Practice Midterm - Math 274
April 29, 2008

1. Let V, W be vector spaces (over $\mathbb{R}$, as usual).

a. Define $V \otimes W$ and $\bigwedge^p V$.

b. If V has basis $\{e_i\}_{i=1,\dots,m}$ and W has basis $\{f_j\}_{j=1,\dots,n}$, show that $V \otimes W$ is spanned by the vectors $e_i \otimes f_j$. (In fact, these vectors form a basis, but you do not need to show that.)

c. Show $\bigwedge^m V$ has dimension ≤ 1 and explicit a spanning element. (In fact, $\bigwedge^m V$ has dimension 1.)

d. Define a linear transformation

$$\Big(\bigwedge^{p} V\Big) \otimes \Big(\bigwedge^{m-p} V\Big) \to \bigwedge^{m} V.$$

2. Let $U \subset \mathbb{R}^k$ be open, and let $f : U \to \mathbb{R}^n$. Assume f has continuous partial derivatives.

a. Define and state the main properties of the derivative Df.

b. Consider the function $f(x, y) = (x^2, xy, y^2)$. What is the best linear approximation to f near the point $(1, 1)$? Justify your answer citing results about Df.

3 a. State carefully the inverse function theorem and the implicit function theorem.

b. Appply the inverse function theorem to the function

$$f(r, \theta) = (r \cos\theta, r \sin\theta).$$

For what values of r, θ does the theorem fail? Indicate what happens at these points.

c. Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ and suppose the implicit function theorem applies to f at a point $(x^0, y^0) \in \mathbb{R}^{k+n}$. Use the implicit function theorem to define a map $L : \mathbb{R}^k \to \mathbb{R}^{k+n}$ in such a way that $Df(x^0, y^0) \circ L = 0$ as a map $\mathbb{R}^k \to \mathbb{R}^n$.

4 a. Define partitions of unity and state the main existence theorem.

b. Let $S \subset \mathbb{R}^n$ be a subset, and let $f : S \to \mathbb{R}$. f is said to be $\mathcal{C}^r$ on S if for each $s \in S$ there exists an open neighborhood U_s of s in $\mathbb{R}^n$ and a function g_s which is defined and $\mathcal{C}^r$ on U_s such that $g_s = f$ on $U_s \cap S$. Show that if S is closed in $\mathbb{R}^n$ then f extends to a $\mathcal{C}^r$ function on all of $\mathbb{R}^n$. Give an example where this is not true for S open in $\mathbb{R}^n$.

5. Define paralleopipeds in $\mathbb{R}^n$ and state the main theorem about computing volumes of parallelopipeds.

Compute the volume of the parallelopiped spanned by $(1, 2, 3, 4), (1, 1, 1, 1)$ in $\mathbb{R}^4$.