} \mathbb{Z}^V \rightarrow H_0(G, \mathbb{Z}) \rightarrow 0.$$

The homology of G with coefficients in an abelian group A is defined similarly.

DEFINITION 3.2. *The graph polynomial $\psi_G(\{A_e\}_{e \in E(G)})$ (also sometimes called the Kirchoff polynomial or the first Symanzik polynomial of G) for G is the first Symanzik polynomial for the configuration $H_1(G, \mathbb{Z}) \subset \mathbb{Z}^E$.*

Let G be a graph and let $e \in G$ be an edge. The graph $G - e$ (resp. G/e) is obtained by cutting (resp. contracting) the edge e in G . The general notion of cutting or contracting in a configuration is explained by the diagram

$$(3.3) \quad \begin{array}{ccccc} 0 & \longrightarrow & H(\text{cut } e) & \longrightarrow & k^{E-\{e\}} \\ & & \downarrow \text{inject} & & \downarrow \text{inject} \\ 0 & \longrightarrow & H & \longrightarrow & k^E \\ & & \downarrow & & \downarrow \text{surject} \\ 0 & \longrightarrow & H(\text{shrink } e) & \longrightarrow & k^{E-e} \end{array}$$

PROPOSITION 3.3. (i) $\psi(H(\text{cut } e)) = \frac{\partial}{\partial A_e} \psi(H)$.
 (ii) $\psi(H(\text{shrink } e)) = \psi(H)|_{A_e=0}$ unless e is a tadpole (i.e. the edge e has only one vertex). If e is a tadpole, $\psi(H)|_{A_e=0} = 0$.
 (iii) $\psi(H)$ has degree ≤ 1 in every A_e .

PROOF. (i) and (ii) are straightforward from (3.3). For (iii), note ψ depends only upto scale on the choice of basis of H . We choose a basis h_1, \dots, h_g such that $e^\vee(h_i) = 0$ for $2 \leq i \leq g$. The matrix M_e is then $g \times g$ diagonal with $g - 1$ zeroes and a single 1 in position (1, 1). The matrix $\sum_{\varepsilon \in E} A_\varepsilon M_\varepsilon$ then involves A_e only in position (1, 1), and (iii) follows. \square

In the case of the graph polynomial of a graph G , one can be more precise. A *spanning tree* $T \subset G$ is a connected subgraph of G with $V(T) = V(G)$ and $H_1(T, \mathbb{Z}) = (0)$.

PROPOSITION 3.4. *Let G be a graph. Then the graph polynomial can be written*

$$(3.4) \quad \psi_G = \sum_{T \subset G} \prod_{e \notin T} A_e.$$

(Here T runs through all spanning trees of G .)

PROOF. We know by proposition 3.3 (iii) that every monomial in ψ_G is a product of $g = h_1(G)$ distinct edge variables A_e . For a given set $\{e_1, \dots, e_g\}$ of distinct edges, we know by proposition 3.3 (ii) that this term is exactly the graph polynomial of G with all the edges $\varepsilon \notin \{e_1, \dots, e_g\}$ shrunk. If the edges not in $\{e_1, \dots, e_g\}$ form a spanning tree for the graph, shrinking them will yield

a *rose* with g loops, i.e. a union of g tadpoles. The graph polynomial of such a graph is simply $\prod_1^g A_{e_i}$ so the coefficient of this product in ψ_G is 1. On the other hand, if the edges A_ε not in $\{e_1, \dots, e_g\}$ do not form a spanning tree, they must necessarily contain a loop and so setting them to 0 kills ψ_G and there is no term in A_{e_1}, \dots, A_{e_g} . \square

We continue to assume G is a graph. We write $n = \#E(G)$ and $g = \dim H_1(G)$. The hypersurface $X_G : \psi_G = 0$ in \mathbb{P}^{n-1} is the *graph hypersurface*. The symmetric matrix $M := \sum_e A_e M_e$ defines a linear map $\bigoplus_g \mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow \bigoplus_g \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Define

$$(3.5) \quad \Lambda_G := \{(a, \beta) \in \mathbb{P}^{n-1} \times \mathbb{P}^{g-1} \mid M_a(\beta) = 0\}$$

Note $\Lambda_G \subset X_G \times \mathbb{P}^{g-1} \subset \mathbb{P}^{n-1} \times \mathbb{P}^{g-1}$.

PROPOSITION 3.5. (i) *There exist coherent sheaves \mathcal{E} on X_G and \mathcal{F} on \mathbb{P}^{g-1} such that $\Lambda_G \cong \text{Proj}(\text{Sym}(\mathcal{E})) \cong \text{Proj}(\text{Sym}(\mathcal{F}))$.*

(ii) *Λ_G is a reduced, irreducible variety of dimension $n - 2$ which is a complete intersection of codimension g in $\mathbb{P}^{n-1} \times \mathbb{P}^{g-1}$. The projection $\pi : \Lambda_G \rightarrow X_G$ is birational.*

PROOF. Define \mathcal{E} by the presentation

$$(3.6) \quad H_1(G) \otimes \mathcal{O}_{X_G} \xrightarrow{M} H^1(G) \otimes \mathcal{O}_{X_G}(1) \rightarrow \mathcal{E} \rightarrow 0.$$

Here for $a \in X_G$, $M_a = \sum_e a_e M_e$.

For \mathcal{F} , the map which is given over $\beta \in \mathbb{P}^{g-1}$ by $a \mapsto \sum_E a_e (e^\vee(\beta)) e^\vee$ dualizes to a presentation

$$(3.7) \quad \mathcal{O}_{\mathbb{P}^{g-1}}^g \rightarrow \mathcal{O}_{\mathbb{P}^{g-1}}^n(1) \rightarrow \mathcal{F} \rightarrow 0.$$

The fibre \mathcal{F}_β is the quotient of $k^{n,\vee}$ by the space of functionals of the form $a \mapsto \sum_e a_e e^\vee(\beta) e^\vee$. We have $\dim \mathcal{F}_\beta = n - g + \varepsilon(\beta)$ where $\varepsilon(\beta)$ is the codimension in $k^{g,\vee}$ of the span of $\{e^\vee \mid e^\vee(\beta) \neq 0\}$. Since the e^\vee span $H_1(G)^\vee$ it follows that for β general we have $\varepsilon(\beta) = 0$ so $\Lambda_G = \text{Proj}(\text{Sym})(\mathcal{F})$ has dimension $n-2$. Since the fibre of \mathcal{E} over a point a is the kernel of M_a it is non-zero for $a \in X_G$, whence $\Lambda_G \rightarrow X_G$ with fibres projective spaces. Since the two varieties have the same dimension, it follows that $\Lambda_G \rightarrow X_G$ is birational.

Finally, to realize Λ_G as a complete intersection, let A_e (resp. B_j) be a basis for the homogeneous coordinates on \mathbb{P}^{n-1} (resp. \mathbb{P}^{g-1}). We can think of $B_j \in H_1(G)$. Write $w_{e,j} = e^\vee(B_j)$. Then the defining equations are

$$(3.8) \quad 0 = \sum_{j=1}^g \sum_E A_e w_{e,j} B_j w_{e,i} = 0; \quad 1 \leq i \leq g.$$

\square

We will further investigate the motives of X_G and Λ_G in the sequel. There is an interesting analogy between X_G and Θ , the theta divisor of a genus g Riemann surface C . Both are determinantal varieties, and $\Lambda_G \rightarrow X_G$ corresponds to $\text{Sym}^{g-1} C \xrightarrow{\pi} \Theta$. One has that Θ is a divisor on the jacobian $J(C)$, and a classical theorem of Riemann states that the multiplicity of Θ at a point x is equal to $1 + \dim \pi^{-1}(x)$. We will show below that the same result holds for graph hypersurfaces X_G .

EXAMPLE 3.6. Consider a determinantal variety defined by the determinant of a diagonal variety

$$X : \det \begin{pmatrix} f_1 & 0 & \dots & 0 \\ 0 & f_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & f_g \end{pmatrix} = 0$$

Then X satisfies Riemann's theorem iff the zero sets of the f_i are smooth and transverse.

THEOREM 3.7 (E. Patterson [17]). Let G be a graph, and let $\pi : \Lambda_G \rightarrow X_G$ be the birational cover of X_G defined above. For $x \in X_G$ the fibre $\pi^{-1}(x)$ has dimension equal to $\text{mult}_x(X_G) - 1$.

PROOF. Define $X_p := \{x \in X_G \mid \dim(\pi^{-1}(x)) \geq p\}$ and $X(p) := \{x \in X_G \mid \text{mult}_x(X_G) \geq p + 1\}$. We want to show $X_p = X(p)$.

LEMMA 3.8. $X_p \subset X(p)$.

PROOF OF LEMMA. We have

$$x \in X_p \Leftrightarrow \dim(\ker(M_x = \sum x_e M_e : H_1(G) \rightarrow H_1(G))) \geq p + 1.$$

On the other hand, $X(p)$ is defined by the vanishing of all p -fold derivatives $\frac{\partial^p}{\partial A_{e_1} \dots \partial A_{e_p}} \psi_G = \psi_{G - \{e_1, \dots, e_p\}}$. If we associate to G the quadratic form $\sum x_e e^{\vee, 2}$ on $H_1(G)$, then X_p is the set of x for which the null space of this form has dimension $\geq p + 1$. The quadratic form associated to $G - \{e_1, \dots, e_p\}$ is simply $\sum x_e e^{\vee, 2} |_{H_1(G) \cap \bigcap_{i=1}^p \{e_i^{\vee} = 0\}}$. This restricted form cannot be nondegenerate if the null space of the form on $H_1(G)$ had dimension $\geq p + 1$. \square

LEMMA 3.9. Let Q be a quadratic form on a vector space H . Let $N \subset H$ be the null space of Q . We assume $\dim N = s > 0$. Suppose we are given an embedding $H \hookrightarrow k^E$ for a finite set E . For $e \in E$ write $e^{\vee} : H \rightarrow k$ for the corresponding functional. Then there exists a subset $\{e_1, \dots, e_s\} \subset E$ such that $Q|_{\{e_1^{\vee} = \dots = e_s^{\vee} = 0\}}$ is non-degenerate.

PROOF OF LEMMA. It suffices to take e_1, \dots, e_s such that $N \cap \{e_1^{\vee} = \dots = e_s^{\vee} = 0\} = (0)$. Indeed, if L is any codimension s subspace with $L \cap N = (0)$ we will necessarily have $Q|_L$ nondegenerate. Since $H = L \oplus N$, any ℓ in the null space of $Q|_L$ will necessarily be orthogonal to $L \oplus N = H$. \square

We return to the proof of the theorem. By the first lemma we have $X_i \subset X(i)$. As a consequence of the last lemma we see that

$$(3.9) \quad X_i - X_{i+1} \subset X(i) - X(i+1).$$

In other words, if the nullspace has dimension exactly $i + 1$, then there exists some $(i + 1)$ -st order partial which doesn't vanish. Taking the disjoint union we get $X_0 - X_j \subset X(0) - X(j)$ for any j . Since $X_0 = X(0)$ it follows that $X(j) \subset X_j$. This completes the proof. \square

4. X_G and Λ_G

It turns out that the hypersurface $X_G : \psi_G = 0$ in \mathbb{P}^{n-1} is quite subtle and complicated, while the variety Λ_G introduced above is rather simple. Recall we have a birational map $\pi : \Lambda_G \rightarrow X_G$ and the fibre $\pi^{-1}(a)$ is the projectivized kernel of $M_a = \sum a_e M_e$.

Our arguments at this point are completely geometric but for simplicity we focus on the the case of Betti cohomology and varieties over the complex numbers. The key point is

PROPOSITION 4.1. *The Hodge structure on Betti cohomology $H^*(\Lambda_G, \mathbb{Q})$ is mixed Tate.*

Recall Betti cohomology of a variety over \mathbb{C} carries a Hodge structure.

DEFINITION 4.2. *A Hodge structure on a finite dimensional \mathbb{Q} -vector space H is a pair of filtrations (W_*, F^*) with $W_* H_{\mathbb{Q}}$ a finite increasing filtration (separated and exhaustive) and $F^* = F^* H_{\mathbb{C}}$ a finite (separated and exhaustive) decreasing filtration. The filtration induced by F on $gr_p^W H_{\mathbb{C}}$ should be p -opposite to its complex conjugate, meaning that*

$$(4.1) \quad gr_p^W H_{\mathbb{C}} = \bigoplus F^q gr_p^W H_{\mathbb{C}} \cap \overline{F}^{p-q} gr_p^W H_{\mathbb{C}}$$

A Hodge structure is *pure* if its weight filtration has a single non-trivial weight.

EXAMPLE 4.3. *The Tate Hodge structures $\mathbb{Q}(n)$ are one dimensional \mathbb{Q} -vector spaces with weight $\mathbb{Q}(n) = W_{-2n} \mathbb{Q}(n)$ and Hodge filtration $F^{-n} \mathbb{Q}(n)_{\mathbb{C}} = \mathbb{Q}(n)_{\mathbb{C}} \supset F^{-n+1} = (0)$.*

DEFINITION 4.4. *A Hodge structure H is mixed Tate if $gr_{-2n}^W H = \bigoplus \mathbb{Q}(n)$ for all n .*

LEMMA 4.5. *Let V be a variety over \mathbb{C} . Assume V admits a finite stratification $V = \coprod V_i$ by Zariski locally closed sets such that $H_c^*(V_i)$ is mixed Tate for all i . (Here H_c is cohomology with compact supports.) Then $H_c^*(V)$ is mixed Tate.*

PROOF. The functor $H^* \mapsto gr_p^W H^*$ is exact on the category of Hodge structures. We apply this functor to the spectral sequence which relates $H_c^*(V_i)$ to $H_c^*(V)$ and deduce a spectral sequence converging to $gr_p^W H^*(V)$ with initial terms direct sums of $\mathbb{Q}(p)$. Since extensions of $\mathbb{Q}(p)$ are all split, it follows that $gr_p^W H^*(V)$ is a direct sum of $\mathbb{Q}(p)$ so $H_c^*(V)$ is mixed Tate. \square

Note of course that $H_c^*(V) = H^*(V)$ if the variety V is proper.

PROOF OF PROPOSITION 4.1. Let $\varepsilon : \mathbb{P}^{g-1} \rightarrow \mathbb{N}$ be as in the proof of Proposition 3.5. Define

$$(4.2) \quad T^m = \{\beta \mid \varepsilon(\beta) \geq m\}.$$

It is clear that $T^m \subset \mathbb{P}^{g-1}$ is closed, and $T^{m+1} \subset T^m$. The sets $S^m := T^m - T^{m+1}$ form a locally closed stratification on \mathbb{P}^{g-1} . Let \mathcal{F} be the constructible sheaf on \mathbb{P}^{g-1} defined in Proposition 3.5. The fibres of \mathcal{F} over S^m have constant rank, so $\mathcal{F}|_{S^m}$ is a vector bundle and $\Lambda|_{S^m}$ is a projective bundle. It will suffice by the lemma to show $H_c^*(\Lambda|_{S^m})$ is mixed Tate, and by the projective bundle theorem this will follow if we show $H_c^*(S^m)$ is mixed Tate.

The set T^m can be described as follows. Let $Z \subset 2^E$ be the set of all subsets $z \subset E$ such that the span of $e^\vee|_{H_1(G)}$, $e \in z$ has codimension $< m$ in $H^1(G)$. Then T^m is the set of β such that $e^\vee(\beta) = 0$ for at least one $e^\vee \in z$. Said another way, for any subset $W \subset E$ containing at least one edge from each $z \in Z$, let $L_W \subset \mathbb{P}^{g-1}$ be the set of those β such that $e^\vee(\beta) = 0$ for all $e \in W$. Then $T^m = \bigcup L_W$ is the union of the L_W . since the cohomology of a union of linear spaces is mixed Tate, we see that $H^*(T^m)$ is mixed Tate. Finally, from the long exact sequence relating the cohomologies of T^m, T^{m+1} to the compactly supported cohomology of S^m we deduce that $H_c^*(S^m)$ is mixed Tate as well. \square

We had mentioned the analogy between X_G and the theta divisor $\Theta \subset J(C)$ of an algebraic curve C of genus $n - 1$. From this point of view, the birational map $\pi : \Lambda_G \rightarrow X_G$ is analogous to the map $\text{Sym}^{n-2}C \rightarrow \Theta$. Unlike the curve case, however, Λ_G is not usually smooth. To understand this, we consider partitions $E(G) = E' \amalg E''$. Let $G', G'' \subset G$ be the unions of the corresponding edge sets. We say our partition is *non-trivial on loops* if neither $\{e^\vee\}_{e \in E'}$ nor $\{e^\vee\}_{e \in E''}$ span $H^1(G)$.

We have seen that $\Lambda_G = \text{Proj}(\mathcal{F})$ is a projective fibre space over \mathbb{P}^{g-1} . The general fibres have dimension $n - g - 1$. Of course, over the open set with fibres of dimension exactly $n - g - 1$, Λ_G is a projective bundle, hence smooth. Singularities can occur only when the fibre dimension jumps.

PROPOSITION 4.6. (i) *The fibre of Λ_G over $\beta \in \mathbb{P}^{g-1}$ has dimension $> n - g - 1$ iff there exists a partition $E = E' \amalg E''$ which is non-trivial on loops such that $e^\vee(\beta) = 0$ for all $e \in E'$.*
(ii) *Λ_G is singular iff there exists a partition $E = E' \amalg E''$ which is non-trivial on loops.*

PROOF. For (i), we may take $E'' = \{e \mid e^\vee(\beta) \neq 0\}$. Assertion (i) is now straightforward from the definition of $\varepsilon(\beta)$ in the proof of Proposition 3.5.

For (ii), note that if no such partition exists, then Λ_G is a projective bundle over \mathbb{P}^{g-1} , hence smooth, so the existence of a partition is certainly necessary. Assume such a partition exists. The equation for Λ_G can be written in vector form $\sum_{e \in E} a_e e^\vee(\beta) e^\vee|_{H_1(G)}$ (compare (3.8)). We take β so $e^\vee(\beta) = 0$ for all $e \in E'$ and we take $a_e = 0$ for all $e \in E'$. (The value of the equation for such β does not depend on the choice of a_e , $e \in E'$, so this is a free choice.) It is clear that the partial derivatives at such a point with respect to the a_e and also with respect to coordinates on $H_1(G)$ all vanish, so this is a singular point. \square

REMARK 4.1. The singular structure of Λ_G is more complicated than the above argument suggests because it may happen that there exists $\emptyset \neq F \subsetneq E'$ such that $e^\vee, e \in F \amalg E''$ still does not span $H^1(G)$. In such a case, it suffices to take $a_e = 0$ for $e \in E' - F$.

EXAMPLE 4.7 (Wheel and spoke graphs with 3 and 4 edges.). (i) *The wheel with 3 spokes graph G_3 has vertices 1, 2, 3, 4 and edges*

$$(4.3) \quad \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 4\}, \{2, 4\}, \{3, 4\}.$$

It has 3 loops, but it is easy to check there are no partitions of the edges which are non-trivial on the loops. It follows from Prop. 4.6 that Λ_{G_3} is a \mathbb{P}^2 -bundle over \mathbb{P}^2

and hence non-singular.

(ii) The wheel with 4 spokes G_4 has 5 vertices, 4 loops, and 8 edges:

$$(4.4) \quad \{1, 2\}, \{2, 3\}, \{3, 1\}, \{4, 1\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}.$$

With the aid of a computer, one can show that $\Lambda_{G_4} \rightarrow \mathbb{P}^3$ is a \mathbb{P}^3 -bundle over $\mathbb{P}^3 - 4$ points. Over the 4 points, the fibre jumps to \mathbb{P}^4 .

We next calculate $gr^W H^*(\Lambda_G)$ for a general graph G . Let $f : \Lambda_G \rightarrow \mathbb{P}^{n-1}$ be the projection.

LEMMA 4.8. *The sheaves $R^a f_* \mathbb{Q}_\Lambda$ are zero for a odd. For $a = 2b$, we have*

$$(4.5) \quad R^a f_* \mathbb{Q}_\Lambda \cong \mathbb{Q}(-b)|_{S_b}$$

where $S_b \subset \mathbb{P}^{g-1}$ is the closed set where the fibre dimension of f is $\geq b$. In particular, $S_b = \mathbb{P}^{g-1}$ for $b \leq n - g - 1$.

PROOF. Let $p : \Lambda_G \rightarrow \mathbb{P}^{n-1}$ be the other projection (recall $\Lambda_G \subset \mathbb{P}^{g-1} \times \mathbb{P}^{n-1}$.) We have a map of sheaves on Λ_G

$$(4.6) \quad p^* : H^a(\mathbb{P}^{n-1}, \mathbb{Q})_{\mathbb{P}^{g-1}} \rightarrow R^a f_* \mathbb{Q}_{\Lambda_G}.$$

The lemma follows from the fact that p^* is surjective with support on S_b . (Both assertions are checked fibrewise.) \square

Consider the Leray spectral sequence

$$(4.7) \quad E_2^{pq} = H^p(\mathbb{P}^{g-1}, R^q f_* \mathbb{Q}_{\Lambda_G}) \Rightarrow H^{p+q}(\Lambda_G, \mathbb{Q}).$$

It follows from the lemma that $E_2^{pq} = H^p(S_{q/2}, \mathbb{Q}(-q/2))$ (zero for q odd) has weights $\leq p + q$ with equality if either $p = 0$ or $q \leq 2(n - g - 1)$. Since E_s , $s \geq 2$ is a subquotient of E_2 we get the same assertion for E_s . From the complex

$$(4.8) \quad E_s^{p-s, q+s-1} \rightarrow E_s^{pq} \rightarrow E_s^{p+s, q-s+1}$$

we deduce

PROPOSITION 4.9. *For the spectral sequence (4.7) we find in the range $q \leq 2(n - g - 1)$ or $p = 0$, $q \leq 2(n - g)$ that $E_\infty^{pq} = \mathbb{Q}(-(p+q)/2)$ if both p, q are even, and $E_\infty^{pq} = (0)$ otherwise. In particular, the pullback $H^s(\mathbb{P}^{n-1} \times \mathbb{P}^{g-1}, \mathbb{Q}) \rightarrow H^s(\Lambda_G, \mathbb{Q})$ is an isomorphism for $s \leq 2(n - g)$.*

An interesting special case is that of *log divergent graphs*. By definition, G is log divergent if $n = 2g$.

COROLLARY 4.10. *For G log divergent, $W_{n-3} H^{n-2}(X_G, \mathbb{Q})$ dies in $H^{n-2}(\Lambda_G, \mathbb{Q})$.*

Concerning the motive of the graph hypersurface X_G for an arbitrary graph G we deduce

THEOREM 4.11. *$gr_a^W H^a(X_G, \mathbb{Q})$ is pure Tate for any $a \in \mathbb{Z}$.*

PROOF. Consider the maps $\tilde{\Lambda}_G \xrightarrow{\rho} \Lambda_G \xrightarrow{\pi} X_G$, where $\tilde{\Lambda}_G$ is a resolution of singularities of Λ_G . By [6], Prop. 8.2.5, the image $\rho^* \pi^* H^a(X_G) \subset H^a(\tilde{\Lambda}_G)$ is isomorphic to $gr_a^W H^a(X_G, \mathbb{Q})$. This image is a subquotient of $H^a(\Lambda_G, \mathbb{Q})$ which is mixed Tate by Prop. 4.1. Since the image is pure of weight a , the theorem follows. \square

5. The Second Symanzik Polynomial for Graphs

The second Symanzik polynomial for a graph G depends on masses and external momenta. More precisely, to each vertex v one associates $p_v \in \mathbb{R}^D$ where D is the dimension of space-time. The conservation of momentum condition is

$$(5.1) \quad \sum_v p_v = 0.$$

In addition, to each edge e is attached a mass $m_e \in \mathbb{R}$. The *propagator* f_e associated to an edge e is defined via the diagram

$$(5.2) \quad \begin{array}{ccc} H_1(G, \mathbb{R}^D) & \longrightarrow & (\mathbb{R}^D)^E \xrightarrow{e^{\vee,2} - m_e^2} \mathbb{R} \\ & & \downarrow \partial \\ & & p \in (\mathbb{R}^D)^{V,0} \end{array}$$

By definition f_e is the function $e^{\vee,2} - m_e^2$ restricted to $\partial^{-1}(p) \cong H_1(G, \mathbb{R}^D)$. The amplitude is

$$(5.3) \quad A_G(p, m) := \int_{\partial^{-1}(p)} \frac{d^D g x}{\prod_{e \in E} f_e}$$

Of course, $\partial^{-1}(p)$ can be identified (non-canonically) with $H_1(G, \mathbb{R}^D)$ and the integral can be viewed as an integral over $H_1(G, \mathbb{R}^D)$.

Let $n = \#E(G)$. We consider edge variables A_e as homogeneous coordinates on \mathbb{P}^{n-1} . (Sometimes it is convenient to order the variables and write A_i rather than A_e .) Write

$$(5.4) \quad \Omega := \sum (-1)^{i-1} A_i dA_1 \wedge \cdots \wedge \widehat{A_i} \wedge \cdots \wedge dA_n$$

For $F(A_e)$ homogeneous of degree n , the ratio Ω/F is a meromorphic form of top degree $n-1$ on \mathbb{P}^{n-1} . A chain of integration σ is defined by

$$(5.5) \quad \sigma = \{(\dots, a_e, \dots) \mid a_e \geq 0\} \subset \mathbb{P}^{n-1}(\mathbb{R})$$

A choice of ordering of the edges orients σ .

EXAMPLE 5.1. *Suppose G is a tree, i.e. $g = 0$. Then $\partial^{-1}(p)$ is a point, and the integral simply becomes evaluation of $\frac{1}{\prod_e f_e}$ at this point. If, for example, G is just a string with vertices $\{1, 2, \dots, n\}$ and edges $(i, i+1)$, $1 \leq i \leq n-1$, then with evident orientation we have $\partial(i, i+1) = (i+1) - (i)$. Write p_i for the external momentum at the vertex i . We must find $q_{i,i+1} \in \mathbb{R}^D$, $1 \leq i \leq n$ such that $q_{i-1,i} - q_{i,i+1} = p_i$, $1 \leq i \leq n$ where $q_{0,1} = q_{n,n+1} = 0$. This yields $q_{i,i+1} = -p_1 - p_2 - \cdots - p_i$ and the amplitude is*

$$(5.6) \quad A_G(p, m) = \frac{1}{(p_1^2 - m_1^2)((p_1 + p_2)^2 - m_2^2) \cdots ((p_1 + \cdots + p_{n-1})^2 - m_{n-1}^2)}.$$

LEMMA 5.2 (Schwinger parameters). *Viewing the f_i as independent coordinates, we have*

$$(5.7) \quad \frac{1}{\prod_{i=1}^n f_i} = (n-1)! \int_{\sigma} \frac{\Omega}{(\sum A_i f_i)^n}.$$

PROOF. In affine coordinates $a_i = A_i/A_n$ the assertion becomes

$$(5.8) \quad \frac{1}{\prod_{i=1}^n f_i} = (n-1)! \int_{\mathbb{R}^{n-1}} \frac{da_1 \cdots da_{n-1}}{(a_1 f_1 + \cdots + a_{n-1} f_{n-1} + f_n)^n}.$$

We have

$$(5.9) \quad d\left(\frac{da_2 \cdots da_{n-1}}{(a_1 f_1 + \cdots + a_{n-1} f_{n-1} + f_n)^{n-1}}\right) = \\ - (n-1) f_1 \frac{da_1 \cdots da_{n-1}}{(a_1 f_1 + \cdots + a_{n-1} f_{n-1} + f_n)^n},$$

and the result follows by induction. \square

The amplitude (5.3) can thus be rewritten

$$(5.10) \quad A_G = \frac{1}{(n-1)!} \int_{\mathbb{R}^{Dg}} d^{Dg} x \int_{\sigma} \frac{\Omega}{(\sum A_e f_e)^n}.$$

We would like to interchange the two integration operations. Following the physicists, we take the metric on \mathbb{R}^D to be Euclidean. There is still an issue of convergence because the quadratic form is only positive semi-definite, but as mathematicians we are looking for interesting motives to study. The issue of convergence of a particular period integral is of secondary concern.

We must evaluate

$$(5.11) \quad \int_{\mathbb{R}^{Dg}} \frac{d^{Dg} x}{(\sum A_e f_e)^n}.$$

To this end, we first complete the square for the quadratic form $\sum A_e f_e$. We identify $H_1(G, \mathbb{R}^D) = (\mathbb{R}^D)^g$ and write

$$x_i = (x_i^1, \dots, x_i^D) : H_1(G, \mathbb{R}^D) \rightarrow \mathbb{R}^D; \quad 1 \leq i \leq g; \quad \mathbf{x} = ({}^t x_1, \dots, {}^t x_g).$$

Similarly, $\mathbf{p} = (\dots, {}^t p_v, \dots)_{v \neq v_0}$ where $p_v \in \mathbb{R}^D$ are the external momenta and we omit one external vertex v_0 . We write

$$(5.12) \quad \sum A_e f_e = \mathbf{x} M {}^t \mathbf{x} - 2 \mathbf{x} B \mathbf{p} + \mathbf{p} \Gamma {}^t \mathbf{p} - \mu$$

Here M (resp. B , resp. Γ) is a $g \times g$ (resp. $g \times (\#V - 1)$, resp. $(\#V - 1) \times (\#V - 1)$) matrix with entries which are linear forms in the A_e ; and $\mu = \sum_e m_e^2 A_e$. Note that M is the symmetric $g \times g$ matrix associated to the configuration $H_1(G, \mathbb{R}) \subset \mathbb{R}^E$ as in (2.2). In particular, $\psi_G = \det(M)$. Note also that the matrix operations in (5.12) are a bit exotic. Whenever column D -vectors in \mathbf{x} and \mathbf{p} are to be multiplied, the multiplication is given by the quadratic form on \mathbb{R}^D .

To complete the square write $\mathbf{x} = \mathbf{x}' + M^{-1} B \mathbf{p}$ and $\mathbf{x}' = \mathbf{x}'' R$ where R is orthogonal with $R M {}^t R = \mathbb{D}$ diagonal. We get

$$(5.13) \quad \sum A_e f_e = \mathbf{x}'' \mathbb{D} {}^t \mathbf{x}'' - (B \mathbf{p})(M^{-1}) {}^t (B \mathbf{p}) + \mathbf{p} \Gamma {}^t \mathbf{p} - \mu$$

By definition, the second Symanzik polynomial is

$$(5.14) \quad \phi_G(\{A_e\}, \mathbf{p}, \{m_e\}) := \\ (B \mathbf{p})(\text{adj}(M)) {}^t (B \mathbf{p}) + (\mathbf{p} \Gamma {}^t \mathbf{p} - \mu) \psi_G(\{A_e\}).$$

Here $\text{adj}(M)$ is the adjoint matrix, so $M^{-1} = \text{adj}(M) \det(M)^{-1}$. We can rewrite (5.13)

$$(5.15) \quad \sum A_e f_e = \mathbf{x}'' \mathbb{D}^t \mathbf{x}'' - \frac{\phi_G(A, p, m)}{\psi_G(A)}$$

Using the elementary identity

$$(5.16) \quad \int_{-\infty^N}^{\infty^N} \frac{du_1 \cdots du_N}{(C_1 u_1^2 + \cdots + C_N u_N^2 + L)^n} = \pi^N \prod_{i=1}^N C_i^{-1/2} L^{-n+(N/2)},$$

we now find (taking $N = gD$)

$$(5.17) \quad \int_{\mathbb{R}^{Dg}} \frac{d^{Dg} x}{(\sum A_e f_e)^n} = \frac{\pi^{Dg} \psi_G^{n-(g+1)D/2}}{\phi_G^{n-gD/2}}$$

$$(5.18) \quad A_G = \frac{\pi^{Dg}}{(n-1)!} \int_{\sigma \subset \mathbb{P}^{n-1}(\mathbb{R})} \frac{\psi_G^{n-(g+1)D/2} \Omega}{\phi_G^{n-gD/2}}.$$

REMARK 5.1. Of particular interest is the *log divergent* case $D = 2n/g$, when $A_G = \frac{\pi^{Dg}}{(n-1)!} \int_{\sigma} \frac{\Omega}{\psi_G^{D/2}}$ is independent of masses and external momenta.

To summarize, we now have three formulas for the amplitude

$$(5.19) \quad A_G = \int_{\mathbb{R}^{Dg}} \frac{d^{Dg} x}{\prod_{e \in E} f_e}$$

$$(5.20) \quad A_G = \frac{1}{(n-1)!} \int_{\mathbb{R}^{Dg} \times \sigma} \frac{d^{Dg} x \Omega}{(\sum A_e f_e)^n}$$

$$(5.21) \quad A_G = \frac{\pi^{Dg}}{(n-1)!} \int_{\sigma \subset \mathbb{P}^{n-1}(\mathbb{R})} \frac{\psi_G^{n-(g+1)D/2} \Omega}{\phi_G^{n-gD/2}}.$$

To these, we add without proof a fourth ([1], formula (6-89))

$$(5.22) \quad A_G = \frac{1}{(i(4\pi)^2)^g} \int_{\tilde{\sigma}} \frac{\exp(i\phi_G/\psi_G) \prod_E dA_e}{\psi_G^{D/2}}.$$

Here $\tilde{\sigma} = [0, \infty]^E$ so $\sigma = \tilde{\sigma} - \{0\}/\mathbb{R}_+^\times$. Philosophically, we can think of $\tilde{\sigma}$ as the space of metrics (i.e. lengths of edges) on G . The integral then looks like a path integral on a space of metrics, with the action ϕ_G/ψ_G . We will see in what sense this action is a limit of string theory action.

Here is a useful way to think about the second Symanzik polynomial when the metric on space-time is euclidean. Let G be a connected graph, and assume the metric on \mathbb{R}^D is positive definite (i.e. euclidean). Then for $a_e > 0$ the metric $\sum_e a_e e^{\vee, 2}$ on $(\mathbb{R}^D)^E$ is positive definite as well, so there are induced metrics on $H_1(G, \mathbb{R}^D)$ and $(\mathbb{R}^D)^{V, 0}$. For $p \in (\mathbb{R}^D)^{V, 0}$ let $m_a(p)$ be the value of the metric.

PROPOSITION 5.3. *We have $m_a(p) = \phi_G(a, p, 0)/\psi_G(a)$.*

PROOF. The symmetric matrix M above is positive definite when the edge coordinates $A_e > 0$. It is then clear from (5.15) that the minimum of the metric in the fibre $\partial^{-1}(p)$ is given by $-\frac{\phi_G(A, p, 0)}{\psi_G(A)}$. This is how the metric on the quotient is defined. (In general, if V has a positive definite metric, the metric on a quotient V/W is defined by identifying $V/W \cong W^\perp \subset V$.) \square

6. Riemann Surfaces

In sections 6 - 9 we will reinterpret formula (5.22) for the amplitude. The graph G becomes the dual graph of a singular rational curve C_0 of arithmetic genus g which we view as lying at infinity on a moduli space of pointed curves of genus g . Vertices of G correspond to irreducible components of C_0 , and the external momentum associated to a vertex is interpreted in terms of families of points meeting the given irreducible component of C_0 . The chain of integration in (5.22) is identified with the nilpotent orbit associated to the degenerating Hodge structures, and the action $\exp(i\phi_G/\psi_G)$ is shown to be a limit of actions involving heights. This is joint work with José Burgos Gil, Javier Fresan, and Omid Amini.

As a first step, in this section we will interpret the rank 1 symmetric matrices M_e on $H_1(G)$, (2.1), in terms of the monodromy of the degenerating family of genus g curves. We continue to assume G is a connected graph with g loops and n edges. Stable rational curves C_0 associated to G arise taking quotients of $\coprod_{V(G)} \mathbb{P}^1$ identifying a chosen point of \mathbb{P}_v^1 with a chosen point of \mathbb{P}_w^1 whenever there exists an edge e with $\partial e = \{v, w\}$. For the moment we assume that every vertex of G meets at least 3 edges, so every $\mathbb{P}^1 \subset C_0$ has at least three “distinguished” points which are singularities of C_0 . Note that if there is a vertex meeting ≥ 4 edges, the corresponding \mathbb{P}^1 will have ≥ 4 distinguished points and C_0 will have moduli.

DEFINITION 6.1. *Let $C = \bigcup \mathbb{P}^1$ be a curve obtained by identifying a finite set of pairs of points in $\coprod_V \mathbb{P}^1$ for some finite set V . The dual graph of C is the graph with vertex set V and edge set E the set of pairs of points being identified. If $e \in E$ corresponds to $\{p_1, p_2\}$ with $p_i \in \mathbb{P}_{v_i}^1$, then the edge e is taken to connect the vertices v_1, v_2 .*

EXAMPLE 6.2. *The dual graph of the curve C_0 constructed above is G .*

PROPOSITION 6.3. *There is a canonical identification $H^1(C_0, \mathcal{O}_{C_0}) \cong H^1(G, \mathbb{Q}) \cong H^1(C_0, \mathbb{Q})$. In particular, the arithmetic genus of C_0 is equal to g , the loop number of G .*

PROOF. Let $p : \coprod_{V(G)} \mathbb{P}^1 \rightarrow C_0$ be the identification map. We have an exact sequence of sheaves

$$(6.1) \quad 0 \rightarrow \mathcal{O}_{C_0} \rightarrow p_* \mathcal{O}_{\coprod \mathbb{P}^1} \rightarrow \mathcal{S} \rightarrow 0$$

where \mathcal{S} is a skyscraper sheaf with stalk k over each singular point. Since p is a finite map, taking cohomology commutes with p_* and we find

$$(6.2) \quad 0 \rightarrow k \rightarrow \bigoplus_{V(G)} k \xrightarrow{\delta} \bigoplus_{E(G)} k \rightarrow H^1(C_0, \mathcal{O}_{C_0}) \rightarrow 0$$

It is straightforward to check that δ in the above can be identified with the dual to the boundary map calculating $H_1(G)$, so $\text{coker}(\delta) \cong H^1(G, k)$. The proof that $H^1(G, \mathbb{Q}) \cong H^1(C_0, \mathbb{Q})$ is similar. One simply replaces the exact sequence of coherent sheaves (6.1) with an analogous sequence of constructible sheaves calculating Betti cohomology. \square

We recall some basic results about deformation theory for C_0 , [8]. There exists a smooth formal scheme $\widehat{S} = \text{Spf } k[[t_1, \dots, t_p]]$ and a (formal) family of curves $\widehat{\mathcal{C}} \xrightarrow{\pi} \widehat{S}$ such that the fibre \mathcal{C}_0 over $0 \in \widehat{S}$ is identified with the curve C_0 above, and such that the family is in some sense maximal. In particular, $\widehat{\mathcal{C}}$ is formally smooth over k and

the tangent space T to \widehat{S} at 0 is identified with $\text{Ext}^1(\Omega_{C_0}^1, \mathcal{O}_{C_0})$. Étale locally at the singular points $C_0 \cong \text{Spec } k[x, y]/(xy) =: \text{Spec } R$ so $\Omega_{C_0}^1 \cong Rdx \oplus Rdy/(xdy + ydx)$. Thus $xdy \in \Omega_{C_0}^1$ is killed by both x and y and so Ω^1 has a non-trivial torsion subsheaf supported at the singular points. The 5-term exact sequence of low degree terms for the local to global Ext spectral sequence yields in this case a short exact sequence

$$(6.3) \quad 0 \rightarrow H^1(C_0, \underline{\text{Hom}}(\Omega_{C_0}^1, \mathcal{O}_{C_0})) \rightarrow \text{Ext}^1(\Omega_{C_0}^1, \mathcal{O}_{C_0}) \rightarrow \Gamma(C_0, \underline{\text{Ext}}^1(\Omega_{C_0}^1, \mathcal{O}_{C_0})) \rightarrow 0.$$

The local ext sheaf on the right is easily calculated using the local presentation at the singular point as above

$$(6.4) \quad 0 \rightarrow R \xrightarrow{1 \rightarrow xdy + ydx} Rdx \oplus Rdy \rightarrow \Omega_R^1 \rightarrow 0.$$

One identifies in this way $\underline{\text{Ext}}^1(\Omega_{C_0}^1, \mathcal{O}_{C_0})$ with the skyscraper sheaf having one copy of k supported at each singular point. The subspace $H^1(C_0, \underline{\text{Hom}}(\Omega_{C_0}^1, \mathcal{O}_{C_0})) \subset T$ corresponds to deformations which keep all the double points, i.e. only the chosen points on the \mathbb{P}_v^1 move. In general, r points on \mathbb{P}^1 have $r - 3$ moduli, so

$$(6.5) \quad \dim H^1(C_0, \underline{\text{Hom}}(\Omega_{C_0}^1, \mathcal{O}_{C_0})) = \sum_{v \in V(G)} (\#\text{edges through } v - 3)$$

$$(6.6) \quad \dim T = \sum_{v \in V(G)} (\#\text{edges through } v - 3) + \#E(G) = -3V(G) + 3\#E(G) = 3h_1(G) - 3 = 3g_a(C_0) - 3.$$

Here $g_a(C_0)$ is the arithmetic genus which coincides with the usual genus of a smooth deformation of C_0 . The last identity follows from proposition 6.3. Note $3g - 3$ is the dimension of the moduli space of genus g curves.

The map $T = \text{Tan}_{\widehat{S}, 0} \rightarrow \bigoplus_E k$ arises as follows. For each $e \in E$ there is a singular point $p_e \in C_0$. The deformations of C_0 which preserve the singularity at p_e give a divisor $\widehat{D}_e \subset \widehat{S}$. Let $g_e \in \mathcal{O}_{\widehat{S}}$ define \widehat{D}_e . The functional $T \rightarrow \bigoplus_E k \xrightarrow{\text{pr}_e} k$ is defined by dg_e . The geometric picture is then a collection of principal divisors $\widehat{D}_e \subset \widehat{S}$ meeting transversally. The subvariety cut out by the divisors is the locus of equisingular deformations of C_0 given by moving the singular points. If G is trivalent, i.e. if every vertex of G has exactly three adjacent edges, then $\{0\} = \bigcap \widehat{D}_e$.

Deformation theory leads to a formal *versal* deformation $\widehat{\mathcal{C}} \rightarrow \widehat{S}$, but these formal schemes can be spread out to yield an analytic deformation $\mathcal{C} \rightarrow S$. Here S is a polydisk of dimension $3g - 3$. The divisors lift to analytic divisors $D_e : g_e = 0$ on S . We fix a basepoint $s_0 \in S - \bigcup D_e$, and we wish to study the monodromy action on $H_1(C_{s_0}, \mathbb{Q})$. We choose simple loops $\ell_e \subset S - \bigcup_{e \neq e} D_e$ based at s_0 looping around D_e . We assume ℓ_e is contractible in $S - \bigcup_{e \neq e} D_e$. The *Picard Lefschetz formula* gives the monodromy for the action of ℓ_* on $H_1(C_{s_0}, \mathbb{Q})$

$$(6.7) \quad b \mapsto b + \langle b, a_e \rangle a_e$$

where $a_e \in H_1(C_{s_0}, \mathbb{Q})$ is the vanishing cycle associated to the double point on the curve which remains as we deform along D_e .

A classical result in differential topology says that, possibly shrinking the polydisk S , the inclusion $C_0 \hookrightarrow \mathcal{C}$ admits a homotopy retraction $\mathcal{C} \rightarrow C_0$ in such a way

that the composition $\mathcal{C} \rightarrow C_0 \rightarrow \mathcal{C}$ is homotopic to the identity. It follows that $C_0 \hookrightarrow \mathcal{C}$ is a homotopy equivalence. In this way, one defines the specialization map

$$(6.8) \quad sp : H_1(C_{s_0}, \mathbb{Q}) \rightarrow H_1(\mathcal{C}, \mathbb{Q}) \cong H_1(C_0, \mathbb{Q}).$$

LEMMA 6.4. *The specialization map sp above is surjective.*

PROOF. Intuitively, a loop in $H_1(C_0, \mathbb{Q})$ can be broken up into segments connecting double points of the curve. These double points arise from shrinking vanishing cycles on C_{S_0} so the segments can be modeled by segments in C_{s_0} connecting the vanishing cycles. These segments connect to yield a loop in C_{s_0} which specializes to the given loop in C_0 . (One can give a more formal proof based on the Clemens-Schmid exact sequence, [19].) \square

LEMMA 6.5. *The subspace $A \subset H_1(C_{s_0}, \mathbb{Q})$ spanned by the vanishing cycles a_e is maximal isotropic.*

PROOF. As the base point s_0 approaches $0 \in S$, the various a_e approach the singular points $p_e \in C_0$. In particular, if s_0 is taken close to 0, the a_e are physically disjoint, so $\langle a_e, a_{e'} \rangle = 0$. Since the pairing on H_1 is symplectic, one has $\langle a_e, a_e \rangle = 0$, so the subspace A spanned by the vanishing cycles is isotropic. To see it is maximal, note we can express C_0 as a topological colimit $C_0 = C_{s_0} / \coprod S^1$ so we get an exact sequence $A \rightarrow H_1(C_{s_0}) \xrightarrow{sp} H_1(C_0)$. In particular,

$$(6.9) \quad g = \dim H_1(C_0) \geq \dim H_1(C_{s_0}) - \dim A = 2g - \dim A.$$

It follows that $\dim A \geq g$ so A is maximal isotropic. In terms of a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ we write

$$(6.10) \quad a_e = \sum_{i=1}^g c_{e,i} a_i.$$

\square

The link between the combinatorics of the graph polynomial and the monodromy is given by the following proposition. Write $N_e = \ell_e - id$ so by (6.7) we have $N_e(b) = \langle b, a_e \rangle a_e$. By lemma 6.5 we get $N_e^2 = 0$ so $N_e = \log(\ell_e)$. We consider the composition

$$(6.11) \quad H_1(G) \cong H_1(C_0) \cong H_1(C_{s_0})/A \xrightarrow{N_e} A \cong (H_1(C_{s_0})/A)^\vee \cong H_1(G)^\vee$$

PROPOSITION 6.6. *The bilinear form on $H_1(G)$ given by (6.11) coincides with the bilinear form M_e in (2.1).*

PROOF. Let $b \in H_1(C_{s_0})$. We can identify $sp(b) \in H_1(C_0) \cong H_1(G)$ with a loop $\sum_e n_e e$. Here $n_e = \langle b, a_e \rangle$ is the multiplicity of intersection of b with the vanishing cycle a_e . The quadratic form on $H_1(C_{s_0})$ corresponding to N_e sends $b \mapsto \langle b, \langle b, a_e \rangle a_e \rangle = n_e^2$. The quadratic form on $H_1(G)$ corresponding to M_e maps the loop $\sum n_e \varepsilon$ to n_e^2 . \square

REMARK 6.1. In terms of the basis b_i for $B \cong H_1(G)$, we can write $M_e = (c_{e,i} c_{e,j})$ using the notation of (6.10). We will generalize this to relate the monodromy for punctured curves to the combinatorics of the second Symanzik in (9.11) through (9.15).

7. Biextensions and Heights

Our objective in this section will be to link the second Symanzik polynomial (definition 2.2) to geometry. Recall for a graph G with edges E and vertices V , the second Symanzik $\phi(H, w, \{A_e\})$ depends on $H = H_1(G) \subset \mathbb{Q}^E$ and on $w \in \mathbb{R}^{V,0}$. (We will see later how to extend the construction and take $w \in (\mathbb{R}^D)^{V,0}$ where D is the dimension of space-time.) Recall that ϕ is quadratic in w . We will work with the corresponding bilinear function

$$(7.1) \quad \phi(H, w, w', \{A_e\}) := \phi(H, w + w', \{A_e\}) - \phi(H, w, \{A_e\}) - \phi(H, w', \{A_e\}).$$

We change notation and assume $\mathcal{C} \xrightarrow{\pi} S$ is a family of pointed curves. More precisely, we suppose given two collections $\sigma_{v,i} : S \rightarrow \mathcal{C}$, $v \in V$, $i = 1, 2$ of sections of π . We assume $\sigma_{v,i}(S) \cap C_0 \in \mathbb{P}_v^1$. Since \mathcal{C} is taken to be regular over the ground field, the sections cannot pass through double points of C_0 . We assume further that the $\sigma_{v,1}$ and $\sigma_{v,2}$ are disjoint. It follows after possibly shrinking S that the multi-sections σ_1 and σ_2 are disjoint as well. Let $W, (*, *)$ be an \mathbb{R} -vector space with a symmetric quadratic form. (In fact, for us $W = \mathbb{R}$ or $W = \mathbb{R}^D$ with the Minkowski metric.) We fix W -divisors

$$(7.2) \quad \mathfrak{A}_i := \sum r_{v,i} \sigma_{v,i}; \quad r_{v,i} \in W, \quad \sum_v r_{v,i} = 0, \quad i = 1, 2.$$

More generally, we should work with a diagram

$$(7.3) \quad \begin{array}{ccc} T_1 \amalg T_2 & \xrightarrow{\text{closed immersion}} & \mathcal{C} \\ \parallel & & \downarrow \\ T_1 \amalg T_2 & \xrightarrow{\tau_1 \amalg \tau_2} & S \end{array}$$

where τ_i are finite étale. The labels $r_{v,i}$ would be replaced by sections of local systems of \mathbb{R} -vector spaces R_i on T_i equipped with trace maps $R_i \rightarrow \tau_i^* \mathbb{R}_S$ so we can talk about sections of degree 0. We have no use for this generalization, but we mention it exists.

The W -divisors will play the role of external momenta. We need to define the action which we write

$$(7.4) \quad \mathbf{S}[C_{s_0}, \mathfrak{A}_1, \mathfrak{A}_2]$$

and which will tend to $\phi(H_1(G), \sum r_{v,1}v, \sum r_{v,2}v)$ as $s_0 \rightarrow 0$. This action is given by the archimedean height pairing which is defined as follows. We identify the section σ_1 on C_{s_0} with its image $\sigma_1 \subset C_{s_0}$. We consider the exact sequence of Hodge structures

$$(7.5) \quad 0 \rightarrow H^1(C_{s_0}, \mathbb{Z}(1)) \rightarrow H^1(C_{s_0} - \sigma_1, \mathbb{Z}(1)) \xrightarrow{\text{res}} \prod_{\sigma_1}^0 \mathbb{Z} \rightarrow 0.$$

This sequence is canonically split as an exact sequence of \mathbb{R} -Hodge structures. I.e. there exists a canonical splitting

$$(7.6) \quad \prod_{\sigma_1}^0 \mathbb{R} \rightarrow F^0(H^1(C_{s_0} - \sigma_1)(1)) \cap H^1(C_{s_0} - \sigma_1, \mathbb{R}(1)); \quad \mathfrak{A} \mapsto \omega_{\mathfrak{A}}.$$

Here $\omega_{\mathfrak{A}} \in \Gamma(C_{s_0}, \Omega^1(\log \sigma_1))$. For \mathfrak{B} another \mathbb{R} -divisor of degree 0 on C_{s_0} with support away from the support of \mathfrak{A} , we can write $\mathfrak{B} = \partial\beta$ where β is a 1-chain on $C_{s_0} - \text{Supp}(\mathfrak{A})$ and define the archimedean height

$$(7.7) \quad \langle \mathfrak{A}, \mathfrak{B} \rangle := \int_{\beta} \omega_{\mathfrak{A}} \in \mathbb{C}/i\mathbb{R} = \mathbb{R}.$$

Because $\omega_{\mathfrak{A}}$ is an $\mathbb{R}(1)$ -class, changing β by a class in $H_1(C_0, \mathbb{R})$ does not change the real part of the integral so the height is well-defined.

Finally, we can couple this pairing to the quadratic form on W and define $\langle \mathfrak{A}_1, \mathfrak{A}_2 \rangle$. Indeed, if we choose basepoints $\sigma_{0,i} \in C_{s_0}$ we can write $\sum_v r_{v,i} \sigma_{v,i} = \sum_v r_{v,i} (\sigma_{v,i} - \sigma_{0,i})$ and define

$$(7.8) \quad \mathbf{S}[C_{s_0}, \mathfrak{A}_1, \mathfrak{A}_2] = \langle \mathfrak{A}_1, \mathfrak{A}_2 \rangle = \sum_{v,v'} (r_{v,1}, r_{v',2}) \langle \sigma_{v,1} - \sigma_{0,1}, \sigma_{v',2} - \sigma_{0,2} \rangle.$$

This is independent of the choice of $\sigma_{0,i}$. Changing e.g. $\sigma_{0,1}$ to $\sigma_{0,1}'$ changes the above by

$$\sum_{v,v'} (r_{v,1}, r_{v',2}) \langle \sigma_{0,1}' - \sigma_{0,1}, \sigma_{v',2} - \sigma_{0,2} \rangle$$

which vanishes because $\sum_v r_{v,1} = 0$.

8. The Poincaré Bundle

To understand the behavior of the height in a degenerating family, it is convenient to use the Poincaré bundle. We first recall the Poincaré bundle for a single compact complex torus $T := V/\Lambda$, (V finite dimensional \mathbb{C} -vector space, $\Lambda \subset V$ a cocompact lattice.) Let $\widehat{V} := \text{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ be the \mathbb{C} -vector space of \mathbb{C} -antilinear functionals on V , and let $\widehat{\Lambda} := \{\phi \in \widehat{V} \mid \phi(\Lambda) \subset \mathbb{Z}\}$. By definition, the dual torus $\widehat{T} := \widehat{V}/\widehat{\Lambda}$.

The Poincaré bundle is a \mathbb{C}^\times -bundle \mathcal{P}^\times on $T \times \widehat{T}$. (NB. It will be more convenient to work with the principal \mathbb{G}_m or \mathbb{C}^\times -bundle rather than the corresponding line bundle.) It is characterized by two properties:

- (i) $\mathcal{P}^\times|_{\{0\} \times \widehat{T}}$ is trivial with a given trivialization.
- (ii) $\mathcal{P}^\times|_{T \times \{\phi\}} = L_\phi$; where L_ϕ is the \mathbb{C}^\times -bundle on T associated to the representation of the fundamental group

$$(8.1) \quad \pi_1(T) = \Lambda \subset V \xrightarrow{\phi} \mathbb{C} \xrightarrow{\exp(2\pi i \cdot)} \mathbb{C}^\times.$$

To construct the Poincaré bundle over the family of all principally polarized abelian varieties of dimension g , we recall the Siegel domain

$$(8.2) \quad \mathbb{H}_g := \{M \text{ } g \times g \text{ complex symmetric matrix} \mid \text{Im}(M) > 0\}.$$

(The following description of the Poincaré bundle is taken from an unpublished manuscript of J. Burgos Gil. Details are omitted here. They will appear in a forthcoming paper with Burgos Gil, Fresan and Amini.) The group $Sp_{2g}(\mathbb{R})$ acts on \mathbb{H}_g by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} M = (AM + B)(CM + D)^{-1}$. The quotient $\mathcal{A}_g := \mathbb{H}_g/Sp_{2g}(\mathbb{Z})$ is the Siegel moduli space parametrizing principally polarized abelian varieties of dimension g . We can also think of \mathcal{A}_g as parametrizing polarized Hodge structures

of weight 1. To $\Omega \in \mathbb{H}_g$ we associate the map $H_{\mathbb{Z}} := \mathbb{Z}^{2g} \rightarrow \mathbb{C}^g$ given by the $2g \times g$ matrix $\begin{pmatrix} \Omega \\ I_g \end{pmatrix}$. The symplectic form on $H_{\mathbb{Z}}$ is given by $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

The space

$$(8.3) \quad \tilde{X} := \mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C}$$

is a homogeneous space for the group

$$(8.4) \quad \tilde{G} = \left\{ \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \alpha \\ 0 & A & B & \mu_1 \\ 0 & C & D & \mu_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \lambda_i \in \text{Row}_g(\mathbb{R}), \mu_j \in \text{Col}_g(\mathbb{R}), \alpha \in \mathbb{C}, \right. \\ \left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R}) \right\}.$$

The action is determined by formulas

$$(8.5) \quad \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & A & B & 0 \\ 0 & C & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = \\ ((A\Omega + B)(C\Omega + D)^{-1}, W(C\Omega + D)^{-1}, (C\Omega + D)^{-t}Z, \rho - WC^t(C\Omega + D)^{-t}Z)$$

$$(8.6) \quad \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = (\Omega, W + \lambda_1\Omega + \lambda_2, Z, \rho + \lambda_1Z)$$

$$(8.7) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & \mu_1 \\ 0 & 0 & I & \mu_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = (\Omega, W, Z + \mu_1 - \Omega\mu_2, \rho - W\mu_2)$$

$$(8.8) \quad \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\Omega, W, Z, \rho) = (\Omega, W, Z, \rho + \alpha).$$

Write $\tilde{G}(\mathbb{Z}) \subset \tilde{G}$ for the subgroup with entries in \mathbb{Z} .

THEOREM 8.1. *The quotient*

$$(8.9) \quad \tilde{G}(\mathbb{Z}) \backslash (\mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C})$$

is the dual of the Poincaré bundle \mathcal{P}^\times .

PROOF. Omitted. □

THEOREM 8.2. *The Poincaré bundle admits a translation-invariant metric $\log \|\cdot\| : \mathcal{P}^\times \rightarrow \mathbb{R}$. For $(\Omega, W, Z, \alpha) \in \mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C}$, we have*

$$(8.10) \quad \log \|(\Omega, W, Z, \alpha)\| = 4\pi \left(\text{Im}(\alpha) + \text{Im}(W)(\text{Im}\Omega)^{-1}\text{Im}(Z) \right).$$

PROOF. Omitted. □

The Poincaré bundle has a Hodge-theoretic interpretation as the moduli space for biextensions, which are mixed Hodge structures M with weights $-2, -1, 0$. We assume

$$(8.11) \quad W_{-2}M = \mathbb{Z}(1), \quad gr_{-1}^W M = H, \quad gr_0^W M = \mathbb{Z}(0),$$

where H is a rank g principally polarized Hodge structure of weight -1 . To see this, we remark that $\Lambda_{\mathbb{C}} = H_1(T, \mathbb{C})$ has a Hodge structure of weight -1 with $F^0\Lambda_{\mathbb{C}} := \ker(\Lambda_{\mathbb{C}} \rightarrow V)$ and $gr_F^{-1}\Lambda_{\mathbb{C}} = V$. For $0 \neq t \in T$ the relative homology $H_1(T, \{0, t\}, \mathbb{Z})$ yields an extension of Hodge structures

$$(8.12) \quad 0 \rightarrow H_1(T, \mathbb{Z}) \rightarrow H_1(T, \{t, 0\}) \rightarrow \mathbb{Z}(0) \rightarrow 0.$$

A point $\phi \in \widehat{T} = \widehat{Hom}_{\mathbb{C}}(V, \mathbb{C})/\widehat{\Lambda}$ yields a character $\exp(2\pi i\phi) : \Lambda \rightarrow \mathbb{C}^{\times}$ and hence a principal \mathbb{C}^{\times} -bundle L_{ϕ}^{\times} over T . The corresponding sequence of homology groups looks like

$$(8.13) \quad 0 \rightarrow \mathbb{Z}(1) \rightarrow H_1(L_{\phi}^{\times}, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z}) \rightarrow 0$$

It follows from exactness of the Hodge filtration functor that $F^0 H_1(L_{\phi}^{\times}, \mathbb{C}) = F^0 H_1(T, \mathbb{C})$. Thus

$$(8.14) \quad L_{\phi}^{\times} \cong gr_F^{-1} H_1(L_{\phi}^{\times}, \mathbb{C}) / H_1(L_{\phi}^{\times}, \mathbb{Z}) = \text{Ext}_{MHS}^1(\mathbb{Z}(0), H_1(L_{\phi}^{\times}, \mathbb{Z})).$$

The projection $L_{\phi}^{\times} \rightarrow T$ yields a diagram of Hodge structures coming from $\ell \in L_{\phi}^{\times}$ lying over $t \in T$

$$(8.15) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H_1(L_{\phi}^{\times}, \mathbb{Z}) & \longrightarrow & H_1(T, \mathbb{Z}) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & M_{\ell} & \longrightarrow & M_t \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathbb{Z}(0) & \longlongequal{\quad} & \mathbb{Z}(0) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

The biextension corresponding to $\ell \in L_{\phi}^{\times} = \mathcal{P}|_{T \times \{\phi\}}$ is then M_{ℓ} with $W_{-1}M_{\ell} = H_1(L_{\phi}^{\times}, \mathbb{Z})$ and $M_{\ell}/W_{-2} = M_t$.

Let $\mathfrak{A}, \mathfrak{B}$ be divisors of degree 0 on a smooth curve C . Assume the supports $|\mathfrak{A}|$ and $|\mathfrak{B}|$ are disjoint. We associate to $\mathfrak{A}, \mathfrak{B}$ a biextension Hodge structure $H^1(C - \mathfrak{A}, \mathfrak{B}; \mathbb{Z})$. (Notation like $H^1(C - \mathfrak{A}, \mathfrak{B}; \mathbb{Z})$ or $H^1(C, \mathfrak{B}; \mathbb{Z})$ is of course abusive. These

are subquotients of $H^1(C - |\mathfrak{A}|, |\mathfrak{B}|; \mathbb{Z})$ and $H^1(C, |\mathfrak{B}|; \mathbb{Z})$ which fit into a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H^1(C, \mathfrak{B}; \mathbb{Z}(1)) & \longrightarrow & H^1(C, \mathbb{Z}(1)) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 (8.16) & 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H^1(C - \mathfrak{A}, \mathfrak{B}; \mathbb{Z}(1)) & \longrightarrow & H^1(C - \mathfrak{A}, \mathbb{Z}(1)) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathbb{Z}(0) & = & \mathbb{Z}(0) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Diagrams (8.15) and (8.16) are related as follows. Take $T = J(C)$ and let $t \in T$ be the image of a 0-cycle \mathfrak{A} on C of degree 0. The bundle L_ϕ^\times is a group which is an extension of T by \mathbb{C}^\times . We view C as embedded in $T = J(C)$. There exists a 0-cycle \mathfrak{B} on C of degree 0 such that $L_\phi^\times|_C \cong \mathcal{O}_C(\mathfrak{B}) - 0$ -section. Then $L_\phi^\times|_{C-|\mathfrak{B}|} \cong \mathbb{C}^\times \times (C - |\mathfrak{B}|)$, and this trivialization is canonical upto $c \in \mathbb{C}^\times$. Fix such a trivialization $\mu : C - |\mathfrak{B}| \rightarrow L_\phi^\times$. Write $\mathfrak{A} = \sum n_i a_i$, and let $\ell = \sum \mu(a_i)^{n_i}$. Then $\ell \in L_\phi^\times$ is well-defined independent of the choice of μ , and diagram (8.15) for $\ell \in L_\phi^\times$ coincides with (8.16) for \mathfrak{A} and \mathfrak{B} .

Note that L_ϕ^\times is an abelian Lie group. As such, it has a unique maximal compact subgroup K , and

$$(8.17) \quad L_\phi^\times / K \cong \mathbb{C}^\times / S^1 \cong \mathbb{R}.$$

Let $\rho_\phi : L_\phi^\times \rightarrow \mathbb{R}$ be the induced map.

PROPOSITION 8.3. *With notation as above, the following quantities are equal*

- (i) $\langle \mathfrak{A}, \mathfrak{B} \rangle$.
- (ii) $\log \|H^1(C - \mathfrak{A}, \mathfrak{B}; \mathbb{Z}(1))\|$.
- (iii) $\rho_\phi(\ell)$.

PROOF. Omitted. □

9. The main theorem; nilpotent orbit and passage to the limit

Write $U^* = (\Delta^*)^E \times \Delta^{3g-3-E+N}$, where N parameters are added to accommodate markings associated to external momenta. Consider our family $\mathcal{C}^* \xrightarrow{\pi} U^*$. Let $\tilde{U} = \mathbb{H}^E \times \Delta^{3g-3-E+N} \rightarrow U^*$ be the universal cover with coordinate map $t_e = \exp(2\pi i z_e)$. We fix a basepoint $t_0 \in U^*$. We choose a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ in $H_1(C_{t_0}, \mathbb{Z}) = A \oplus B$ with the vanishing cycles $a_e \in A$. We fix divisors $\delta = \delta_t, \mu = \mu_t$ which have degree 0 on each C_t and are disjoint. We choose 1-chains $\sigma_{\delta,t}$ and $\sigma_{\mu,t}$ on C_t which have δ_t and μ_t as boundary. We fix a basis $\{\omega_{i,t}\}$ of the holomorphic 1-forms on C_t such that $\int_{a_i} \omega_j = \delta_{ij}$. The classical period matrix is then $(\int_{b_i} \omega_{j,t})$. Finally, we choose 1-forms of the third kind $\omega_{\delta,t}, \omega_{\mu,t}$ with log poles on δ_t and μ_t respectively.

We define the period map $\tilde{\phi} : \tilde{U} \rightarrow \mathbb{H}_g \times \mathbb{C}^g \times \mathbb{C}^g \times \mathbb{C}$

$$(9.1) \quad \tilde{\phi}(z) = \left(\left(\int_{b_i} \omega_{j,t} \right)_{i,j}, \left(\int_{\sigma_{\delta,t}} \omega_{j,t} \right)_j, \left(\int_{\sigma_{\mu,t}} \omega_{j,t} \right)_j, \int_{\sigma_{\delta,t}} \omega_{\mu,t} \right)$$

We check that this is the correct map as follows. It suffices to work pointwise, so we drop the t . We identify $J(C) = \text{Ext}_{MHS}^1(H^1(C, \mathbb{Z}), \mathbb{Z}(0))$ so e.g. μ defines an extension

$$(9.2) \quad 0 \rightarrow \mathbb{Z}(0) \rightarrow H_\mu \rightarrow H^1(C, \mathbb{Z}) \rightarrow 0.$$

Dually, we can think of δ as defining an extension of $\mathbb{Z}(0)$ by $H^1(C, \mathbb{Z}(1))$. We want to think of δ as defining an extension of $\mathbb{Z}(0)$ by $H_\mu(1)$ which ‘‘lifts’’ this extension. The diagram looks like

$$(9.3) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{Z}(1) & = & \mathbb{Z}(1) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & (H_\delta)^\vee(1) & \rightarrow & M_{\mu,\delta} & \rightarrow & \mathbb{Z}(0) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & H^1(C, \mathbb{Z}(1)) & \rightarrow & H_\mu & \rightarrow & \mathbb{Z}(0) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

The exact sequence of ext groups is

$$(9.4) \quad 0 \rightarrow \text{Ext}_{MHS}^1(\mathbb{Z}(0), \mathbb{Z}(1)) \rightarrow \text{Ext}_{MHS}^1(\mathbb{Z}(0), H_\delta^\vee(1)) \rightarrow \text{Ext}_{MHS}^1(\mathbb{Z}(0), H^1(C, \mathbb{Z}(1))) \rightarrow 0$$

More concretely, we can think of \int_{σ_μ} as a functional on $\Gamma(C, \Omega^1) = F^1 H^1(C, \mathbb{C})$. Let $\Gamma(C, \Omega^1(\log \delta)) \supset \Gamma(C, \Omega^1)$ be spanned by the ω_j together with the form of the third kind ω_δ . We can define $H_1(C - \delta, \mathbb{Z})$ as a quotient of $H_1(C - \text{Supp}(\delta), \mathbb{Z})$ in such a way that we get an exact sequence

$$(9.5) \quad 0 \rightarrow \mathbb{C}/\mathbb{Z} \rightarrow \Gamma(C, \Omega^1(\log \delta))^\vee / H_1(C - \delta, \mathbb{Z}) \rightarrow \Gamma(C, \Omega^1)^\vee / H_1(C, \mathbb{Z}) \rightarrow 0$$

The exact sequences (9.4) and (9.5) coincide. The description of $\tilde{\psi}$ in (9.1) follows from this; lifting \int_{σ_μ} to a functional on $\Gamma(C, \Omega^1(\log \delta))$. Indeed, $(\int_{\sigma_{\delta,t}} \omega_{j,t})_j$ and $(\int_{\sigma_{\mu,t}} \omega_{j,t})_j$ yield the points in $J(C)$ associated to δ_t and μ_t respectively. The term $\Gamma(C, \Omega^1(\log \delta))^\vee / H_1(C - \delta, \mathbb{Z})$ in (9.5) shows that the class of δ in the generalized jacobian extension corresponding to μ is obtained by taking $\int_{\sigma_{\delta,t}} \omega_{\mu,t}$ as entry in \mathbb{C} .

The idea will be to use the nilpotent orbit theorem to understand the limiting behavior of (9.1). This theorem in the case of variations of polarized pure Hodge structures was proven by W. Schmid, [21]. We need the more general case of a

variation of mixed Hodge structures, [18],[10],[11],[12]. In our case, we have the diagram

$$(9.6) \quad \begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{\psi}} & \mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C} \\ \downarrow & & \downarrow \\ U^* & \longrightarrow & \tilde{G}(\mathbb{Z}) \backslash \left(\mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C} \right) \end{array}$$

The action of the fundamental group \mathbb{Z}^E of U^* is unipotent, and we write \mathcal{N}_e for the logarithm of the generator $1_e \in \mathbb{Z}^E$. The expression $\tilde{\psi}(z) := \exp(-\sum_E z_e \mathcal{N}_e) \tilde{\phi}(z)$ takes values in a compact dual $\tilde{\mathcal{M}}$ which is essentially a flag variety parametrizing filtrations $F^* \mathbb{C}^{g+2}$ satisfying the conditions to be the Hodge filtration on a biextension of genus g . It is invariant under $z_e \mapsto z_e + 1$ and so descends to a map $\psi : U^* \rightarrow \mathcal{M}$.

THEOREM 9.1. *[Nilpotent orbit theorem] (i) The limit*

$$F_\infty := \lim_{s \rightarrow 0} \psi(s) \in \tilde{\mathcal{M}}$$

exists.

(ii) We have

$$\exp\left(\sum_E z_e \mathcal{N}_e\right) F_\infty \in \mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C}$$

whenever $\text{Im}(z_e) \gg 0, \forall e$.

(iii) For an invariant metric d on $\mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C}$ there exist constants b, K such that for $\min_e(\text{Im}(z_e)) \gg 0$ we have

$$d(\tilde{\phi}(z), \exp\left(\sum_E z_e \mathcal{N}_e\right) F_\infty) \leq K (\min_e(\text{Im}(z_e)))^b \exp(-2\pi \min_e(\text{Im}(z_e))).$$

We want to understand how the logarithm of monodromy maps N_e act on the entries in (9.1). We have using (6.10) and $\int_{a_i} \omega_j = \delta_{ij}$

$$(9.7) \quad N_e \left(\int_{b_i} \omega_{j,t} \right) = \langle b_i, a_e \rangle \int_{a_e} \omega_{j,t} = c_{e,i} c_{e,j}.$$

By remark 6.1 we conclude

$$(9.8) \quad N_e \left(\int_{b_i} \omega_{j,t} \right)_{i,j} = M_e.$$

Said another way, if we view $b_i, b_j \in H_1(G)$, we have

$$(9.9) \quad N_e \left(\int_{b_i} \omega_{j,t} \right)_{i,j} = (\dots, b_i M_e b_j^t \dots)$$

To define the Picard-Lefschetz transformation on the path $\sigma_\delta \in H_1(C, \delta)$ we define vanishing cycles $a_e^\delta \in H_1(C - |\delta|)$. With this refinement, the P.-L. formula

$$(9.10) \quad N_e(\sigma_\delta) = \langle \sigma_\delta, a_e^\delta \rangle a_e$$

holds. Here N_e is viewed as an endomorphism of $H_1(C, \delta)$. Thus

$$(9.11) \quad N_e \left(\int_{\sigma_\delta, t} \omega_{j,t} \right)_j = (\langle \sigma_\delta, a_e^\delta \rangle c_{e,j})_j$$

To understand this, notice that the specialization identification $sp : B \cong H_1(G) \subset \mathbb{R}^E$ maps $b_j \mapsto \sum_e \langle b_j, a_e \rangle e = \sum_e c_{e,j} e$. Note that $\langle b_j, a_e \rangle$ counts how many times (with orientation) the closed chain $sp(b_j)$ passes through the singular point $pt_e \in C_0$. Similarly we can think of $sp(\sigma_\delta)$ as a 1-chain on C_0 . It is not closed, but it bounds the 0-chain corresponding to δ . Thus, it corresponds to $\sum_e \langle \sigma_\delta, a_e^\delta \rangle e \in \mathbb{R}^E$. In this way we can embed $H_1(C_s, \delta_s)/A \hookrightarrow \mathbb{R}^E$ with the path σ_δ mapping as indicated. Formula (9.11) then can be interpreted as the pairing applied to σ_δ and b_j .

Finally we have

$$(9.12) \quad N_e \left(\int_{\sigma_{\delta,t}} \omega_{\mu,t} \right) = \langle \sigma_{\delta,t}, a_e^\delta \rangle \int_{a_e^\mu} \omega_\mu$$

Notice that $\omega_\mu \in H^1(C_t - \mu_t) \cong H_1(C_t, \mu_t)$. Modifying by a linear combination of the ω_j we can assume under that identification that ω_μ corresponds to σ_μ . Thus, we have

$$(9.13) \quad N_e \left(\int_{\sigma_{\delta,t}} \omega_{\mu,t} \right) = \langle \sigma_{\delta,t}, a_e^\delta \rangle \langle \sigma_{\mu,t}, a_e^\mu \rangle$$

We can summarize our computations as follows. Consider the relative homology $H_1(C_t, \delta_t, \mu_t)$ which contains classes for the two paths $\sigma_{\delta,t}$ and $\sigma_{\mu,t}$. Let $A_{\sigma,\mu} \subset H_1(C_t, \delta_t, \mu_t)$ be the kernel of the specialization map

$$(9.14) \quad sp : H_1(C_t, \delta_t, \mu_t)/A_{\sigma,\mu} \rightarrow H_1(C_0, \delta_0, \mu_0) \subset \mathbb{R}^E.$$

We can now consider the composition

$$(9.15) \quad H_1(C_t, \delta_t, \mu_t)/A_{\sigma,\mu} \xrightarrow{sp} H_1(C_0, \delta_0, \mu_0) \subset \mathbb{R}^E \\ \xrightarrow{(\dots, y_e, \dots)} \mathbb{R}^E \rightarrow (H_1(C_t, \delta_t, \mu_t)/A_{\sigma,\mu})^\vee$$

Replacing N_e by $\sum y_e N_e$, the formulas (9.8), (9.11), and (9.13) give the pairings $\langle b_i, b_j \rangle, \langle \sigma_\delta, b_i \rangle, \langle \sigma_\delta, \sigma_\mu \rangle$ for the pairing defined by (9.15).

In our case, the nilpotent orbit theorem 9.1 says that the family (9.1) is well-approximated by

$$(9.16) \quad \left(\Omega_\infty + \sum_E z_e M_e, W_\infty + \sum_E z_e (\dots, \langle \sigma_\delta, a_e^\delta \rangle c_{ej}, \dots), \right. \\ \left. Z_\infty + \sum_E z_e (\dots, \langle \sigma_\mu, a_e^\mu \rangle c_{ej}, \dots), \alpha_\infty + \sum_E z_e \langle \sigma_{\delta,t}, a_e^\delta \rangle \langle \sigma_{\mu,t}, a_e^\mu \rangle \right).$$

We write $z_e = x_e + iy_e$ and $t_e = \exp(2\pi i z_e)$ so $y_e = -(\log |t_e|)/2\pi$. We imagine

$$(9.17) \quad y_e = Y_e / \alpha'$$

where $Y_e > 0$ is constant and the *string tension* $\alpha' \rightarrow 0$. Substituting from (8.10) yields the following expression for the height

$$(9.18) \quad \frac{1}{\alpha'} \left[(\alpha' \text{Im} W_0 + \sum_e Y_e (\dots, \langle \sigma_\delta, a_e^\delta \rangle c_{ej}, \dots)) \cdot \right. \\ \left. (\alpha' \text{Im} \Omega_0 + \sum_e Y_e M_e)^{-1} (\alpha' \text{Im} Z_0 + \sum_e Y_e (\dots, \langle \sigma_\mu, a_e^\mu \rangle c_{ej}, \dots)) \right. \\ \left. - \alpha' \text{Im} \alpha_0 - \sum_e Y_e \langle \sigma_\delta, a_e^\delta \rangle \langle \sigma_\mu, a_e^\mu \rangle \right]$$

Now, using the determinant formula for the second Symanzik polynomial (remark 2.3) we conclude

THEOREM 9.2. *Let $\tilde{\phi} : \tilde{U} \rightarrow \mathcal{P}$ be as in (9.1). Then*

$$(9.19) \quad \lim_{\alpha' \rightarrow 0} \alpha' \log \|\tilde{\phi}(z)\| = \phi_G(p, p', \{Y_e\}) / \psi_G(\{Y_e\}).$$

Here $z_e = X_e + i\frac{Y_e}{\alpha'}$, ψ and ϕ are the first and second Symanzik polynomials, and p, p' denote external momenta (see (7.1)).

PROOF. The proof is simply a question of pulling together the linear algebra. We split the exact sequence of homology of the graph to write $\mathbb{R}^E = H_1(G, \mathbb{R}) \oplus \mathbb{R}^{V,0}$. With respect to this basis, the symmetric matrix corresponding to the quadratic form $\sum y_e e^{V,2}$ is

$$(9.20) \quad \begin{pmatrix} M_y & B_y^{\delta,t} \\ B_y^\mu & p^\mu \Gamma_y p^{\delta,t} \end{pmatrix}$$

Here the pairing on the b_i is $M_y = \sum y_e M_e$ with M_e as in remark 6.1. One external momentum maps to $\sum_e \langle \sigma_\delta, a_e^\delta \rangle e$. Pairing this with the b_j yields the column vector $B_y^{\delta,t} = (\dots, \sum_e y_e \langle \sigma_\delta, a_e^\delta \rangle c_{e,j}, \dots)^t$. The other external momentum pairs with the b_j to yield a row vector $B_y^\mu = (\dots, \sum_e y_e \langle \sigma_\mu, a_e^\delta \rangle c_{e,j}, \dots)$. Finally, the entry in the bottom right corner is $\sum_e y_e \langle \sigma_\delta, a_e^\delta \rangle \langle \sigma_\mu, a_e^\mu \rangle$. The theorem follows from remark 2.3. \square

To summarize, our recipe for computing the Feynman amplitude (5.22) is the following. Given the external momentum $p := \sum_{v \in V(G)} r_v v$ with $r_v \in \mathbb{R}^D$ and $\sum_v r_v = 0$, choose an analytic neighborhood S of C_0 in a suitable local moduli space of marked curves and two relative \mathbb{R}^D -divisors $\mathfrak{A}_i := \sum_v r_v \sigma_{v,i}$, $i = 1, 2$ supported on the markings. We assume the markings $\sigma_{v,i}$ are all disjoint. Further, we require that the intersection numbers with components of the rational curve C_0 coincide, and that $\sigma_{v,1} \cdot C_{0,v'} = \sigma_{v,2} \cdot C_{0,v'} = \delta_{v,v'}$. (Kronecker delta.) (The issue here is that the height $\langle \mathfrak{A}_1, \mathfrak{A}_2 \rangle$ is only defined if the supports of $\mathfrak{A}_1, \mathfrak{A}_2$ are disjoint.) Recall from the beginning of this section we have

$$\begin{aligned} \tilde{U} &= \mathbb{H}^{E(G)} \times \Delta^{3g-3-E(G)+N} \rightarrow U^* = \Delta^{*,E(G)} \times \Delta^{3g-3-E(G)+N}; \\ X_e + iY_e/\alpha' &= z_e = \frac{\log(t_e)}{2\pi i}. \end{aligned}$$

Then (compare (7.8). The factor 2 occurs because of (7.1).)

$$(9.21) \quad \lim_{\alpha' \rightarrow 0} \alpha' \mathbf{S}[C_{t_e}, \mathfrak{A}_1, \mathfrak{A}_2] = 2\phi_G(p, \{Y_e\}) / \psi_G(\{Y_e\}).$$

The Feynman amplitude arises by plugging this limit into the exponential in (5.22) and integrating over the the space of nilpotent orbits which is parametrized by $Y_e \geq 0$.

Let me suggest two directions for further research.

1. The rational curve C_0 with dual graph G corresponds to a maximally degenerate boundary point on the moduli space of (marked) curves of genus g . Are there physically meaningful quantities arising when we specialize the height action (7.8) to less degenerate boundary points. The classical interpretation of Feynman amplitudes as coefficients of Green's functions arising from path integrals is really more of a metaphor than a mathematically rigorous calculus. Perhaps studying partial

degenerations might throw more light.

2. The height $\langle \mathfrak{A}_1, \mathfrak{A}_2 \rangle$ is just one example of an \mathbb{R} -valued function associated to a variation of mixed Hodge structures. (In this case, we consider variations of biextensions.) Are there other interesting quantities arising from integration over the space of nilpotent orbits when a variation of Hodge structures degenerates?

10. Cutkosky Rules

The final topic in the notes is a report on some aspects of joint work with Dirk Kreimer on Cutkosky rules. The amplitude $A_G = A_G(q)$ is a multi-valued function of external momenta q , as one sees e.g. from (5.19). (For simplicity the masses are kept fixed.) The following beautiful physical interpretation of the *variation* $\text{Var}(A_G)$ as q winds around the threshold locus was given in [5], [7]:

$$(10.1) \quad \text{Var}(A_G) = (-2\pi i)^r \int \frac{\delta^+(q_1^2 - m_1^2) \cdots \delta^+(q_r^2 - m_r^2) d^{Dg}x}{(q_{r+1}^2 - m_{r+1}^2) \cdots (q_n^2 - m_n^2)}.$$

Here A_G is the amplitude associated to a graph G with n edges and g loops. The $q_i^2 - m_i^2$ are propagators, and $\delta^+(q_i^2 - m_i^2)$ means to take the residue of the original form $\omega := \frac{d^{Dg}x}{\prod_1^n (q_j^2 - m_j^2)}$ along the divisor $q_i^2 - m_i^2 = 0$ and then to integrate over the part of the real locus of that intersection of divisors where the energy $q_i^{(0)} > 0$. External momenta are placed near the threshold divisor associated to a pinch point of the intersection $\bigcap_1^r \{q_i^2 - m_i^2 = 0\}$.

The basic monodromy calculation was related to the classical Picard-Lefschetz formula in mathematics by Pham, [20]. Pham gives a complete description of the local topology of the union of propagator quadrics as the external momentum approaches a general threshold point. However, the physical formula (10.1) involves the real structure of the quadrics, and more work is necessary to link the topology to the Cutkosky formula. In the following I sketch the arguments involved in proving Cutkosky's formula for the case of *physical singularities* (definition 12.1).

The quadrics have the form $q^2 - m^2 = (e^\vee - c_e)^2 - m^2$ where $e \in E = E(G)$ is an edge, $c_e \in \mathbb{C}^D$ is a D -tuple of constants, and e^\vee is a D -tuple of functions on \mathbb{C}^{Dg} :

$$(10.2) \quad \mathbb{C}^{Dg} = H_1(G, \mathbb{C}^D) \hookrightarrow (\mathbb{C}^D)^E \xrightarrow{e^\vee - c_e} \mathbb{C}^D \xrightarrow{(\cdot)^2 - m^2} \mathbb{C}.$$

Write Q_e for the quadric in \mathbb{C}^{Dg} defined by (10.2). Given edges e_1, \dots, e_r , it is straightforward to check that the rD linear functions $(e_1^\vee, \dots, e_r^\vee) : H_1(G, \mathbb{C}^D) \rightarrow \mathbb{C}^{rD}$ form part of a system of coordinates on \mathbb{C}^{Dg} if and only if this map is surjective, and that this condition is independent of D . Taking $D = 1$, one sees that the linear functions do not form part of a system of coordinates if and only if the graph obtained from G by cutting e_1, \dots, e_r is not connected. On the other hand, if the linear functions form part of a system of coordinates and if none of the masses $m_i = 0$, it is clear that the intersection of the propagator quadrics cannot have a pinch point (irrespective of the c_{e_i}). We conclude

PROPOSITION 10.1. *With notation as above, assuming all masses $m_i \neq 0$, a necessary condition for the intersection of the quadrics*

$$Q_{e_1} \cap \cdots \cap Q_{e_r}$$

to have a pinch point for suitable values of the c_{e_i} is that the graph obtained from G by cutting e_1, \dots, e_r be disconnected.

For simplicity, in what follows I consider only the case $r = n$, i.e. the case when all the edges are placed on shell. The resulting cut graph is just the collection $V(G)$ of vertices and is certainly disconnected (assuming G has at least 2 vertices).

EXAMPLE 10.2 (Banana graphs). *Suppose G has 2 vertices connected by n edges e_1, \dots, e_n . The only possibility to disconnect G is to cut all the edges. Since cutting e_1, \dots, e_{n-1} leaves G connected, it follows as above that $e_i^\vee : \mathbb{C}^{D(n-1)} \rightarrow \mathbb{C}^D$, $1 \leq i \leq n-1$ are independent coordinates, and upto orientation $e_n^\vee = \sum_1^{n-1} e_i^\vee - a$ for some $a \in \mathbb{C}^D$. We can replace the e_i^\vee by $e_i^\vee - c_i$ and arrange that the jacobian matrix for the n quadrics looks like*

$$(10.3) \quad 2 \begin{pmatrix} e_1^\vee & 0 & 0 & \dots \\ 0 & e_2^\vee & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & e_{n-1}^\vee \\ \sum_1^{n-1} e_i^\vee + a & \sum_1^{n-1} e_i^\vee + a & \sum_1^{n-1} e_i^\vee + a & \sum_1^{n-1} e_i^\vee + a \end{pmatrix}$$

(Note each entry is a D -vector.) We want this matrix to have rank $n-1$ at some point where the n propagator quadrics vanish. A small exercise in linear algebra (for details, see [1], section 12) yields

$$(10.4) \quad |a| = \sum_1^n \mu(i) m_i; \quad \mu(i) = \pm 1.$$

(Here $|a| := \sqrt{a^2}$.)

11. Cutkosky rules: Pham's vanishing cycles

In this section I summarize briefly the calculus of vanishing cycles; first the classical theory and then the theory of F. Pham. Classically, $f : \mathcal{X} \rightarrow \Delta$ is a proper family of varieties over a disk $\Delta \subset \mathbb{C}$. We assume \mathcal{X} is non-singular, and f is smooth away from $0 \in \Delta$. Further, $\mathcal{X}_0 = f^{-1}(0)$ has a single ordinary double point s_0 and is non-singular elsewhere. With these hypotheses, we can find analytic coordinates x_1, \dots, x_n near x_0 and a coordinate t on Δ such that locally the family is defined by

$$(11.1) \quad x_1^2 + \dots + x_n^2 = t.$$

For $0 < \varepsilon \ll 1$, the homology of a ball B around s_0 intersected with the fibre \mathcal{X}_ε is generated by the class of the real sphere $S_\varepsilon : \sum x_i^2 = \varepsilon$, $x_i \in \mathbb{R}$. The monodromy action on the smooth fibre $H^*(\mathcal{X}_\varepsilon, \mathbb{Q})$ is given by the Picard-Lefschetz formula

$$(11.2) \quad c \mapsto c + (-1)^{n(n+1)/2} \langle ex(c), S_\varepsilon \rangle i_* S_\varepsilon.$$

Here

$$(11.3) \quad ex : H_*(\mathcal{X}_\varepsilon) \rightarrow H_*(B \cap \mathcal{X}_\varepsilon, \partial(B \cap \mathcal{X}_\varepsilon))$$

denotes excision, and $i_* : H_*(B \cap \mathcal{X}_\varepsilon) \rightarrow H_*(\mathcal{X}_\varepsilon)$ is the pushforward. The pairing $\langle ex(c), S_\varepsilon \rangle$ refers to the natural pairing

$$(11.4) \quad H_{n-1}(B \cap \mathcal{X}_\varepsilon, \partial(B \cap \mathcal{X}_\varepsilon)) \otimes H_{n-1}(B \cap \mathcal{X}_\varepsilon) \rightarrow \mathbb{Q}.$$

To explain Pham's vanishing cycles, we change notation. Let $f : M \rightarrow P$ be a smooth, proper map of algebraic varieties. Let $S = \bigcup_{i=1}^n S_i \subset M$ be a normal crossings divisor. Let $s_0 \in \bigcap_{i=1}^n S_i \subset S \subset M$, and let $p_0 = f(s_0)$. We will work

locally near s_0 on M , so we may fix analytic coordinates t_1, \dots, t_k around p_0 on P and x_1, \dots, x_ℓ around s_0 on M in such a way that $t_1, \dots, t_k, x_1, \dots, x_\ell$ form a full set of coordinates on M near s_0 . We assume that locally

$$(11.5) \quad S_i : x_i = 0; \quad i = 1, \dots, n-1,$$

$$(11.6) \quad S_n : t_1 - (x_1 + \dots + x_{n-1} + x_n^2 + \dots + x_\ell^2) = 0.$$

For such a configuration of divisors, the point s_0 (origin) is called a *pinch point*. Notice that on the smallest stratum $\bigcap_1^n S_i$ of S , viewed as a variety fibred over P , we have a classical Picard-Lefschetz vanishing cycle local equation, which is a family of spheres $x_n^2 + \dots + x_\ell^2 = t_1$ degenerating as $t \rightarrow 0$. For $t_1 = \varepsilon > 0$, the *vanishing sphere* $\text{vsphere}_\varepsilon$ is the real sphere

$$(11.7) \quad \text{vsphere}_\varepsilon : \varepsilon = x_n^2 + \dots + x_\ell^2; \quad x_i \in \mathbb{R}, \quad n \leq i \leq \ell.$$

Classically, $\text{vsphere}_\varepsilon$ is referred to as the vanishing cycle, but we follow Pham here and distinguish 3 different topological chains, the vanishing sphere, the vanishing cell, and the vanishing cycle.

DEFINITION 11.1. *With notation as above, the vanishing cell,*

$$(11.8) \quad \text{vcell}_\varepsilon : x_i \geq 0, \quad 1 \leq i \leq n-1; \quad \varepsilon - (x_n^2 + \dots + x_\ell^2) \geq 0.$$

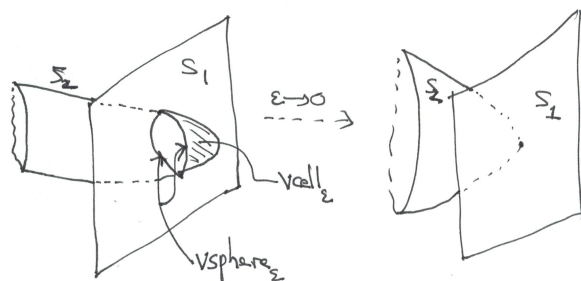
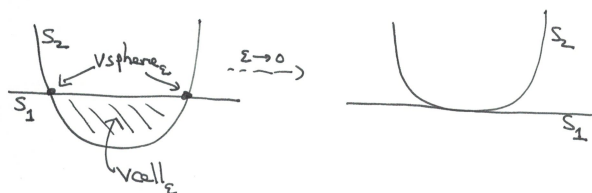
The vanishing cycle $\text{vcycle}_\varepsilon$ is the iterated tube

$$\tau_{1,\varepsilon_1}^* \tau_{2,\varepsilon_2}^* \cdots \tau_{n-1,\varepsilon_{n-1}}^* (\text{vsphere}_\varepsilon).$$

Here $\varepsilon \gg \varepsilon_{n-1} \gg \dots \gg \varepsilon_1 > 0$. The notation τ_{i,ε_i}^ refers to pulling back to the circle bundle of radius ε_i inside the (metrized) normal bundle for $S_1 \cap \dots \cap S_i \subset S_1 \cap \dots \cap S_{i-1}$. This circle bundle is viewed as embedded in $S_1 \cap \dots \cap S_{i-1}$ and not meeting $S_1 \cap \dots \cap S_i$. The inequalities $\varepsilon_i \ll \varepsilon_{i+1}$ insure that $\text{vcycle}_\varepsilon$ is a closed chain on $M - \bigcup_{i=1}^n S_i$.*

Here are two pictures in the case $n = 2$.

Pham's pictures for $n=2$



$vcycle_\epsilon = S^1 \times S^1$ -bundle over $vsphere_\epsilon$
 contained in $B - (S_1 \cup S_2)$.

The local structure of homology around s_0 is computed by Pham to be

THEOREM 11.2. *Let B be a small ball around s_0 in M . Let $p \in P$ be near p_0 and assume $t_1(p) = \epsilon > 0$. Write $d = \dim_{\mathbb{C}} M_p$.*

(i) *We have for reduced homology*

$$\tilde{H}_j \left(\bigcap_1^n S_{i,p} \cap B, \mathbb{Z} \right) = (0), \quad j \neq d - n := \dim_{\mathbb{C}} \left(\bigcap_1^n S_{i,p} \right).$$

$$\tilde{H}_{d-n} \left(\bigcap_1^n S_{i,p} \cap B, \mathbb{Z} \right) = \mathbb{Z} \cdot vsphere_\epsilon.$$

(ii) *In relative homology*

$$H_j(B_p, S_p \cap B) = (0), \quad j \neq d; \quad H_d(B_p, S_p \cap B) = \mathbb{Z} \cdot vcell_\epsilon.$$

(iii) *For the homology of the complement*

$$H_j(B_p - S_p \cap B) = (0), \quad j \neq d; \quad H_d(B_p - S_p \cap B) = \mathbb{Z} \cdot vcycle_\epsilon.$$

Local Poincaré duality yields a pairing

$$(11.9) \quad \langle \cdot, \cdot \rangle : H_d(B_p - S_p \cap B) \otimes H_d(B_p, (S_p \cap B) \cup \partial B_p) \rightarrow H_{2d}(B_p, \partial B_p) \cong \mathbb{Z}.$$

Let γ be a simple closed path on P based at p , supported in a neighborhood of p_0 , and looping once around the divisor $t_1 = 0$. Using Thom's isotopy lemma [15] and assuming that f is a submersion on strata except at the point s_0 , Pham shows that the variation of monodromy

$$(11.10) \quad var := \gamma_* - Id : H_*(M_p - S_p) \rightarrow H_*(M_p - S_p)$$

factors through excision and a local variation map var_{loc}

$$(11.11) \quad H_d(M_p - S_p) \xrightarrow{excision} H_d(B_p - S_p \cap B_p, \partial B_p) \xrightarrow{var_{loc}} H_d(B_p - S_p) \xrightarrow{i_*} H_d(M_p - S_p).$$

(The variation map is zero in homological degrees $\neq d$.)

The Picard-Lefschetz theorem in this setup is

THEOREM 11.3. *We have*

$$(11.12) \quad var_{loc}(excision(c)) = excision(c) + (-1)^{(n+1)(n+2)/2} \langle excision(c), vcell_\varepsilon \rangle vcycle_\varepsilon$$

PROOF. See [20]. □

EXAMPLE 11.4. *In this example we compute the Picard-Lefschetz transformation arising in physics. One has a set of quadrics $Q_i : f_i = 0$, $1 \leq i \leq n$ indexed by the edges of a graph. Our normal crossings divisor in this case is $\bigcup Q_i$. We consider the situation locally around a pinch point s_0 , and we write in Pham's coordinates, $Q_i : x_i = 0$, $1 \leq i \leq n-1$ and $Q_n : t_1 - (x_1 + \cdots + x_{n-1} + \sum_n x_i^2) = 0$. Notice that in these coordinates, the bad fibre over p_0 (where $t_1 = 0$) of $\bigcap_1^n Q_{i,p_0}$ has an isolated \mathbb{R} -point at the origin. We will see (proposition 12.4) in the case of physical singularities that this is also the case for the space-time coordinates. Hence it is plausible to assume that at least for physical singularities Pham's local coordinates are defined over \mathbb{R} . We are interested in computing the monodromy on $M - \bigcup Q_i$ of the chain given by taking space-time coordinates in \mathbb{R} . As it stands this makes no sense, because our quadrics are defined using the Minkowski metric so the Q_i meet the real locus. The standard ploy to avoid this problem is to replace the defining equations $f_j = 0$ by $f_j + ia_j = 0$ with $1 \gg a_j > 0$. Using theorem 11.3, we need to compute*

$$\langle excision(\mathbb{R}^{Dg}), vcell_\varepsilon \rangle,$$

where D is the dimension of space-time and g is the number of loops of the graph.

In Pham's local coordinates, the deformed quadrics are defined by

$$x_j = -ia_j, \quad 1 \leq j \leq n-1 \quad \text{and} \quad \varepsilon + ia_n - \left(\sum_1^{n-1} x_j + \sum_n^{Dg} x_j^2 \right) = 0.$$

We let x_j run over the path

$$x_j = (1 - \rho_j)ia_j + \rho_j(\varepsilon + ia_n)/(n-1), \quad 1 \leq j \leq n-1.$$

Here $0 \leq \rho_j \leq 1$. For points $x^0 = (x_1^0, \dots, x_{n-1}^0)$ on those paths, the quadratic equation becomes

$$\sum_n^{Dg} x_j^2 = \varepsilon + ia_n - \sum_1^{n-1} x_j^0 =: r(x^0)e^{i\theta(x^0)}.$$

The locus

$$(11.13) \quad \{(e^{i\theta(x^0)/2}u_n, \dots, e^{i\theta(x^0)/2}u_{Dg}) \mid \sum_{k=n}^{Dg} u_k^2 \leq r(x^0)\}$$

is a solid sphere, and the union of those solid spheres as the x_j run over those paths is the chain $vcell_\varepsilon$ for the deformed quadrics.

According to theorem 11.3, the multiplicity of the vanishing cycle in the variation of the cycle \mathbb{R}^{Dg} (or more precisely of $\mathbb{P}^{Dg}(\mathbb{R})$) around the given pinch point is the intersection of this real locus with $vcell_\varepsilon$.

Note that x_j is real only at the point $\rho_j = a_j/(a_j + a_n/(n-1)) \in (0, 1)$. At the point with these coordinates, the quadratic equation reads $\sum_n^{Dg} x_j^2 = R + ia_n$ for some $R \in \mathbb{R}$. since $a_n \neq 0$, the only real point on the corresponding solid sphere (11.13) lies at the origin. Thus, $vcell_\varepsilon \cap \mathbb{R}^{Dg}$ is precisely one point. The intersection looks locally like the intersection $\mathbb{R}^N \cap e^{i\theta}\mathbb{R}^N \subset \mathbb{C}^N$, so it is transverse, and

$$\langle \text{excision}(\mathbb{P}^{Dg}(\mathbb{R})), v\text{cycle} \rangle = \pm 1.$$

We do not try to compute the sign.

This example is important because it proves that there really is non-trivial monodromy associated to the physical situation. Note the computation is very dependent on taking the $a_j > 0$.

12. Cutkosky's Theorem

We want to prove (10.1) in the case of *physical singularities*. (Recall, we are also assuming for simplicity that we cut all edges so $r = n$ in (10.1).) To understand physical singularities, consider formula (5.20) for the amplitude A_G . Viewed as a function of external momenta, A_G will be non-singular unless the chain of integration $\mathbb{R}^{Dg} \times \sigma$ meets the polar locus $\sum_{e \in E} A_e = 0$. When it does meet, it may still happen that the chain can be deformed away from the polar locus so A_G has no singularity. The logical place to look for branchpoints of A_G is thus at external momenta where $\sigma \times \mathbb{R}^{Dg}$ contains singular points of the universal quadric

$$(12.1) \quad Q : \sum_{e \in E} A_e f_e = 0.$$

DEFINITION 12.1. A point $p = (c_1, \dots, c_n, p') \in Q(\mathbb{R}) \subset \mathbb{R}^n \times \mathbb{R}^{Dg}$ is a *physical singularity* if p is a singular point of Q and all $c_i > 0$. By extension, a point $p' \in \mathbb{R}^{Dg}$ is a *physical singularity* if there exists a physical singularity $(c, p') \in Q$.

LEMMA 12.2. Let $p = (c_1, \dots, c_n, p')$ be a physical singularity of the universal quadric Q . Then $p' \in \bigcap_1^n \{f_i = 0\}$ and p' is a singular point in that intersection of quadrics.

PROOF. Let $f = \sum A_i f_i$. By assumption, p is singular so $f_i(p') = \frac{\partial}{\partial A_i} f(p) = 0$. Also, vanishing of the partials of f with respect to coordinates on \mathbb{R}^{Dg} yields the

relation

$$(12.2) \quad \sum_1^n c_i \text{grad}_{p'}(f_i) = 0,$$

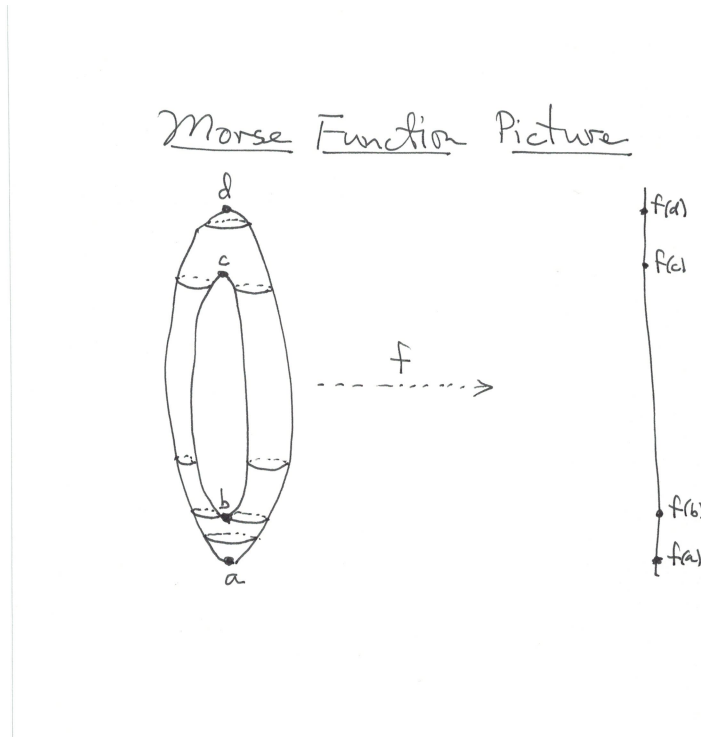
which implies that p' is singular on the intersection of quadrics. \square

DEFINITION 12.3. A physical singularity p will be called non-degenerate if two conditions hold:

- (i) The relation (12.2) is the only non-trivial relation among the gradients at p' .
- (ii) The Hessian matrix at the singular point for the intersection of the quadrics is non-degenerate.

Condition (i) in the above definition means we can choose a partial set of local coordinates x_1, \dots, x_{n-1} at the singular point p' in such a way that the intersection of the quadrics is cut out locally near p' by $x_1 = \dots = x_{n-1} = g = 0$ for some function g . Condition (ii) says that the matrix of second order partials of $g|_{\{x_1 = \dots = x_{n-1} = 0\}}$ is non-degenerate at p' .

In order to understand the \mathbb{R} -structure we borrow a standard picture for a Morse function f , [16].



The Hessian matrix of f at the points a, d is definite, while the Hessian matrix at b, c is indefinite. Note $f^{-1}f(a) = a$ and the fibre $f^{-1}(x)$ for x slightly above $f(a)$ is a sphere (a circle in the picture). This is the vanishing cycle which contracts to the point a as $x \rightarrow f(a)$. The picture at d is similar for y slightly below $f(d)$. On the other hand, $f^{-1}f(b)$ and $f^{-1}f(c)$ are figure-eights, and there are no vanishing cycles. In our case, if we know the Hessian for the function g at p' is either positive

or negative definite, then Pham's vanishing cycle will coincide with the real sphere given by Morse theory.

PROPOSITION 12.4. *Let $p = (c_1, \dots, c_n, p')$ be a non-degenerate physical singularity. Assume $m_i^2 > 0$ for all masses. Then the Hessian associated to $p' \in \bigcap_1^n \{f_i = 0\}$ is negative definite. In particular, Pham's vanishing sphere is real, and Cutkosky's theorem (10.1) holds for the variation of A_G .*

PROOF. Order the edges e_1, \dots, e_n such that $e_1^\vee, \dots, e_g^\vee$ give coordinates on $H_1(G, \mathbb{R}^D)$. (Note each $e_i^\vee = (e_i^{\vee,1}, \dots, e_i^{\vee,D})$ is a D -tuple of coordinate functions.) We change notation slightly and write

$$p = (c_1, \dots, c_n, e_1^\vee(p'), \dots, e_g^\vee(p')) \in \mathbb{R}^n \times \mathbb{R}^{Dg}.$$

The quadrics have the form $f_i = (e_i^\vee - s_i)^2 - m_i^2$ for some $s_i \in \mathbb{R}^D$. By a Euclidean change of variables, we can take $s_i = 0$ for $1 \leq i \leq g$. Because p is singular on the universal quadric Q , the sum $\sum c_i f_i$ is pure quadratic when expanded about p' , i.e. constant and linear terms vanish. Notice that the quadratic part of f_i is $(e_i^\vee)^2$ which is a Minkowski square. The quadratic form associated to the Hessian is then simply $\sum c_i (e_i^\vee)^2$. Of course, these are Minkowski squares, so it is not possible to say anything about the sign of the form on \mathbb{R}^{Dg} . However, what we want is the sign of the form restricted to the intersection of the tangent spaces at p' of the f_i . Among these are the tangent spaces at p' to $f_i = (e_i^\vee)^2 - m_i^2$ for $1 \leq i \leq g$. The tangent space at p' for such a f_i is $(e_i^\vee)^{-1}(e_i^\vee(p')^\perp) \subset \mathbb{R}^{Dg}$. (Here $e_i^\vee(p')^\perp \subset \mathbb{R}^D$. Note $(e_i^\vee(p'))^2 = m_i^2 > 0$, $1 \leq i \leq g$.) Since the Minkowski metric has sign $(1, -1, \dots, -1)$ on \mathbb{R}^D , we conclude that the sign of the metric on the intersection of these spaces is negative definite. In particular, $\sum_1^n c_i (e_i^\vee)^2$ is certainly negative semi-definite on the intersection of the tangent spaces. (Indeed, we can write $e_j^\vee = \sum_{i=1}^g \mu_j^i e_i^\vee$ with $\mu_j^i \in \mathbb{R}$. This decomposition is orthogonal, so $(e_j^\vee)^2 = \sum_{i=1}^g (\mu_j^i)^2 (e_i^\vee)^2$. Since the $(e_i^\vee)^2$ on the right are negative definite on the intersection of the tangent spaces, $(e_j^\vee)^2$ is negative definite as well.) The only way $\sum_1^n c_i (e_i^\vee)^2$ could fail to be negative definite would be if there was a non-zero tangent vector t such that $e_i^\vee(t) = 0$, $1 \leq i \leq n$. But this is not possible since the e_i^\vee , $1 \leq i \leq g$, give coordinates. \square

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