Counting the changes of random Δ^0_2 sets

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Abstract. We study the number of changes of the initial segment $Z_s\upharpoonright_n$ for computable approximations of a Martin-Löf random Δ^0_2 set Z. We establish connections between this number of changes and various notions of computability theoretic lowness, as well as the fundamental thesis that, among random sets, randomness is antithetical to computational power. We introduce a new randomness notion, called balanced randomness, which implies that for each computable approximation and each constant c, there are infinitely many n such that $Z_s\upharpoonright_n$ changes more than $c2^n$ times. We establish various connections with ω-c.e. tracing and ω-c.e. jump domination, a new lowness property. We also examine some relationships to randomness theoretic notions of highness, and give applications to the study of (weak) Demuth cuppability.

1 Introduction

Randomness and the number of changes

A *computable approximation* of a set $Z \subseteq \mathbb{N}$ is a computable sequence $(Z_s)_{s\in\mathbb{N}}$ of finite sets such that $Z(x) = \lim_s Z_s(x)$ for each x. The Shoenfield Limit Lemma states that a set $Z \subseteq \mathbb{N}$ is Δ^0_2 iff Z has a computable approximation. In this paper, we will look at the number of changes of the initial segment $Z_s\upharpoonright_n$ for computable approximations of a Martin-Löf random Δ^0_2 set Z, and establish connections between this number of changes and various notions of computability theoretic lowness, as well as the fundamental thesis that, among random sets, randomness is antithetical to computational power. We assume familiarity with

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basic notions and results in computability theory and algorithmic randomness. For definitions not given here, and additional background, see [2] or [11].

In Section 3 we give some lower bounds on the number of changes for computable approximations of an ML-random set. In Section 4 we prove a hierarchy theorem saying that allowing more changes yields new ω-c.e. ML-random sets. In Section 5 we prove the "$o(2^n)$ changes" low basis theorem, which says that each nonempty Π^0_1 class has a low member Z with a computable approximation such that $Z\restriction_n$ changes only $o(2^n)$ times. We conclude that there is a low ML-random set with a computable approximation that changes only $o(2^n)$ times.

In Section 6 we briefly consider the notion of computable randomness, which is weaker than ML-randomness. We show that we can computably approximate some computably random set with far fewer changes than are necessary to approximate ML-random sets.

Our results suggest calibrating the randomness content of Δ^0_2 sets by the number of changes needed to computably approximate them, rather than by the growth of the initial segment complexity, as has been traditionally done. Intuitively, the more random a set, the more changes are needed. It would be interesting to establish further results along these lines for Schnorr randomness, or for partial computable randomness.

This intuition is reinforced by a notion at the opposite end of the randomness spectrum, the K-trivial sets, which are far from random. The theory of cost functions (see [11, Section 5.3]) shows that a K-trivial set can be computably approximated with a finite total amount of changes, as measured by an appropriate cost function, which means that the approximation changes very little.

This article is an extended version of [3]. In particular, definitions and results of sections 2, 3, 4, 5, and 7 (except Theorem 23 and Corollary 24) were presented there. Sections 6, 8, and 9 are new.

Randomness versus computational complexity for ML-random sets

Martin-Löf [9] introduced a notion of randomness that has been widely accepted in the field. A ML-test is a uniformly c.e. sequence $(G_i)_{i\in\mathbb{N}}$ of open sets such that $\lambda G_i \le 2^{-i}$ for all i, where λ denotes the uniform measure on Cantor space, which assigns the quantity $2^{-|\sigma|}$ to each basic open cylinder $[\sigma] = \{Z : \sigma \prec Z\}$. A set $Z \in 2^\omega$ *fails* the test if $Z \in \bigcap_i G_i$; otherwise Z *passes* the test. The set Z is *ML-random* if Z passes each ML-test.

In [11, Section 8.6] evidence was presented for the following fundamental thesis: Among ML-random sets,

being computationally less complex is equivalent to being more random.

For instance, an ML-random set forms a minimal pair with $\emptyset'$ iff it is weakly 2-random. In Section 7 we give further evidence for this thesis when the ML-random set is Δ^0_2.

To specify what we mean by being more random, we consider variants of Demuth randomness, a notion that strengthens ML-randomness but is still compatible with being Δ^0_2. Demuth tests (see [11, Def. 3.6.24]) generalize Martin-Löf tests $(G_m)_{m\in\mathbb{N}}$ in that one can exchange the m-th component a computably bounded number of times. A set $Z \subseteq \mathbb{N}$ passes a Demuth test if Z is in only finitely many final versions of the G_m.

The passing condition that at least one of the G_m does not contain Z yields *weak* Demuth randomness. In this case, we can require as well that $G_m \supseteq G_{m+1}$ for each m, since we can replace G_m by $\bigcap_{i\leq m} G_i$ if necessary. A test with this property is called *monotonic*. Note that the number of version changes is still computably bounded. Thus Z is weakly Demuth random iff it passes all monotonic Demuth tests (where passing the test can be taken in either sense).

We introduce balanced randomness, an even more restricted form of weak Demuth randomness where the bound on the number of changes of the m-th version is $O(2^m)$. Every balanced random set is ML-random and Turing incomplete. In fact, we show that every balanced random set is difference random, a notion introduced recently by Franklin and Ng [4].

For evidence of the direction from left to right in the thesis above, we show that every superlow ML-random set is balanced random. Being ω-c.e.-tracing (Definition 21) is a highness property, i.e., a property saying that a set is close to being Turing above $\emptyset'$, due to Greenberg and Nies [5]. This notion is incompatible with superlowness. In fact, we show that every ML-random set that is not ω-c.e.-tracing is balanced random. One consequence of the results of Section 7 is that, while, as mentioned above, there is a computable approximation to a low ML-random set that changes only $o(2^n)$ times, no superlow ML-random set has a computable approximation that changes only $O(2^n)$ times.

Evidence for the direction from right to left in the thesis above is given by the fact that a Demuth random set bounds only generalized low sets, and the result of [7] that a c.e. set Turing below a Demuth random set must be strongly jump-traceable. In [7] further evidence for this direction is given by showing that a weakly Demuth random set Z is not superhigh, that is, $Z' \not\geq_{\mathrm{tt}} \emptyset''$. (However, such a set can be high.)

Being ω-c.e.-tracing, and ω-c.e.-jump domination

Sections 8 and 9 are somewhat independent of the preceding sections, and relate only indirectly to our main topic, counting the changes of random Δ^0_2 sets. We relate the highness property of being ω-c.e.-tracing introduced above to a new lowness property, being ω-c.e.-jump dominated.

Recall the lowness property of jump traceability, and the weaker property of array computability: We say that A is *jump traceable* if there is a c.e. trace with computable bound for the universal partial A-computable function J^A. We say that A is *array computable* if some ω-c.e. function dominates all A-computable functions. See [11, Chapter 8] for background on these notions.

Being ω-c.e.-jump dominated (Definition 27) is strictly weaker than jump traceability, but strictly stronger than array computability. In Section 8 we extend our result that no superlow set is ω-c.e.-tracing by showing that every superlow set is ω-c.e.-jump dominated, while no set that is ω-c.e.-tracing can be ω-c.e.-jump dominated. In Section 9 we examine some relationships between being ω-c.e.-tracing, being ω-c.e. jump dominated, and randomness theoretic notions of highness. We also show that these results can be applied to study the question of how much computational strength is necessary to cup a (weakly) Demuth random set to $\emptyset'$.

Notation

We denote the usual uniform measure on Cantor space 2^ω by λ. For a binary string x, we let $[x]$ denote the subset of 2^ω consisting of all extensions of x. For a set of strings S, we let $[S]^{\prec} = \bigcup_{x \in S}[x]$. By W_e we mean the e-th c.e. set of binary strings. We denote the class of ML-random sets by MLR.

2 Counting the changes of a Δ^0_2 set

For a computable approximation $(Z_s)_{s\in\mathbb{N}}$, unless otherwise stated, we will assume that $Z_s(x) = 0$ for each $x \geq s$. Given such an approximation, for a number n and a stage number $s > 0$, to say that $Z\restriction_n$ *changes* at stage s means that $Z_s\restriction_n \neq Z_{s-1}\restriction_n$.

When we say that we bound the number of changes for a Δ^0_2 set Z from above, we mean that the changes of *some* approximation can be bounded from above.

Definition 1. Let $f : \mathbb{N} \to \mathbb{N}$. We say a set $Z \subseteq \mathbb{N}$ is *f-c.e.* if there is a computable approximation $(Z_s)_{s\in\mathbb{N}}$ of Z such that for each n, the segment $Z\restriction_n$ changes at most $f(n)$ times via this approximation. Terminology such as $O(f)$-c.e. set, $o(f)$-c.e. set, and so on has the obvious meaning. For instance, Z is $o(f)$-c.e. if there is a function $g \in o(f)$ such that Z is g-c.e.

Note that the above definition is not quite standard. Other authors have defined a set Z to be f-c.e. if there is a computable approximation of Z such that for each n, the approximation to the single value $Z(n)$ changes at most $f(n)$ times. For almost all of our results, this distinction will not matter, the exception being Theorem 9.

We identify reals and sets of natural numbers, so we refer to sets corresponding to left-c.e. reals (that is, reals that are computably approximable from below) as left-c.e. sets. Each left-c.e. set is $o(2^n)$-c.e.:

Fact 2. *Let Z be a left-c.e. set as shown by the computable approximation $(Z_s)_{s\in\mathbb{N}}$. Then Z is $o(2^n)$-c.e. via this computable approximation.*

Proof. Given k, let t be the least stage such that $Z_t\restriction_{k+1}$ has the final value. Let $n \geq t+k+1$. By our convention that $Z_s(x) = 0$ for each $x \geq s$, the approximation to $Z\restriction_n$ changes at no more than $2^t \leq 2^{n-k-1}$ stages that are $\leq t$. Furthermore, since the approximation cannot return to previous states, $Z\restriction_n$ changes at no more than 2^{n-k-1} stages that are greater than t. Thus $Z\restriction_n$ changes at no more than 2^{n-k} stages. $\square$

By this argument, the above fact still holds if we merely require that the approximation to $Z\restriction_n$ can never return to a previous value.

3 Some lower bounds on the number of changes of an ML-random set

In this section we assume that Z is an ML-random Δ^0_2 set with a fixed computable approximation $(Z_s)_{s\in\mathbb{N}}$. We give some lower bounds on the number of times $Z\restriction_n$ can change. We confirm the intuition that the number of changes cannot be far below 2^n.

First we look at computable functions bounding the number of changes of $Z\restriction_n$ for infinitely many n; thereafter, we look for bounds that work for all n.

Proposition 3. *Let $q:\mathbb{N}\to\mathbb{Q}^+$ be computable. If $Z\restriction_n$ changes fewer than $\lfloor 2^n q(n)\rfloor$ times for infinitely many n, then $\sum_n q(n)=\infty$.*

Proof. Assume for a contradiction that $\sum_n q(n)<\infty$. We define an effective sequence $(\mathcal{S}_i)_{i\in\mathbb{N}}$ of Σ^0_1 classes in the following way. For each n, we put into $\mathcal{S}_n$ the first $\lfloor 2^n q(n)\rfloor$ versions of $[Z\restriction_n]$. Clearly $(\mathcal{S}_i)_{i\in\mathbb{N}}$ is a sequence of uniformly c.e. open sets and $\lambda\mathcal{S}_n\le q(n)$ for all n. Thus $(\mathcal{S}_i)_{i\in\mathbb{N}}$ is a Solovay test. By hypothesis $Z\in S_n$ for infinitely many n. Thus Z fails the test $(\mathcal{S}_i)_{i\in\mathbb{N}}$ and therefore is not ML-random. $\square$

Example 4. There is no ML-random Δ^0_2 set Z such that $Z\restriction_n$ changes fewer than $\lfloor 2^n/n^2\rfloor$ times for infinitely many n.

The proof of the foregoing proposition can easily be extended to the case that the function q is effectively approximable from below, that is, $q(n)=\sup_s q_s(n)$ for an effective sequence of rationals that is nondecreasing in s. For instance, we can let $q(n)=2^{-K(n)}$, where K is prefix-free Kolmogorov complexity. Thus, in the example above, in fact we have a lower bound of $2^{n-K(n)}$.

If for almost every n the number of changes of $Z\restriction_n$ is bounded above by $2^n q(n)$, then the function q is in fact bounded away from 0.

Proposition 5. *Let $q:\mathbb{N}\to\mathbb{Q}^+$ be computable. If $Z\restriction_n$ changes fewer than $\lfloor 2^n q(n)\rfloor$ times for almost every n, then $\inf_n q(n)>0$.*

Proof. Let n^* be a number such that the bound holds from n^* on. Assume for a contradiction that $\inf_n q(n)=0$. We show that $\exists^\infty n\, K(Z\restriction_n)\le^+ n$, contrary to the assumption that Z is ML-random. To do so we build a bounded request (aka Kraft-Chaitin) set L. Let $(n_i)_{i>0}$ be a computable sequence of numbers greater than n^* such that $q(n_i)<2^{-i}$ for each i. For each s, we put the request

$$\langle n_i, Z_s\restriction_{n_i}\rangle$$

into L. For each $i > 0$, the weight put into L is at most $2^{-n_i}2^{n_i}q(n_i) \leq 2^{-i}$. Thus L is a bounded request set. Hence by the usual machine existence theorem (aka Kraft-Chaitin Theorem), we have $\exists^\infty n\, K(Z\restriction_n) \leq^+ n$ as required. □

The proof of the foregoing proposition can easily be extended to the case that the function q is effectively approximable from *above*. For each i, we can search for an s and an n_i such that $q_s(n_i) < 2^{-i}$.

It is natural to ask what else we can say about the number of times $Z\restriction_n$ can change for a Δ^0_2 ML-random Z. In particular, we consider strengthening Propositions 3 and 5 simultaneously: whenever $Z\restriction_n$ changes fewer than $\lfloor 2^n q(n)\rfloor$ times for infinitely many n, then $q(n)$ is bounded away from zero on these n. By the following proposition this is true if q is a computable nonincreasing function, but by Corollary 12 this fails in general.

Proposition 6. *Let $q : \mathbb{N} \to \mathbb{Q}^+$ be computable and nonincreasing. If $Z\restriction_n$ changes fewer than $\lfloor 2^n q(n)\rfloor$ times for infinitely many n, then* $\inf_n q(n) > 0$.

Proof. Suppose the contrary, that $\inf_n q(n) = 0$. Let $(n_i)_{i\in\mathbb{N}}$ be a computable sequence of natural numbers such that for every i, we have that n_i is the least number larger than n_{i-1} such that $q(n_i) < 2^{-i-1}$. We build a Solovay test $(\mathcal{S}_i)_{i\in\mathbb{N}}$ in the following way. For each i enumerate into $\mathcal{S}_i$ the first 2^{n_i-i} different versions of $[Z\restriction_{n_i}]$. Then $\lambda\mathcal{S}_i \leq 2^{-i}$ for every i. Since Z is ML-random and $Z\restriction_n$ changes fewer than $\lfloor 2^n q(n)\rfloor$ times for infinitely many n, we fix $m > n_0$ and $i > 0$ such that $Z\restriction_m$ changes fewer than $\lfloor 2^m q(m)\rfloor$ times, $Z \notin \mathcal{S}_i$, and i is least such that $n_i \geq m$. Since $Z \notin \mathcal{S}_i$, there must be at least $2^{n_i-i}+1$ many distinct elements in the set $\{Z_s\restriction_{n_i} : s \in \mathbb{N}\}$. Now since $n_{i-1} < m$ we have $q(m) \leq q(n_{i-1}) < 2^{-i}$. Hence $Z\restriction_m$ changes fewer than 2^{m-i} times. Thus $Z\restriction_{n_i}$ changes fewer than $2^{n_i-m}2^{m-i} = 2^{n_i-i}$ times. This is a contradiction. □

We can now improve Example 4. For instance:

Example 7. There is no ML-random Δ^0_2 set Z such that $Z\restriction_n$ changes fewer than $\lfloor 2^n/\log\log n\rfloor$ times for infinitely many n.

4 A hierarchy theorem for ML-random ω-c.e. sets

Using a method of Kučera one can code a given set into a member of a Π^0_1 class of positive measure. The method rests on the following lemma (see [11, Lem. 3.3.1]), where $\lambda(\mathcal{C} \mid x)$ denotes $2^{|x|}\lambda(\mathcal{C} \cap [x])$.

Lemma 8. *Let $\mathcal{C} \subseteq 2^\omega$ be measurable and $\lambda(\mathcal{C} \mid x) \geq 2^{-(r+1)}$. Then for every $n \geq |x| + r + 2$ there are distinct strings $y_0, y_1 \succ x$ with $|y_i| = n$ such that $\lambda(\mathcal{C} \mid y_i) > 2^{-(r+2)}$ for $i = 0, 1$.*

An *order function* is a nondecreasing unbounded computable function. Recall the notion of f-c.e. set from Definition 1.

Theorem 9. *Let b be an order function such that $\forall n\, b(n) \geq \epsilon 2^n$ for some positive real ϵ. Then for each order function s there is an ML-random Z that is $s \cdot b$-c.e. but not b-c.e.*

We can restate Proposition 5 as follows: if the ML-random set Z is b-c.e. for some computable function b, then there is $\epsilon > 0$ such that $\forall n\, b(n) \geq \epsilon 2^n$. Thus the additional hypothesis $\forall n\, b(n) \geq \epsilon 2^n$ in this hierarchy theorem does not restrict its generality.

Proof (of Theorem 9). The idea is the following. To make Z ML-random, we ensure that it belongs to an appropriate Π^0_1-class. To make Z non b-c.e., let $(f_e)_{e\in\mathbb{N}^+}$ be an enumeration of total computable functions f mapping pairs of natural numbers to strings such that for all n, it is the case that $|\{t : f(n,t) \neq f(n,t+1)\}| \leq b(n)$, that $|f(n,t)| = n$, and that $f(n,t) \prec f(n+1,t)$. Each such f is the approximation of some b-c.e. set. Conversely, though our enumeration cannot contain all such functions, we may assume it contains enough so that if a set is b-c.e., then there is an f in our enumeration giving the set in the limit. Thus it suffices to ensure that for every e there is an n such that $\lim_t f_e(n,t) \neq Z\restriction_n$.

Here are the details. Recall that s is the given order function. Choose a computable sequence $(n_e)_{e\in\mathbb{N}^+}$ such that $n_1 = 0$,

$$s(n_e) > e + 1/\epsilon, \quad \text{and} \quad n_{e+1} \geq n_e + e + 2.$$

Let $\mathcal{P}$ be a Π^0_1-class such that $\mathcal{P} \subseteq \mathsf{MLR}$, where we recall that MLR is the class of ML-random sets, and $\lambda\mathcal{P} > 1/2$. Let $\widehat{\mathcal{P}}$ be the Π^0_1 class of paths through the Π^0_1 tree

$$T = \{y : (\forall i)[n_i \leq |y| \to \lambda(\mathcal{P} \mid (y\restriction_{n_i})) \geq 2^{-(i+1)}]\}.$$

Note that $\widehat{\mathcal{P}} \subseteq \mathcal{P}$. Since $\lambda\mathcal{P} \geq 1/2$, by Lemma 8, $\widehat{\mathcal{P}}$ is nonempty.

We define $z_0 \prec z_1 \prec z_2 \prec \cdots$ in such a way that $|z_e| = n_e$ and $z_e \neq \lim_t f_e(n_e,t)$. We also define $Z = \bigcap_e [z_e]$. In this way, we ensure that $Z\restriction_{n_e} \neq \lim_t f_e(n_e,t)$ for all $e \geq 1$, and therefore Z is not b-c.e. At the same time, we ensure that $Z \in \widehat{\mathcal{P}}$, and hence Z is ML-random.

The definition of z_e proceeds by steps. Recall that each Π^0_1 class $\mathcal{Q}$ has an effective approximation by descending clopen sets $\mathcal{Q}_s$; see [11, Sect. 1.8]. Let $z_{0,s} = \emptyset$ and for $e > 0$ let

$$z_{e+1,s} = \min\{z : [z] \subseteq \widehat{\mathcal{P}}_s \ \wedge\ |z| = n_{e+1} \ \wedge\ z \succ z_{e,s} \ \wedge\ f_e(n_e,s) \neq z\}, \tag{1}$$

where the minimum is taken with respect to the lexicographic ordering.

Suppose $z_{e,s}$ has already been defined. By Lemma 8 and the definition of $\widehat{\mathcal{P}}$, there are two distinct strings $y_0, y_1 \succ z_{e,s}$ such that $|y_i| = n_e$ and $[y_i] \cap \widehat{\mathcal{P}} \neq \emptyset$. Hence $z_{e+1,s}$ is well defined in equation (1).

To show that Z is $s \cdot b$-c.e., define a computable approximation $(Z_i)_{i\in\mathbb{N}^+}$ with $Z_s = z_{s,s}$. Suppose $n_e \leq n < n_{e+1}$.

If $Z_{s+1}\restriction_n \neq Z_s\restriction_n$ then

$$[Z_s\restriction_n] \not\subseteq \widehat{\mathcal{P}}_{s+1} \text{ or } \exists i \leq e\, f_i(n,s+1) \neq f_i(n,s).$$

The former may occur at most 2^n times, and the latter at most $e \cdot b(n)$ times. For all $e \geq 1$, the number of changes of $Z \upharpoonright_n$ is at most

$$\begin{aligned} 2^n + e \cdot b(n) &\leq b(n)/\epsilon + e \cdot b(n) \\ &\leq b(n)(e + 1/\epsilon) \\ &\leq b(n) \cdot s(n_e) \leq b(n) \cdot s(n). \end{aligned}$$

□

5 Counting changes for sets given by the (super)low basis theorem

The low basis theorem of Jockusch and Soare [6] says that every nonempty Π^0_1 class has a member Z that is low, that is, $Z' \leq_{\mathrm{T}} \emptyset'$. The proof actually makes Z *superlow*, that is, $Z' \leq_{\mathrm{tt}} \emptyset'$. Here we study possible bounds on the number of changes for a low member of the class. We find that to make the member superlow will in general take more changes, not fewer. This result may seem surprising, but at least in the case of a Π^0_1 class of ML-random sets, it is in fact in line with the discussion in the introduction, as we should expect that it takes more changes to make a Δ^0_2 set more random.

Theorem 10. *Let $\mathcal{P}$ be a nonempty Π^0_1 class. For each order function h, the class $\mathcal{P}$ has a superlow $2^{n+h(n)}$-c.e. member.*

Proof. The idea is to run the proof of the superlow basis theorem with a c.e. operator W^X that codes X' only at a sparse set of positions, and simply copies X for the other bit positions. Let R be the infinite computable set $\{n : h(n+1) > h(n)\}$. Define the c.e. operator W by

$$W^X(n) = \begin{cases} X(i) & \text{if } n \text{ is the } i\text{-th smallest element in } \mathbb{N} - R \\ X'(j) & \text{if } n \text{ is the } j\text{-th smallest element in } R \end{cases} \tag{2}$$

By the proof of the superlow basis theorem as in [11, Thm. 1.8.38], there is a $Z \in \mathcal{P}$ such that $B = W^Z$ is left-c.e. via some approximation (B_s). Let Z_s be the computable approximation of Z given by $Z_s(i) = B_s(n)$ where n is the i-th smallest element in $\mathbb{N} - R$. If $Z_s \upharpoonright_n$ changes then $B_s \upharpoonright_{n+h(n)}$ changes. Thus $Z_s \upharpoonright_n$ changes at most $2^{n+h(n)}$ times. Furthermore, $Z' \leq_{\mathrm{m}} B$. Since B is ω-c.e. we have $B \leq_{\mathrm{tt}} \emptyset'$, so Z is superlow. □

Theorem 23 below shows that if $\mathcal{P} \subseteq \mathsf{MLR}$, no superlow member of $\mathcal{P}$ can be $O(2^n)$-c.e. On the other hand, if we merely want a low member, we can actually get away with $o(2^n)$ changes. For the case $\mathcal{P} \subseteq \mathsf{MLR}$, this result shows that $o(2^n)$-c.e. ML-random sets can be very different from the Turing complete ML-random set Ω, even though Ω is also $o(2^n)$-c.e. by Fact 2.

Theorem 11. *Each nonempty Π^0_1 class $\mathcal{P}$ contains a low $o(2^n)$-c.e. member.*

Proof. We combine the construction in the proof of Theorem 10 with a dynamic coding of the jump. At each stage we have movable markers γ_k at the positions where $X'(k)$ is currently coded. Thus, the positions where X' is coded become sparser and sparser as the construction proceeds.

Construction. At stage 0 let $\gamma_{0,0} = 1$ and B_0 be the empty set.
Stage $t > 0$.
(i). Let $W^X[t]$ be the c.e. operator such that

$$W^X[t](v) = \begin{cases} X(i) & \text{if } v \text{ is the } i\text{-th smallest element} \\ & \text{not of the form } \gamma_{k,t-1} \\ X'(k) & \text{if } v = \gamma_{k,t-1}\,. \end{cases} \tag{3}$$

Uniformly in the stage number t, we define a sequence of Π^0_1 classes $\mathcal{Q}_n[t]$ ($n \in \mathbb{N}$). We follow the proof of the low basis theorem as in [11, Thm. 1.8.38], but at stage t we use the operator $W[t]$ instead of the jump operator.

Let $\mathcal{Q}_0[t] = \mathcal{P}$. If $\mathcal{Q}_n[t]$ has been defined, let

$$\mathcal{Q}_{n+1}[t] = \begin{cases} \mathcal{Q}_n[t] & \text{if for all } X \in \mathcal{Q}_{n,t}[t], \\ & \text{we have } n \in W^X[t] \\ \left\{X \in \mathcal{Q}_n[t] \,:\, n \notin W^X[t]\right\} & \text{otherwise.} \end{cases}$$

In the first case, define $B_t(n) = 1$; in the second case, define $B_t(n) = 0$.

(ii). Let k be least such that $k = t$ or $B_t \restriction_{2k} \neq B_{t-1} \restriction_{2k}$. Define $\gamma_{r,t} = \gamma_{r,t-1}$ for $r < k$, and $\gamma_{r,t} = t + 2r$ for $t \geq r \geq k$.

Verification.
Claim 1. *B is left-c.e. via the computable approximation $(B_t)_{t\in\mathbb{N}}$.*
Suppose i is least such that $B_t(i) \neq B_{t-1}(i)$. Since $\gamma_{r,t-1} > 2r$ for each r, we have $\gamma_{r,t} = \gamma_{r,t-1}$ for all r such that $\gamma_{r,t-1} \leq i$. Thus the construction up to $\mathcal{Q}_i[t]$ behaves like the usual construction to prove the low basis theorem, whence we have $B_{t-1}(i) = 0$ and $B_t(i) = 1$.

We conclude that $\gamma_k = \lim_t \gamma_{k,t}$ exists for each k, and therefore $\mathcal{Q}_n = \lim_t \mathcal{Q}_n[t]$ exists as well.

By the compactness of 2^ω there is $Z \in \bigcap_n \mathcal{Q}_n$. Clearly Z is low because $Z'(k) = B(\gamma_k)$ and the expression on the right can be evaluated by $\emptyset'$. It remains to show the following.
Claim 2. *Z is $o(2^n)$-c.e.*
We have a computable approximation to Z given by

$$Z_t(i) = B_t(v) \text{ where } v \text{ is the } i\text{-th smallest number not of the form } \gamma_{k,t}.$$

Given n, let k be largest such that $\gamma_k \leq n$. We show that $Z\restriction_n$ changes at most 2^{n-k+1} times.

For $n \geq r \geq k$, let t_r be the least stage t such that $\gamma_{r+1,t} > n$. Then $B_t \restriction_{2r}$ is stable for $t_r \leq t < t_{r+1}$. Since $(B_t)_{t\in\mathbb{N}}$ is a computable approximation via which B is left-c.e., $B\restriction_{n+r}$ changes at most 2^{n-r} times for $t \in [t_r, t_{r+1})$. Hence $Z\restriction_n$

changes at most 2^{n-r} times for such t. The total number of changes is therefore bounded by $\sum_{k\leq r\leq n} 2^{n-r} < 2^{n-k+1}$. □

As a consequence, we see that in Proposition 6 it was necessary to assume that the function q is nonincreasing.

Corollary 12. *There are an ML-random set Z and a computable $q : \mathbb{N} \to \mathbb{Q}^+$ such that $Z\restriction_n$ changes fewer than $\lfloor 2^n q(n)\rfloor$ times for infinitely many n, and* $\lim_n q(n) = 0$.

Proof. In the proof of Theorem 11, let $\mathcal{P}$ be a Π^0_1 class containing only ML-randoms. We define $q(m) = 2^{-r+1}$, where r is least such that $\gamma_{r,m} \geq m$. Then q is computable and $\lim_n q(n) = 0$ because each marker reaches a limit. Also, $q(\gamma_r) = 2^{-r+1}$ for every r. By the proof of Theorem 11, for every r, the approximation to $Z\restriction_{\gamma_r}$ changes fewer than $2^{\gamma_r - r+1} = \lfloor 2^{\gamma_r} q(\gamma_r)\rfloor$ times. □

6 A computably random set that changes little

The lower bounds on changes obtained in the previous sections actually relied on the given Δ^0_2 set being ML-random. For the weaker notion of computable randomness, we can get away with far fewer changes.

We thank Frank Stephan for helpful discussions.

Theorem 13. *For each order function h, there is a computably random h-c.e. set Z. Moreover, Z can be chosen to be left-c.e.*

Proof. For the first statement, we check that the construction from [13], in the version of [11, Remark 7.4.13], yields a set Z as required. We use the notation from [11], which we briefly summarize here.

We view the k-th Turing functional Φ_k as a map from $2^{<\omega}$ to the non-negative dyadic rationals. We let B_k be the k-th partial computable martingale; it copies Φ_k as long as Φ_k looks like a martingale. For each $n \in \mathbb{N}$, we obtain a partial computable martingale $B_{k,n}$, which succeeds on the same sequences as B_k, by modifying B_k in such a way that $B_{k,n}(x) = 1$ for each string x of length $< n$.

We define the computable sequence $n_0 < n_1 < \cdots$ by $n_0 = 0$ and

$$n_{k+1} = \mu n > n_k \left[h(n) > (k+1)^2\right].$$

Let $I_k = [n_k, n_{k+1})$. For a string z, we let $\mathsf{Int}(z)$ denote the number k such that $|z| \in I_k$.

We let $B^*_k = B_{k,n_{k+1}}$, and define the following supermartingale that copies B_k with some restrictions:

$$V_k(x) = \begin{cases} B^*_k(x) & \text{if } \mathsf{Int}(x) < k \text{ or } \forall y\, [\mathsf{Int}(y) = \mathsf{Int}(x) \to B^*_k(y)\downarrow] \\ 0 & \text{otherwise.} \end{cases}$$

Let L be the supermartingale given by $L(x) = \sum_k 2^{-k} V_k(x)$. Note that by definition, $L(x)$ for $\mathsf{Int}(x) \le r$ depends only on $B^*_k(x)$ where $k < r$; the remaining components B^*_k for $k \ge r$ together contribute $\sum_{k \ge r} 2^{-k} = 2^{-r+1}$. In particular, $L(x)$ is rational.

Let Z be the leftmost non-ascending path of L, i.e., Z is the leftmost bit sequence in 2^ω such that $\forall n\ L(Z\upharpoonright_{n+1}) \le L(Z\upharpoonright_n)$ As shown in [11, 7.4.13], L multiplicatively dominates each computable martingale, whence Z is computably random. It remains to show Z is h-c.e.

For each stage s we have a computable approximation $B_k[s]$, which copies $\Phi_k[s]$ as long as $\Phi_k[s]$ looks like a martingale. From this we get $B^*_k[s]$ and $V_{k,s}$, the computable approximations of B^*_k and V_k respectively. Hence L is a c.e. supermartingale, as shown by the supermartingale approximation (see [11, Def. 7.2.3]) $L_s(x) = \sum_k 2^{-k} V_{k,s}(x)$.

Now let Z_s be the leftmost non-ascending path of L_s, i.e., Z_s is the leftmost bit sequence in 2^ω such that $\forall n\ L_s(Z_s\upharpoonright_{n+1}) \le L_s(Z_s\upharpoonright_n)$. Clearly $(Z_s)_{s\in\mathbb{N}}$ is a computable approximation of Z. We show that Z is h-c.e. via $(Z_s)_{s\in\mathbb{N}}$.

Let n be given, where $n \in I_k$. Thus $n_k \le n$, whence $k^2 \le h(n)$. For $i < k$ and stage s, let $r(i,s)$ be the maximal number $r \le k$ such that $B^*_i[s](x)$ is defined whenever $\mathsf{Int}(x) \le r$. If $Z_s\upharpoonright_n \ne Z_{s-1}\upharpoonright_n$ then $L_s(x) \ne L_{s-1}(x)$ for some x of length $\le n$. By the remark following the definition of L, this inequality implies that $r(i,s) > r(i,s-1)$ for some $i < k$, which can happen at most $k^2 \le h(n)$ times.

To make Z left-c.e., we modify its definition in a manner similar to [11, Remark 7.4.17]. The V_k are uniformly c.e. supermartingales, hence L is a c.e. supermartingale as well. Thus, since $L(\emptyset) \le 2$, the paths of the tree $T = \{x : \forall y \preceq x\ L(y) \le 2\}$ form a nonempty Π^0_1 class. Now let Z be the leftmost path of T. Let $T_s = \{x : \forall y \preceq x\ L_s(y) \le 2\}$, and let Z_s be the leftmost path of T_s. Then $(Z_s)_{s\in\mathbb{N}}$ is a computable approximation of Z showing that Z is left-c.e.; furthermore, it also shows that Z is h-c.e. by an argument similar to the preceding one. □

7 Balanced randomness

Basics on balanced randomness

We study a restricted form of weak Demuth randomness (which was defined in the introduction). The bound on the number of version changes for the m-th component of a test is now $O(2^m)$.

Definition 14. A *balanced test* is a sequence of c.e. open sets $(G_m)_{m\in\mathbb{N}}$ such that $\forall m\, \lambda G_m \le 2^{-m}$ and, furthermore, there is a function f such that $G_m = [W_{f(m)}]^\prec$ and $f(m) = \lim_s g(m,s)$ for a computable function g such that the function mapping m to the size of the set $\{s : g(m,s) \ne g(m,s-1)\}$ is in $O(2^m)$.

A set Z *passes* the test if $Z \notin G_m$ for some m. We call Z *balanced random* if it passes each balanced test.

We denote $[W_{g(m,s)}]^\prec$ by $G_m[s]$ and call it the *version* of G_m at stage s.

Example 15. No $O(2^m)$-c.e. set is balanced random.

To see that this is the case, given an $O(2^m)$-c.e. set Z, simply let $G_m[s] = [Z_s \upharpoonright_m]$; then Z fails the balanced test $(G_m)_{m\in\mathbb{N}}$.

Before we proceed, we make three remarks on Definition 14. The second remark will explain our choice of terminology.

Remark 16. Again, we may monotonize a test and thus assume $G_m \supseteq G_{m+1}$ for each m, because the number of changes of $\bigcap_{i\le m} G_i[s]$ is also $O(2^m)$.

Remark 17. Let $(\alpha_i)_{i\in\mathbb{N}}$ be a nonincreasing computable sequence of rationals that converges effectively to 0, for instance $\alpha_i = 1/i$. If we build monotonicity into the definition of balanced tests, we can replace the bound 2^{-m} on the measure of the m-th component by α_m, and bound the number of changes by $O(1/\alpha_m)$. Thus, the important condition is being balanced in the sense that the measure bound times the bound on the number of changes is $O(1)$. In this case, we can emulate a test $(G_m)_{m\in\mathbb{N}}$ by a test $(H_i)_{i\in\mathbb{N}}$ as in Definition 14 by letting $H_i[s] = G_m[s]$, where m is least such that $2^{-i} \ge \alpha_m > 2^{-i-1}$.

Remark 18. $O(2^m)$ in Definition 14 can be replaced by 2^m. Suppose we are given a balanced test $(G_m)_{m\in\mathbb{N}}$ with at most $(N+1)2^m$ many changes to the version of G_m. Assume that for infinitely many m we have at least $N2^m$ many changes to the version of G_m (otherwise we can repeat with $N2^m$ many changes). Since $G_m[s]$ is given uniformly there is a computable increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that G_{m_i} changes at least $N2^{m_i}$ times. It is easy to use this sequence to build a new test $(V_m)_{m\in\mathbb{N}}$ with at most 2^m changes to the version V_m, and $\bigcap_m V_m \supseteq \bigcap_m G_m$. In fact, being balanced random is equivalent to passing every $c2^m$-change test for a fixed rational $c > 0$.

Difference randomness and Turing incompleteness

Franklin and Ng [4] have recently introduced *difference randomness*, where the m-th component of a test is a class of the form $A_m - B_m$ with measure at most 2^{-m}, for uniformly given Σ^0_1 classes A_m, B_m. To pass such a test means to be not in $A_m - B_m$ for some m. (We could replace the individual B_m in each component by $B = \bigcup B_m$. We may also assume that the test is monotonic after replacing $A_m - B_m$ by $\bigcap_{i\le m} A_i - B$ if necessary.)

Proposition 19. *Each balanced random set is difference random.*

Proof. Given a test $(A_m - B_m)_{m\in\mathbb{N}}$, we may assume that $\lambda(A_{m,t} - B_{m,t}) \le 2^{-m}$ for each t (these are the clopen sets effectively approximating A_m, B_m). At stage t let i be greatest such that $\lambda B_{m,t} \ge i2^{-m}$, and let $t^* \le t$ be least such that $\lambda B_{m,t^*} \ge i2^{-m}$. Let $G_m[t] = A_m - B_{m,t^*}$. Then G_m changes at most 2^m times. Clearly $A_m - B_m$ is contained in the last version of G_m. For each t we have $\lambda G_m[t] \le 2^{-m+1}$, so after omitting the first component we have a balanced test. □

Franklin and Ng [4] proved that for ML-random sets, being difference random is equivalent to being Turing incomplete. It is instructive to give a direct proof of this fact for balanced randomness.

Proposition 20. *Each balanced random set is Turing incomplete.*

Proof. Suppose Z is ML-random and Turing complete. Then $\Omega = \Gamma(Z)$ for some Turing functional Γ. By a result of Miller and Yu (see [11, Prop. 5.1.14]), there is a constant c such that $2^{-m} \geq \lambda\{Z : \Omega\upharpoonright_{m+c} \prec \Gamma(Z)\}$ for each m. Now let the version $G_m[t]$ copy $\{Z : \Omega_t\upharpoonright_{m+c} \prec \Gamma_t(Z)\}$ as long as the measure does not exceed 2^{-m}. Then Z fails the balanced test $(G_m)_{m\in\mathbb{N}}$. □

Balanced randomness and being ω-c.e.-tracing

The following (somewhat weak) highness property was introduced by Greenberg and Nies [5]; it coincides with the class $\mathcal{G}$ in [11, Proof of 8.5.17].

Definition 21. A set Z is *ω-c.e.-tracing* if each function $f \leq_{\mathrm{wtt}} \emptyset'$ has a Z-c.e. trace $(T^Z_x)_{x\in\mathbb{N}}$ such that $|T^Z_x| \leq 2^x$ for each x.

Since we trace only total functions, by a method of Terwijn and Zambella (see [11, Thm. 8.2.3]), the bound 2^x can be replaced by any order function without changing the class. Greenberg and Nies [5] showed that there is a single benign cost function such that each c.e. set obeying it is Turing below each ω-c.e.-tracing ML-random set. In particular, each strongly jump traceable, c.e. set is below each ω-c.e.-tracing set.

Fact 22. *No superlow set is ω-c.e.-tracing.*

Proof. Let Z be superlow. Let $(T^Z_{e,x})_{e,x\in\mathbb{N}}$ be an effective list of all Z-c.e. traces such that $|T^Z_{e,x}| \leq 2^x$ for each e, x. Since $Z' \leq_{\mathrm{tt}} \emptyset'$, the ternary relation "$n \in T^Z_{e,x}$" is truth-table below $\emptyset'$. Now let $f(x)$ be the least number not in $\bigcup_{e\leq x} T^Z_{e,x}$. Then $f \leq_{\mathrm{tt}} \emptyset'$, and f is not traced by any Z-c.e. trace $(T^Z_x)_{x\in\mathbb{N}}$ such that $|T^Z_x| \leq 2^x$ for each x. □

In the following result we characterize a notion slightly stronger than balanced randomness via the thesis in the introduction: within the ML-randoms, computationally less complex means more random. Let g be a computable function. To be g-weakly Demuth random means to pass all monotonic Demuth tests where the number of changes of version m is bounded by $g(m)$.

Theorem 23. *Let Z be an ML-random set. Then Z is $O(h(m)2^m)$-weakly Demuth random for some order function h iff Z is not ω-c.e.-tracing.*

Proof. ($\Leftarrow$) Suppose that Z is not $O(h(m)2^m)$-weakly Demuth random for any order function h. Suppose we are given a function $f \leq_{\mathrm{wtt}} \emptyset'$ with computable use bound $\tilde{h}$. Thus there is a computable approximation $f(x) = \lim_s f_s(x)$ with at most $\tilde{h}(x)$ changes. We will show that f is traced by a Z-c.e. trace $(T^Z_x)_{x\in\mathbb{N}}$ such that $|T^Z_x| \leq 2^x$ for each x.

Let $(m_i)_{i\in\mathbb{N}}$ be a computable sequence of numbers such that

$$\sum_i \tilde{h}(i)2^{-m_i} < \infty,$$

for instance $m_i = \lfloor \log \tilde{h}(i) + 2\log(i+1) \rfloor$. Let h be an order such that $h(m_i) < 2^{\frac{1}{2}i}$ for every i. Fix a monotonic Demuth test $(G_m)_{m\in\mathbb{N}}$ with an $O(h(m)2^m)$ bound on the number of version changes such that $Z \in \bigcap_m G_m$.

To obtain the required trace for f, we define an auxiliary Solovay test $\mathcal{S}$ of the form $\bigcup \mathcal{S}_i$. We put $[\sigma]$ into $\mathcal{S}_i$ if there are 2^i many versions $G_{m_i}[t]$ such that $[\sigma] \subseteq G_{m_i}[t]$. Clearly the $\mathcal{S}_i$ are uniformly Σ^0_1. We show that $\lambda\mathcal{S}_i = O(2^{-i}h(m_i))$ for each i. Let $(\sigma_k)_{k\in\mathbb{N}}$ be a prefix-free set of strings such that $\bigcup_k [\sigma_k] = \mathcal{S}_i$. Let C be the set of stages s such that $G_{m_i}[s]$ is a new version.

$$O(h(m_i)) = \sum_{s\in C} \lambda G_{m_i}[s] \geq \sum_{s\in C}\sum_k \lambda(G_{m_i}[s] \cap [\sigma_k]) \geq 2^i \sum_k \lambda[\sigma_k] = 2^i \lambda\mathcal{S}_i.$$

The total measure of all clopen sets put in $\mathcal{S}$ is at most

$$\sum_i 2^{-i}h(m_i) < \sum_i 2^{-\frac{1}{2}i} < \infty.$$

To define the c.e. operators T^Z_i, when Z enters $G_{m_i}[s]$, put $f_s(i)$ into T^Z_i. Since Z passes the Solovay test $\mathcal{S}$, for almost every i we put at most 2^i numbers into T^Z_i.

To show that f is traced, we define a further Solovay test $\mathcal{R}$. When $f_s(i) \neq f_{s-1}(i)$, put the current version $G_{m_i}[s]$ into $\mathcal{R}$. Note that $\mathcal{R}$ is a Solovay test because $\sum_i \tilde{h}(i)2^{-m_i} < \infty$. Since Z passes the Solovay test $\mathcal{R}$ but fails the monotonic Demuth test $(G_m)_{m\in\mathbb{N}}$, we have $f(i) \in T^Z_i$ for almost every i. For, if $f_s(i) \neq f_{s-1}(i)$ then Z must enter a further version $G_m[t]$ for some $t \geq s$, so we can put the new value $f_s(i)$ into T^Z_i.

($\Rightarrow$) We use the method of [11, Lem. 8.5.18]. Suppose Z is ω-c.e.-tracing. Given an order function h, we want to build an $O(h(m)2^m)$ weak Demuth test failed by Z. We may assume that $h(m) > 0$ for each m. Let $\widehat{h}(m) = \lfloor \sqrt{h(m)} \rfloor$, and let $(T_{e,x})_{e,x\in\mathbb{N}}$ be an effective list of all oracle c.e. traces (i.e., families of uniformly c.e. operators) such that for each oracle Y, we have $|T^Y_{e,x}| \leq \widehat{h}(x)$ for each e, x. For each Y, let $V^Y_m = \bigcup_{e<\hat{h}(m)} T^Y_{e,m}$. Then $|V^Y_m| \leq h(m)$ for each Y and m, and if $p \leq_{\text{wtt}} \emptyset'$ then $p(m) \in V^Z_m$ for almost all m.

We will build a function $p \leq_{\text{wtt}} \emptyset'$ such that

$$2^{-m} \geq \lambda\{Y : p(m) \in V^Y_m\} \text{ for each } m.$$

To do so, we define a computable approximation for p. Let $p_0(m) = 1$ for each m. For $s > 0$, if $2^{-m} < \lambda\{Y : p_{s-1}(m) \in V^Y_{m,s}\}$ then let $p_s(m) = p_{s-1}(m) + 1$; otherwise let $p_s(m) = p_{s-1}(m)$.

Claim. *For each m, s we have $p_s(m) \leq 2^m h(m)$.*

We apply [11, Exercise 1.9.15] for $\epsilon = 2^{-m}$. Suppose $p_s(m)$ is incremented $N > 2^m h(m)$ many times. Let $\mathcal{C}_i = \{Y : i \in V^Y_m\}$. For each $i \leq N$, since i is not the final value of $p_s(m)$, we have $\lambda\mathcal{C}_i \geq \epsilon$. Also $N\epsilon > h(m)$, so by that exercise there is an $F \subseteq \{1, \ldots, N\}$ such that $|F| = h(m) + 1$ and $\bigcap_{i\in F} \mathcal{C}_i \neq \emptyset$. If $Y \in \bigcap_{i\in F} \mathcal{C}_i$ then $F \subseteq V^Y_m$, contradicts the fact that $|V^Y_m| \leq h(m)$.

Now let $p(m) = \lim_t p_t(m)$ and $G_m = \{Y : p(m) \in V^Y_m\}$. Then $(G_m)_{m\in\mathbb{N}}$ is an $O(h(m)2^m)$ weak Demuth test as shown by the double sequence of versions $G_m[t] = \{Y : p_t(m) \in V^Y_{m,t}\}$. Furthermore, $Z \in G_m$ for almost all m, which is sufficient. ☐

Corollary 24. *Every superlow ML-random set is balanced random. Hence no superlow ML-random set is $O(2^n)$-c.e.*

By Theorem 11 we have the following result.

Corollary 25. *There is an ω-c.e.-tracing low ML-random set.*

Proof. Applying Theorem 11 to a Π^0_1 class $\mathcal{P} \subseteq \mathsf{MLR}$, we obtain a low ML-random set that is $o(2^n)$-c.e. This set is not balanced random. Then, by Theorem 23 the set is ω-c.e.-tracing. ☐

Recall that every Turing incomplete ML-random set is difference random. So the above proof also shows that some difference random set is not balanced random.

Remark 26. It is a persistent open question [10] whether each K-trivial A is ML-noncuppable. If A is a K-trivial set that can be cupped above $\emptyset'$ by a ML-random set Y (that is, $\emptyset' \leq_{\mathrm{T}} A \oplus Y$), then Y is LR-complete by work of Hirschfeldt and Nies (see the proof of (iii)⇒(iv) of [11, Thm. 8.5.18]).

Every LR-complete set is ω-c.e.-tracing by [11, Thm. 8.4.15]. Thus, by Theorem 23, no K-trivial set can be cupped above $\emptyset'$ by a set that is $O(h(m)2^m)$-weakly Demuth random for some order function h. We do not know at present whether every balanced random set already satisfies this condition for some h (equivalently, whether every balanced random set fails to be ω-c.e.-tracing). We do not even know whether a balanced random set can be LR-complete. We note that a balanced random set Z can be superhigh (i.e., $\emptyset'' \leq_{tt} Z'$) by [12, Thm. 4.4] where it is shown that each Demuth test is passed by a superhigh set. For, given an order function h there is a single Demuth test such that each set passing it passes all Demuth tests with h bounding the number of version changes.

8 Dominating the jump

Recall the discussion of certain lowness properties in the introduction. We now give the formal definition of being ω-c.e.-jump dominated.

Definition 27. A set A is *ω-c.e.-jump dominated* if there is an ω-c.e. function $g(x)$ such that $J^A(x) \leq g(x)$ for every x such that $J^A(x)$ is defined.

The following implications are easy to verify:

jump traceable ⇒ ω-c.e.-jump dominated ⇒ array computable.

We will see that both implications are proper.

There is a superlow set that is not jump traceable (for instance, a superlow ML-random set), so the first implication is proper by the following result. In [11, Thm. 3.6.26] it is shown that each Demuth random set is GL_1. This proof actually shows that each Demuth random set is ω-c.e.-jump dominated. This fact gives further examples of ω-c.e.-jump dominated sets that are not jump traceable. (We do not need full Demuth randomness as a hypothesis, because the number of version changes for the Demuth test $(S_m)_{m\in\mathbb{N}}$ constructed in this proof is bounded by 2^m; however, the test is necessarily not monotonic by Remark 30 below.)

Proposition 28. *(i) Every superlow set is ω-c.e.-jump dominated.*
(ii) For c.e. sets, the converse implication holds as well.

Proof. (i) Suppose that $\{A_s\}_{s\in\mathbb{N}}$ is a computable approximation of a superlow set A, and f is a computable function such that $\lim_s f(x,s) = A'(x)$ for every x, with computably bounded many mind changes. Let φ_e be the e-th partial computable function. We define uniformly c.e. sets $U_{i,x,e}$ as follows. For each s such that

1. $\varphi_e(x)[s]\downarrow$,
2. $f(\varphi_e(x),s) = 0$,
3. $|\{t < s : f(\varphi_e(x),t) \neq f(\varphi_e(x),t+1)\}| \leq 2i$, and
4. $J^{A_s}(x)[s]\downarrow$,

we enumerate the shortest initial segment σ of A_s such that $J^\sigma(x)[s]\downarrow$ into $U_{i,x,e}$. There is a computable function r such that $J^\sigma(r(x,e))\downarrow$ iff some $\tau \preceq \sigma$ is in $\bigcup_i U_{i,x,e}$. By the recursion theorem, we can fix an e such that $r(x,e) = \varphi_e(x)$ for all x.

We define a function g as follows. If $r(x,e) \notin A'$ then let $g(x) = 0$. Otherwise, there is an i such that there are exactly $2i+1$ many $f(r(x,e),-)$-changes. Let $g(x)$ be the maximum of all $J^\sigma(x)$ such that $\sigma \in U_{i,x,e}$. Since $U_{i,x,e}$ stabilizes by the least stage t at which $f(r(x,e),t)$ has changed exactly $2i+1$ many times, it is easy to see that g is ω-c.e.

Suppose that $J^A(x)\downarrow$, and let σ be the shortest initial segment of A such that $J^\sigma(x)\downarrow$. If $r(x,e) \notin A'$ then $f(\varphi_e(x),s) = f(r(x,e),s) = 0$ for all sufficiently large s, so we eventually put σ into $U_{i,x,e}$ for some i. But then $J^\sigma(r(x,e))\downarrow$, so $r(x,e) \in A'$, which is a contradiction. Thus we must have $r(x,e) \in A'$. So there is an i such that there are exactly $2i+1$ many $f(r(x,e),-)$-changes. Note that $\bigcup_j U_{j,x,e} = U_{i,x,e}$, because if $|\{t < s : f(r(x,e),t) \neq f(r(x,e),t+1)\}| > 2i$ then $f(r(x,e),s) = 1$. Since $J^\sigma(r(x,e))\downarrow$ but $J^\tau(r(x,e))\uparrow$ for all $\tau \prec \sigma$, we must have $\sigma \in \bigcup_j U_{j,x,e} = U_{i,x,e}$. Thus $J^A(x) \leq g(x)$.

(ii) Suppose that A is c.e. and there is an ω-c.e. function g such that $J^A(x) \leq g(x)$ for all x such that $J^A(x)\downarrow$. Let $g(x,s)$ be an approximation to g with a computably bounded number of mind changes. Let c be a computable function such that $J^A(c(x))$ is the least s for which $J^A(x)[s]\downarrow$ with an A-correct use. Let $p(x,s) = 1$ if $J^A(x)[g(c(x),s)]\downarrow$; otherwise, let $p(x,s) = 0$. It is easy to see that

$p(x,-)$ changes at most as often as $g(c(x),-)$, and that $A'(x) = \lim_s p(x,s)$, so A is superlow. $\square$

To show that the second implication above is proper, note that every ω-c.e.-jump dominated set is Δ^0_2-jump dominated, i.e., Definition 27 holds with a Δ^0_2 function g. This notion, which has also been called "weakly jump traceable", implies that the set is GL_1. Some array computable set is not GL_1, and hence not ω-c.e.-jump dominated.

The following result extends Fact 22.

Fact 29. *If a set is ω-c.e.-jump dominated, then it is not ω-c.e.-tracing.*

Proof. As before, let $(T^Z_{e,x})_{e,x\in\mathbb{N}}$ be an effective list of all Z-c.e. traces such that $|T^Z_{e,x}| \le 2^x$ for each e, x. There is a Turing functional Γ such that $\Gamma^Z(e,x,r)$ is the r-th element enumerated into $T^Z_{e,x}$. Hence we can choose a ternary computable function p such that $\Gamma^X(e,x,r) \simeq J^X(p(e,x,r))$ for each oracle X, where $\simeq$ means that the two functions have the same domain and are equal where defined.

Now suppose Z is ω-c.e.-jump dominated via a function $g \le_{\mathrm{wtt}} \emptyset'$. Let

$$\hat{g}(x) = 1 + \max\{g(p(e,x,r)) : e \le x \wedge r \le 2^x\}.$$

Then $\hat{g} \le_{\mathrm{wtt}} \emptyset'$. Furthermore, $\hat{g}(x) > \max T^Z_{e,x}$ for each $e \le x$. Hence $\hat{g}$ is not traced by any Z-c.e. trace of the appropriate size. $\square$

Remark 30. Every weakly 2-random set is weakly Demuth random. A weakly 2-random set that is not GL_1 is neither ω-c.e.-jump dominated nor ω-c.e.-tracing, by Theorem 23. Such a set was proved to exist by Lewis, Montalbán, and Nies [8]. (Miller and Nies later proved that no weakly 2-random set of hyperimmune-free degree can be GL_1 [11, Thm. 8.1.19].)

Remark 31. We say that Z is *Δ^0_2-tracing* if each function $f \le_{\mathrm{T}} \emptyset'$ has a Z-c.e. trace $(T^Z_x)_{x\in\mathbb{N}}$ such that $|T^Z_x| \le 2^x$ for each x. Barmpalias [1, Cor. 1] proved a result related to Fact 29 but harder: if a set is array computable, then it is not Δ^0_2-tracing. The hypothesis is weaker, and so is the conclusion. We do not know at present whether there is an array computable ω-c.e.-tracing set.

9 Tracing, jump domination, and randomness

In this section we examine some relationships between being ω-c.e.-tracing, being ω-c.e.-jump dominated, and randomness theoretic notions of highness. As corollaries, we obtain upper bounds on the class of sets that can be cupped to $\emptyset'$ by a Demuth random set, and the class of sets that can be cupped to $\emptyset'$ by a weakly Demuth random set.

We say that a set A is Demuth cuppable if there is a Demuth random set X such that $\emptyset' \le_{\mathrm{T}} A \oplus X$. We say that A is High(ML-random, Demuth random) if every set that is ML-random relative to A is Demuth random. The analogous definitions apply to weak Demuth randomness.

Proposition 32. *Every ω-c.e.-tracing set is High(ML-random, Demuth random).*

Proof. Suppose A is ω-c.e.-tracing. Fix a Demuth test $(G_m)_{m\in\mathbb{N}}$. Let the ω-c.e. function f be such that $[W_{f(m)}]^{\prec} = G_m$ for all m. Let $(T^A_m)_{m\in\mathbb{N}}$ be a c.e. trace relative to A such that $|T^A_m| \leq m$, and for each m, the component T^A_m contains the least s such that $f(m) = f(m, s)$.

Define an A-Solovay test $(S^A_m)_{m\in\mathbb{N}}$ as follows: for each $s \in T^A_m$, enumerate $[W_{f(m,s)}]^{\prec}$ into $\mathcal{S}^A_m$. Then $\sum_m \mu(\mathcal{S}^A_m) \leq \sum_m m2^{-m} < \infty$. Thus no set that is in infinitely many G_m is ML-random relative to A. ◻

For every set A, we have $\emptyset' \leq_{\mathrm{T}} A \oplus \Omega^A$. So every ω-c.e.-tracing set is Demuth cuppable. We do not know whether the converse of Proposition 32 holds.

Proposition 33. *If A is not ω-c.e.-jump dominated then A is High(ML-random, weakly Demuth random).*

Proof. Suppose A is not High(ML random, weakly Demuth random). Let Z be ML-random relative to A and fix a weak Demuth test such that $Z \in G_m$ for every m. Let the ω-c.e. function f be such that $[W_{f(m)}]^{\prec} = G_m$ for all m. We define an ω-c.e. function g and an oracle Solovay test $(\mathcal{S}_m)_{m\in\mathbb{N}}$.

Fix m. Let $0 = s_0 < s_1 < \cdots < s_N$ list all s such that $f(m, s) \neq f(m, s-1)$. At each stage t, proceed as follows. Let i be largest such that $s_i \leq t$. For each σ such that $J^\sigma(m)$ converges for the first time at stage t, put σ into an auxiliary set C_i and put each $[\tau] \subseteq [W_{f(m,s_i)}]^{\prec}$ into $\mathcal{S}^\sigma_m$. Let $g(m) = \max\{J^\sigma(m) : \sigma \in C_i \ \wedge\ i < N\}$.

Clearly, g is an ω-c.e. function. The total weight of strings enumerated into S^A_m is at most 2^{-m} for each m, so $(S^A_m)_{m\in\mathbb{N}}$ is an A-Solovay test. Thus, for almost every m, we have $Z \notin \mathcal{S}^A_m$. For any such m, if $J^A(m)\downarrow$, then the first stage at which $J^A(m)$ converges must be less than s_N, where N is as above, as otherwise we would have $Z \in G_m = [W_{f(m)}]^{\prec} = [W_{f(m,s_n)}]^{\prec} \subseteq \mathcal{S}^A_m$. It follows that $J^A(m) \leq g(m)$. ◻

An immediate corollary is that every set A that is not ω-c.e.-jump dominated is cuppable via a weakly Demuth random set, namely Ω^A. In particular, every c.e. non-superlow set is cuppable via a weakly Demuth random set.

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