Algebro-Geometric Aspects of Feynman Graphs

Joint work with H. Esnault and D. Kreimer

1. Rank 1 quadrics, $\omega(q)$, Schwinger period.

2. Renormalization Hopf algebra in the category of motives.

3. Limiting mixed Hodge structures and renormalization.

Rank 1 Quadrics

Γ connected graph, n loops, N edges.

$$0 \to H_1(\Gamma) \to \mathbb{Q}[Edges] \to \mathbb{Q}[Vert] \to \mathbb{Q} \to 0$$

$$0 \to H \to \mathbb{Q}[Edge] \to Mom \to 0$$

$H = H_1$; $Mom = \mathbb{Q}[Vert]^0 :=$ Momentum .

$$H \subset \mathbb{Q}[Edge] \xrightarrow{e^\vee} \mathbb{Q}; \quad e \in Edge$$

$\sum_1^N A_i(e_i^\vee)^2$ linear series of rank 1 quadrics on $\mathbb{P}(H)$.

$M_H = M_H(A)$ corresponding symmetric matrix.

Graph polynomial $\Psi_H = \Psi_\Gamma := \det M$.

Dualize

$$Mom^{\vee} \subset \mathbb{Q}[Edges]^{\vee} = \mathbb{Q}[Edges]$$

Same construction yields symmetric matrix $M_{Mom^{\vee}}(A)$, $\Psi_{Mom^{\vee}} = \det M_{Mom^{\vee}}(A)$

$$\Psi_H(A) = (\prod_1^N A_i)\Psi_{Mom^{\vee}}(A^{-1})$$

Q non-degenerate quadratic form on $V \leftrightarrow$ symmetric matrix S. Then S^{-1} defines a quadratic from Q^{-1} on $V^{\vee}$ (gauge invariant)

$R(A) := M_{Mom^{\vee}}(A^{-1})^{-1} =$ quadratic form on $(Mom^{\vee})^{\vee} = Mom$. $R(cA) = cR(A)$. Entries homogeneous of degree 1.

$$R_{ij} = \Theta_{ij}/\Psi_H; \quad \deg \Theta_{ij} = \deg \Psi_H + 1 = n + 1$$

Differential Forms

Value of interest to physicists:

$$I_\Gamma(q) :=$$

$$\frac{1}{(4\pi)^{2n}} \int_{A_1,\ldots,A_N=0}^{\infty} \frac{\exp(-qRq^t)dA_1\ldots dA_N}{\Psi_\Gamma^2}$$

$$q \in Mom$$

Motives?

$$\Omega_{N-1} := \sum_1^N (-1)^i A_i dA_1 \ldots \widehat{dA_i} \ldots dA_n$$

Substitute $A_i = \lambda B_i$

$$(4\pi)^{2n} I_\Gamma(q) =$$

$$\int_{A_i \geq 0;\ \sum A_i = 1} \frac{\Omega_{N-1}}{\Psi_\Gamma^2} \int_{\lambda=0}^{\infty} \exp(\lambda q R q^t) \lambda^{N-1-2n} d\lambda$$

Substitute $\nu = \lambda q R q^t$, $\sigma : A_i \geq 0, \forall i$

$$I_\Gamma(q) = \frac{\Gamma(2n-N)}{(4\pi)^{2n}} \int_\sigma \frac{(qRq^t)^{2n-N}\Omega}{\Psi_\Gamma^2}$$

Periods and Motives

$\mathbb{P}^{N-1}$, coordinates $A_1, \ldots, A_N$, $X_\Gamma : \Psi_\Gamma = 0$.

$\Delta : A_1 A_2 \cdots A_N = 0$ coordinate simplex.

σ real $N-1$-chain with $\partial\sigma \subset \Delta$. Assume $2n - N \geq 0$. (Divergent case)

$[\sigma] \in H_{N-1}(\mathbb{P}^{N-1}, \Delta; \mathbb{Z})$;

$$\omega(q) := \frac{(qRq^t)^{2n-N}\Omega}{\Psi_\Gamma^2},$$

$[\omega(q)] \in H_{DR}^{N-1}(\mathbb{P}^{N-1} - X_\Gamma)$

$\int_\sigma \omega(q) \stackrel{?}{=}$ period (depending on q) associated to motive

$$H^{N-1}(\mathbb{P}^{N-1} - X_\Gamma, \Delta - \Delta \cap X_\Gamma)$$

Log divergent case, $N = 2n$, no q dependence.

The Structure of X_Γ

Spanning tree $T \subset \Gamma$: $h_0(T) = 1,\ h_1(T) = 0$, T contains all vertices of Γ.

$$\Psi_\Gamma(A) = \sum_{T\, spn.tr.} \prod_{e \notin T} A_e$$

Example: Γ "banana graph" N lines between 2 vertices. Spanning trees = single lines.
$\Psi_{banana} = \sum_i A_1 A_2 \cdots \widehat{A_i} \cdots A_N$

Ψ_Γ sum of monomials with coefficient $+1$.

Singularities of $\int_\sigma \omega(q)$:

$\sigma : A_i \geq 0$, $\sigma \cap X_\Gamma = \bigcup L_\gamma$

L_γ coordinate linear space, $L_\gamma \subset X_\Gamma$

$$L_\gamma \subset X_\Gamma \leftrightarrow \{\gamma \subset \Gamma \mid h_1(\gamma) > 0\}$$

$L_\gamma : A_e = 0,\ e \in \gamma$.

Blowing Up; Renormalization

$L \subset X_\Gamma,\ L = L_\gamma,\ \gamma \subset \Gamma.$

$$\begin{array}{ccc} \pi^{-1}L = E \cup Y & \longrightarrow & P = BL(L \subset \mathbb{P}^{N-1}) \\ \downarrow & & \downarrow \pi \\ L & \longrightarrow & \mathbb{P}^{N-1} \end{array}$$

$\#(Edge\ \gamma) = r,\quad E \cong L \times \mathbb{P}^{r-1}$. Homogeneous coordinates on $\mathbb{P}^{r-1} \leftrightarrow e \in \gamma$

Normal cone of $L \subset X_\Gamma$:

$$\Psi_\Gamma = \Psi_\gamma \Psi_{\Gamma//\gamma} + R$$

$\deg_{A_e, e\in\gamma} R > h_1(\gamma) = \deg \Psi_\gamma.$

$$Y \cap E = (X_{\Gamma//\gamma} \times \mathbb{P}^{r-1}) \cup (L \times X_\gamma)$$

$$E - E \cap Y = (L - X_{\Gamma//\gamma}) \times (\mathbb{P}^{r-1} - X_\gamma)$$

Motivic Realization of the Renormalization Hopf Algebra

$\Delta \subset \mathbb{P}^{N-1}$; $\Delta' \subset P$ strict transform of $\Delta \supset L$

$\Delta' \cap E = (\Delta_L \times \mathbb{P}^{r-1}) \cup (L \times \Delta_{\mathbb{P}^{r-1}})$

$$Mot(\Gamma) := H^{N-1}(\mathbb{P}^{N-1}-X_\Gamma, \Delta-\Delta\cap X_\Gamma)(N-1)$$
$$= H^{N-1}(P-(Y\cup E), \Delta' - \Delta' \cap (Y \cup E))(N-1)$$
$$\xrightarrow{Res} H^{N-2}(E - E\cap Y, \Delta' \cap E)(N-2) \cong$$
$$Mot(\gamma) \otimes Mot(\Gamma//\gamma)$$

Also

$$Mot(\Gamma_1 \amalg \Gamma_2) = Mot(\Gamma_1) \otimes Mot(\Gamma_2)$$

This is because $\Psi_{\Gamma_1 \amalg \Gamma_2}(A, B) = \Psi_{\Gamma_1}(A)\Psi_{\Gamma_1}(B)$, and

$$\mathbb{P}^{N_1+N_2-1} - Cone(X_{\Gamma_1}) - Cone(X_{\Gamma_2})$$
$$\xrightarrow{\mathbb{G}_m\text{-bundle}} (\mathbb{P}^{N_1-1} - X_{\Gamma_1}) \times (\mathbb{P}^{N_2-1} - X_{\Gamma_2})$$

Renormalization Hopf Algebra

$$H = \bigoplus \mathbb{Q} \cdot \Gamma; \quad \Gamma \cdot \Gamma' = \Gamma \amalg \Gamma'$$

$$\Delta(\Gamma) = \sum_{\substack{\gamma \subset \Gamma \\ diverg.}} \gamma \otimes \Gamma/\gamma$$

Motivic version

$$H_{Mot} := \bigoplus Mot(\Gamma),$$

Hopf algebra in the tannakian category of motives.

Periods - ω

Γ connected graph, *log divergent* ($2n$ edges, n loops)

$$H := H^{2n-1}(\mathbb{P}^{2n-1} - X_\Gamma, \Delta - X \cap \Delta) = Mot(\Gamma).$$

Rational structures: H_{Betti} and H_{DR}.

$$H_{Betti} \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR} \otimes_{\mathbb{Q}} \mathbb{C}$$

Recall $\omega := \Omega/\Psi_\Gamma^2 \in H_{DR}^{2n-1}(\mathbb{P}^{2n-1} - X_\Gamma)$.

ω top degree form on affine $\mathbb{P}^{2n-1} - X_\Gamma$ implies ω lifts canonically to H_{DR}.

Periods - σ

Assume Γ log divergent, n loops, $2n$ edges. Recall

$$\sigma : A_i \geq 0,\ \sigma \cap X_\Gamma = \bigcup L_\gamma$$

$$\pi : P = BL(L_\gamma \subset \mathbb{P}^{2n-1}) \to \mathbb{P}^{2n-1}$$

$$E \subset P \text{ exceptional divisor}$$

Proposition 1 *Subgraph $\gamma \subset \Gamma$ divergent iff $\pi^*\omega$ has a pole along E.*

Definition 2 Γ *(assumed log-divergent) is* primitive *if it has no divergent subgraphs $\gamma \subsetneq \Gamma$.*

Proposition 3 *Assume Γ primitive. Then there exists a sequence of blowups $\pi : P \to \mathbb{P}^{2n-1}$ of (strict transforms of) L_γ such that the strict transform of σ doesn't meet the strict transform Y of X_Γ. In particular, $\int_\sigma \omega$ becomes a well-defined period for $H^{2n-1}(P - Y, \Delta_P - Y \cap \Delta_P)$. Here $\Delta_P = \pi^{-1}(\Delta)$.*

Renormalization (Work in Progress)

Γ log divergent, n loops, $2n$ edges.

Basic problem: If Γ has divergent subgraphs, then $\int_\sigma \omega$ diverges.

$\ell_i(A_1, \ldots, A_{2n})$ independent linear forms with coefs in $\mathbb{R}$.

$\Delta_\ell : \ell_1 \ell_2 \cdots \ell_{2n} = 0; \quad \sigma_\ell : \ell_i \geq 0.$

Example 4

$\ell_i : A_i - tA_{2n} = 0, \ \ell_{2n} : A_{2n} - \sum_1^{2n-1} A_i = 0$

$$0 < t < \frac{1}{2n-1} \Rightarrow \sigma_\ell \subset \{A_i > 0 \mid 1 \leq i \leq 2n\}$$
$$\Rightarrow \sigma_\ell \cap X_\Gamma = \emptyset.$$

$$\int_{\sigma_\ell} \omega \text{ period for}$$
$$H^{2n-1}(\mathbb{P}^{2n-1} - X_\Gamma, \Delta_\ell - X \cap \Delta_\ell)$$

Limiting Mixed Hodge Structures

$H_t = H^{2n-1}(\mathbb{P}^{2n-1} - X_\Gamma, \Delta_t - X_\Gamma \cap \Delta_t), \quad t \in D^*$

Suitable variation of mixed Hodge structure over punctured disk D^*. (Think of Δ_t general for $t \neq 0$.)

Limiting mixed Hodge structure H_{lim}.

Monodromy of H_t quasi-unipotent. (Ramifying in t, we may assume unipotent.)

Monodromy matrix $\exp(N \log t)$

Period matrix $\mathcal{P}_t = (\int_{c_{i,t}} \omega_{j,t})$

$c_{i,t}$ horizontal basis of $H_t^\vee$;

$\omega_{j,t}$ *suitable* algebraic basis of H_t.

Basic result(Deligne): $\exp(-N \log t)\mathcal{P}_t$ is single-valued in a neighborhood of $t = 0$, and

$$M_0 := \lim_{t \to 0} \exp(-N \log t)\mathcal{P}_t$$

exists. H_{lim} defined using M_0.

Good Lattices

$$\omega = \frac{\Omega_{2n-1}}{\Psi_\Gamma^2} \stackrel{?}{=} \omega_{1,t}$$

Need

$$|\int_{c_{i,t}} \omega| = O(|\log(t)|^N).$$

(Good Lattice condition.)

$\gamma \subset \Gamma$ subgraph, edges $e_1, \ldots, e_p$. Assume $h_1(\gamma) > 0$.

$$L : A_1 = \ldots = A_p = 0; \quad L \subset X_\Gamma$$

$$\pi : P = BL(L \subset \mathbb{P}^{2n-1}) \to \mathbb{P}^{2n-1}; E \subset P$$

Conclusion

Conjecture 5 *Assume* Γ *has only logarithmically divergent subgraphs. Then for a suitable variation* Δ_t *and chains* $c_{i,t}$*, the column vector*

$$\exp(-N\log t)\left(\int_{\sigma_t}\omega, \int_{c_{2,t}}\omega, \ldots, \int_{c_{N,t}}\omega\right)^t$$

is single-valued in a neighborhood of $t = 0$ *and admits a limit as* $t \to 0$.