1. Let $V, W$ be vector spaces (over $\mathbb{R}$, as usual).

a. Define $V \otimes W$ and $\bigwedge^p V$.

b. If $V$ has basis $\{e_i\}_{i=1,...,m}$ and $W$ has basis $\{f_j\}_{j=1,...,n}$, show that $V \otimes W$ is spanned by the vectors $e_i \otimes f_j$. (In fact, these vectors form a basis, but you do not need to show that.)

c. Show $\bigwedge^m V$ has dimension $\leq 1$ and explicit a spanning element. (In fact, $\bigwedge^m V$ has dimension 1.)

d. Define a linear transformation

$$\left(\bigwedge^p V\right) \otimes \left(\bigwedge^{m-p} V\right) \to \bigwedge^m V.$$ 

2. Let $U \subset \mathbb{R}^k$ be open, and let $f : U \to \mathbb{R}^n$. Assume $f$ has continuous partial derivatives.

a. Define and state the main properties of the derivative $Df$.

b. Consider the function $f(x, y) = (x^2, xy, y^2)$. What is the best linear approximation to $f$ near the point $(1,1)$? Justify your answer citing results about $Df$.

3 a. State carefully the inverse function theorem and the implicit function theorem.

b. Apply the inverse function theorem to the function

$$f(r, \theta) = (r \cos \theta, r \sin \theta).$$

For what values of $r, \theta$ does the theorem fail? Indicate what happens at these points.

c. Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ and suppose the implicit function theorem applies to $f$ at a point $(x^0, y^0) \in \mathbb{R}^{k+n}$. Use the implicit function theorem to define a map $L : \mathbb{R}^k \to \mathbb{R}^{k+n}$ in such a way that $Df(x^0, y^0) \circ L = 0$ as a map $\mathbb{R}^k \to \mathbb{R}^n$. 

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4 a. Define partitions of unity and state the main existence theorem.

b. Let $S \subset \mathbb{R}^n$ be a subset, and let $f : S \to \mathbb{R}$. $f$ is said to be $C^r$ on $S$ if for each $s \in S$ there exists an open neighborhood $U_s$ of $s$ in $\mathbb{R}^n$ and a function $g_s$ which is defined and $C^r$ on $U_s$ such that $g_s = f$ on $U_s \cap S$. Show that if $S$ is closed in $\mathbb{R}^n$ then $f$ extends to a $C^r$ function on all of $\mathbb{R}^n$. Give an example where this is not true for $S$ open in $\mathbb{R}^n$.

5. Define parallelepipeds in $\mathbb{R}^n$ and state the main theorem about computing volumes of parallelepipeds.

Compute the volume of the parallelepiped spanned by $(1, 2, 3, 4), (1, 1, 1, 1)$ in $\mathbb{R}^4$. 