A FEINER LOOK AT THE INTERMEDIATE DEGREES

DENIS R. HIRSCHFELDT, ASHER M. KACH, AND ANTONIO MONTALBÁN

Abstract. We say that a set S is $\Delta^0_{(n)}(X)$ if membership of n in S is a $\Delta^0_n(X)$ question, uniformly in n. A set X is *low for Δ-Feiner* if every set S that is $\Delta^0_{(n)}(X)$ is also $\Delta^0_{(n)}(\emptyset)$. It is easy to see that every low$_n$ set is low for Δ-Feiner, but we show that the converse is not true by constructing an intermediate c.e. set that is low for Δ-Feiner. We also study variations on this notion, such as the sets that are $\Delta^0_{(bn+a)}(X)$, $\Sigma^0_{(bn+a)}(X)$, or $\Pi^0_{(bn+a)}(X)$, and the sets that are low, intermediate, and high for these classes. In doing so, we obtain a result on the computability of Boolean algebras, namely that there is a Boolean algebra of intermediate c.e. degree with no computable copy.

1. Introduction

In [2], Feiner introduced a hierarchy of complexities that we term the *Feiner Δ-hierarchy*, for sets computable in $\emptyset^{(\omega)}$. His original motivation was to build a computably enumerable Boolean algebra that has no computable copy. We can relativize this hierarchy to an arbitrary set X, and define the analogous *Feiner Σ-hierarchy* and *Feiner Π-hierarchy*, as follows.

Definition 1.1. Fix a set $X \subseteq \omega$ and $a, b \in \omega$ with $b \geqslant 1$.

A set $S \subseteq \omega$ is $\Delta^0_{(bn+a)}(X)$ *in the Feiner Δ-hierarchy*, denoted $S \in \Delta^0_{(bn+a)}(X)$, if there is a Turing functional Φ_e such that

$$\Phi_e^{X^{(bn+a-1)}}(n) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S, \end{cases}$$

where $X^{(m)}$ is the mth Turing jump of X (and $X^{(-1)} = X^{(0)} = X$).

A set $S \subseteq \omega$ is $\Sigma^0_{(bn+a)}(X)$ *in the Feiner Σ-hierarchy*, denoted $S \in \Sigma^0_{(bn+a)}(X)$, if there is a computably enumerable operator W_e such that

$$n \in S \iff n \in W_e^{X^{(bn+a-1)}}.$$

A set $S \subseteq \omega$ is $\Pi^0_{(bn+a)}(X)$ *in the Feiner Π-hierarchy*, denoted $S \in \Pi^0_{(bn+a)}(X)$, if its complement is $\Sigma^0_{(bn+a)}(X)$.

In other words, a set S is $\Delta^0_{(bn+a)}(X)$ (respectively, $\Sigma^0_{(bn+a)}(X)$ or $\Pi^0_{(bn+a)}(X)$) if membership of n in S is a $\Delta^0_{bn+a}(X)$ (respectively, $\Sigma^0_{bn+a}(X)$ or $\Pi^0_{bn+a}(X)$) question, uniformly in n. It is easy to see that every $\Delta^0_{(bn+a)}(X)$ set is $\Sigma^0_{(bn+a)}(X)$

Date: October 11, 2021.

Key words and phrases. Feiner hierarchy, low for Feiner, intermediate for Feiner, high for Feiner, intermediate degrees, computable Boolean algebras.

Hirschfeldt was partially supported by NSF grant DMS-1600543. Montalbán was partially supported by NSF grant DMS-1954062. The authors thank Dan Rosendorf for helping typeset the hierarchy figure in Proposition 2.6.

and $\Pi^0_{(bn+a)}(X)$, and that every $\Sigma^0_{(bn+a)}(X)$ and $\Pi^0_{(bn+a)}(X)$ set is computable in $X^{(\omega)}$ (see Proposition 2.6).

We study the classes of sets that are low, intermediate, and high with respect to these complexity classes. Given a relativizable complexity class Γ (for example, Δ^0_2 or $\Delta^0_{(bn+a)}$), we say that a set $X \subseteq \omega$ is *low for* Γ if every set S belonging to $\Gamma(X)$ belongs to $\Gamma(\emptyset)$; that a set $X \subseteq \omega$ is *high for* Γ if every set S belonging to $\Gamma(\emptyset')$ belongs to $\Gamma(X)$; and that a set S is *intermediate for* Γ if it is neither low nor high for Γ. Thus, for example, a set X is low for Δ^0_2 if and only if it is low in the usual sense. More generally, the low$_n$ sets are the sets that are low for Δ^0_{n+1}. It is not hard to see that every low$_n$ set is low for $\Delta^0_{(bn+a)}$ for all a and b, for example (see Proposition 3.1).

Since $S \in \Sigma^0_{(an+b)}$ if and only if $\overline{S} \in \Pi^0_{(an+b)}$, being low for $\Sigma^0_{(an+b)}$ is equivalent to being low for $\Pi^0_{(an+b)}$, so from now on we do not consider the latter notion explicitly, and similarly for being high or intermediate for $\Pi^0_{(an+b)}$.

In Section 2, we show that for all $a, b, a', b' \in \omega$ with $b, b' \geqslant 1$, a set is low for $\Delta^0_{(bn+a)}$ if and only if it is low for $\Delta^0_{(b'n+a')}$. Consequently, when a set is low for $\Delta^0_{(bn+a)}$ for some (and hence all) a, b, we say it is *low for Δ-Feiner*. Similar results in that section justify analogous definitions of *high for Δ-Feiner*, *low for Σ-Feiner*, and *high for Σ-Feiner*. We also show that being low for Σ-Feiner implies being low for Δ-Feiner, and similarly for highness.

Our Results. In Section 3, we prove that there exists a computably enumerable set of intermediate Turing degree (in the usual sense of being neither low$_n$ nor high$_n$ for any n) that is low for Σ-Feiner, and hence low for Δ-Feiner. We therefore find that (in general and for c.e. sets),

$$\text{low}_1 \subsetneq \text{low}_2 \subsetneq \cdots \subsetneq \text{low}_n \subsetneq \cdots \subsetneq \text{low for } \Sigma\text{-Feiner} \subseteq \text{low for } \Delta\text{-Feiner}.$$

We conjecture that the last containment is also proper.

We also examine the classes of sets that are high for Δ-Feiner and high for Σ-Feiner. We obtain similar results, and in particular find that (in general and for c.e. sets),

$$\text{high}_1 \subsetneq \text{high}_2 \subsetneq \cdots \subsetneq \text{high}_n \subsetneq \cdots \subsetneq \text{high for } \Sigma\text{-Feiner} \subseteq \text{high for } \Delta\text{-Feiner}.$$

Again we conjecture that the last containment is proper.

Finally, we show that there is a c.e. set that is intermediate for Δ-Feiner, and hence intermediate for Σ-Feiner. Thus, assuming our conjectures above hold, the intermediate (c.e.) degrees can be split into five nonempty classes: low for Σ-Feiner, low for Δ-Feiner but not for Σ-Feiner, intermediate for Δ-Feiner, high for Δ-Feiner but not for Σ-Feiner, and high for Σ-Feiner.

An Application to Computable Structures. By extending the ideas in [2], Thurber obtained the following result.

Theorem 1.2 (Thurber [6]; see also [1, § 18.3]). *There is a sequence of infinitary sentences $\psi_0, \psi_1, \ldots$ in the language of Boolean algebras such that for every set $S \subseteq \omega$, the following are equivalent.*

(1) *There exists a computable Boolean algebra $\mathcal{B}$ such that $S = \{n : \mathcal{B} \vDash \psi_n\}$.*
(2) *The set S is $\Pi^0_{(2n+4)}(\emptyset)$.*

Corollary 1.3. *If X is not low for Σ-Feiner, then there is an X-computable Boolean algebra $\mathcal{B}$ that has no computable copy.*

Proof. Let $S \in \Pi^0_{(2n+4)}(X) \setminus \Pi^0_{(2n+4)}(\emptyset)$. Relativizing Theorem 1.2 to X, there is an X-computable Boolean algebra $\mathcal{B}$ such that S satisfies (1) of Theorem 1.2. But then this Boolean algebra cannot be computable, or else we would have that S is $\Pi^0_{(2n+4)}(\emptyset)$ by Theorem 1.2. □

The same result follows from the work of Kach in [3], where he studied the complexity of the Ketonen invariants on a certain class of Boolean algebras: the class of *depth zero* Boolean algebras. He proved that a depth zero, rank ω Boolean algebra has a computable copy if and only if its Ketonen invariant is $\Sigma^0_{(2n+3)}(\emptyset)$. As this result relativizes, it follows that if a depth zero, rank ω Boolean algebra has a presentation in a low for Σ-Feiner set, then it has a computable copy. We similarly obtain Corollary 1.3 from Kach's result.

Our results below (Theorem 3.5 or Theorem 3.6) show that there exists an intermediate c.e. degree that is not low for Σ-Feiner, so we obtain the following corollary, which contrasts with Knight and Stob's result in [4] that every low_4 Boolean algebra has a computable copy. (Whether every low_5 Boolean algebra has a computable copy remains a well-known open question.)

Corollary 1.4. *There is a Boolean algebra of intermediate c.e. degree that has no computable copy.*

Notation. Though our notation for the most part follows [5], we review certain aspects of it briefly. We use upper case Greek letters (e.g., Φ, Ψ, Ξ, Θ, etc.) to denote Turing functionals and lower case Greek letters (e.g., φ, ψ, ξ, θ, etc.) to denote the corresponding use functions. We use W_e to denote the domain of the eth functional Φ_e and $(W_e)^{[i]}$ to denote the subset $\{\langle x, y\rangle \in W_e : y = i\}$. We write $X =^* Y$ to denote that the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ is finite.

2. Parameter Independence and Other Basic Results

Before studying which sets are low/intermediate/high for Δ-Feiner and Σ-Feiner, we eliminate the need for working with $\Delta^0_{(bn+a)}$ and $\Sigma^0_{(bn+a)}$ sets for varying a and b, through a sequence of quick lemmas. Though we state and prove these lemmas only for being low for $\Delta^0_{(bn+a)}$, all still work (with obvious modifications to their proofs) for being high for $\Delta^0_{(bn+a)}$, low for $\Sigma^0_{(bn+a)}$, and high for $\Sigma^0_{(bn+a)}$. Throughout this section, $b \geqslant 1$.

Lemma 2.1. *If X is low for $\Delta^0_{(bn+a)}$, then X is low for $\Delta^0_{(bn+a')}$.*

Proof. Suppose that X is low for $\Delta^0_{(bn+a)}$. Let $S \in \Delta^0_{(bn+a')}(X)$. Then $\pi(S) := \{x \in \omega : x + a - a' \in S\} \in \Delta^0_{(bn+a)}(X)$, so $\pi(S) \in \Delta^0_{(bn+a)}(\emptyset)$. It follows that $S \in \Delta^0_{(bn+a')}(\emptyset)$. □

Lemma 2.2. *If X is low for $\Delta^0_{(bn)}$, then X is low for $\Delta^0_{(n)}$.*

Proof. Suppose that X is low for $\Delta^0_{(bn)}$. Let $S \in \Delta^0_{(n)}(X)$. Define sets S_i for $i < b$ by $S_i := \{x \in \omega : bx + i \in S\}$. Then $S_i \in \Delta^0_{(bn+i)}(X)$. By Lemma 2.1, X is low for $\Delta^0_{(bn+i)}$, so $S_i \in \Delta^0_{(bn+i)}(\emptyset)$. It follows that $S \in \Delta^0_{(n)}(\emptyset)$. □

Lemma 2.3. *If X is low for $\Delta^0_{(n)}$, then X is low for $\Delta^0_{(bn)}$.*

Proof. Suppose that X is low for $\Delta^0_{(n)}$. Let $S \in \Delta^0_{(bn)}(X)$. Then $\pi(S) := \{bx : x \in S\} \in \Delta^0_{(n)}(X)$, and hence $\pi(S) \in \Delta^0_{(n)}(\emptyset)$. It follows that $S \in \Delta^0_{(bn)}(\emptyset)$. □

Combining Lemmas 2.1, 2.2, and 2.3, we obtain the desired invariance.

Proposition 2.4. *If X is low (respectively, intermediate or high) for $\Delta^0_{(bn+a)}$ for some a, b, then X is low (respectively, intermediate or high) for $\Delta^0_{(b'n+a')}$ for all a', b'. If X is low (respectively, intermediate or high) for $\Sigma^0_{(bn+a)}$ for some a, b, then X is low (respectively, intermediate or high) for $\Sigma^0_{(b'n+a')}$ for all a', b'.*

This proposition justifies our use of terms like low for Δ-Feiner. We also have the following relationship.

Proposition 2.5. *If X is low (respectively, high) for Σ-Feiner, then X is low (respectively, high) for Δ-Feiner.*

Proof. Suppose X is low for Σ-Feiner. Let $S \in \Delta^0_{(n)}(X)$. Then $S \in \Sigma^0_{(n)}(X)$ and $S \in \Pi^0_{(n)}(X)$, as $\Delta^0_{(n)}(X) \subset \Sigma^0_{(n)}(X), \Pi^0_{(n)}(X)$. Since X is low for $\Sigma^0_{(n)}$, and hence low for $\Pi^0_{(n)}$, we have $S \in \Sigma^0_{(n)}(\emptyset)$ and $S \in \Pi^0_{(n)}(\emptyset)$. It follows that $S \in \Delta^0_{(n)}(\emptyset)$.

The proof for highness is analogous. □

As noted above, we conjecture that the converse to this proposition does not hold.

We finish by noting the relationships between the classes of sets that are $\Delta^0_{(bn+a)}$, $\Sigma^0_{(bn+a)}$, and $\Pi^0_{(bn+a)}$ for varying a, b. These relationships essentially follow from Feiner's work in [2]; see [1] or [6], for example.

Proposition 2.6. *The classes of sets that are $\Delta^0_{(bn+a)}$, $\Sigma^0_{(bn+a)}$, and $\Pi^0_{(bn+a)}$ satisfy the inclusions*

$$\begin{array}{ccccccccccc}
 & & \Sigma^0_{(n)} & & & & \Sigma^0_{(n+1)} & & & & \\
 & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & \\
\Delta^0_{(n)} & & & & \Delta^0_{(n+1)} & & & & \Delta^0_{(n+2)} & & \cdots \\
 & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & \\
 & & \Pi^0_{(n)} & & & & \Pi^0_{(n+1)} & & & &
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & \Sigma^0_{(2n)} & & & & \Sigma^0_{(2n+1)} & & & & \\
 & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & \\
\Delta^0_{(2n)} & & & & \Delta^0_{(2n+1)} & & & & \Delta^0_{(2n+2)} & & \cdots \\
 & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & \\
 & & \Pi^0_{(2n)} & & & & \Pi^0_{(2n+1)} & & & &
\end{array}$$

In particular, a set cannot be both low for Δ-Feiner and high for Δ-Feiner, and similarly for Σ-Feiner.

3. The Intermediate Turing Degrees

We now turn our attention to studying which Turing degrees can be low, intermediate, or high for Δ-Feiner and which can be low, intermediate, or high for Σ-Feiner. The following proposition was essentially noted in [3].

Proposition 3.1. *Every low$_k$ Turing degree is low for Δ-Feiner and low for Σ-Feiner. Every high$_k$ Turing degree is high for Δ-Feiner and high for Σ-Feiner.*

Proof. Let X be a low$_k$ set. Provided $n \geqslant k$, we have

$$X^{(n)} = \left(X^{(k)}\right)^{(n-k)} \equiv_{\mathrm{T}} \left(\emptyset^{(k)}\right)^{(n-k)} = \emptyset^{(n)},$$

and the Turing reductions are uniform in n. It follows that X is low for $\Delta^0_{(n)}$ and low for $\Sigma^0_{(n)}$, and thus low for Δ-Feiner and low for Σ-Feiner.

If instead X is a high$_k$ set, the equivalence becomes $X^{(n)} \equiv_{\mathrm{T}} \emptyset^{(n+1)}$ for $n \geqslant k$. It follows that X is high for $\Delta^0_{(n)}$ and high for $\Sigma^0_{(n)}$, and thus high for Δ-Feiner and high for Σ-Feiner. □

The intermediate degrees have greater complexity: some are low for Δ-Feiner and Σ-Feiner, some are intermediate for Δ-Feiner and Σ-Feiner, some are high for Δ-Feiner and Σ-Feiner, and we conjecture that some behave differently for Δ-Feiner and for Σ-Feiner. In order to construct examples, we will modify the construction of an intermediate c.e. degree. We note the following properties of this construction as given in [5].

Remark 3.2. Let $q(x)$ be a total computable function as in the proof of Corollary VIII.3.5 in [5], i.e., a total computable function satisfying

$$Y <_{\mathrm{T}} W^Y_{q(x)} <_{\mathrm{T}} Y' \qquad \text{and} \qquad (W^Y_{q(x)})' \equiv_{\mathrm{T}} (W^{Y'}_x) \oplus Y' \tag{3.1}$$

for all x and Y. We can ensure that $(W^Y_{q(x)})^{[0]}$ is equal to $Y \times \{0\}$ for all x and Y. As noted in that proof, if we take a fixed point m such that $W^Y_{q(m)} = W^Y_m$ for all Y, which exists by the relativized form of the Recursion Theorem, then the degree of $W^{\emptyset}_m$ is intermediate. This fact relies only on the properties in (3.1), so we will be able to produce intermediate degrees with additional properties by modifying q while preserving these properties.

It is easy to check from the proof of the Jump Theorem in [5] (Theorem VIII.3.1) that the second equivalence in (3.1) is uniform in Y; i.e., for each x there are functionals Ψ and Ξ such that $(W^Y_{q(x)})' = \Psi^{(W^{Y'}_x)\oplus Y'}$ and $(W^{Y'}_x) \oplus Y' = \Xi^{(W^Y_{q(x)})'}$ for all Y. It also follows from that proof that we can define Ξ so that for all Y and $X =^* W^Y_{q(x)}$, we have $(W^{Y'}_x) \oplus Y' = \Xi^{X'}$.

We will also need the following lemma.

Lemma 3.3. *There is a partial computable function $f : 2^\omega \to \omega$ such that $f(\emptyset^{(n)}) = n$ for all $n \geqslant 0$.*

Proof. Let $\{e_i\}_{i\in\omega}$ be a computable sequence of integers such that $\Phi^X_{e_0}(e_0)\downarrow$ for all X and $\Phi^X_{e_{i+1}}(e_{i+1})\downarrow$ if and only if $e_i \in X$. Then $e_i \in \emptyset^{(n)}$ if and only if $i < n$, so we can define $f(X)$ to be the least i, if any, such that $e_i \notin X$. □

We begin by showing the existence of an intermediate c.e. degree that is low for Σ-Feiner, and hence for Δ-Feiner.

Theorem 3.4. *There is a computably enumerable set A of intermediate Turing degree such that A is low for Σ-Feiner.*

Proof. Let q be as in Remark 3.2, and let f be as in Lemma 3.3. We will define a total computable function r such that $W^Y_{r(x)} =^* W^Y_{q(x)}$ whenever $f(Y)$ is defined, which suffices to ensure that if m is a fixed point of r, then the degree of $W^{\emptyset}_m$ is intermediate. We will also have $(W^Y_{r(x)})^{[0]} = (W^Y_{q(x)})^{[0]}$ (which recall is equal to $Y \times \{0\}$). It thus makes sense to establish the convention that $\Phi_e^{W^Y_{r(x),t}}(k)[t]\downarrow$ means that this computation converges and

$$\left(W^Y_{r(x),t}\right)^{[0]} \upharpoonright \varphi_e^{W^Y_{r(x),t}}(k) = \left(Y \upharpoonright \varphi_e^{W^Y_{r(x),t}}(k)\right) \times \{0\}.$$

Define r so that on oracle Y:

(a) $(W^Y_{r(x)})^{[0]} = (W^Y_{q(x)})^{[0]}$.
(b) If $f(Y)\uparrow$ and $i \notin \omega^{[0]}$ then $i \notin W^Y_{r(x)}$.
(c) If $f(Y)\downarrow = n$ and $i \notin \omega^{[0]}$ enters $W^Y_{q(x)}$ at stage t, then $i \in W^Y_{r(x)}$ unless $\Phi_e^{W^Y_{r(x),t}}(n)[t]\downarrow$ and $i < \varphi_e^{W^Y_{r(x),t}}(n)$ for some $e \leqslant n$.
(d) No other numbers are in $W^Y_{r(x)}$.

The point of item (c) is that (given our convention above) it ensures that if $e \leqslant n = f(Y)$ and $\Phi_e^{W^Y_{r(x)}}(n)\downarrow$, then $\Phi_e^{W^Y_{r(x)}}(n) = \Phi_e^{W^Y_{r(x),t}}(n)[t]$ for the least t such that $\Phi_e^{W^Y_{r(x),t}}(n)[t]\downarrow$. Thus $\Phi_e^{W^Y_{r(x)}}(n)\downarrow$ if and only if there is a t such that $\Phi_e^{W^Y_{r(x),t}}(n)[t]\downarrow$, which is a Y-c.e. condition.

We have $W^Y_{r(x)} =^* W^Y_{q(x)}$ whenever $f(Y)$ is defined, so r satisfies (3.1) for all such Y. Thus, if we let m be a fixed point of r and let $A = W^{\emptyset}_m$, then A has intermediate degree as in Corollary VIII.3.5 of [5]. We are left with showing that A is low for Σ-Feiner. We do so by showing that there is a uniform procedure for computing $A^{(n)}$ from $W^{\emptyset^{(n)}}_m$, and then applying the previous paragraph.

If $f(Y)$ is defined then the difference between $W^Y_{r(x)}$ and $W^Y_{q(x)}$ can be computed uniformly from Y', so for each x there is a functional Θ (defined using the functional Ψ in Remark 3.2) for which $(W^Y_{r(x)})' = \Theta^{(W^{Y'}_x) \oplus Y'}$ for all Y such that $f(Y)$ is defined. Since Y' is encoded into the 0th column of $W^{Y'}_m = W^{Y'}_{r(m)}$, taking $x = m$ we actually have a functional Θ such that $(W^Y_m)' = \Theta^{(W^{Y'}_m)}$ for all Y such that $f(Y)$ is defined.

Thus $A' = (W^{\emptyset}_m)' = \Theta^{(W^{\emptyset'}_m)}$. Similarly, $A'' = (W^{\emptyset}_m)'' = (\Theta^{(W^{\emptyset'}_m)})'$, which is computable from $(W^{\emptyset'}_m)' = \Theta^{(W^{\emptyset''}_m)}$ via a reduction that can be found uniformly from an index for Θ. Continuing in this manner, we obtain a uniform procedure for computing $A^{(n)}$ from $W^{\emptyset^{(n)}}_m$.

Now let $S \in \Sigma^0_{(n)}(A)$. Then, by the above, there is a functional Φ_e such that $n \in S$ if and only if $\Phi_e^{W^{\emptyset^{(n)}}_m}(n)\downarrow$. For $n \geqslant e$, we can use the oracle $\emptyset^{(n)}$ to search for a t such that $\Phi_e^{W^{\emptyset^{(n)}}_{m,t}}(n)[t]\downarrow$, enumerating n into a set $\widehat{S}$ when such a t is found. By construction, $\Phi_e^{W^{\emptyset^{(n)}}_m}(n)\downarrow$ if and only if there is such a t, so $\widehat{S} =^* S$. Since $\widehat{S}$ is in $\Sigma^0_{(n)}(\emptyset)$, so is S. Thus A is low for Σ-Feiner. □

We now show the existence of an intermediate c.e. degree that is high for Σ-Feiner, and hence for Δ-Feiner.

Theorem 3.5. *There is a computably enumerable set A of intermediate Turing degree such that A is high for Σ-Feiner.*

Proof. Again let q be as in Remark 3.2 and let f be as in Lemma 3.3.

It is not difficult to see that there is a c.e. operator V so that on oracle Y:

(a) If $f(Y)\uparrow$, then $V^Y = \emptyset$.
(b) If $f(Y)\downarrow = n$, then:
  - If $e > n$, then the eth column of V^Y is empty.
  - If $e \leqslant n$ and $\Phi_e^{Y'}(n)\uparrow$, then the eth column of V^Y is $\omega \times \{e\}$.
  - Otherwise, the eth column of V^Y is $[0,t] \times \{e\}$, where t is minimal such that $\Phi_e^{Y'_t}(n)[t]\downarrow$ and $Y'_t \upharpoonright \varphi_e^{Y'_t}(n) = Y' \upharpoonright \varphi_e^{Y'_t}(n)$.

Here we are thinking of a standard Y-enumeration of Y'.

Let $s : \omega \to \omega$ be a total computable function such that, on oracle Y,

$$W^Y_{s(x)} = \begin{cases} \emptyset & \text{if } f(Y)\uparrow \\ W^Y_{q(x)} \oplus V^Y & \text{otherwise.} \end{cases}$$

The definition of s ensures that if $f(Y)$ is defined, then $W^Y_{s(x)} \equiv_{\mathrm{T}} W^Y_{q(x)}$, so (3.1) holds for s in place of q. Moreover, for each x there is a single functional Θ (defined using the functional Ξ in Remark 3.2) such that $W_x^{Y'} = \Theta^{(W^Y_{s(x)})'}$. Furthermore, for any x and e, we can compute $\Phi_e^{Y'}(n)$ (in the sense of computing a partial function) uniformly from $W^Y_{s(x)}$ for Y such that $f(Y)\downarrow = n$.

Let m be a fixed point of s, let $A = W_m^\emptyset$, and let Θ be as above with $x = m$. Then A is intermediate as in Corollary VIII.3.5 of [5]. It follows from the definition of m, A, and Θ that $W_m^{\emptyset'} = \Theta^{(W_m^\emptyset)'} = \Theta^{A'}$. Similarly, $(W_m^{\emptyset'})'$ can be obtained from $A'' = (W_m^\emptyset)''$ via a reduction that can be found from an index for Θ, and hence so can $W_m^{\emptyset''} = \Theta^{(W_m^{\emptyset'})'}$. Continuing in this manner, we obtain a uniform procedure for computing $W_m^{\emptyset^{(n)}}$ from $A^{(n)}$.

Now suppose that $W_e = \operatorname{dom} \Phi_e$ witnesses that $S \in \Sigma^0_{(n)}(\emptyset')$. Then there is a uniform procedure for computing $\Phi_e^{\emptyset^{(n+1)}}(n)$ from $W_m^{\emptyset^{(n)}}$, and hence from $A^{(n)}$. Thus $S \in \Sigma^0_{(n)}(A)$, and so A is high for Σ-Feiner. ☐

We next show the existence of a c.e. degree that is intermediate for Δ-Feiner, and hence for Σ-Feiner (and hence is also intermediate in the usual sense).

Theorem 3.6. *There is a computably enumerable set that is intermediate for Δ-Feiner.*

Proof. We will define sets A_0 and A_1 such that $\Delta^0_{(n)}(A_0) \not\subseteq \Delta^0_{(n)}(A_1)$ and $\Delta^0_{(n)}(A_1) \not\subseteq \Delta^0_{(n)}(A_0)$. Then both A_0 and A_1 must be intermediate for Δ-Feiner.

Let q be as in Remark 3.2, let f be as in Lemma 3.3, and let r be as in the proof of Theorem 3.4. Let V be a c.e. operator so that on oracle Y:

(a) If $f(Y)\uparrow$, then $V^Y = \emptyset$.
(b) If $f(Y)\downarrow = n$, then if $\Phi_e^{W^Y_{r(x),t-1}}(n)[t-1]\uparrow$ and $\Phi_e^{W^Y_{r(x),t}}(n)[t]\downarrow$ (under the same convention as in the proof of Theorem 3.4) for some $e \leqslant n$, then all numbers less than t are enumerated into V^Y.

Note that if $f(Y)\downarrow = n$ then V^Y is finite (by item (c) in the definition of r), and for the least $t \notin V^Y$ and every $e \leqslant n$, we have $\Phi_e^{W^Y_{r(x)}}(n) = \Phi_e^{W^Y_{r(x),t}}(n)[t]$. Let $s : \omega \to \omega$ be a total computable function such that $W^Y_{s(x)} = W^Y_{q(x)} \oplus V^Y$ for all Y.

For $i \in \{0, 1\}$, let $p_i : \omega \to \omega$ be a total computable function such that

$$W^Y_{p_i(x)} = \begin{cases} \emptyset & \text{if } f(Y)\uparrow \\ W^Y_{s(x)} & \text{if } f(Y)\downarrow = i \bmod 2 \\ W^Y_{r(x)} & \text{otherwise.} \end{cases}$$

The definition of the p_i ensures that if $f(Y)$ is defined, then $W^Y_{p_i(x)} \equiv_{\mathrm{T}} W^Y_{q(x)}$, the set Y is effectively coded in $W^Y_{p_i(x)}$, and the difference between $W^Y_{p_i(x)}$ and $W^Y_{q(x)}$ is uniformly computable in Y'. Thus, for each x and i, there is a functional Θ such that $(W^Y_{p_i(x)})' = \Theta^{W^{Y'}_x \oplus Y'}$ for all Y such that $f(Y)$ is defined. Furthermore, if $f(y)\downarrow \neq i \bmod 2$, then $W^Y_{p_i(x)} =^* W^Y_{q(x)}$, and if $f(Y)\downarrow = i \bmod 2$, then $W^Y_{p_i(x)}$ codes $W^Y_{q(x)}$ in its even bits. So, for each x and i, there are functionals Γ and Λ such that $W^{Y'}_x = \Gamma^{(W^Y_{p_i(x)})'}$ if $f(Y) = i \bmod 2$ and $W^{Y'}_x = \Lambda^{(W^Y_{p_i(x)})'}$ otherwise, for all Y such that $f(Y)$ is defined.

Let m_i be a fixed point of p_i and let $A_i = W^{\emptyset}_{m_i}$. By the previous paragraph, and arguing as in the previous two proofs, there are uniform procedures for computing $A_i^{(n)}$ from $W^{\emptyset^{(n)}}_{m_i}$ and vice-versa. As mentioned above, it suffices to show that $\Delta^0_{(n)}(A_0) \nsubseteq \Delta^0_{(n)}(A_1)$ and $\Delta^0_{(n)}(A_1) \nsubseteq \Delta^0_{(n)}(A_0)$.

Towards a contradiction, assume that $\Delta^0_{(n)}(A_0) \subseteq \Delta^0_{(n)}(A_1)$. Define a total function $h : \omega \to 2$ as follows. If n is odd, then $h(n) = 0$. Otherwise, let t be least such that $2t + 1 \notin W^{\emptyset^{(n)}}_{m_0}$ (which must exist by the definition of the operator V), let $e = n/2$, and compute $\Phi_e^{W^{\emptyset^{(n)}}_{r(m_1),t}}(n)[t]$. If this value is defined, then let $h(n)$ be different from it; otherwise let $h(n) = 0$.

The value $h(n)$ can be computed uniformly from $W^{\emptyset^{(n)}}_{m_0}$, and hence from $A_0^{(n)}$, so $h \in \Delta^0_{(n)}(A_0)$. Thus, by assumption, $h \in \Delta^0_{(n)}(A_1)$. Consequently, there is a functional Φ_e such that $\Phi_e^{W^{\emptyset^{(n)}}_{p_1(m_1)}}(n) = \Phi_e^{W^{\emptyset^{(n)}}_{m_1}}(n) = h(n)$ for all n. Let $n = 2e$. Then, by the definition of r and p, for the least t such that $2t + 1 \notin W^{\emptyset^{(n)}}_{m_0}$, we have $\Phi_e^{W^{\emptyset^{(n)}}_{p_1(m_1)}}(n) = \Phi_e^{W^{\emptyset^{(n)}}_{r(m_1)}}(n) = \Phi_e^{W^{\emptyset^{(n)}}_{r(m_1),t}}(n)[t]$, so $h(n) \neq \Phi_e^{W^{\emptyset^{(n)}}_{p_1(m_1)}}(n)$, yielding a contradiction.

The symmetric argument shows that $\Delta^0_{(n)}(A_1) \nsubseteq \Delta^0_{(n)}(A_0)$. It follows that both A_0 and A_1 are intermediate for Δ-Feiner. □

We finish with the following conjectures.

Conjecture 3.7. There is a computably enumerable set that is intermediate for Σ-Feiner but low for Δ-Feiner.

Conjecture 3.8. There is a computably enumerable set that is intermediate for Σ-Feiner but high for Δ-Feiner.

References

[1] C. J. Ash and J. Knight, Computable Structures and the Hyperarithmetical Hierarchy, Studies in Logic and the Foundations of Mathematics 144, North-Holland Publishing Co., Amsterdam, 2000.
[2] Lawrence Feiner, Hierarchies of Boolean algebras, J. Symbolic Logic 35:365–374, 1970.
[3] Asher M. Kach, Depth zero Boolean algebras, Trans. Amer. Math. Soc. 362(8):4243–4265, 2010.
[4] Julia F. Knight and Michael Stob, Computable Boolean algebras, J. Symbolic Logic 65(4):1605–1623, 2000.
[5] Robert I. Soare, Recursively Enumerable Sets and Degrees, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.
[6] John J. Thurber, Recursive and r.e. quotient Boolean algebras, Arch. Math. Logic 33(2):121–129, 1994.

Department of Mathematics, University of Chicago
Email address: drh@math.uchicago.edu

Department of Mathematics, University of California, Berkeley
Email address: antonio@math.berkeley.edu

Google LLC
Email address: asher.kach@gmail.com