Degree Spectra of Relations on Computable Structures

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0 Introduction

There has been increasing interest over the last few decades in the study of the effective content of Mathematics. One field whose effective content has been the subject of a large body of work, dating back at least to the early 1960's, is model theory. (A valuable reference is the handbook [7]. In particular, the introduction and the articles by Ershov and Goncharov and by Harizanov give useful overviews, while the articles by Ash and by Goncharov cover material related to the topic of this communication.)

Several different notions of effectiveness of model-theoretic structures have been investigated. This communication is concerned with *computable* structures, that is, structures with computable domains whose constants, functions, and relations are uniformly computable.

In model theory, we identify isomorphic structures. From the point of view of computable model theory, however, two isomorphic structures might be very different. For example, under the standard ordering of ω, the successor relation is computable, but it is not hard to construct a computable linear ordering of type ω in which the successor relation is not computable. In fact, for every computably enumerable (c.e.) degree $\mathbf{a}$, we can construct a computable linear ordering of type ω in which the successor relation has degree $\mathbf{a}$. It is also possible to build two isomorphic computable groups, only one of which has a computable center, or two isomorphic Boolean algebras, only one of which has a computable set of atoms. Thus, for the purposes of computable model theory, studying structures up to isomorphism is not enough. Instead, we study structures up to *computable* isomorphism. This leads naturally to the idea of a *computable presentation* of a structure, which is, roughly speaking, a computable copy of this structure. (Formal definitions of this and other concepts will be given in the next section.)

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What the above examples, as well as many other natural ones, have in common is the idea of attempting to understand the differences between noncomputably isomorphic computable presentations of a structure $\mathcal{M}$ by comparing (from a computability-theoretic point of view) the images in these presentations of a particular relation on the domain of $\mathcal{M}$. (Of course, this is only interesting if this relation is not the interpretation in $\mathcal{M}$ of a relation in the language of $\mathcal{M}$.) The study of additional relations on computable structures began with the work of Ash and Nerode [2] and has been continued in a large number of papers. (References can be found in the aforementioned articles in [7].)

One approach to the study of relations on computable structures, which began with the work of Harizanov [17], is to look at the collection of (Turing) degrees of the images of a relation in different computable presentations of a structure, which is known as the *degree spectrum* of the relation. This communication is mainly concerned with the question of which sets of degrees can be realized as degree spectra of relations on computable structures, both in the general case and with certain restrictions imposed on the relation or the structure. The latter case will bring us to the intersection of computable model theory and computable algebra.

After discussing basic definitions and notation in Section 1, in Section 2 we give a few examples of possible degree spectra of relations. In Section 3, we discuss finite degree spectra of relations, and in Section 4, we present an application of the techniques used to construct relations with certain particular finite degree spectra to the question of what can happen to the number of computable presentations of a structure when the structure is expanded by a constant. Finally, Sections 5 and 6 deal with degree spectra of relations on structures which belong to well-known classes of structures such as linear orderings, groups, rings, and so forth.

1 Definitions and Notation

For basic notions of computability theory and model theory, the reader is referred to [33] and [23], respectively. By *degree*, we will mean Turing degree unless otherwise specified. We will denote the join of degrees $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \cup \mathbf{b}$.

Since α-c.e. sets and degrees for arbitrary computable ordinals α may not be familiar to all readers, we define them here. It is slightly cumbersome to give a definition of α-c.e. sets that works for both $\alpha < \omega$ (where we want to agree with the definition of n-c.e. sets, $n \in \omega$, given by the difference hierarchy) and $\alpha \geqslant \omega$. Furthermore, for $\alpha \geqslant \omega^2$, which sets and degrees are α-c.e. depends on the choice of ordinal notation system (see [6] for details). The following (slightly nonstandard) definition works well for our purposes, and is easily seen to be equivalent to standard definitions of n-c.e. and α-c.e. sets and degrees.

1.1 Definition. Let α be a computable ordinal and assume we have fixed a univalent, computably related ordinal notation system with a notation for α. Let $\ulcorner\beta\urcorner$ denote the unique notation for $\beta \leqslant \alpha$ in this system. A set A is *α-c.e.* if there exists a partial computable binary function Ψ satisfying the following conditions for all $x \in \omega$.

1. $\Psi(\ulcorner\alpha\urcorner, x)\downarrow = 0$.
2. If $\alpha \geqslant \omega$ then there exists a $\beta < \alpha$ such that $\Psi(\ulcorner\beta\urcorner, x)$ converges.
3. For the least $\beta \leqslant \alpha$ such that $\Psi(\ulcorner\beta\urcorner, x)$ converges, $\Psi(\ulcorner\beta\urcorner, x) = A(x)$.

A degree is α-c.e. if it contains an α-c.e. set.

One of the central notions of computable model theory is that of a *computable structure.* We will always assume that we are working with computable languages.

1.2 Definition. A structure $\mathcal{A}$ is *computable* if both its domain $|\mathcal{A}|$ and the atomic diagram of $\langle \mathcal{A}, a\rangle_{a\in|\mathcal{A}|}$ are computable.

If, in addition, the n-quantifier diagram of $\langle \mathcal{A}, a\rangle_{a\in|\mathcal{A}|}$ is computable then $\mathcal{A}$ is *n-decidable*, while if the full first-order diagram of $\langle \mathcal{A}, a\rangle_{a\in|\mathcal{A}|}$ is computable then $\mathcal{A}$ is *decidable.*

As we have discussed above, the following definition is a natural one to make in the context of computable model theory.

1.3 Definition. An isomorphism from a structure $\mathcal{M}$ to a computable structure is called a *computable presentation* of $\mathcal{M}$. (We often abuse terminology and refer to the image of a computable presentation as a computable presentation.)

If $\mathcal{M}$ has a computable presentation then it is *computably presentable.*

Another important notion is the number of computable presentations of a computably presentable structure.

1.4 Definition. The *computable dimension* of a computably presentable structure $\mathcal{M}$ is the number of computable presentations of $\mathcal{M}$ up to computable isomorphism.

A structure of computable dimension 1 is said to be *computably categorical.*

We will also have occasion to consider structures that, while not computably categorical, have relatively simple isomorphisms between their various computable presentations.

1.5 Definition. A computably presentable structure is *Δ^0_2-categorical* if any two of its presentations are isomorphic via a Δ^0_2 map.

In Section 3, we will also mention c.e. presentations. We will take the more general of two possible definitions of c.e. structure, in which equality is c.e. rather than computable. It will be clear that the result involving c.e. structures in Section 3 also holds for the less general definition.

1.6 Definition. A structure $\mathcal{A}$ is *c.e.* if its domain $|\mathcal{A}|$ is computable and the atomic diagram of $\langle \mathcal{A}, a\rangle_{a\in|\mathcal{A}|}$ is c.e..

An isomorphism from a structure $\mathcal{M}$ to a c.e. structure is called a *c.e. presentation* of $\mathcal{M}$. If $\mathcal{M}$ has a c.e. presentation then it is *c.e. presentable.* The *c.e. dimension* of a c.e. presentable structure $\mathcal{M}$ is the number of c.e. presentations of $\mathcal{M}$ up to computable isomorphism.

As we have mentioned above, the study of additional relations on computable structures began with the work of Ash and Nerode [2], who were concerned with relations that maintain some degree of effectiveness in different computable presentations of a structure.

1.7 Definition. Let U be a relation on the domain of a computable structure $\mathcal{A}$ and let $\mathfrak{C}$ be a class of relations. U is *intrinsically* $\mathfrak{C}$ on $\mathcal{A}$ if the image of U in any computable presentation of $\mathcal{A}$ is in $\mathfrak{C}$.

In [2], conditions that guarantee that a relation is intrinsically computable or intrinsically c.e. were given. More recent work has led to a number of other conditions guaranteeing that a relation is intrinsically $\mathfrak{C}$ for various classes $\mathfrak{C}$ (see [3], for example).

An important class of relations is that of *invariant* relations. All relations mentioned below are invariant unless otherwise noted.

1.8 Definition. A relation U on the domain of a structure $\mathcal{M}$ is *invariant* if, for every automorphism $f : \mathcal{M} \cong \mathcal{M}$, $f(U) = U$.

The following definition is due to Harizanov [17].

1.9 Definition. Let U be a relation on the domain of a computable structure $\mathcal{A}$. The *degree spectrum* of U on $\mathcal{A}$, $\mathrm{DgSp}_{\mathcal{A}}(U)$, is the set of degrees of the images of U in all computable presentations of $\mathcal{A}$.

It is also interesting to consider degree spectra of relations with respect to other reducibilities.

1.10 Definition. Let r be a reducibility, such as many-one reducibility (m-reducibility) or weak truth-table reducibility (wtt-reducibility). Let U be a relation on the domain of a computable structure $\mathcal{A}$. The *r-degree spectrum* of U on $\mathcal{A}$, $\mathrm{DgSp}^{\mathrm{r}}_{\mathcal{A}}(U)$, is the set of r-degrees of the images of U in all computable presentations of $\mathcal{A}$.

2 Examples of Degree Spectra of Relations

In this section, we give a few examples of sets of degrees that can be realized as degree spectra of relations. We begin with three observations that often allow us to modify and combine examples of possible degree spectra of relations to yield further examples.

The first one is rather simple: For any class of relations $\mathfrak{C}$, if U is an intrinsically $\mathfrak{C}$ k-ary relation on the domain of a computable structure $\mathcal{A}$ then $V = (|\mathcal{A}|)^k - U$ is intrinsically co-$\mathfrak{C}$ and $\mathrm{DgSp}_{\mathcal{A}}(V) = \mathrm{DgSp}_{\mathcal{A}}(U)$.

Our second observation is contained in the following result.

2.1 Proposition. *Let A and B be sets of degrees and let $\mathfrak{C}$ be a class of relations closed under m-equivalence and finite disjoint unions. Let $C = \{\mathbf{c} \mid \exists \mathbf{a}, \mathbf{b}(\mathbf{a} \in A \wedge \mathbf{b} \in B \wedge \mathbf{c} = \mathbf{a} \cup \mathbf{b})\}$. If there exist intrinsically $\mathfrak{C}$ relations U and V on computable structures $\mathcal{A}$*

and $\mathcal{B}$, respectively, such that $\mathrm{DgSp}_{\mathcal{A}}(U) = A$ *and* $\mathrm{DgSp}_{\mathcal{B}}(V) = B$*, then there exists an intrinsically* $\mathfrak{C}$ *relation* W *on a computable structure* $\mathcal{C}$ *such that* $\mathrm{DgSp}_{\mathcal{C}}(W) = C$*. Furthermore, if both* U *and* V *are invariant then* W *can be chosen to be invariant, and if both* $\mathcal{A}$ *and* $\mathcal{B}$ *are* Δ^0_2*-categorical then* $\mathcal{C}$ *can be chosen to be* Δ^0_2*-categorical.*

Our final observation is about degree spectra of relations with respect to different reducibilities. Let r and s be reducibilities such that r is stronger than s, and let U be a relation on the domain of a computable structure $\mathcal{A}$. Then $\mathrm{DgSp}^{\mathrm{s}}_{\mathcal{A}}(U)$ is equal to the set of s-degrees that contain at least one r-degree in $\mathrm{DgSp}^{\mathrm{r}}_{\mathcal{A}}(U)$.

Now let C_0 be the directed graph consisting of a single node and no edges and let C_1 be the directed graph consisting of two nodes x and y with an edge from x to y. Consider the directed graph $\mathcal{G} = \langle |\mathcal{G}|, E \rangle$ that is the disjoint union of infinitely many copies of each of C_0 and C_1. Let U be the unary relation on the domain of $\mathcal{G}$ that holds of x if and only if there is a y such that $E(x, y)$. Since U is defined by an existential formula in the language of directed graphs, U is intrinsically c.e.. Furthermore, it is not hard to check that $\mathrm{DgSp}_{\mathcal{G}}(U)$ contains all c.e. degrees. (In fact, $\mathrm{DgSp}^{\mathrm{m}}_{\mathcal{G}}(U)$ contains all c.e. m-degrees other than the m-degrees of $\emptyset$ and ω.)

By modifying this example in a natural way, it is possible to realize, for any $n \in \omega$, the set of all n-c.e. degrees as the degree spectrum of an intrinsically n-c.e. relation on the domain of a computable structure. Another way to generalize the above example is given in the following result.

2.2 Theorem. *Let* $n > 0$*. There exists an intrinsically* Σ^0_n *relation* U *on the domain of a computable structure* $\mathcal{A}$ *such that* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *consists of all* Σ^0_n *degrees.*

By the first observation above, we can replace Σ^0_n by Π^0_n in the statement of Theorem 2.2. The next result shows that we can also replace Σ^0_n by Δ^0_n.

2.3 Theorem. *Let* $n > 0$*. There exists an intrinsically* Δ^0_n *relation* U *on the domain of a computable structure* $\mathcal{A}$ *such that* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *consists of all* Δ^0_n *degrees.*

It is worth noting that the above results stand in contrast to the following theorem, proved independently by Ash, Cholak, and Knight [1] and Harizanov [20], thus illustrating the potential differences between the general case, in which we are trying to realize certain sets of degrees as degree spectra of relations on computable structures with no additional restrictions, and cases in which we impose extra conditions on some aspect of this realization. (See Section 6 for more on this theme.)

2.4 Theorem (Ash, Cholak, and Knight; Harizanov)**.** *Let* U *be a relation on the domain of a computable structure* $\mathcal{A}$*. Suppose that for each* Δ^0_3 *set* C *there is an isomorphism* f *from* $\mathcal{A}$ *to a computable structure* $\mathcal{B}$ *such that* $f \leqslant_T C$ *and* $C \leqslant_T f(U)$*. Then for each set* C *there is an isomorphism* f *from* $\mathcal{A}$ *to a computable structure* $\mathcal{B}$ *such that* $f \leqslant_T C$ *and* $C \leqslant_T f(U)$*. In particular,* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *contains every degree.*

As mentioned above, it is not hard to construct, for each $n > 0$, an intrinsically n-c.e. relation on the domain of a computable structure whose degree spectrum consists of all n-c.e. degrees. A little more work can get us a similar result with α-c.e. in place of n-c.e. for any computable ordinal α. The following theorem is a generalization of this result, although the proof is somewhat more complicated and involves building graphs that represent computations in which we are computably approximating a given c.e. oracle.

2.5 Theorem. *Let α be a computable ordinal and let* $\mathbf{a}$ *be a c.e. degree. There exists an intrinsically α-c.e. invariant relation U on the domain of a Δ^0_2-categorical computable structure $\mathcal{A}$ such that* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *consists of all α-c.e. degrees less than or equal to* $\mathbf{a}$.

It is also possible to realize all degrees below a given c.e. degree as the degree spectrum of a relation on the domain of a computable structure.

2.6 Theorem. *Let* $\mathbf{a}$ *be a c.e. degree. There exists an invariant relation U on the domain of a Δ^0_2-categorical computable structure $\mathcal{A}$ such that* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *consists of all degrees less than or equal to* $\mathbf{a}$.

The fact that, in Theorems 2.5 and 2.6, U is invariant and $\mathcal{A}$ is Δ^0_2-categorical is interesting because, as we will see in Section 6, there are certain restrictions on what the degree spectrum of an invariant computable relation on the domain of a Δ^0_2-categorical computable structure can be.

It is easy to give an example of a relation on the domain of a computable structure whose degree spectrum contains all degrees, and for any degree $\mathbf{a}$, it is equally easy to give an example of a relation on the domain of a computable structure whose degree spectrum is $\{\mathbf{a}\}$. Thus, realizing all degrees above a given (not necessarily c.e.) degree as the degree spectrum of a relation on the domain of a computable structure is a simple application of Proposition 2.1.

Similarly, combining Theorem 2.6 with Proposition 2.1, we see that if $\mathbf{a} < \mathbf{b}$ are degrees and $\mathbf{b}$ is c.e. then there exists a relation U on the domain of a computable structure $\mathcal{A}$ such that $\mathrm{DgSp}_{\mathcal{A}}(U)$ consists of all degrees in the interval $[\mathbf{a}, \mathbf{b}]$. The analogous argument shows that, for each computable ordinal α, there exists an intrinsically α-c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $\mathrm{DgSp}_{\mathcal{A}}(U)$ consists of all α-c.e. degrees in the interval $[\mathbf{a}, \mathbf{b}]$.

Theorems 2.5 and 2.6 remain true with *degree* replaced by *wtt-degree* and $DgSp_{\mathcal{A}}(U)$ replaced by $DgSp^{wtt}_{\mathcal{A}}(U)$.

Realizing sets of degrees of finite cardinality greater than 1 as degree spectra of relations on computable structures, which is the topic of the next section, normally requires fairly complicated constructions. It is possible, however, to obtain finite degree spectra of relations as easy corollaries to two results, one in computable model theory, and the other in classical computability theory.

First of all, the existence of a relation with a two-element degree spectrum that includes $\mathbf{0}$ follows from the existence of a rigid structure of computable dimension 2, which was shown by Goncharov [10]. (This has been noted by Harizanov (see [18]).)

It is also possible to give an example of an intrinsically d.c.e. relation on the domain of a Δ^0_2-categorical structure with a two-element degree spectrum, but one that does not include $\mathbf{0}$. Let $\mathbf{d}$ be a maximal incomplete d.c.e. degree, as constructed in [5]. (That is, $\mathbf{d} \neq \mathbf{0}'$ is d.c.e. and there are no d.c.e. degrees in $(\mathbf{d}, \mathbf{0}')$.) It is easy to define an invariant d.c.e. relation V on the domain of a computably categorical structure $\mathcal{B}$ whose degree spectrum is the singleton $\{\mathbf{d}\}$. It is likewise easy to define an invariant intrinsically d.c.e. relation U on the domain of a Δ^0_2-categorical computable structure $\mathcal{A}$ whose degree spectrum is the set of all d.c.e. degrees.

By Proposition 2.1, there exists an intrinsically d.c.e. invariant relation W on the domain of a Δ^0_2-categorical computable structure $\mathcal{C}$ whose degree spectrum is

$$\{\mathbf{c} \mid \exists \mathbf{a}, \mathbf{b}(\mathbf{a} \in \mathrm{DgSp}_{\mathcal{A}}(U) \wedge \mathbf{b} \in \mathrm{DgSp}_{\mathcal{B}}(V) \wedge \mathbf{c} = \mathbf{a} \cup \mathbf{b})\} = \\ \{\mathbf{c} \mid \mathbf{d} \leqslant \mathbf{c} \text{ and } \mathbf{c} \text{ is d.c.e.}\} = \{\mathbf{d}, \mathbf{0}'\}.$$

The fact that $\mathcal{C}$ is Δ^0_2-categorical and W is invariant is particularly interesting in light of the result, discussed in Section 6, that no finite set of degrees containing $\mathbf{0}$ can be the degree spectrum of an invariant relation on the domain of a Δ^0_2-categorical computable structure.

3 Finite Degree Spectra of Relations

The Ash-Nerode type conditions mentioned in Section 1 usually imply that the degree spectrum of a relation is either a singleton or infinite. Indeed, for various classes of degrees, conditions have been formulated that guarantee that the degree spectrum of a relation consists of all the degrees in the given class (see [1], for example). Motivated by these considerations, as well as by Goncharov's examples [10] of structures of finite computable dimension, Harizanov and Millar suggested the study of relations with finite degree spectra.

Harizanov [19] was the first to give an example of an intrinsically Δ^0_2 relation with a two-element degree spectrum that includes $\mathbf{0}$.

3.1 Theorem (Harizanov)**.** *There exist a Δ^0_2 but not c.e. degree $\mathbf{a}$ and a relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension 2 such that $DgSp_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{a}\}$.*

Khoussainov and Shore and Goncharov [15],[24] showed the existence of an intrinsically c.e. relation with a two-element degree spectrum.

3.2 Theorem (Khoussainov and Shore, Goncharov)**.** *There exist a c.e. degree $\mathbf{a}$ and an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension 2 such that $DgSp_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{a}\}$.*

This left open the question, asked explicitly in [15], of which (c.e.) degrees can be the nonzero element of a two-element degree spectrum. This is answered for c.e. degrees in [21], where it is shown that every c.e. degree belongs to some two-element degree spectrum whose other element is $\mathbf{0}$.

3.3 Theorem. *Let $\mathbf{a} > \mathbf{0}$ be a c.e. degree. There exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $DgSp_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{a}\}$.*

The proof of this theorem uses techniques from [24], which in turn builds on work of Goncharov [9],[10] and Cholak, Goncharov, Khoussainov, and Shore [4]. This proof can be modified to obtain the following result, which is also due independently to Khoussainov and Shore [25] (as are its extensions, Theorems 3.7 and 3.9 below), whose proof uses a complicated modification of their proof of Theorem 3.2.

3.4 Theorem. *Let $\mathbf{a} > \mathbf{0}$ be a c.e. degree. There exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension 2 such that $DgSp_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{a}\}$. In addition, $\mathcal{A}$ can be chosen so that every c.e. presentation of $\mathcal{A}$ is computable, which implies that $\mathcal{A}$ has c.e. dimension 2.*

In [24], Khoussainov and Shore also proved the following theorem.

3.5 Theorem (Khoussainov and Shore)**.** *For each computable partial ordering $\mathcal{P}$ there exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $\langle DgSp_{\mathcal{A}}(U), \leqslant_T\rangle \cong \mathcal{P}$. If $\mathcal{P}$ has a least element then we can pick U and $\mathcal{A}$ so that $\mathbf{0} \in DgSp_{\mathcal{A}}(U)$.*

The proof of Theorem 3.3 can be modified to establish the following extension of Theorem 3.5.

3.6 Theorem. *Let $\{A_i\}_{i\in\omega}$ be a uniformly c.e. collection of sets. There exists an intrinsically c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $DgSp_{\mathcal{A}}(U) = \{\deg(A_i) \mid i \in \omega\}$.*

Another way in which we can extend Theorem 3.3 is by broadening our focus from the c.e. degrees to larger classes of degrees.

3.7 Theorem. *Let $\alpha \in \omega \cup \{\omega\}$ and let $\mathbf{a} > \mathbf{0}$ be an α-c.e. degree. There exists an intrinsically α-c.e. relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension 2 such that $DgSp_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{a}\}$.*

Remark. One interesting consequence of Theorem 3.7 is that there exists a minimal degree $\mathbf{a}$ such that $\{\mathbf{0}, \mathbf{a}\}$ is realized as the degree spectrum of a relation on the domain of a computable structure.

Theorems 3.6 and 3.7 can be conflated to produce the following results.

3.8 Theorem. *Let $\alpha \in \omega \cup \{\omega\}$ and let $\{A_i\}_{i\in\omega}$ be a uniformly α-c.e. collection of sets. There exists an intrinsically α-c.e. relation U on the domain of a computable structure $\mathcal{A}$ such that $DgSp_{\mathcal{A}}(U) = \{\deg(A_i) \mid i \in \omega\}$.*

3.9 Theorem. *Let $\alpha \in \omega \cup \{\omega\}$ and let $\mathbf{a}_0, \ldots, \mathbf{a}_n$ be α-c.e. degrees. There exists an intrinsically α-c.e. relation U on the domain of a computable structure $\mathcal{A}$ of computable dimension $n+1$ such that $DgSp_{\mathcal{A}}(U) = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$.*

The proofs of Theorems 3.3, 3.4, and 3.7 are such that these theorems remain true with *degree* replaced by *m-degree* and $DgSp_{\mathcal{A}}(U)$ replaced by $DgSp^m_{\mathcal{A}}(U)$. Thus, for any reducibility r weaker than m-reducibility, these theorems remain true with *degree* replaced by *r-degree* and $DgSp_{\mathcal{A}}(U)$ replaced by $DgSp^r_{\mathcal{A}}(U)$. The same holds of Theorems 3.6 and 3.8 if we require that $A_i \neq \emptyset$ and $A_i \neq \omega$ for all $i \in \omega$, and of Theorem 3.9 if we require that the m-degrees of $\emptyset$ and ω not be on the list $\mathbf{a}_0, \ldots, \mathbf{a}_n$.

4 Expansions of Computably Categorical Structures

In classical model theory, it follows from the Ryll-Nardzweski Theorem that a countably categorical structure remains countably categorical when expanded by finitely many constants. It is natural to ask whether the same is true in the analogous situation in computable model theory. That is, does every computably categorical structure remain computably categorical when expanded by finitely many constants?

Millar [29] showed that, with a relatively small additional amount of decidability, computable categoricity is preserved under expansion by finitely many constants.

4.1 Theorem (Millar)**.** *If $\mathcal{A}$ is computably categorical and 1-decidable then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical.*

However, preservation of categoricity does not hold in general, as was shown by Cholak, Goncharov, Khoussainov, and Shore [4].

4.2 Theorem (Cholak, Goncharov, Khoussainov, and Shore)**.** *For each $k > 0$ there exists a computably categorical structure $\mathcal{A}$ and an $a \in |\mathcal{A}|$ such that $\langle \mathcal{A}, a \rangle$ has computable dimension k.*

This raises the following question, left open in [4], as well as in [24], where an easier proof of Theorem 4.2 is given: Does there exist a computably categorical structure whose expansion by some set of finitely many constants has computable dimension ω? In joint work with Bakhadyr Khoussainov and Richard A. Shore, the methods of the proof of Theorem 3.4 have been applied to give the following positive answer to this question.

4.3 Theorem (Hirschfeldt, Khoussainov, and Shore)**.** *There exists a computably categorical structure $\mathcal{A}$ and an $a \in |\mathcal{A}|$ such that $\langle \mathcal{A}, a \rangle$ has computable dimension ω.*

5 Relations on Algebraic Structures I: Positive Results

Whenever a computable structure with a particularly interesting property is found, it is natural to ask whether similar examples can be found within well-known classes of algebraic structures, such as groups, rings, lattices, and so forth. As an example, let us consider the computable dimension of computable structures.

It is easy to construct computable structures with computable dimension 1 or ω. Indeed, most familiar structures and even all members of many classes of familiar structures have computable dimension 1 or ω. For instance, Nurtazin [31] showed that all decidable structures fall into this category. Goncharov [8] later extended this result to 1-decidable structures, and there have been several other familiar classes of structures for which similar results have been established.

5.1 Theorem. *All structures in each of the following classes have computable dimension* 1 *or* ω.

- **(Nurtazin; Metakides and Nerode)** *algebraically closed fields*
- **(Nurtazin)** *real closed fields*
- **(Goncharov)** *Abelian groups*
- **(Goncharov and Dzgoev; Remmel)** *linear orderings*
- **(Goncharov; LaRoche)** *Boolean algebras*
- **(Goncharov)** Δ^0_2*-categorical structures*

The result for algebraically closed and real closed fields is implied by the results in [31]; the result for algebraically closed fields was also independently proved in [28]. The result for Abelian groups appears in [11], that for linear orderings independently in [14] and [32], and that for Δ^0_2-categorical structures in [12]. The result for Boolean algebras appears in full in [13], though it is implicit in earlier work of Goncharov and, independently, in [27].

Thus, an important question early in the development of computable model theory was whether there exist computable structures of finite computable dimension greater than 1. As previously mentioned, this question was answered positively by Goncharov [10].

5.2 Theorem (Goncharov). *For each* $n > 0$ *there is a computable structure with computable dimension* n.

Further investigation led to examples of computable structures with finite computable dimension greater than 1 in several classes of algebraic structures. In each case, the proof consists of coding families of c.e. sets with a finite number of computable enumerations (up to a suitable notion of computable equivalence of enumerations) in a sufficiently effective way.

5.3 Theorem. *For each $n > 0$ there are structures with computable dimension n in each of the following classes.*

- **(Goncharov)** *graphs, partial orderings, and lattices*
- **(Goncharov, Molokov, and Romanovskii)** *2-step nilpotent groups*
- **(Kudinov)** *integral domains*

The results for partial orderings and (implicitly) graphs appear in [10], and the result for lattices is an easy consequence of the results in that paper. The result for 2-step nilpotent groups (which improves a result in [11]) appears in [16], and that for integral domains in [26].

In the original proofs of Theorems 3.2 and 4.2, the structures in question are directed graphs, and the relation mentioned in Theorem 3.2 is unary. The same holds of the other results mentioned in Sections 3 and 4. It is natural to ask, in the spirit of what was done for structures of finite computable dimension, for which theories these theorems remain true if we also require that $\mathcal{A}$ be a model of the given theory.

In this section, we present a method for showing that Theorems 3.2 and 4.2, as well as related results, including the results of Sections 2, 3, and 4, remain true if we also require that $\mathcal{A}$ be a model of a given theory. In joint work with Bakhadyr Khoussainov, Richard A. Shore, and Arkadii M. Slinko [22], this method is applied to several classes of algebraic structures.

5.4 Theorem (Hirschfeldt, Khoussainov, Shore, and Slinko)**.** *For each theory in the following list, Theorems 2.2, 2.3, 2.5, 2.6, 3.1–3.9, 4.2, and 4.3 remain true if we also require that the structures mentioned in these theorems be models of the given theory, and that the relations mentioned in these theorems be submodels: symmetric, irreflexive graphs; partial orderings; lattices; rings (with zero-divisors); integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups.*

Notice that, by Theorem 5.1, most of the results mentioned above cannot be extended from partial orderings to linear orderings, from lattices to Boolean algebras, or from commutative semigroups and 2-step nilpotent groups to Abelian groups. We will say more about this in Section 6.

The proof of Theorem 5.4 is based on coding computable graphs with the desired properties into models of the given theories in a way that is effective enough to preserve these properties. This approach is much simpler than attempting to adapt the original proofs of the theorems under consideration. Furthermore, the codings in [22] are sufficiently effective to make several other existing theorems, as well as similar results that might be proved for graphs in the future, carry over to the classes of structures mentioned above without additional work.

The following theorem gives a sufficient condition for a coding of a graph into a structure to be effective enough for the purpose of establishing Theorem 5.4. It corresponds to an especially effective version of interpretations of theories (in the standard model-theoretic sense). (See Chapter 5 of [23] for more on interpretations of theories.)

If Q is an equivalence relation on a set D then by a *set of Q-representatives* we mean a set of elements of D containing exactly one member of each Q-equivalence class.

5.5 Theorem (Hirschfeldt, Khoussainov, Shore, and Slinko)**.** *Let $\mathcal{G}$ be a computably presentable directed graph and let $\mathcal{A}$ be a computably presentable structure. Suppose there exist intrinsically computable, invariant relations $D(x)$, $Q(x,y)$, and $R(x,y)$ on $|\mathcal{A}|$ and a map $G \mapsto A_G$ from the set of computable presentations of $\mathcal{G}$ to the set of computable presentations of $\mathcal{A}$ with the following properties.*

(P1) For each computable presentation G of $\mathcal{G}$, there exists a computable map $g_G : D(A_G) \xrightarrow{onto} |G|$ such that, for $x,y \in D(A_G)$, $R^{A_G}(x,y) \Leftrightarrow E^G(g_G(x), g_G(y))$ and $Q^{A_G}(x,y) \Leftrightarrow g_G(x) = g_G(y)$. (Note that this implies that Q is an equivalence relation and that if $Q(x,x')$ and $Q(y,y')$ then $R(x,y) \Leftrightarrow R(x',y')$.)

(P2) For every pair S, S' of sets of Q-representatives, if $f : S \xrightarrow[onto]{1\text{–}1} S'$ is such that for every $x,y \in S$, $R(x,y) \Leftrightarrow R(f(x),f(y))$, then f can be extended to an automorphism of $\mathcal{A}$.

(P3) If G is a computable presentation of $\mathcal{G}$ and S is a computable set of Q^{A_G}-representatives then there is a computable set of existential formulas $\{\varphi_0(\vec{a}, \vec{b}_0, x), \varphi_1(\vec{a}, \vec{b}_1, x), \ldots\}$ such that $\vec{a}$ is a tuple of elements of $|A_G|$, for each $i \in \omega$, $\vec{b}_i$ is a tuple of elements of S, each $x \in |A_G|$ satisfies some φ_i, and no two elements of $|A_G|$ satisfy the same φ_i. (Such a set of formulas is known as a defining family *for $\langle A_G, a\rangle_{a \in S}$.)*

Then the following hold.

1. *$\mathcal{A}$ has the same computable dimension as $\mathcal{G}$.*

2. *If $x \in |\mathcal{G}|$ then there exists an $a \in D(\mathcal{A})$ such that $\langle \mathcal{A}, a\rangle$ has the same computable dimension as $\langle \mathcal{G}, x\rangle$.*

3. *If $V \subseteq |\mathcal{G}|$ then there exists a $U \subseteq D(\mathcal{A})$ such that* $\mathrm{DgSp}_{\mathcal{A}}(U) = \mathrm{DgSp}_{\mathcal{G}}(V)$.

6 Relations on Algebraic Structures II: Negative Results

As remarked in Section 5, results such as Theorem 3.4 cannot be extended to any of the classes of structures mentioned in Theorem 5.1. However, since it is certainly possible for a relation on the domain of a computable structure of infinite computable dimension to have a degree spectrum of finite cardinality, this does not rule out the possibility that, for one or more of these classes, results such as Theorem 3.3 still hold if we also require that the structures mentioned in these results be in the given class. In this section, we present conditions that guarantee that the degree spectrum of a relation on the domain of a computable structure is either a singleton or infinite.

It should be pointed out that, in the general case, there are no known restrictions on the sets of degrees that can be realized as degree spectra of relations on computable structures other than the ones that follow from the fact that the set of images of a relation on the domain of a computable structure in different computable presentations of the structure is (by definition) Σ^1_1.

Goncharov [12] has shown that if two computable structures are Δ^0_2-isomorphic but not computably isomorphic then their computable dimension is ω. This theorem is quite useful in establishing results such as those in Theorem 3.3, since it reduces the task of building infinitely many noncomputably isomorphic computable presentations of a computable structure to that of building a single computable presentation that is Δ^0_2-isomorphic but not computably isomorphic to the original structure. It is thus desirable to have an analog of this result in the case of degree spectra of relations. Such an analog is a corollary to the following result.

6.1 Theorem. *Let $k \in \omega$. Let U^0 and U^1 be k-ary relations on the domains of computable structures $\mathcal{A}^0$ and $\mathcal{A}^1$, respectively, and let $B_0, \ldots, B_{n-1} \subset \omega^k$ be Δ^0_2 but not computable. Suppose that U^0 is not computable, U^1 is computable, and there exists a Δ^0_2 isomorphism $f : \mathcal{A}^0 \to \mathcal{A}^1$ such that $f(U^0) = U^1$. Then there exists a Δ^0_2 function $h : |\mathcal{A}^0| \to \omega$ such that $h(\mathcal{A}^0)$ is a computable structure, $h(U^0)$ is not computable, and for all $m < n$, $B_m \nleq_T h(U^0)$.*

6.2 Corollary. *Let U^0 and U^1 be relations on the domains of computable structures $\mathcal{A}^0$ and $\mathcal{A}^1$, respectively. Suppose that U^0 is not computable, U^1 is computable, and there exists a Δ^0_2 isomorphism $f : \mathcal{A}^0 \cong \mathcal{A}^1$ such that $f(U^0) = U^1$. Then $\mathrm{DgSp}_{\mathcal{A}^0}(U^0)$ is infinite.*

The following is an obvious application of Corollary 6.2.

6.3 Corollary. *Let U be an invariant computable relation on the domain of a Δ^0_2-categorical computable structure $\mathcal{A}$. Either U is intrinsically computable or $\mathrm{DgSp}_{\mathcal{A}}(U)$ is infinite.*

Remark. In the above corollary, both conditions on U are necessary. In Section 2, we saw that there exists an invariant relation on the domain of a Δ^0_2-categorical computable structure whose degree spectrum consists of exactly two degrees, neither of them computable. Now let $\mathcal{A}^0$, $\mathcal{A}^1$, U^0, and U^1 be the structures and relations built in [24] to prove the theorem that we have numbered Theorem 3.2. We can assume that $|\mathcal{A}^0| \cap |\mathcal{A}^1| = \emptyset$. Let P be the predicate $\{(x, y) \mid x \in U^0 \wedge y \in U^1 \wedge$ there is an isomorphism from $\mathcal{A}^0$ to $\mathcal{A}^1$ that extends the map $x \mapsto y\}$ and let E be the equivalence relation whose equivalence classes are $|\mathcal{A}^0|$ and $|\mathcal{A}^1|$. In the proof of Theorem 4.2 of [24], it is shown that if $\mathcal{B}$ is the computable structure obtained by taking the union of $\mathcal{A}^0$ and $\mathcal{A}^1$ and expanding it by P and E then $\mathcal{B}$ is computably categorical. Since $\mathcal{B}$ has exactly one nontrivial automorphism, which sends U^1 to U^0, $\mathrm{DgSp}_{\mathcal{B}}(U^1) = \{\mathbf{0}, \deg(U^0)\}$.

Even when $\mathcal{A}$ is not necessarily Δ^0_2-categorical (and U is not necessarily invariant), it is sometimes possible to use Corollary 6.2 to show that either U is intrinsically computable or $\mathrm{DgSp}_{\mathcal{A}}(U)$ is infinite.

In [30], Moses showed that, for any computable relation U on a linear ordering $\mathcal{L}$, either U is definable by a quantifier free formula in the language of $\mathcal{L}$ expanded by finitely many constants (in which case it is obviously intrinsically computable) or there is a function f such that $f(\mathcal{L})$ is a computable structure and $f(U)$ is not computable. It is clear from the proof of this result that, in the latter case, f can be chosen to be Δ^0_2. Corollary 6.2 can now be invoked to establish the following result.

6.4 Theorem. *Let U be a computable relation on the domain of a computable linear ordering $\mathcal{L}$. Either U is intrinsically computable or* $\mathrm{DgSp}_{\mathcal{L}}(U)$ *is infinite.*

The analogous result for Boolean algebras has recently been established by Downey, Goncharov, and Hirschfeldt (in preparation).

As mentioned in Section 5, every 1-decidable structure has computable dimension 1 or ω. A roughly analogous situation holds in the context of degree spectra of relations. This follows from a result of Harizanov [18], but Corollary 6.2 allows us to conclude it from the proof of an earlier result of Ash and Nerode [2].

6.5 Definition. Let U be a relation on a computable structure $\mathcal{A}$. We say that U *satisfies condition* $(*)$ if there is a computable procedure for determining, given $a_0, \ldots, a_n \in |\mathcal{A}|$ and an existential formula $\psi(\vec{x})$ in the language of $\mathcal{A}$ expanded by constants for $a_0, \ldots, a_n$, whether $\langle \mathcal{A}, U, a_0, \ldots, a_n \rangle \vDash \forall \vec{x}(\psi(\vec{x}) \rightarrow U(\vec{x}))$.

Notice that if U is a nonempty relation on a computable structure $\mathcal{A}$ satisfying condition $(*)$ then U is computable and $\mathcal{A}$ is 1-decidable. Notice also that a sufficient condition for both U and its complement to satisfy condition $(*)$ is that $\langle \mathcal{A}, U \rangle$ be 1-decidable.

6.6 Theorem (Ash and Nerode)**.** *Let U be a relation on a computable structure satisfying condition $(*)$. Then U is formally c.e. if and only if it is intrinsically c.e..*

The proof of Theorem 6.6 shows that if a relation U on a computable structure $\mathcal{A}$ satisfying condition $(*)$ is not formally c.e. then there is a Δ^0_2 function f such that $f(\mathcal{A})$ is a computable structure and $f(U)$ is not c.e.. Given a computable relation U on a computable structure $\mathcal{A}$ such that both it and its complement satisfy condition $(*)$, we can apply this result to U and its complement to conclude that either U is formally computable (in which case it is intrinsically computable) or there is a Δ^0_2 function f such that $f(\mathcal{A})$ is a computable structure and $f(U)$ is not computable. The result below now follows from Corollary 6.2.

6.7 Theorem (Harizanov). *Let U be a computable relation on the domain of a computable structure $\mathcal{A}$ such that both U and its complement satisfy condition $(*)$. Either U is intrinsically computable or $\mathrm{DgSp}_{\mathcal{A}}(U)$ is infinite.*

References

[1] C. J. Ash, P. Cholak, and J. F. Knight, Permitting, forcing, and copying of a given recursive relation, Ann. Pure Appl. Logic 86 (1997) 219–236.

[2] C. J. Ash and A. Nerode, Intrinsically recursive relations, in J. N. Crossley (ed.), Aspects of Effective Algebra (Clayton, 1979) (Upside Down A Book Co., Yarra Glen, Australia, 1981) 26–41.

[3] E. Barker, Intrinsically Σ^0_α relations, Ann. Pure Appl. Logic 39 (1988) 105–130.

[4] P. Cholak, S. S. Goncharov, B. Khoussainov, and R. A. Shore, Computably categorical structures and expansions by constants, J. Symbolic Logic 64 (1999) 13–37.

[5] S. B. Cooper, L. Harrington, A. H. Lachlan, S. Lempp, and R. I. Soare, The d.r.e. degrees are not dense, Ann. Pure Appl. Logic 55 (1991) 125–151.

[6] R. L. Epstein, R. Haas, and R. L. Kramer, Hierarchies of sets and degrees below $\mathbf{0}'$, in M. Lerman, J. H. Schmerl, and R. I. Soare (eds.), Logic Year 1979–80 (Proc. Seminars and Conf. Math. Logic, Univ. Connecticut, Storrs, Conn., 1979/80), vol. 859 of Lecture Notes in Mathematics (Springer–Verlag, Heidelberg, 1981) 32–48.

[7] Y. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel (eds.), Handbook of Recursive Mathematics, vol. 138–139 of Studies in Logic and the Foundations of Mathematics (Elsevier Science, Amsterdam, 1998).

[8] S. S. Goncharov, The quantity of nonautoequivalent constructivizations, Algebra and Logic 16 (1977) 169–185.

[9] S. S. Goncharov, Computable single-valued numerations, Algebra and Logic 19 (1980) 325–356.

[10] S. S. Goncharov, Problem of the number of non-self-equivalent constructivizations, Algebra and Logic 19 (1980) 401–414.

[11] S. S. Goncharov, Groups with a finite number of constructivizations, Soviet Math. Dokl. 23 (1981) 58–61.

[12] S. S. Goncharov, Limit equivalent constructivizations, in Mathematical Logic and the Theory of Algorithms (“Nauka” Sibirsk. Otdel., Novosibirsk, 1982) 4–12, in Russian.

[13] S. S. Goncharov, Countable Boolean Algebras and Decidability, Siberian School of Algebra and Logic (Consultants Bureau, New York, 1997).

[14] S. S. Goncharov and V. D. Dzgoev, Autostability of models, Algebra and Logic 19 (1980) 28–37.

[15] S. S. Goncharov and B. Khoussainov, On the spectrum of degrees of decidable relations, Doklady Math. 55 (1997) 55–57, research announcement.

[16] S. S. Goncharov, A. V. Molokov, and N. S. Romanovskii, Nilpotent groups of finite algorithmic dimension, Siberian Math. J. 30 (1989) 63–68.

[17] V. S. Harizanov, Degree spectrum of a recursive relation on a recursive structure, PhD Thesis, University of Wisconsin, Madison, WI (1987).

[18] V. S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, Ann. Pure Appl. Logic 55 (1991) 51–65.

[19] V. S. Harizanov, The possible Turing degree of the nonzero member in a two element degree spectrum, Ann. Pure Appl. Logic 60 (1993) 1–30.

[20] V. S. Harizanov, Turing degrees of certain isomorphic images of computable relations, Ann. Pure Appl. Logic 93 (1998) 103–113.

[21] D. R. Hirschfeldt, Degree spectra of intrinsically c.e. relations, to appear in J. Symbolic Logic.

[22] D. R. Hirschfeldt, B. Khoussainov, R. A. Shore, and A. M. Slinko, Degree spectra and computable dimension in algebraic structures, to appear.

[23] W. Hodges, Model Theory, vol. 42 of Encyclopedia Math. Appl. (Cambridge University Press, Cambridge, 1993).

[24] B. Khoussainov and R. A. Shore, Computable isomorphisms, degree spectra of relations, and Scott families, Ann. Pure Appl. Logic 93 (1998) 153–193.

[25] B. Khoussainov and R. A. Shore, Solutions of the Goncharov-Millar and degree spectra problems in the theory of computable models, Dokl. Akad. Nauk SSSR (to appear).

[26] O. Kudinov, An integral domain with finite algorithmic dimension, personal communication.

[27] P. LaRoche, Recursively represented Boolean algebras, Notices Amer. Math. Soc. 24 (1977) A–552, research announcement.

[28] G. Metakides and A. Nerode, Effective content of field theory, Ann. Math. Logic 17 (1979) 289–320.

[29] T. Millar, Recursive categoricity and persistence, J. Symbolic Logic 51 (1986) 430–434.

[30] M. Moses, Relations intrinsically recursive in linear orders, Z. Math. Logik Grundlag. Math. 32 (1986) 467–472.

[31] A. T. Nurtazin, Strong and weak constructivizations and enumerable families, Algebra and Logic 13 (1974) 177–184.

[32] J. B. Remmel, Recursively categorical linear orderings, Proc. Amer. Math. Soc. 83 (1981) 387–391.

[33] R. I. Soare, Recursively Enumerable Sets and Degrees, Perspect. Math. Logic (Springer–Verlag, Heidelberg, 1987).