# Degree Spectra of Relations on Computable Structures in the Presence of $\Delta^0_2$ Isomorphisms

Denis R. Hirschfeldt
School of Mathematical and Computing Sciences
Victoria University of Wellington, New Zealand

**Abstract**

We give some new examples of possible degree spectra of invariant relations on $\Delta^0_2$-categorical computable structures, which demonstrate that such spectra can be fairly complicated. On the other hand, we show that there are nontrivial restrictions on the sets of degrees that can be realized as degree spectra of such relations. In particular, we give a sufficient condition for a relation to have infinite degree spectrum that implies that every invariant computable relation on a $\Delta^0_2$-categorical computable structure is either intrinsically computable or has infinite degree spectrum. This condition also allows us to use the proof of a result of Moses [23] to establish the same result for computable relations on computable linear orderings.

We also place our results in the context of the study of what types of degree-theoretic constructions can be carried out within the degree spectrum of a relation on a computable structure, given some restrictions on the relation or the structure. From this point of view we consider the cases of $\Delta^0_2$-categorical structures, linear orderings, and 1-decidable structures, in the last case using the proof of a result of Ash and Nerode [3] to extend results of Harizanov [13].

Most of the results in this paper are part of the author's doctoral dissertation, written at Cornell University under the supervision of Richard A. Shore. The author thanks Professor Shore for many useful comments and suggestions. The author was partially supported by an Alfred P. Sloan Doctoral Dissertation Fellowship.

Current Address: Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, U.S.A.. `drh@math.uchicago.edu`

# 1 Introduction

The study of properties of computable structures has formed an important and fertile branch of computable model theory. (A valuable reference is the handbook [9]. In particular, the introduction and the articles by Ershov and Goncharov [8] and Harizanov [14] give useful overviews, while the articles by Ash [1] and Goncharov [11] cover material related to the topic of this paper. Another relevant survey article is [21].) One direction this study has taken, beginning with the work of Ash and Nerode [3] in the early 1980's, concerns the question of what can be said about the images of an additional relation $U$ on the domain of a computable structure $\mathcal{M}$ (that is, one that is not the interpretation in $\mathcal{M}$ of a relation in the language of $\mathcal{M}$) in different computable copies of $\mathcal{M}$.

For example, if $\mathcal{L}$ is a linear ordering of type $\omega$ and $S$ is the successor relation on $\mathcal{L}$ then there is a computable copy of $\mathcal{L}$ in which the image of $S$ is computable, namely $\omega$ with its standard ordering. But there are also computable copies of $\mathcal{L}$ in which the images of $S$ are not computable (see, for instance, [5]). In fact, for every computably enumerable (c.e.) degree $\mathbf{a}$, we can construct a computable linear ordering of type $\omega$ in which the successor relation has degree $\mathbf{a}$. On the other hand, the successor relation on any linear ordering is always co-c.e., so this is the most we can do.

In the previous example we see two ways of approaching the question of what happens to a relation in different copies of a structure. One, taken in [3], is to begin with a relation $U$ on a computable structure $\mathcal{A}$ that is in some particular class of relations, such as the computable relations or the c.e. relations, and ask when is it the case that the image of $U$ in different computable copies of $\mathcal{A}$ always remains in the given class.

A different approach, which was first taken by Harizanov [12] (although there are portents of this approach in earlier work such as [24]), is to look at the collection of (Turing) degrees of the images of a relation in different computable copies of a structure, which is known as the *degree spectrum* of the relation.

The question of which sets of degrees can be realized as degree spectra of relations on computable structures, both in the general case and with certain restrictions imposed on the relation or the structure, has received increasing attention from a number of researchers. (References can be found in the aforementioned articles in [9], as well as in [21].) Several natural classes of degrees can be so realized, for example the $\alpha$-c.e. degrees (see Section 3) and the $\Sigma^0_\alpha$ and $\Delta^0_\alpha$ degrees (see [18]) for any computable ordinal $\alpha$. However, not all degree spectra of relations are natural classes of degrees. For instance, in [16] it is shown that, for every uniformly c.e. collection $\mathfrak{S}$ of sets of natural numbers, there is an invariant relation on a computable structure whose degree spectrum coincides with the degrees of elements of $\mathfrak{S}$.

In [17] it is shown that for various classes of structures, such as nilpotent groups and integral domains, there are no more restrictions on the possible degree spectra of relations on structures belonging to these classes than there are in the general case. In this paper, we discuss classes of structures of which this is not true. Our main focus is on $\Delta^0_2$-categorical structures, that is, structures for which any two computable copies are isomorphic via a $\Delta^0_2$ map. This class of structures includes several natural classes of structures, such as algebraically closed fields, graphs in which all connected components are finite, and structures in relational languages all of whose relations are unary. We show, for example, that every invariant computable relation on a $\Delta^0_2$-categorical computable structure is either intrinsically computable (that is, computable in every computable copy of the structure) or has infinite degree spectrum. This is in contrast to the general case, since invariant computable relations with degree spectra of finite cardinality greater than one have been known to exist since the work of Harizanov [12].

First, however, in Section 3, we give some new examples of possible degree spectra of invariant relations on $\Delta^0_2$-categorical computable structures. In Section 4, we establish results that, besides their independent interest discussed below, can be used to give a sufficient condition for a relation to have infinite degree spectrum. This condition is given in Section 5, and is applied to $\Delta^0_2$-categorical structures and linear orderings, in the last case making use of the proof of a result of Moses [23].

There is a sense in which the study of degree spectra of relations on computable structures is as a generalization of the classical study of the degrees, since the degrees can be seen as the degree spectrum of an infinite and coinfinite relation (of any arity) on the domain of a computable structure in the empty language. (The view of the study of relations on structures as a form of generalized computability theory, and of the study of relations on computable structures as an effective version of this approach, is well-known in abstract computability theory (see [22] or [26], for instance), but the focus there has been principally on questions of definability and computability-theoretic hierarchies, rather than on degrees. Of course, it is only in the effective case that concentrating on degrees becomes natural.) From this point of view, it is interesting to consider the question of which of the constructions that have been developed in the context of the degrees as a whole (or, more to the point, the c.e. degrees, since we are dealing with computable structures and thus with issues of effectivity that do not arise in the case of the degrees as a whole) are still possible in subcases of this more general setting.

Of course, in general, nothing is possible, since there are relations whose degree spectra are singletons, and, even if we ignore this case, the existence of structures of finite computable dimension shows that not much can be done in the most general case. However, when restrictions on the relation or the structure are imposed, certain constructions become possible. Results that give sufficient conditions for the degree spectrum of a relation to contain all degrees in a given natural class of degrees (see [2] or [15], for example) can be thought of as examples of this approach, but a study of degree spectra along the lines suggested above can be carried out even when no such conditions hold.

In the case of invariant computable relations on $\Delta^0_2$-categorical computable structures whose degree spectra are not singletons, the results of Section 4 will show that techniques for constructing infinite sets of pairwise incomparable c.e. degrees and for avoiding the cone of degrees above a given degree are always applicable. (We will show that this is also the case for computable relations on linear orderings and for relations satisfying certain decidability conditions from [3].) On the other hand, the results of Section 3 will imply that the same is not true of techniques for building noncomputable sets below a given (c.e.) degree.

# 2 Basic Definitions and Notation

For basic notions of computability theory and model theory, the reader is referred to [25] and [19], respectively. By *degree*, we will mean Turing degree unless otherwise specified. We will denote the join of degrees $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \cup \mathbf{b}$.

Since $\alpha$-c.e. sets and degrees for arbitrary computable ordinals $\alpha$ may not be familiar to all readers, we define them here. The definition of $\alpha$-c.e. sets and degrees depends on the choice of ordinal notation system; see [7] for details. When we talk about $\alpha$-c.e. sets and degrees, where $\alpha$ is a computable ordinal, we assume that we have fixed a univalent, computably related ordinal notation system with a notation for $\alpha$ (and hence for all ordinals less than $\alpha$).

It is slightly cumbersome to give a definition of $\alpha$-c.e. sets that works for both $\alpha < \omega$ (where we want to agree with the definition of $n$-c.e. sets, $n \in \omega$, given by the difference hierarchy) and $\alpha \geqslant \omega$. The following (slightly nonstandard) definition works well for our purposes, and is easily seen to be equivalent to standard definitions of $n$-c.e. and $\alpha$-c.e. sets and degrees (as in [7]).

**2.1 Definition.** Let $\alpha$ be a computable ordinal and assume we have fixed a univalent, computably related ordinal notation system with a notation for $\alpha$. Let $\ulcorner\beta\urcorner$ denote the unique notation for $\beta \leqslant \alpha$ in this system.

A set $A$ is *$\alpha$-c.e.* if there exists a partial computable binary function $\Psi$ satisfying the following conditions for all $x \in \omega$. (We will say that $\Psi$ *witnesses* the fact that $A$ is $\alpha$-c.e..)

1. $\Psi(\ulcorner\alpha\urcorner, x)\downarrow = 0$.
2. If $\alpha \geqslant \omega$ then there exists a $\beta < \alpha$ such that $\Psi(\ulcorner\beta\urcorner, x)\downarrow$.
3. For the least $\beta \leqslant \alpha$ such that $\Psi(\ulcorner\beta\urcorner, x)\downarrow$, $\Psi(\ulcorner\beta\urcorner, x) = A(x)$.

A degree is $\alpha$-c.e. if it contains an $\alpha$-c.e. set.

Whenever we mention a c.e. set $X$, we assume we have fixed some computable enumeration of $X$ and let $X[s]$ denote the part of $X$ enumerated after $s+1$ many steps. Similarly, whenever we mention a $\Delta^0_2$ set $Y$, we assume we have fixed some computable approximation of $Y$ and let $Y[s]$ denote the result of performing $s+1$ many steps of this approximation.

When we mention a *fresh large number* in one of our constructions, we mean a number larger than any appearing in the construction up to that point.

For any set $X$, let $X \upharpoonright m = X \cap \{0, \ldots, m-1\}$.

The $e^{\text{th}}$ partial computable function with oracle $X$ is denoted by $\Phi^X_e$. The evaluation of $\Phi^{X[s]}_e$ at stage $s$ is denoted by $\Phi^X_e[s]$ and its value at $x$ by $\Phi^X_e(x)[s]$. The use of the computations $\Phi^X_e(x)$ and $\Phi^X_e(x)[s]$ are denoted by $\varphi^X_e(x)$ and $\varphi^X_e(x)[s]$, respectively.

Fix a computable one-to-one function from $\omega \times \omega$ onto $\omega$ and let $\langle a, b\rangle$ denote the image under this function of the ordered pair consisting of $a \in \omega$ and $b \in \omega$. We will write $\langle a, b, c\rangle$ instead of $\langle a, \langle b, c\rangle\rangle$, and similarly for longer sequences of natural numbers. Let $\pi_1(\langle a, b\rangle) = a$.

If $\vec{x} = (x_0, \ldots, x_m)$ and $\vec{y} = (y_0, \ldots, y_n)$ are sequences of natural numbers then $\vec{x}(i) = x_i$, $\max(\vec{x}) = \max\{x_i \mid i < k\}$, and $\vec{x}^\frown\vec{y}$ is the sequence $(x_0, \ldots, x_m, y_0, \ldots, y_n)$. We will write $\vec{x}^\frown z$ instead of $\vec{x}^\frown(z)$, where $(z)$ is the sequence consisting of the single element $z$. If $f$ is a function on $\omega$ then we will write $f(\vec{x})$ to mean $(f(x_0), \ldots, f(x_m))$. If $z \in \omega$ then we will write $\vec{x} < z$ to mean that $\max(\vec{x}) < z$. The notations $\vec{x} \leqslant z$ and $z \leqslant \vec{x}$ are defined analogously.

One of the central notions of computable model theory is that of a *computable structure*. We will always assume we are working with computable languages.

**2.2 Definition.** A structure $\mathcal{A}$ is *computable* if its domain $|\mathcal{A}|$ is a computable subset of $\omega$ and the atomic diagram of $(\mathcal{A}, a)_{a\in|\mathcal{A}|}$ is computable.

If, in addition, the existential diagram of $(\mathcal{A}, a)_{a\in|\mathcal{A}|}$ is computable then $\mathcal{A}$ is *1-decidable*.

Of course, there are often many different ways of making a given structure computable. From the point of view of computable model theory, two computably isomorphic computable copies of a structure are the same, since they share all computability-theoretic properties. However, two noncomputably isomorphic computable copies of a structure can have very different computability-theoretic properties. This leads naturally to the following definitions.

**2.3 Definition.** An isomorphism from a structure $\mathcal{M}$ to a computable structure is called a *computable presentation* of $\mathcal{M}$. (We often abuse terminology and refer to the image of a computable presentation as a computable presentation.)

If $\mathcal{M}$ has a computable presentation then it is *computably presentable.*

**2.4 Definition.** The *computable dimension* of a computably presentable structure $\mathcal{M}$ is the number of computable presentations of $\mathcal{M}$ up to computable isomorphism.

A structure of computable dimension one is said to be *computably categorical.*

The main focus of this paper is on structures that, while perhaps not computably categorical, do have relatively simple isomorphisms between their various computable presentations.

**2.5 Definition.** A computably presentable structure is $\Delta^0_2$*-categorical* if any two of its presentations are isomorphic via a $\Delta^0_2$ map.

As we have mentioned above, the study of additional relations on computable structures began with the work of Ash and Nerode [3], who were concerned with relations that maintain some degree of effectiveness in different computable presentations of a structure.

**2.6 Definition.** Let $U$ be a relation on the domain of a computably presentable structure $\mathcal{M}$ and let $\mathfrak{C}$ be a class of relations. The relation $U$ is *intrinsically* $\mathfrak{C}$ on $\mathcal{M}$ if the image of $U$ in any computable presentation of $\mathcal{M}$ is in $\mathfrak{C}$.

In particular, for any computable ordinal $\alpha$, the relation $U$ is intrinsically $\alpha$-c.e. on $\mathcal{M}$ if the image of $U$ in any computable presentation of $\mathcal{M}$ is $\alpha$-c.e..

For example, any relation on the domain of a computably presentable structure that is definable by an existential formula is intrinsically c.e., although the converse does not hold. For more details, see Section 15 of [14].

It is often useful to consider relations that are preserved under all automorphisms of a structure.

**2.7 Definition.** A relation $U$ on the domain of a structure $\mathcal{M}$ is *invariant* if $f(U) = U$ for every automorphism $f : \mathcal{M} \cong \mathcal{M}$.

The following definition is due to Harizanov [12].

**2.8 Definition.** Let $U$ be a relation on the domain of a computably presentable structure $\mathcal{M}$. The *degree spectrum* of $U$ on $\mathcal{M}$, denoted by $\mathrm{DgSp}_{\mathcal{M}}(U)$, is the set of degrees of the images of $U$ in all computable presentations of $\mathcal{M}$.

It is also interesting to consider degree spectra of relations with respect to other reducibilities.

**2.9 Definition.** Let r be a reducibility, such as many-one reducibility (m-reducibility) or weak truth-table reducibility (wtt-reducibility). Let $U$ be a relation on the domain of a computably presentable structure $\mathcal{M}$. The *r-degree spectrum* of $U$ on $\mathcal{M}$, denoted by $\mathrm{DgSp}^{\mathrm{r}}_{\mathcal{M}}(U)$, is the set of r-degrees of the images of $U$ in all computable presentations of $\mathcal{M}$.

# 3 Examples of Degree Spectra of Relations on $\mathbf{\Delta^0_2}$-Categorical Structures

We begin with a simple observation. Let $U$ and $V$ be $k$-ary relations on the domains of computable graphs $\mathcal{A} = (|\mathcal{A}|, E)$ and $\mathcal{B} = (|\mathcal{B}|, F)$, respectively. Let $\mathcal{C} = (|\mathcal{C}|, R, Q)$ be the computable structure in the language with one binary and one unary relation defined by

$$|\mathcal{C}| = \{2x \mid x \in |\mathcal{A}|\} \cup \{2x+1 \mid x \in |\mathcal{B}|\},$$

$$R = \{(2x, 2y) \mid E(x, y)\} \cup \{(2x+1, 2y+1) \mid F(x, y)\},$$

and

$$Q = \{2x \mid x \in |\mathcal{A}|\}.$$

Let

$$W = \{(2x_0, \ldots, 2x_{k-1}) \mid (x_0, \ldots, x_{k-1}) \in U\} \cup \\ \{(2x_0 + 1, \ldots, 2x_{k-1} + 1) \mid (x_0, \ldots, x_{k-1}) \in V\}.$$

It is easy to check that

$$\mathrm{DgSp}_{\mathcal{C}}(W) = \{\mathbf{c} \mid \exists \mathbf{a}, \mathbf{b}(\mathbf{a} \in \mathrm{DgSp}_{\mathcal{A}}(U) \wedge \mathbf{b} \in \mathrm{DgSp}_{\mathcal{B}}(V) \wedge \mathbf{c} = \mathbf{a} \cup \mathbf{b})\}.$$

Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are $\Delta^0_2$-categorical then so is $\mathcal{C}$, if $U$ and $V$ are invariant then so is $W$, and, for any class of relations $\mathfrak{C}$ closed under m-equivalence and finite disjoint unions, if $U$ and $V$ are intrinsically $\mathfrak{C}$ then so is $W$.

It is not hard to modify this construction to handle the case in which $U$ and $V$ do not necessarily have the same arity and $\mathcal{A}$ and $\mathcal{B}$ are arbitrary computable structures, and hence establish the following result.

**3.1 Proposition.** *Let $A$ and $B$ be sets of degrees and let $\mathfrak{C}$ be a class of relations closed under m-equivalence and finite disjoint unions. Let $C = \{\mathbf{c} \mid \exists \mathbf{a}, \mathbf{b}(\mathbf{a} \in A \wedge \mathbf{b} \in B \wedge \mathbf{c} = \mathbf{a} \cup \mathbf{b})\}$. If both $A$ and $B$ can be realized as degree spectra of intrinsically $\mathfrak{C}$ invariant relations on the domains of $\Delta^0_2$-categorical computable structures then so can $C$.*

Now let $C_0$ be the directed graph consisting of a single node and no edges and let $C_1$ be the directed graph consisting of two nodes $x$ and $y$ with an edge from $x$ to $y$. Consider the directed graph $\mathcal{G} = (|\mathcal{G}|, E)$ that is the disjoint union of infinitely many copies of each of $C_0$ and $C_1$. Let $U$ be the unary relation on the domain of $\mathcal{G}$ that holds of $x$ if and only if there is a $y$ such that $E(x, y)$. Since $U$ is defined by an existential formula in the language of directed graphs, $U$ is intrinsically c.e.. We claim that $\mathrm{DgSp}_{\mathcal{G}}(U)$ consists of all c.e. degrees. (In fact, $\mathrm{DgSp}^{\mathrm{m}}_{\mathcal{G}}(U)$ consists of all c.e. m-degrees other than the m-degrees of $\emptyset$ and $\omega$.)

Indeed, let $A$ be an infinite and coinfinite c.e. set and let $a_0, a_1, \ldots$ be a computable enumeration of $A$. Define a directed graph $G$ with edge relation $F$ as follows. Let $|G| = \omega$ and, for $x, y \in \omega$, let $F(x, y)$ hold if and only if $x = 2a_k$ and $y = 2k + 1$ for some $k \in \omega$. It is easy to check that $G$ is a computable presentation of $\mathcal{G}$. Furthermore, $U^G(x) \Leftrightarrow x = 2a \wedge a \in A$, and hence $U^G \equiv_{\mathrm{m}} A$.

By modifying this example, it is possible to construct, for each $n > 0$, an intrinsically $n$-c.e. invariant relation on a $\Delta^0_2$-categorical computable structure whose degree spectrum consists of all $n$-c.e. degrees. A little more work can get us a similar result with $\alpha$-c.e. in place of $n$-c.e. for any computable ordinal $\alpha$.

In this section, we show that, in fact, for any c.e. degree **a** and any computable ordinal $\alpha$, there is an intrinsically $\alpha$-c.e. invariant relation on a $\Delta^0_2$-categorical computable structure whose degree spectrum consists of all $\alpha$-c.e. degrees less than or equal to **a**, and we extend this result to certain other nonprincipal ideals of $\Delta^0_2$ degrees. We begin with a theorem that has a similar but simpler proof.

**3.2 Theorem.** *Let* **a** *be a c.e. degree. There exists an invariant relation $U$ on the domain of a $\Delta^0_2$-categorical computably presentable structure $\mathcal{M}$ such that* $\mathrm{DgSp}_{\mathcal{M}}(U)$ *consists of all degrees less than or equal to* **a**.

*Proof.* The structure $\mathcal{M}$ will be a directed graph. We begin by defining our basic building blocks.

**3.3 Definition.** Let $n \in \omega$. The directed graph $[n]$ consists of $n+3$ many nodes $x_0, x_1, \ldots, x_{n+2}$ with an edge from $x_0$ to itself, an edge from $x_{n+2}$ to $x_0$, and an edge from $x_i$ to $x_{i+1}$ for each $i \leqslant n+1$. We call $x_0$ the *top* of $[n]$.

Figure 3.1 shows [2] as an example.

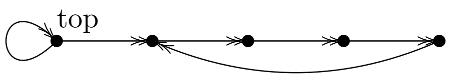


Figure 3.1: [2]

**3.4 Definition.** Let $\vec{m} = (m_0, m_1, \ldots, m_k) \in \omega^{k+1}$ and $S \subseteq \{0, 1, \ldots, k\}$. The directed graph $[\vec{m}, S]$ consists of the following nodes and edges.

1. $k+1$ many nodes $x_0, x_1, x_2, \ldots, x_k$ with an edge from $x_i$ to $x_{i+1}$ for each $i < k$.

2. For each $i \in S$, a copy of $[2m_i + 1]$ with $x_i$ as its top.

3. For each $i \in \{0, 1, \ldots, k\} - S$, a copy of $[2m_i]$ with $x_i$ as its top.

We call $x_0$ the *principal node* of $[\vec{m}, S]$. The *height* of $[\vec{m}, S]$ is defined to be $|\vec{m}|$ and its *length* is defined to be $\max\{|\sigma_m| \mid m \in \vec{m}\}$.

Figure 3.2 shows $[(1, 2, 3), \{2\}]$ as an example.

Let $A$ be a c.e. set in **a** and let $\sigma_0, \sigma_1, \ldots$ be a computable list of all finite binary strings. The idea behind Definition 3.4 is that the $[\vec{m}, S]$ can be used to represent computations in which we are computably approximating a $\Delta^0_2$ oracle, with the strings

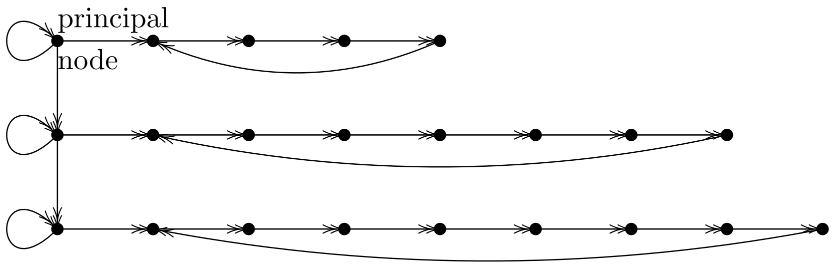


Figure 3.2: $[(1,2,3),\{2\}]$

$\sigma_m$, $m \in \vec{m}$, representing initial segments of approximations of the oracle and the elements of $S$ representing stages at which the computation changes its mind about its output at a particular input. Of course, we are only interested in the case in which the oracle is $A$. This leads to the following definition.

**3.5 Definition.** We say that $[\vec{m}, S]$, with $\vec{m} = (m_0, m_1, \ldots, m_k)$ and $S \subseteq \{0, 1, \ldots, k\}$, is $A$-*acceptable* if it satisfies both of the following conditions.

1. $A \upharpoonright |\sigma_{m_k}| = \sigma_{m_k}$.

2. If $i < k$ then $\sigma_{m_i} \neq A \upharpoonright |\sigma_{m_i}|$ and $\sigma_{m_i}(j) = 1 \Rightarrow A(j) = 1$ for every $j < |\sigma_{m_i}|$.

We define $A[s]$-acceptability analogously.

Note that, if we think of an $A$-acceptable $[\vec{m}, S]$ as an approximation of some computation relative to $A$ in the manner described above, then condition 2 in Definition 3.5 makes sense because $A$ is c.e.. This condition is important for two reasons. As we will see, together with condition 1 it ensures that, given a copy of an $A$-acceptable $[\vec{m}, S]$ in some computable graph, we can $A$-computably determine $\vec{m}$ and $S$. Furthermore, it guarantees that if $[\vec{m}, S]$ is $A[s]$-acceptable, $\vec{n}$ is a proper initial segment of $\vec{m}$, and $T \subseteq \{0, \ldots, |\vec{n}| - 1\}$, then $[\vec{n}, T]$ is not $A[t]$-acceptable for any $t \geqslant s$, and hence is not $A$-acceptable.

We now define $\mathcal{M}$ and $U$.

**3.6 Definition.** Let $\mathcal{M}'$ be the disjoint union of infinitely many copies of each $A$-acceptable $[\vec{m}, S]$. Let $T$ be the set of principal nodes of these copies.

The directed graph $\mathcal{M}$ consists of $\mathcal{M}'$ and one additional *root node* $x$, with an edge from $x$ to each element of $T$. We call the connected components of $\mathcal{M}'$ the *components* of $\mathcal{M}$. For any computable presentation $M$ of $\mathcal{M}$, we call the image of $x$ in $M$ the root node of $M$.

The unary relation $U$ on the domain of $\mathcal{M}$ is the set of all elements of $T$ that are principal nodes of components of $\mathcal{M}$ of the form $[\vec{m}, S]$ with $|S|$ odd.

We need to show that $U$ and $\mathcal{M}$ have the desired properties. Let $[\vec{m}, S]$ have length $l$. The following facts, which will be used below without explicit mention, follow easily from the definitions.

1. Let $s \in \omega$ be such that $A[s] \upharpoonright l = A \upharpoonright l$. Then $[\vec{m}, S]$ is $A$-acceptable if and only if it is $A[s]$-acceptable.

2. If $[\vec{m}, S]$ is $A[s]$-acceptable, $A[s+1] \upharpoonright l \neq A[s] \upharpoonright l$, and $m$ is such that $\sigma_m = A[s+1] \upharpoonright l$, then $[\vec{m}^\frown m, S]$ is $A[s+1]$-acceptable.

3. Let $m \in \omega$ and either $T = S$ or $T = S \cup \{|\vec{m}|\}$. Then $[\vec{m}, S]$ can be extended to $[\vec{m}^\frown m, T]$ by adding new nodes and edges.

**3.7 Lemma.** $\mathcal{M}$ *is* $\Delta^0_2$*-categorical.*

*Proof.* This follows from the fact that each connected component of $\mathcal{M}'$ is finite. □

**3.8 Lemma.** *If* $M$ *is a computable presentation of* $\mathcal{M}$ *then* $U^M \leqslant_T A$.

*Proof.* Let $T$ be set of all nodes $y$ of $M$ such that there is an edge from $y$ to itself. Let $y \in T$. Then $y$ is the top of a copy of $[k]$ for some $k \in \omega$. Let $m$ be such that $k = 2m$ or $k = 2m+1$. Define $\sigma(y) = \sigma_m$ and $c(y) = k - 2m$. Note that $T$ is computable, and so are the maps taking $y \in T$ to $\sigma(y)$ and $c(y)$.

To $A$-computably determine whether $x \in U^M$, we can proceed as follows. First, check whether there is an edge from the root node of $M$ to $x$. If not then $x \notin U^M$. Otherwise, $x$ is the principal node of a copy of some $[\vec{m}, S]$. In this case, by the definition of $\mathcal{M}$, there is a unique list $x_0, \dots, x_n$ of elements of $T$ such that $x = x_0$, for all $i < n$ there is an edge from $x_i$ to $x_{i+1}$, and $\sigma(x_n) = A \upharpoonright |\sigma(x_n)|$. Clearly, we can $A$-computably find $x_0, \dots, x_n$, and hence $A$-computably determine $c = \sum_{i=0}^{n} c(x_i)$. It follows from the definition of $U$ that $x \in U^M$ if and only if $c$ is odd. □

To complete the proof of the theorem, given a set $B = \Phi^A_e$, we need to build a computable presentation $M$ of $\mathcal{M}$ such that $U^M \equiv_T B$. (In fact, we will build $M$ so that $U^M \equiv_m B$.) We take advantage of the fact that $\mathcal{M}$ contains infinitely many copies of each of its components and proceed as follows. We first construct a computable presentation $N$ of $\mathcal{M}$ such that $U^N$ is computable. We then add to this presentation $A$-acceptable components $C_n$, $n \in \omega$, such that the principal node of $C_n$ is in $U^M$ if and only if $n \in B$.

At each stage $s+1$ in the construction of $M$, we will have approximations $C_n[s+1]$ for each $n$ such that $\Phi_e^A(n)[t]\downarrow$ for some $t \leqslant s$. Each such $C_n[s+1]$ will be a copy of some $[\vec{m}, S]$ such that, for the last element $m$ of $\vec{m}$ and the largest $t \leqslant s$ such that $\Phi_e^A(n)[t]\downarrow$, we have $\sigma_m = A[t] \upharpoonright \varphi_e^A(n)[t]$ and $|S| \equiv \Phi_e^A(n)[t] \bmod 2$.

Every time the computation $\Phi_e^A(n)$ changes, we change the approximation of $C_n$ to reflect this. Since $\Phi_e^A(n)$ is total, this will guarantee that $C_n = \lim_s C_n[s]$ is $A$-acceptable and is a copy of some $[\vec{m}, S]$ such that $|S| \equiv \Phi_e^A(n) \bmod 2$.

**3.9 Lemma.** *There exists a computable presentation $N$ of $\mathcal{M}$ such that $U^N$ is computable.*

*Proof.* We build $N$ in stages. By the beginning of each stage $s+1$, we will have built components $C_0[s], \dots, C_{k_s-1}[s]$ for some $k_s \in \omega$, where each $C_i[s]$ will be a copy of some $A[s]$-acceptable $\big[\vec{m}_i[s], S_i\big]$. For each $i$, $\vec{m}_i[s]$ will have a limit $\vec{m}_i$, and thus $C_i[s]$ will have a limit $C_i$.

*stage* 0. Choose 0 as the root node of $N$. Let $k_0 = 0$.

*stage* $s+1$. We break the stage up into two phases.

1. Define $k_{s+1}$, and $\vec{m}_i[s+1]$ and $S_i$ for $k_s \leqslant i < k_{s+1}$, so that the set $\big\{\big[\vec{m}_i[s+1], S_i\big] \mid k_s \leqslant i < k_{s+1}\big\}$ contains every $A[s+1]$-acceptable $[\vec{m}, S]$ whose height and length are less than or equal to $s$. For each $k_s \leqslant i < k_{s+1}$, build a new copy $C_i[s+1]$ of $\big[\vec{m}_i[s+1], S_i\big]$ using fresh large numbers and add an edge from 0 to the principal node of $C_i[s+1]$.

2. For each $C_i[s]$, $i < k_s$, if $C_i[s]$ is not $A[s+1]$-acceptable then proceed as follows. Let $m$ be such that $\sigma_m = A[s+1] \upharpoonright l$, where $l$ is the length of $C_i[s]$. Let $\vec{m}_i[s+1] = \vec{m}_i[s]^\frown m$. Extend $C_i[s]$ to a copy $C_i[s+1]$ of $\big[\vec{m}_i[s+1], S_i\big]$ using fresh large numbers. Note that, since $C_i[s]$ is $A[s]$-acceptable but not $C_i[s+1]$-acceptable, $C_i[s+1]$ is $A[s+1]$-acceptable.

   On the other hand, if $C_i[s]$ is $A[s+1]$-acceptable then let $\vec{m}_i[s+1] = \vec{m}_i[s]$ and $C_i[s+1] = C_i[s]$.

Let $i \in \omega$ and let $s$ be such that $C_i[s]$ is defined. It is easy to check that if $t \geqslant s$ then $C_i[t]$ is $A[t]$ acceptable and has the same length as $C_i[s]$. Thus there exists a $t \geqslant s$ such that $C_i[u] = C_i[t]$ for all $u \geqslant t$, and hence $C_i = \lim_u C_i[u]$ is well-defined and $A$-acceptable. So every $C_i$ is a copy of some component of $\mathcal{M}$.

Now suppose that $[\vec{m}, S]$ is $A$-acceptable and has length $l$ and let $s \geqslant l$ be such that $A[s+1] \upharpoonright l = A \upharpoonright l$. Then for every $t \geqslant s$ there exists $k_{t-1} \leqslant i < k_t$ such that $C_i[t]$ is a copy of $[\vec{m}, S]$, and $C_i = C_i[t]$ by the choice of $s$. Thus each component of $\mathcal{M}$ has infinitely many copies in $N$. Together with the result of the previous paragraph, this shows that $N$ is a computable presentation of $\mathcal{M}$.

To determine whether $x \in U^N$, all we need to do is to look for a stage $s$ in the construction during which numbers greater than $x$ are used. Then $x \in U^N$ if and only if it is the principal node of some $\left[\vec{m}_i[s], S_i\right]$, $i < k_s$, with $|S_i|$ odd. □

**3.10 Lemma.** *Let $B \leqslant_T A$. There is a computable presentation $M$ of $\mathcal{M}$ such that $U^M \equiv_m B$.*

*Proof.* Let $e$ be such that $\Phi_e^A = B$. By Lemma 3.9, there is a computable presentation $N$ of $\mathcal{M}$ such that $U^N$ is computable. We can assume that $D = \omega - |N|$ is infinite.

We extend $N$ to another computable presentation $M$ of $\mathcal{M}$ in stages. When we make use of fresh numbers in the construction, we take them from $D$ in order. We adopt the conventions that $n \leqslant s \Rightarrow \Phi_e^A(n)[s]\uparrow$ and $A[s+1] \upharpoonright \varphi_e^A(n)[s] \neq A[s] \upharpoonright \varphi_e^A(n)[s] \Rightarrow \Phi_e^A(n)[s+1]\uparrow$.

At the beginning of stage $s+1$, we have copies $C_n[s]$ of graphs $\left[\vec{m}_n[s], S_n[s]\right]$ for each $n < s$ such that $\Phi_e^A(n)[t]\downarrow$ for some $t < s$. For each $n \leqslant s$, we proceed as follows.

If $\Phi_e^A(n)[s]\downarrow$ and $\Phi_e^A(n)[t]\uparrow$ for all $t < s$ then let $m$ be such that $\sigma_m = A[s] \upharpoonright \varphi_e^A(n)[s]$ and let $\vec{m}_n[s+1] = (m)$. If $\Phi_e^A(n)[s] = 0$ then let $S_n[s+1] = \emptyset$; otherwise, let $S_n[s+1] = \{0\}$. Let $C_n[s+1]$ be a new copy of $\left[\vec{m}_n[s+1], S_n[s+1]\right]$, formed using fresh numbers in $D$, and add an edge from the root node of $N$ to the principal node of $C_n[s+1]$.

If $C_n[s]$ is defined, $\Phi_e^A(n)[s]\downarrow$, and $\Phi_e^A(n)[s-1]\uparrow$, then let $m$ be such that $\sigma_m = A[s] \upharpoonright \varphi_e^A(n)[s]$ and let $\vec{m}_n[s+1] = \vec{m}_n[s]^\frown m$. If $\Phi_e^A(n)[s] \equiv |S_n[s]| \bmod 2$ then let $S_n[s+1] = S_n[s]$; otherwise, let $S_n[s+1] = S_n[s] \cup \{|\vec{m}_n[s]|\}$. Extend $C_n[s]$ to a copy $C_n[s+1]$ of $\left[\vec{m}_n[s+1], S_n[s+1]\right]$, using fresh numbers in $D$.

If neither of the previous two cases holds then let $\vec{m}_n[s+1] = \vec{m}_n[s]$, $S_n[s+1] = S_n[s]$, and $C_n[s+1] = C_n[s]$.

It is easy to check that $M$ is a computable presentation of $\mathcal{M}$. In particular, the following facts hold.

1. Whenever $C_n[s]$ changes, it is only to reflect the fact that a number has entered $A$ below the use of the computation $\Phi_e^A(n)$.

2. $C_n[s]$ will necessarily change to reflect the last change in this use.

Thus each $C_n[s]$ comes to a limit $C_n$, and it is then a copy of an $A$-acceptable $[\vec{m}_n, S_n]$.

Let $x_n$ be the principal node of $C_n$. We wish to show that $U^M \equiv_{\mathrm{m}} B$. By our choice of $N$, $U^M \cap |N|$ is computable, so it suffices to show that $n \in B \Leftrightarrow x_n \in U^M$, that is, that $\forall n \in \omega(B(n) \equiv |S_n| \bmod 2)$.

Fix $n$ and let $s$ be the least number such that $\Phi_e^A(n)[t] \downarrow = \Phi_e^A(n)$ for all $t \geqslant s$. By the minimality of $s$, $\Phi_e^A(n)[s-1] \uparrow$, and hence one of the first two cases in the description of the stage $s+1$ action of the construction of $M$ holds for $n$, so that $B(n) = \Phi_e^A(n) = \Phi_e^A(n)[s] \equiv |S_n[s+1]| \bmod 2$. Furthermore, neither of these cases ever holds after stage $s+1$, so that $S_n = S_n[s+1]$. Thus $B(n) \equiv |S_n| \bmod 2$. □

The theorem follows from Lemmas 3.7, 3.8, and 3.10. ■

The above theorem could have been proved by building, instead of a graph, a structure in the language consisting of infinitely many unary relations $R_0, R_1, \dots$, the basic idea being to substitute each graph $[\vec{m}, S]$, with $\vec{m} \in \omega^{k+1}$ and $S \subseteq \{0, \dots, k\}$, by a point $x_{\vec{m},S,f}$ for each function $f : \{0, \dots, k\} \to \omega$, letting $R_n$ hold of $x_{\vec{m},S,f}$ if and only if, for some $i \leqslant k$, either $i \notin S$ and $n = \langle 2\vec{m}(i), f(i) \rangle$, or $i \in S$ and $n = \langle 2\vec{m}(i)+1, f(i) \rangle$. This would probably make the proof less perspicuous, but it is interesting to note that the results of this section also hold in this seemingly simpler case.

We now show how to modify the proof of Theorem 3.2 in order to realize the set of all $\alpha$-c.e. degrees below a given c.e. degree as the degree spectrum of an invariant relation on a $\Delta_2^0$-categorical computable structure.

**3.11 Theorem.** *Let* **a** *be a c.e. degree and let $\alpha$ be a computable ordinal. There exists an intrinsically $\alpha$-c.e. invariant relation $U$ on the domain of a $\Delta_2^0$-categorical computably presentable structure $\mathcal{M}$ such that $\mathrm{DgSp}_{\mathcal{M}}(U)$ consists of all $\alpha$-c.e. degrees less than or equal to* **a**.

*Proof.* This proof is similar to that of Theorem 3.2; we give the necessary changes. Unless otherwise noted, we use the same notation and conventions as in that proof.

Let $A$ be a c.e. set in $\mathbf{a}$. In the proof of Theorem 3.2, we had graphs $[\vec{m}, S]$ that could be used to represent computations in which we computably approximate the oracle $A$. In this proof, we also want to be able to represent the nonincreasing sequences of ordinals less than or equal to $\alpha$ that can be associated to computable approximations of $\alpha$-c.e. sets. This leads to the following definitions.

**3.12 Definition.** Let $\vec{m} = (m_0, m_1, \ldots, m_k) \in \omega^{k+1}$, let $S \subseteq \{0, 1, \ldots, k\}$, and let $\vec{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_k) \in (\alpha+1)^{k+1}$ be nonincreasing. The directed graph $[\vec{m}, S, \vec{\gamma}]$ consists of the following nodes and edges.

1. $k+1$ many nodes $x_0, x_1, x_2, \ldots, x_k$ with an edge from $x_i$ to $x_{i+1}$ for each $i < k$.
2. For each $i \in S$, a copy of $[\langle m_i, \ulcorner\gamma_i\urcorner, 1\rangle]$ with $x_i$ as its top.
3. For each $i \in \{0, 1, \ldots, k\} - S$, a copy of $[\langle m_i, \ulcorner\gamma_i\urcorner, 0\rangle]$ with $x_i$ as its top.

As before, we call $x_0$ the principal node of $[\vec{m}, S, \vec{\gamma}]$. The *height* of $[\vec{m}, S, \vec{\gamma}]$ is defined to be $|\vec{m}|$, its *length* is defined to be $\max\{|\sigma_m| \mid m \in \vec{m}\}$, and its *range* is defined to be $\max\{\ulcorner\gamma_i\urcorner \mid i \leqslant k\}$.

**3.13 Definition.** Let $\vec{m} = (m_0, m_1, \ldots, m_k) \in \omega^{k+1}$, let $S \subseteq \{0, 1, \ldots, k\}$, and let $\vec{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_k) \in (\alpha+1)^{k+1}$ be nonincreasing. We say that $[\vec{m}, S, \vec{\gamma}]$ is *$A$-acceptable* if it satisfies all of the following conditions.

1. $A \upharpoonright |\sigma_{m_k}| = \sigma_{m_k}$.
2. If $i < k$ then $\sigma_{m_i} \neq A \upharpoonright |\sigma_{m_i}|$ and $\sigma_{m_i}(j) = 1 \Rightarrow A(j) = 1$ for every $j < |\sigma_{m_i}|$.
3. For $i > 0$, if $i \in S$ then $\gamma_i \neq \gamma_{i-1}$.
4. If $\alpha \geqslant \omega$ or $0 \in S$ then $\gamma_0 < \alpha$.

We define $A[s]$-acceptability analogously.

Now $\mathcal{M}$ and $U$ are defined much as before.

**3.14 Definition.** Let $\mathcal{M}'$ be the disjoint union of infinitely many copies of each $A$-acceptable $[\vec{m}, S, \vec{\gamma}]$. Let $T$ be the set of principal nodes of these copies.

The directed graph $\mathcal{M}$ consists of $\mathcal{M}'$ and one additional root node $x$, with an edge from $x$ to each element of $T$.

The unary relation $U$ on the domain of $\mathcal{M}$ is the set of all elements of $T$ that are principal nodes of connected components of $\mathcal{M}'$ of the form $[\vec{m}, S, \vec{\gamma}]$ with $|S|$ odd.

The following lemmas hold for the same reasons as the corresponding ones in the proof of Theorem 3.2.

**3.15 Lemma.** *$\mathcal{M}$ is $\Delta^0_2$-categorical.*

**3.16 Lemma.** *If $M$ is a computable presentation of $\mathcal{M}$ then $U^M \leqslant_T A$.*

We also need to check that $U$ is intrinsically $\alpha$-c.e..

**3.17 Lemma.** *If $M$ is a computable presentation of $\mathcal{M}$ then $U^M$ is $\alpha$-c.e..*

*Proof.* Let $S$ be the set of all nodes $y$ of $M$ such that there is an edge from the root node of $M$ to $y$. Let $T$ be set of all nodes $y$ of $M$ such that there is an edge from $y$ to itself. Let $y \in T$. Then $y$ is the top of a copy of $[k]$ for some $k \in \omega$. Let $m \in \omega$, $\beta \leqslant \alpha$, and $i \leqslant 1$ be such that $k = \langle m, \ulcorner\beta\urcorner, i\rangle$. Define $\sigma(y) = \sigma_m$, $\beta(y) = \beta$, and $c(y) = i$. Note that $T$ is computable, and so are the maps taking $y \in T$ to $\sigma(y)$, $\beta(y)$, and $c(y)$.

Define the partial computable binary function $\Psi$ as follows.

*stage* 0. For all $x \in \omega$, let $\Psi(\ulcorner\alpha\urcorner, x) = 0$. If $x \notin S$ then let $\Psi(\ulcorner\beta\urcorner, x) = 0$ for all $\beta < \alpha$. If $x \in S$ then let $\Psi(\ulcorner\beta(x)\urcorner, x) = c(x)$.

*stage* $s+1$. For all $x \in S$, proceed as follows. Let $x_0, \dots, x_n$ be the longest chain of elements of $T \restriction s \cup \{x\}$ such that $x = x_0$ and for all $i < n$ there is an edge from $x_i$ to $x_{i+1}$. Let $c = |\{i < n \mid c(x_i) \neq c(x_{i+1})\}|$. If $\Psi(\ulcorner\beta(x_n)\urcorner, x)$ has not yet been defined then let $\Psi(\ulcorner\beta(x_n)\urcorner, x) = c$.

It is not hard to check that the fact that each $x \in S$ is the principal node of an $A$-acceptable component of $M$ implies that $\Psi$ witnesses that $U^M$ is $\alpha$-c.e. in the sense of Definition 2.1. □

As before, in order to show that $\mathrm{DgSp}_{\mathcal{M}}(U)$ contains every $\alpha$-c.e. degree less than or equal to $\mathbf{a}$, we begin by showing that it contains $\mathbf{0}$.

**3.18 Lemma.** *There exists a computable presentation $N$ of $\mathcal{M}$ such that $U^N$ is computable.*

*Proof.* We build $N$ in stages in much the same way as before.

*stage* 0. Choose 0 as the root node of $N$. Let $k_0 = 0$.

*stage* $s+1$. We break the stage up into two phases.

1. Define $k_{s+1}$, and $\vec{m}_i[s+1]$, $S_i$, and $\vec{\gamma}_i[s+1]$ for $k_s \leqslant i < k_{s+1}$, so that the set $\{[\vec{m}_i[s+1], S_i, \vec{\gamma}_i[s+1]] \mid k_s \leqslant i < k_{s+1}\}$ contains every $A[s+1]$-acceptable $[\vec{m}, S, \vec{\gamma}]$ of height, length, and range less than or equal to $s$. For each $k_s \leqslant i < k_{s+1}$, build a new copy $C_i[s+1]$ of $[\vec{m}_i[s+1], S_i, \vec{\gamma}_i[s+1]]$ using fresh large numbers and add an edge from 0 to the principal node of $C_i[s+1]$.

2. For each $C_i[s]$, $i < k_s$, if $C_i[s]$ is not $A[s+1]$-acceptable then proceed as follows. Let $m$ be such that $\sigma_m = A[s+1] \upharpoonright l$, where $l$ is the length of $C_i[s]$. Let $k$ be the height of $C_i[s]$. Let $\vec{m}_i[s+1] = \vec{m}_i[s]^\frown m$ and $\vec{\gamma}_i[s+1] = \vec{\gamma}_i[s]^\frown\vec{\gamma}_i(k-1)$. Extend $C_i[s]$ to a copy $C_i[s+1]$ of $[\vec{m}_i[s+1], S_i, \vec{\gamma}_i[s+1]]$ using fresh large numbers.

   On the other hand, if $C_i[s]$ is $A[s+1]$-acceptable then let $\vec{m}_i[s+1] = \vec{m}_i[s]$, $\vec{\gamma}_i[s+1] = \vec{\gamma}_i[s]$, and $C_i[s+1] = C_i[s]$.

It is easy to check, as in the proof of Lemma 3.9, that $C_i = \lim_s C_i[s]$ is well-defined and $A$-acceptable for every $i \in \omega$. So every $C_i$ is a copy of some component of $\mathcal{M}$. Moreover, by the same argument as before, each component of $\mathcal{M}$ has infinitely many copies in $N$. Thus $N$ is a computable presentation of $\mathcal{M}$.

As before, to determine whether $x \in U^N$, all we need to do is to look for a stage $s$ in the construction during which numbers greater than $x$ are used. Then $x \in U^N$ if and only if it is the principal node of some $[\vec{m}_i[s], S_i, \vec{\gamma}_i[s]]$ with $i < k_s$ and $|S_i|$ odd. □

**3.19 Lemma.** *Let $B \leqslant_T A$ be $\alpha$-c.e.. There exists a computable presentation $M$ of $\mathcal{M}$ such that $U^M \equiv_m B$.*

*Proof.* This proof is much the same as that of Lemma 3.10; we give the necessary changes.

Let $\Psi$ be a partial computable binary function witnessing the fact that $B$ is $\alpha$-c.e.. It is not hard to see that there exists an $e \in \omega$ with the following properties.

1. $\Phi_e^A = B$.

2. If $\Phi_e^A(n)[s]\downarrow$ then $\Psi(\ulcorner\alpha\urcorner, n)[s]\downarrow$ and $\Phi_e^A(n)[s] = \Psi(\ulcorner\beta\urcorner, n)$ for the least $\beta \leqslant \alpha$ such that $\Psi(\ulcorner\beta\urcorner, n)[s]\downarrow$.

3. For the least number $s$ such that $\Phi_e^A(n)[s]\downarrow$, if either $\alpha \geqslant \omega$ or $\Phi_e^A(n)[s] = 1$ then $\Psi(\ulcorner\beta\urcorner, n)[s]\downarrow$ for some $\beta < \alpha$.

By Lemma 3.18, there is a computable presentation $N$ of $\mathcal{M}$ such that $U^N$ is computable. We can assume that $D = \omega - |N|$ is infinite. We extend $N$ to another computable presentation $M$ of $\mathcal{M}$ in stages. When we make use of fresh numbers in the construction, we take them from $D$ in order.

At the beginning of stage $s+1$, we have copies $C_n[s]$ of graphs $\left[\vec{m}_n[s], S_n[s], \vec{\gamma}_n[s]\right]$ for each $n < s$ such that $\Phi_e^A(n)[t]\downarrow$ for some $t < s$. For each $n \leqslant s$, we proceed as follows.

If $\Phi_e^A(n)[s]\downarrow$ and $\Phi_e^A(n)[t]\uparrow$ for all $t < s$ then let $m$ be such that $\sigma_m = A[s] \upharpoonright \varphi_e^A(n)[s]$ and let $\vec{m}_n[s+1] = (m)$. If $\Phi_e^A(n)[s] = 0$ then let $S_n[s+1] = \emptyset$; otherwise, let $S_n[s+1] = \{0\}$. Let $\beta \leqslant \alpha$ be the least ordinal such that $\Psi(\ulcorner\beta\urcorner, n)[s]\downarrow$ and let $\vec{\gamma}_n[s+1] = (\beta)$. Let $C_n[s+1]$ be a new copy of $\left[\vec{m}_n[s+1], S_n[s+1], \vec{\gamma}_n[s+1]\right]$, formed using fresh numbers in $D$, and add an edge from the root node of $N$ to the principal node of $C_n[s+1]$.

If $C_n[s]$ is defined, $\Phi_e^A(n)[s]\downarrow$, and $\Phi_e^A(n)[s-1]\uparrow$, then let $m$ be such that $\sigma_m = A[s] \upharpoonright \varphi_e^A(n)[s]$ and let $\vec{m}_n[s+1] = \vec{m}_n[s]^\frown m$. If $\Phi_e^A(n)[s] \equiv |S_n[s]| \bmod 2$ then let $S_n[s+1] = S_n[s]$; otherwise, let $S_n[s+1] = S_n[s] \cup \{|\vec{m}_n[s]|\}$. Let $\beta \leqslant \alpha$ be the least ordinal such that $\Psi(\ulcorner\beta\urcorner, n)[s]\downarrow$ and let $\vec{\gamma}_n[s+1] = \vec{\gamma}_n[s]^\frown\beta$. Extend $C_n[s]$ to a copy $C_n[s+1]$ of $\left[\vec{m}_n[s+1], S_n[s+1], \vec{\gamma}_n[s+1]\right]$, using fresh numbers in $D$.

If neither of the previous two cases holds then let $\vec{m}_n[s+1] = \vec{m}_n[s]$, $S_n[s+1] = S_n[s]$, $\vec{\gamma}_n[s+1] = \vec{\gamma}_n[s]$, and $C_n[s+1] = C_n[s]$.

It is easy to check that $M$ is a computable presentation of $\mathcal{M}$, and the proof that $U^M \equiv_{\mathrm{m}} B$ is the same as before. □

The theorem follows from Lemmas 3.15, 3.16, 3.17, and 3.19. ■

It is not hard to replace the single c.e. degree of the previous results by any finite set of degrees.

**3.20 Theorem.** *Let $\mathbf{a}_0, \dots, \mathbf{a}_n$ be c.e. degrees and let $\alpha$ be a computable ordinal.*

1. *There exists an invariant relation $U$ on the domain of a $\Delta^0_2$-categorical computably presentable structure $\mathcal{M}$ such that $\mathrm{DgSp}_{\mathcal{M}}(U)$ consists of all degrees $\mathbf{b}$ such that $\mathbf{b} \leqslant \mathbf{a}_i$ for all $i \leqslant n$.*

2. *There exists an intrinsically $\alpha$-c.e. invariant relation $V$ on the domain of a $\Delta^0_2$-categorical computably presentable structure $\mathcal{N}$ such that $\mathrm{DgSp}_{\mathcal{N}}(V)$ consists of all $\alpha$-c.e. degrees $\mathbf{b}$ such that $\mathbf{b} \leqslant \mathbf{a}_i$ for all $i \leqslant n$.*

*Proof sketch.* The proof of this theorem is very similar to those of Theorems 3.2 and 3.11. We will restrict ourselves to defining $\mathcal{M}$ and $U$. It should be clear how to define $\mathcal{N}$ and $V$ and complete the proof along the lines of the proofs of Theorems 3.2 and 3.11.

Let $A_0, \ldots, A_n$ be c.e. sets in $\mathbf{a}_0, \ldots, \mathbf{a}_n$, respectively. Define the graphs $[\vec{m}, S]$ and the concept of $A_i$-acceptability of such graphs as before. For $\vec{m}_0, \ldots, \vec{m}_n \in \omega^{k+1}$ and $S_0, \ldots, S_n \subseteq \{0, \ldots, k\}$, the directed graph $[\vec{m}_0, \ldots, \vec{m}_n; S_0, \ldots, S_n]$ consists of the following nodes and edges.

1. A *coding node* $x$ and $n+1$ many nodes $x_0, \ldots, x_n$ with an edge from $x$ to $x_i$ for each $i \leqslant n$.

2. For each $i \leqslant n$, a copy of $[i]$ with $x_i$ as its top.

3. For each $i \leqslant n$, a node $y_i$ with an edge from $x_i$ to $y_i$.

4. For each $i \leqslant n$, a copy of $[\vec{m}_i, S_i]$ with $y_i$ as its principal node.

We say that $[\vec{m}_0, \ldots, \vec{m}_n; S_0, \ldots, S_n]$ is *acceptable* if each $[\vec{m}_i, S_i]$ is $A_i$-acceptable and all $|S_i|$ have the same parity. The idea, of course, is that an acceptable component can be used to represent $n+1$ many simultaneous computations, each using a different $A_i$ as an oracle, and all arriving at the same result. Part 2 of the definition of $[\vec{m}_0, \ldots, \vec{m}_n; S_0, \ldots, S_n]$ guarantees that we can tell which representation corresponds to a given $A_i$.

Now let $\mathcal{M}'$ be the disjoint union of infinitely many copies of each acceptable $[\vec{m}_0, \ldots, \vec{m}_n; S_0, \ldots, S_n]$. Let $T$ be the set of coding nodes of these copies. The directed graph $\mathcal{M}$ consists of $\mathcal{M}'$ and one additional node $z$, with an edge from $z$ to each element of $T$. The unary relation $U$ on the domain of $\mathcal{M}$ is the set of all elements of $T$ that are coding nodes of components of $\mathcal{M}$ of the form $[\vec{m}_0, \ldots, \vec{m}_n; S_0, \ldots, S_n]$ with $|S_0|$ odd (equivalently, $|S_i|$ odd for all $i \leqslant n$). $\square$

For any degree $\mathbf{a}$, it is easy to give an example of an invariant relation on the domain of a computably categorical structure whose degree spectrum is $\{\mathbf{a}\}$. Thus, combining Theorem 3.2 with Proposition 3.1, we see that if $\mathbf{a} < \mathbf{b}$ are degrees and $\mathbf{b}$ is c.e. then there exists an invariant relation $U$ on the domain of a $\Delta^0_2$-categorical computable structure $\mathcal{A}$ such that $\mathrm{DgSp}_{\mathcal{A}}(U)$ consists of all degrees in the interval $[\mathbf{a}, \mathbf{b}]$. Similarly, for each computable ordinal $\alpha$, there exists an intrinsically $\alpha$-c.e. invariant relation $U$ on the domain of a $\Delta^0_2$-categorical computable structure $\mathcal{A}$ such that $\mathrm{DgSp}_{\mathcal{A}}(U)$ consists of all

$\alpha$-c.e. degrees in the interval $[\mathbf{a}, \mathbf{b}]$, and similar results can be obtained by combining Theorem 3.20 with Proposition 3.1.

Suppose that we add to Definition 3.5 the condition that $|\sigma_{m_0}| = |\sigma_{m_1}| = \cdots = |\sigma_{m_k}|$ and define $\mathcal{M}$ and $U$ as in Definition 3.6. Then, given a computable presentation $M$ of $\mathcal{M}$, we can compute as a function of $x$ the use of the $A$-computable procedure given in the proof of Lemma 3.8 for determining whether $x \in U^M$, and hence $U^M \leqslant_{\mathrm{wtt}} A$. Furthermore, Lemma 3.9 can be proved as before, as can Lemma 3.10 for $B \leqslant_{\mathrm{wtt}} A$. (In the proof, we need to pick $e$ so that there is a computable bound $f$ on the use of $\Phi_e^A$ and then adopt the convention that $\varphi_e^A(n)[s] = f(n)$ for all $n, s \in \omega$.) Similar changes can be made to the proofs of Theorems 3.11 and 3.20. Thus we have the following result, where we restrict ourselves to $\alpha$-c.e. wtt-degrees, $\alpha \in \omega \cup \{\omega\}$, because every wtt-degree less than or equal to a c.e. wtt-degree is $\omega$-c.e..

**3.21 Theorem.** *Let $\mathbf{a_0}, \ldots, \mathbf{a}_n$ be c.e. wtt-degrees and let $\alpha \in \omega \cup \{\omega\}$. There exists an intrinsically $\alpha$-c.e. invariant relation $U$ on the domain of a $\Delta_2^0$-categorical computably presentable structure $\mathcal{M}$ such that $\mathrm{DgSp}_{\mathcal{M}}^{wtt}(U)$ consists of all $\alpha$-c.e. wtt-degrees $\mathbf{b}$ such that $\mathbf{b} \leqslant \mathbf{a}_i$ for all $i \leqslant n$.*

The following is a natural question in light of the above results.

**3.22 Question.** How far can Theorems 3.2 and 3.11 be extended to the case where $\mathbf{a}$ is not c.e.?

By the results of Section 5, these theorems cannot hold if $\mathbf{a}$ is a minimal degree, and hence *c.e.* cannot be replaced by $\Delta_2^0$, or even by $\omega$-*c.e.*, in their statements. Similarly, the existence of a minimal c.e. tt-degree implies that *wtt* cannot be replaced by *tt*, or by any stronger reducibility, in the statement of Theorem 3.21. As shown in [16], the situation is quite different if the requirement that $\mathcal{M}$ be $\Delta_2^0$-categorical is dropped. For instance, for each $\omega$-c.e. degree $\mathbf{a} > \mathbf{0}$ there is an invariant relation on a computable structure whose degree spectrum is $\{\mathbf{0}, \mathbf{a}\}$, and this remains true if *degree* is replaced by *m-degree*.

As we will see in Section 5, no finite set of degrees containing $\mathbf{0}$ can be the degree spectrum of an invariant relation on the domain of a $\Delta_2^0$-categorical computable structure. However, it is not hard to give an example of an intrinsically d.c.e. invariant relation on the domain of a $\Delta_2^0$-categorical structure with a two-element degree spectrum, but one that does not include $\mathbf{0}$.

Indeed, let $\mathbf{d}$ be a maximal incomplete d.c.e. degree, as constructed in [4]. (That is, $\mathbf{d} \neq \mathbf{0}'$ is d.c.e. and there are no d.c.e. degrees in $(\mathbf{d}, \mathbf{0}')$.) It is easy to construct invariant, intrinsically d.c.e. relations $U$ and $V$ on the domains of $\Delta^0_2$-categorical computable structures $\mathcal{A}$ and $\mathcal{B}$, respectively, so that the degree spectra of $U$ and $V$ are the singleton $\{\mathbf{d}\}$ and the set of all d.c.e. degrees, respectively. By Proposition 3.1, there exists an intrinsically d.c.e. invariant relation $W$ on the domain of a $\Delta^0_2$-categorical computable structure $\mathcal{C}$ whose degree spectrum is

$$\{\mathbf{c} \mid \exists \mathbf{a}, \mathbf{b}(\mathbf{a} \in \mathrm{DgSp}_{\mathcal{A}}(U) \wedge \mathbf{b} \in \mathrm{DgSp}_{\mathcal{B}}(V) \wedge \mathbf{c} = \mathbf{a} \cup \mathbf{b})\} = \{\mathbf{c} \mid \mathbf{d} \leqslant \mathbf{c} \text{ and } \mathbf{c} \text{ is d.c.e.}\} = \{\mathbf{d}, \mathbf{0}'\}.$$

**3.23 Question.** Can the degree spectrum of an intrinsically c.e. invariant relation on a $\Delta^0_2$-categorical computable structure have finite cardinality greater than one?

# 4 Restrictions on Degree Spectra in the Presence of $\Delta^0_2$ Isomorphisms

In this section, we show that, for a computable relation $U$ on a computable structure $\mathcal{A}$ such that the image of $U$ in some $\Delta^0_2$-isomorphic copy of $\mathcal{A}$ is not computable, both upper cone avoidance and the building of infinitely many pairwise incomparable degrees are possible within $\mathrm{DgSp}_{\mathcal{A}}(U)$. We will give the full proof of the possibility of cone avoidance and then comment on the modifications necessary to build infinitely many pairwise incomparable degrees.

We will need some notation to talk about finite portions of a computable structure $\mathcal{A}$ of (possibly infinite) signature $L$. Let $S \subset \omega$ be finite. Define $L_S$ to be the language obtained by restricting $L$ to its first $|S|$ many symbols, substituting all $j$-ary function symbols by $(j+1)$-ary relation symbols in the obvious way, and dropping any constant whose interpretation in $\mathcal{A}$ is not in $S$. Define $\mathcal{A} \upharpoonright S$ to be the finite structure obtained from $\mathcal{A}$ by restricting the domain to $|\mathcal{A}| \cap S$ and restricting the language to $L_S$.

**4.1 Theorem.** *Let $k \in \omega$. Let $U^0$ and $U^1$ be $k$-ary relations on the domains of computable structures $\mathcal{A}^0$ and $\mathcal{A}^1$, respectively, and let $B_0, B_1, \ldots$ be a uniformly $\Delta^0_2$ sequence of noncomputable subsets of $\omega^k$. Suppose that $U^0$ is not computable, $U^1$ is computable, and there exists a $\Delta^0_2$ isomorphism $f : \mathcal{A}^0 \to \mathcal{A}^1$ such that $f(U^0) = U^1$. Then there exists a $\Delta^0_2$ function $h : |\mathcal{A}^0| \xrightarrow[\text{onto}]{1\text{–}1} \omega$ such that $h(\mathcal{A}^0)$ is a computable structure, $h(U^0)$ is not computable, and $B_n \not\leqslant_T h(U^0)$ for all $n \in \omega$.*

*Proof.* Let $\Phi_e$ be the $e^{\text{th}}$ $k$-ary Turing functional. We will build the $\Delta^0_2$ function $h : |\mathcal{A}^0| \xrightarrow[\text{onto}]{1\text{–}1} \omega$ to satisfy the requirements

$$Q_e : \Phi_e \neq h(U^0)$$

and

$$R_{\langle i,j\rangle} : \Phi_i^{h(U^0)} \neq B_j$$

for each $e, i, j \in \omega$, while in addition guaranteeing that $h(\mathcal{A}^0)$ is a computable structure.

Since $f$ is $\Delta^0_2$, there exist sequences $S^i_0 \subset S^i_1 \subset \cdots$, $i = 0, 1$, of subsets of $\omega$ such that $\bigcup_{s\in\omega} S^i_s = |\mathcal{A}^i|$, and a computable sequence $f_0, f_1, \dots$ of maps such that $f_s : \mathcal{A}^0 \restriction S^0_s \cong \mathcal{A}^1 \restriction S^1_s$ for each $s \in \omega$ and $f(x) = \lim_s f_s(x)$ for each $x \in |\mathcal{A}^0|$. For each $s \in \omega$, we will denote $f_s^{-1}(U^1 \cap (S^1_s)^k)$ by $U^0[s]$.

Our construction will be similar to the standard finite injury argument that would be used to satisfy the above requirements with a $\Delta^0_2$ set $A$ in place of $h(U^0)$. Of course, when building a $\Delta^0_2$ set, we can decide at any stage whether we want the value of $A$ at some given element to remain the same or change. In our construction, the only thing we control is $h$.

At each stage $s + 1$, we will define the approximation $h_{s+1}$ of $h$ to extend either $h_s$ or $h_s \circ f_s^{-1} \circ f_{s+1}$. If $h_{s+1}$ extends $h_s \circ f_s^{-1} \circ f_{s+1}$ then, for all $\vec{x}$ in the range of $h_s$, $U^0(h^{-1}_{s+1}(\vec{x}))[s + 1] = U^0(h^{-1}_s(\vec{x}))[s]$, which means that $h(U^0)$ remains unaltered at this stage. On the other hand, if $U^0(h^{-1}_s(\vec{x}))[s + 1] \neq U^0(h^{-1}_s(\vec{x}))[s]$ then we can change the value of $h(U^0)$ at $\vec{x}$ by letting $h_{s+1}$ extend $h_s$. The fact that $f$ is $\Delta^0_2$ means that, for each $\vec{x} \in \omega^k$, there is a stage $s$ such that $f^{-1}_t \circ f_{t+1}(\vec{x}) = \vec{x}$ for all $t \geqslant s$. This will imply that $h_{t+1}(\vec{x}) = h_t(\vec{x})$ for all $t \geqslant s$, thus ensuring that $\lim_t h_t$ exists.

We now proceed with the construction. To simplify our notation, we assume without loss of generality that $|\mathcal{A}| = \omega$. We can do this because every infinite computable structure is computably isomorphic to one with domain $\omega$.

*stage* 0. Let $h_0 = \emptyset$.

*stage* $s + 1$. For each $e < s + 1$, let $i$ and $j$ be such that $\langle i, j\rangle = e$ and define

$$q_{e,s} = \begin{cases} \max(f_s^{-1}(\vec{z})) & Q_e \text{ is currently satisfied through } \vec{z} \text{ (defined below)} \\ \max\{y \mid \forall \vec{z} < y(\Phi_e(\vec{z})[s]\downarrow)\} & \text{otherwise,} \end{cases}$$

$$l_{e,s} = \max\left\{y \mid \forall \vec{z} < y\left(\Phi_i^{h_s(U^0[s])}(\vec{z})[s]\downarrow = B_j(\vec{z})[s]\right)\right\},$$

$$m_{e,s} = \max\{l_{e,t} \mid t \leqslant s\},$$

and

$$r_{e,s} = \max\left\{\varphi_i^{h_s(U^0[s])}(\vec{z})[s] \mid \vec{z} \leqslant m_{e,s}\right\}.$$

In order to define $h_{s+1}$, we will begin by defining an auxiliary function $g$. Let $e < s+1$ be the least number such that

1. $f_{s+1}(y) = f_s(y)$ for all $y$ such that either $y \leqslant e$ or $h_s(y) \leqslant e$ and
2. one of the following holds.
   (a) $Q_e$ is not satisfied and, for some $\vec{x} \in \mathrm{dom}(h_s)$, we have $U^0(\vec{x})[s+1] \neq U^0(\vec{x})[s]$ and $\Phi_e(h_s(\vec{x}))[s]\downarrow$.
   (b) Not 2.a and $f_{s+1}(y) \neq f_s(y)$ for some $y \leqslant q_{e,s}$.
   (c) Not 2.a or 2.b, and $f_{s+1}(y) \neq f_s(y)$ for some $y$ such that $h_s(y) \leqslant r_{e,s}$.

If no such number exists then let $g = h_s$.

If condition 2.a holds then proceed as follows. If $\Phi_e(h_s(\vec{x})) = U^0(\vec{x})[s]$ then let $g = h_s$; otherwise, let $g = h_s \circ f_s^{-1} \circ f_{s+1}$. In either case, declare $Q_e$ to be satisfied through $f_s(\vec{x})$. We say that $Q_e$ is active at stage $s+1$.

If condition 2.b holds then proceed as follows. If $Q_e$ is not satisfied then let $g = h_s$; otherwise, let $g = h_s \circ f_s^{-1} \circ f_{s+1}$. In either case, we say that $Q_e$ is active at stage $s+1$.

If condition 2.c holds then let $g = h_s \circ f_s^{-1} \circ f_{s+1}$. We say that $R_e$ is active at stage $s+1$.

If either $Q_e$ or $R_e$ is active at stage $s+1$ then declare each $Q_i$, $i > e$, to be unsatisfied.

Now define $h_{s+1}$ as follows. For $y \in \mathrm{dom}(g)$, let $h_{s+1}(y) = g(y)$. Let $y_0 < \cdots < y_m$ be the elements of $S_{s+1} - \mathrm{dom}(g)$ and let $z_0 < \cdots < z_m$ be the $m+1$ least numbers not in $\mathrm{rng}(g)$. For $i \leqslant m$, let $h_{s+1}(y_i) = z_i$.

This completes the construction. We now need to show that $h = \lim_s h_s$ and $h^{-1} = \lim_s h_s^{-1}$ are well-defined, all requirements are met, and $h(\mathcal{A}^0)$ is a computable structure. We begin by showing by induction that $h$ and $h^{-1}$ are well-defined; each requirement is active only finitely often; $q_{e,s}$ and $r_{e,s}$ have finite limits for each $e \in \omega$; and for each $e \in \omega$, if $\Phi_e$ is total then $Q_e$ is eventually permanently satisfied.

For the following lemmas, fix $e \in \omega$ and assume by induction that, for all $i < e$, the requirements $Q_i$ and $R_i$ are active only finitely often and $\lim_s h_s^{-1}(i)$ is well-defined.

**4.2 Lemma.** *Both* $h^{-1}(e) = \lim_s h_s^{-1}(e)$ *and* $h(e) = \lim_s h_s(e)$ *are well-defined.*

*Proof.* Let $s$ be a stage such that no requirement $Q_i$ or $R_i$, $i < e$, is active after stage $s$. By construction, for all $t \in \omega$, the map $h_{t+1}$ extends either $h_t$ or $h_t \circ f_t^{-1} \circ f_{t+1}$. One of the conditions for a requirement $Q_j$ or $R_j$, $j \geqslant e$, to be active at stage $t+1$ is that $f_{t+1}(e) = f_t(e)$ and $f_{t+1}(h_t^{-1}(e)) = f_t(h_t^{-1}(e))$. So, for all $t \geqslant s$, if $f_{t+1}(e) \neq f_t(e)$ or $f_{t+1}(h_t^{-1}(e)) \neq f_t(h_t^{-1}(e))$ then no requirement is active at stage $t+1$, and hence $h_{t+1}$ extends $h_t$. Thus $h_t(e) = h_s(e)$ and $h_t^{-1}(e) = h_s^{-1}(e)$ for all $t > s$. □

Let $s_0 > e$ be such that the following conditions hold: no requirement $Q_i$ or $R_i$, $i < e$, is active after stage $s_0$; $h_t^{-1}(i) = h^{-1}(i)$ for all $t \geqslant s_0$ and $i \leqslant e$; and $f_t(y) = f(y)$ for all $t \geqslant s_0$ and all $y$ such that either $y \leqslant e$ or $h^{-1}(i) = y$ for some $i \leqslant e$.

**4.3 Lemma.** *If* $\Phi_e$ *is total then* $Q_e$ *is eventually permanently satisfied.*

*Proof.* It is enough to show that if $\Phi_e$ is total then $Q_e$ is satisfied at some stage $t > s_0$. Suppose otherwise. We claim that we can compute $U^0$, which contradicts the hypothesis that $U^0$ is not computable. Let $\vec{x} \in \omega^k$. Since $\Phi_e$ is total and $Q_e$ is never satisfied after stage $s_0$, we have $\lim_t q_{e,t} = \infty$. Let $t > s_0$ be such that $\vec{x} < q_{e,t}$ and $\vec{x} \in (\mathrm{dom}(h_t))^k$. As mentioned above, for all $u \in \omega$, the map $h_{u+1}$ extends either $h_u$ or $h_u \circ f_u^{-1} \circ f_{u+1}$. Furthermore, for all $u \geqslant t$, if $f_{u+1}(\vec{x}) \neq f_u(\vec{x})$ then $h_{u+1}$ extends $h_u$. So $h_u(\vec{x}) = h_t(\vec{x})$ for all $u > t$. Now let $u \geqslant t$ be such that $\Phi_e(h_t(\vec{x}))[u]\downarrow$. If $U^0(\vec{x})[v+1] \neq U^0(\vec{x})[v]$ for some $v \geqslant u$ then $Q_e$ is satisfied at stage $v+1$. Therefore, $\vec{x} \in U^0 \Leftrightarrow \vec{x} \in U^0[u]$. □

**4.4 Lemma.** $\lim_s q_{e,s} < \infty$ *and* $Q_e$ *is active only finitely often.*

*Proof.* If $Q_e$ is satisfied through $\vec{z}$ after stage $s_0$ then $\lim_s q_{e,s} = \max(f^{-1}(\vec{z}))$. Otherwise, by the previous lemma, $\Phi_e$ is not total, and thus $\lim_s q_{e,s}$ is equal to the largest $y$ such that $\Phi_e(\vec{x})\downarrow$ for all $\vec{x} < y$. If $Q_e$ is satisfied after stage $s_0$ then let $t > s_0$ be such that $Q_e$ is satisfied at stage $t$. Otherwise, let $t > s_0$ be such that for all $u > t$ and $y \leqslant \lim_s q_{e,s}$, we have $q_{e,u} = q_{e,t}$ and $f_u(y) = f_t(y)$. In either case, $Q_e$ is not active after stage $t$. □

Let $s_1 \geqslant s_0$ be such that $Q_e$ is not active after stage $s_1$. Let $i$ and $j$ be such that $\langle i, j \rangle = e$.

**4.5 Lemma.** $\lim_s m_{e,s} < \infty$.

*Proof.* Let $t > s_1$, let $x \leqslant r_{e,t}$, and let $z = f_t \circ h_t^{-1}(x)$. For any $u \geqslant t$, if $f_{u+1}(x) \neq f_u(x)$ then $h_{u+1}$ extends $h_u \circ f_{u+1}^{-1} \circ f_u$. So $f_u \circ h_u^{-1}(x) = z$ for all $u \geqslant t$. Therefore, for all

$u \geqslant t$, we have $h_u(U^0[u])(x) = U^0(h_u^{-1}(x))[u] = U^1(f_u \circ h_u^{-1}(x)) = U^1(f_t \circ h_t^{-1}(x)) = U^0(h_t^{-1}(x))[t] = h_t(U^0[t])(x)$.

Now assume for a contradiction that $\lim_s m_{e,s} = \infty$. Then for each $\vec{x} \in \omega^k$ there is a $t_{\vec{x}} \in \omega$ such that $\vec{x} < l_{e,t_{\vec{x}}}$. Since $B_j$ is not computable and we can computably determine $t_{\vec{x}}$ from $\vec{x}$, there exists an $\vec{x} \in \omega^k$ such that $B_j(\vec{x}) \neq B_j(\vec{x})[t_{\vec{x}}]$. Let $u$ be such that $B_j(\vec{x})[v] = B_j(\vec{x})[u]$ for all $v > u$. Using the result of the previous paragraph, we conclude that, for all $v \geqslant u$, we have $\Phi_i^{h_v(U^0[v])}(\vec{x})[v]\downarrow = \Phi_i^{h_{t_{\vec{x}}}(U^0[t_{\vec{x}}])}(\vec{x})[t_{\vec{x}}]\downarrow = B_j(\vec{x})[t_{\vec{x}}] \neq B_j(\vec{x})[v]$, which implies that $l_{e,v} \leqslant \vec{x}$, contradicting the assumption that $\lim_s m_{e,s} = \infty$. □

**4.6 Lemma.** $\lim_s r_{e,s} < \infty$ *and* $R_e$ *is active only finitely often.*

*Proof.* Let $t > s_1$ be such that $m_{e,u} = m_{e,t}$ for all $u > t$. As we have seen, $h_u(U^0[u])(\vec{y}) = h_t(U^0[t])(\vec{y})$ for all $\vec{y} \leqslant r_{e,t}$ and $u > t$. Thus $\varphi_i^{h_u(U^0[u])}(\vec{x})[u] = \varphi_i^{h_t(U^0[t])}(\vec{x})[t]$ for all $u > t$ and $\vec{x} \leqslant m_{e,t}$. So $r_e = \lim_s r_{e,s} < \infty$.

Now let $t > s_1$ be such that $r_{e,u} = r_{e,t}$ for all $u > t$. For $m \leqslant r_e$, let $z_m = f_t \circ h_t^{-1}(m)$. For all $u > t$ and $m \leqslant r_e$, we have $f_u \circ h_u^{-1}(m) = z_m$. Let $u > t$ be such that $f_v^{-1}(z_m) = f_u^{-1}(z_m)$ for all $v > u$ and $m \leqslant r_e$. For $m \leqslant r_e$, let $y_m = h_u^{-1}(m)$. Now, for all $v > u$ and $m \leqslant r_e$, we have $h_v^{-1}(m) = y_m$ and $f_v(y_m) = f_u(y_m)$. It follows that $R_e$ is not active after stage $u$. □

This completes the induction. We now show that all requirements are met and $h(\mathcal{A}^0)$ is a computable structure.

**4.7 Lemma.** *For all* $e \in \omega$, $\Phi_e \neq h(U^0)$.

*Proof.* If $\Phi_e$ is not total then there is nothing to show, so assume that $\Phi_e$ is total. By Lemma 4.3, there are $\vec{z} \in \omega^k$ and $t \in \omega$ such that $Q_e$ is permanently satisfied through $\vec{z}$ at stage $t+1$. Let $\vec{y} = h_t \circ f_t^{-1}(\vec{z})$. It is easy to check from the definition of $h_{t+1}$ that $\Phi_e(\vec{y}) \neq U^1(\vec{z})$.

We claim that $h_{u+1} \circ f_{u+1}^{-1}(\vec{z}) = \vec{y}$ for all $u \geqslant t$. Indeed, let $u \geqslant t$ and assume by induction that $h_u \circ f_u^{-1}(\vec{z}) = \vec{y}$. There are two cases.

1. If $f_{u+1}^{-1}(\vec{z}) = f_u^{-1}(\vec{z})$ then, no matter which way $h_{u+1}$ is defined, $h_{u+1} \circ f_{u+1}^{-1}(\vec{z}) = h_u \circ f_u^{-1}(\vec{z}) = \vec{y}$.

2. If $f_{u+1}^{-1}(\vec{z}) \neq f_u^{-1}(\vec{z})$ then, since $q_{e,u} = f_u^{-1}(\vec{z})$, it follows that $h_{u+1}$ extends $h_u \circ f_u^{-1} \circ f_{u+1}$, and hence $h_{u+1} \circ f_{u+1}^{-1}(\vec{z}) = h_u \circ f_u^{-1} \circ f_{u+1} \circ f_{u+1}^{-1}(\vec{z}) = h_u \circ f_u^{-1}(\vec{z}) = \vec{y}$.

So, by induction, $h_{u+1} \circ f_{u+1}^{-1}(\vec{z}) = \vec{y}$ for all $u \geqslant t$, and hence $h \circ f^{-1}(\vec{z}) = \vec{y}$. Thus $h(U^0)(\vec{y}) = U^0(h^{-1}(\vec{y})) = U^0(f^{-1}(\vec{z})) = U^1(\vec{z}) \neq \Phi_e(\vec{y})$. □

**4.8 Lemma.** *For all $i \in \omega$ and $j < n$, $\Phi_i^{h(U^0)} \neq B_j$.*

*Proof.* If $\Phi_i^{h(U^0)} = B_j$ then $\lim_s m_{\langle i,j\rangle,s} = \infty$, which we have already shown not to be the case. □

**4.9 Lemma.** *$h(\mathcal{A}^0)$ is a computable structure.*

*Proof.* For all $s \in \omega$, $\mathrm{rng}(h_{s+1}) \supset \mathrm{rng}(h_s)$ and $h_{s+1}^{-1} \circ h_s$ is either the identity or is equal to $f_{s+1}^{-1} \circ f_s$, and hence is an embedding from $\mathcal{A}^0 \restriction S_s$ into $\mathcal{A}^0 \restriction S_{s+1}$, if we restrict the latter structure to the language $L_{S_s}$. Furthermore, $\bigcup_{s\in\omega} \mathrm{rng}(h_s) = \omega$. So the images of $h_s$ form a chain whose limit $h(\mathcal{A}^0)$ is a computable structure. □

The theorem follows from Lemmas 4.7, 4.8, and 4.9. ■

It is straightforward to modify the proof of Theorem 4.1 to build, instead of one $\Delta_2^0$ map $h$, infinitely many such maps $h_0, h_1, \ldots$, to satisfy the requirements

$$Q_e : \Phi_e \neq h_i(U^0)$$

and

$$R_{\langle e,i,j\rangle} : \Phi_e^{h_i(U^0)} \neq B_j$$

for each $e, i, j \in \omega$ and

$$S_{\langle e,i,j\rangle} : \Phi_e^{h_i(U^0)} \neq h_j(U^0)$$

for each $e, i, j \in \omega$, $i \neq j$, while in addition guaranteeing that each $h_j(\mathcal{A}^0)$ is a computable structure. This shows that, as mentioned above, in the context of invariant computable (but not intrinsically computable) relations on $\Delta_2^0$-categorical computable structures, techniques for upper cone avoidance and building of incomparable degrees developed for the c.e. degrees are applicable. We record below this stronger form of Theorem 4.1.

**4.10 Theorem.** *Let $U$ be a computable relation on the domain of a computable structure $\mathcal{A}$. Let $D$ be the set of degrees of a uniformly $\Delta_2^0$ collection of noncomputable sets. If there exists a $\Delta_2^0$, 1–1 function $f$ such that $f(\mathcal{A})$ is a computable structure (isomorphic to $\mathcal{A}$) and $f(U)$ is not computable, then $\mathrm{DgSp}_{\mathcal{A}}(U)$ contains an infinite set $S$ of pairwise incomparable degrees such that $\mathbf{b} \ngeqslant \mathbf{a}$ for each $\mathbf{a} \in D$ and $\mathbf{b} \in S$.*

In the next section, the role of these results in showing that certain kinds of relations always have infinite degree spectra will be explored. It should be noted, however, that they already show that there are limitations on the possible degree spectra of invariant relations on computably categorical structures that do not arise in the general case, even if we restrict ourselves to infinite degree spectra.

For instance, as mentioned in Section 1, it is shown in [16] that, for every uniformly c.e. collection $\mathfrak{S}$ of sets of natural numbers, there is an invariant relation on a computable structure whose degree spectrum coincides with the degrees of elements of $\mathfrak{S}$. An appropriate choice of $\mathfrak{S}$ shows that, for each c.e. degree $\mathbf{a} < \mathbf{0}'$, there is an invariant computable relation on a computable structure whose degree spectrum is infinite and such that every nonzero member is above $\mathbf{a}$. Similarly, there is an invariant computable relation on a computable structure whose degree spectrum is infinite but contains no pairwise incomparable elements. By Theorem 4.10, neither of these cases is possible if the structure is $\Delta^0_2$-categorical.

On the other hand, the results of Section 3 imply that there are certain directions in which Theorem 4.1 cannot be extended. For instance, we might have hoped that some kind of permitting could be used to show that for any c.e. degree $\mathbf{a}$, the degree spectrum of an invariant computable relation on a $\Delta^0_2$-categorical computable structure that is not intrinsically computable has a noncomputable element below $\mathbf{a}$. However, Theorem 3.11 shows that this is not possible. Indeed, let $\mathbf{a}$ and $\mathbf{b}$ be a minimal pair of c.e. degrees. By Theorem 3.11, there is an invariant relation $U$ on a $\Delta^0_2$-categorical computable structure $\mathcal{A}$ such that $\mathrm{DgSp}_{\mathcal{A}}(U)$ consists of all c.e. degrees less than or equal to $\mathbf{a}$. For any $\mathbf{c} \in \mathrm{DgSp}_{\mathcal{A}}(U)$, either $\mathbf{c} = \mathbf{0}$ or $\mathbf{c} \nleq \mathbf{b}$.

# 5 A Sufficient Condition for Infinite Degree Spectra

The results of the previous section (either Theorem 4.10 directly or Theorem 4.1 by repeated applications) have as an immediate consequence the following sufficient condition for a relation to have infinite degree spectrum.

**5.1 Theorem.** *Let $U$ be a computable relation on the domain of a computable structure $\mathcal{A}$. If there exists a $\Delta^0_2$, 1–1 function $f$ such that $f(\mathcal{A})$ is a computable structure (isomorphic to $\mathcal{A}$) and $f(U)$ is not computable, then* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *is infinite.*

It should be noted that this result is an analog of the following theorem of Goncharov [10], which is very useful in showing that certain classes of structures have no members of finite computable dimension greater than one.

**5.2 Theorem** (Goncharov)**.** *If two computable structures are $\Delta^0_2$-isomorphic but not computably isomorphic then their computable dimension is $\omega$.*

The following result follows immediately from Theorem 5.1.

**5.3 Theorem.** *Let $U$ be an invariant computable relation on the domain of a $\Delta^0_2$-categorical computable structure $\mathcal{A}$. Either $U$ is intrinsically computable or* $\mathrm{DgSp}_{\mathcal{A}}(U)$ *is infinite.*

In Theorem 5.3, both conditions on $U$ are necessary. In Section 3, we saw that there exists an invariant relation on the domain of a $\Delta^0_2$-categorical computable structure whose degree spectrum consists of exactly two degrees, neither of them computable. Now let $\mathcal{A}^0$, $\mathcal{A}^1$, $U^0$, and $U^1$ be the structures and relations built by Khoussainov and Shore to prove Theorem 2.1 of [20]. We can assume that $|\mathcal{A}^0| \cap |\mathcal{A}^1| = \emptyset$. Let $P$ be the predicate $\{(x,y) \mid x \in U^0 \wedge y \in U^1 \wedge$ there is an isomorphism from $\mathcal{A}^0$ to $\mathcal{A}^1$ that extends the map $x \mapsto y\}$ and let $E$ be the equivalence relation whose equivalence classes are $|\mathcal{A}^0|$ and $|\mathcal{A}^1|$. The construction of the $\mathcal{A}^i$ ensures that $P$ is computable. In the proof of Theorem 4.2 of [20], it is shown that if $\mathcal{B}$ is the computable structure obtained by taking the union of $\mathcal{A}^0$ and $\mathcal{A}^1$ and expanding it by $P$ and $E$ then $\mathcal{B}$ is computably categorical. Since $\mathcal{B}$ has exactly one nontrivial automorphism, which sends $U^1$ to $U^0$, it follows that $\mathrm{DgSp}_{\mathcal{B}}(U^1) = \{\mathbf{0}, \deg(U^0)\}$.

Even when $\mathcal{A}$ is not necessarily $\Delta^0_2$-categorical and $U$ is not necessarily invariant, it is sometimes possible to use Theorem 5.1 to show that either $U$ is intrinsically computable or $\mathrm{DgSp}_{\mathcal{A}}(U)$ is infinite.

In [23], Moses showed that, for any computable relation $U$ on a linear ordering $\mathcal{L}$, either $U$ is definable by a quantifier free formula in the language of $\mathcal{L}$ expanded by finitely many constants (in which case it is obviously intrinsically computable) or there is a 1–1 function $f$ such that $f(\mathcal{L})$ is a computable structure and $f(U)$ is not computable. It is clear from the proof of this result that, in the latter case, $f$ can be chosen to be $\Delta^0_2$. Theorem 5.1 can thus be invoked to establish the following result.

**5.4 Theorem.** *Let $U$ be a computable relation on the domain of a computable linear ordering $\mathcal{L}$. Either $U$ is intrinsically computable or* $\mathrm{DgSp}_{\mathcal{L}}(U)$ *is infinite.*

**5.5 Question.** Another way of phrasing Theorem 5.4 is that the degree spectrum of a computable relation on the domain of a computable linear ordering is either a singleton or infinite. How far can this result be extended to the case where the relation is not computable? Is there any $\Delta^0_2$ relation on a computable linear ordering whose degree spectrum is finite but not a singleton?

An example of an intrinsically $\Delta^0_3$ relation on a computable linear ordering with a two element degree spectrum is given in [6].

Of course, it also follows from the proof of Moses's result that the comments following Theorem 4.1 are also applicable in the case of linear orderings. That is, upper cone avoidance and the building of infinitely many pairwise incomparable degrees are possible within the degree spectrum of any computable but not intrinsically computable relation on a computable linear ordering.

By the proof of a result of Ash and Nerode [3], the same is true for certain relations on 1-decidable structures. The key here is the following extra decidability condition.

**5.6 Definition.** Let $U$ be a relation on a computable structure $\mathcal{A}$. We say that $U$ *satisfies condition* $(*)$ if there is a computable procedure for determining, given $a_0, \ldots, a_n \in |\mathcal{A}|$ and an existential formula $\psi(\vec{x})$ in the language of $\mathcal{A}$ expanded by constants for $a_0, \ldots, a_n$, whether $(\mathcal{A}, U, a_0, \ldots, a_n) \vDash \forall\vec{x}(\psi(\vec{x}) \to U(\vec{x}))$.

Notice that if $U$ is a nonempty relation on a computable structure $\mathcal{A}$ satisfying condition $(*)$ then $U$ is computable and $\mathcal{A}$ is 1-decidable. Notice also that a sufficient condition for both $U$ and its complement to satisfy condition $(*)$ is that $(\mathcal{A}, U)$ be 1-decidable.

Two other notions introduced in [3] are those of *formally c.e.* and *formally computable* relations.

**5.7 Definition.** A $k$-ary relation $U$ on a computable structure $\mathcal{A}$ is *formally c.e.* if there exists a c.e. sequence $\psi_0, \psi_1, \ldots$ of existential formulas in the language of $\mathcal{A}$ expanded by finitely many constants from $\mathcal{A}$ such that $U(\vec{x}) \Leftrightarrow \bigvee_{n\in\omega} \psi_n(\vec{x})$ for every $\vec{x} \in \omega^k$.

A relation $U$ on a computable structure is *formally computable* if both it and its complement are formally c.e..

In [3] the following result was established.

**5.8 Theorem** (Ash and Nerode). *Let $U$ be a relation on a computable structure satisfying condition $(*)$. Then $U$ is formally c.e. if and only if it is intrinsically c.e..*

The proof of Theorem 5.8 shows that if a relation $U$ on a computable structure $\mathcal{A}$ satisfying condition $(*)$ is not formally c.e. then there is a $\Delta^0_2$, 1–1 function $f$ such that

$f(\mathcal{A})$ is a computable structure and $f(U)$ is not c.e.. Given a computable relation $U$ on a computable structure $\mathcal{A}$ such that both it and its complement satisfy condition $(*)$, we can apply this result to $U$ and its complement to conclude that either $U$ is formally computable (in which case it is intrinsically computable) or there is a $\Delta^0_2$, 1–1 function $f$ such that $f(\mathcal{A})$ is a computable structure and $f(U)$ is not computable.

Thus we have the following results, of which only the cone avoidance part is new, the other parts having been established by Harizanov [13]. For a c.e. but not intrinsically c.e. relation $U$ on a computable structure satisfying condition $(*)$, both upper cone avoidance and the building of infinitely many pairwise incomparable degrees are possible within the degree spectrum of $U$, and the same is true of a computable but not intrinsically computable relation on a computable structure such that both it and its complement satisfy condition $(*)$. In particular, in both these cases the degree spectrum is infinite.

**5.9 Question.** It would be interesting to know whether condition $(*)$ can be replaced in the above results by the weaker condition that $U$ be computable and $\mathcal{A}$ be 1-decidable. In particular, is there a computable but not intrinsically computable relation on a 1-decidable structure whose degree spectrum is finite?

The results of this section suggest the following question.

**5.10 Question.** For what other natural classes of structures is it true that every (invariant) computable but not intrinsically computable relation on a computable structure from the given class has infinite degree spectrum?

The case of Boolean algebras has been handled by Downey, Goncharov, and Hirschfeldt [6]. Another strong possibility, in light of results about computable dimension and well-established structure theorems, is the class of Abelian groups. On the other hand, the results of [17] give several examples of classes of structures that do not satisfy the condition of Question 5.10.

# References


[1] C. J. Ash, Isomorphic recursive structures, in Ershov et al. [9] 167–182.

[2] C. J. Ash, P. Cholak, and J. F. Knight, Permitting, forcing, and copying of a given recursive relation, Ann. Pure Appl. Logic 86 (1997) 219–236.

[3] C. J. Ash and A. Nerode, Intrinsically recursive relations, in J. N. Crossley (ed.), Aspects of Effective Algebra (Clayton, 1979) (Upside Down A Book Co., Yarra Glen, Australia, 1981) 26–41.

[4] S. B. Cooper, L. Harrington, A. H. Lachlan, S. Lempp, and R. I. Soare, The d.r.e. degrees are not dense, Ann. Pure Appl. Logic 55 (1991) 125–151.

[5] R. G. Downey, Computability theory and linear orderings, in Ershov et al. [9] 823–976.

[6] R. G. Downey, S. S. Goncharov, and D. R. Hirschfeldt, Degree spectra of relations on Boolean algebras, to appear.

[7] R. L. Epstein, R. Haas, and R. L. Kramer, Hierarchies of sets and degrees below $\mathbf{0}'$, in M. Lerman, J. H. Schmerl, and R. I. Soare (eds.), Logic Year 1979–80 (Proc. Seminars and Conf. Math. Logic, Univ. Connecticut, Storrs, Conn., 1979/80), vol. 859 of Lecture Notes in Mathematics (Springer–Verlag, Heidelberg, 1981) 32–48.

[8] Y. L. Ershov and S. S. Goncharov, Elementary theories and their constructive models, in Ershov et al. [9] 115–166.

[9] Y. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel (eds.), Handbook of Recursive Mathematics, vol. 138–139 of Studies in Logic and the Foundations of Mathematics (Elsevier Science, Amsterdam, 1998).

[10] S. S. Goncharov, Limit equivalent constructivizations, in Mathematical Logic and the Theory of Algorithms, vol. 2 of Trudy Inst. Mat. ("Nauka" Sibirsk. Otdel., Novosibirsk, 1982) 4–12, in Russian.

[11] S. S. Goncharov, Autostable models and algorithmic dimensions, in Ershov et al. [9] 261–288.

[12] V. S. Harizanov, Degree spectrum of a recursive relation on a recursive structure, PhD Thesis, University of Wisconsin, Madison, WI (1987).

[13] V. S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, Ann. Pure Appl. Logic 55 (1991) 51–65.

[14] V. S. Harizanov, Pure computable model theory, in Ershov et al. [9] 3–114.

[15] V. S. Harizanov, Turing degrees of certain isomorphic images of computable relations, Ann. Pure Appl. Logic 93 (1998) 103–113.

[16] D. R. Hirschfeldt, Degree spectra of intrinsically c.e. relations, to appear in J. Symbolic Logic.

[17] D. R. Hirschfeldt, B. Khoussainov, R. A. Shore, and A. M. Slinko, Degree spectra and computable dimension in algebraic structures, to appear in Ann. Pure Appl. Logic.

[18] D. R. Hirschfeldt and W. M. White, Realizing levels of the hyperarithmetic hierarchy as degree spectra of relations on computable structures, to appear.

[19] W. Hodges, Model Theory, vol. 42 of Encyclopedia Math. Appl. (Cambridge University Press, Cambridge, 1993).

[20] B. Khoussainov and R. A. Shore, Computable isomorphisms, degree spectra of relations, and Scott families, Ann. Pure Appl. Logic 93 (1998) 153–193.

[21] B. Khoussainov and R. A. Shore, Effective model theory: the number of models and their complexity, in S. B. Cooper and J. K. Truss (eds.), Models and Computability, vol. 259 of London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, 1999) 193–239.

[22] Y. N. Moschovakis, Elementary induction on abstract structures, vol. 77 of Studies in Logic and the Foundations of Mathematics (North-Holland Publishing Company, Amsterdam-London, 1974).

[23] M. Moses, Relations intrinsically recursive in linear orders, Z. Math. Logik Grundlag. Math. 32 (1986) 467–472.

[24] J. B. Remmel, Recursive isomorphism types of recursive Boolean algebras, J. Symbolic Logic 46 (1981) 572–594.

[25] R. I. Soare, Recursively Enumerable Sets and Degrees, Perspect. Math. Logic (Springer–Verlag, Heidelberg, 1987).

[26] I. N. Soskov, Intrinsically hyperarithmetical sets, Math. Log. Quart. 42 (1996) 469–480.