(1) Let $I \subset L$ be an ideal in a Lie algebra. Show every member of the derived and descending central series for $I$ is an ideal in $L$.

(2) Prove that the sum of two nilpotent ideals in a Lie algebra is a nilpotent ideal. Conclude that a Lie algebra has a maximal nilpotent ideal.

(3) Show that any irreducible representation of a solvable Lie algebra $L$ is one dimensional, and that $[L, L]$ acts trivially.

(4) Let $I \subset L$ be an ideal in a Lie algebra. Show that $L$ is semi-simple if and only if $I$ and $L/I$ are semisimple.

(5) Let $A$ be a $k$-algebra, where $k$ is a field. Define the notion of $k$-derivation $\delta : A \to A$. Show that the space $\text{Der}_k(A)$ is a Lie algebra. If $A$ is finite dimensional and $D$ is a derivation, show that the semi-simple and nilpotent parts $D_s, D_n$ for the Jordan decomposition are also derivations. Define the notion of inner derivation and show $\text{Inner}_k(A) \subset \text{Der}_k(A)$ is an ideal.

(6) Let $B : V \otimes_k V \to k$ be a bilinear form on a finite dimensional $k$-vector space $V$. Define the orthogonal group $O(B) \subset GL(V)$. Describe the tangent space to $O(B)$, $\mathfrak{o}(B) \subset \text{End}_k(V)$. Verify $\mathfrak{o}(B)$ is a Lie algebra, and write down explicit bases for $\mathfrak{o}(B)$ when $B$ is associated to one of the three matrices

$$
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix};
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & I_n \\
0 & I_n & 0
\end{pmatrix};
\begin{pmatrix}
0 & I_n \\
I_n & 0
\end{pmatrix}.
$$

The resulting Lie algebras are of types $C_n, B_n, D_n$ respectively for the standard classification.

(7) Let $x, y \in \text{End}_k(V)$. Prove that if $[x, y] = 0$, then $(x + y)_s = x_s + y_s$ and $(x + y)_n = x_n + y_n$. Give an example where this fails when $[x, y] \neq 0$. 