REALIZING LEVELS OF THE HYPERARITHMETIC HIERARCHY AS DEGREE SPECTRA OF RELATIONS ON COMPUTABLE STRUCTURES

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Abstract. We construct a class of relations on computable structures whose degree spectra form natural classes of degrees. Given any computable ordinal α and reducibility r stronger than or equal to m-reducibility, we show how to construct a structure with an intrinsically Σ_α invariant relation whose degree spectrum consists of all nontrivial Σ_α r-degrees. We extend this construction to show that Σ_α can be replaced by either Π_α or Δ_α.

1. Introduction

Since the pioneering work of Ash and Nerode [10], the study of additional relations on computable structures has been one of the central topics in computable model theory. Not only is it often possible to understand the differences between the various computable copies of a structure $\mathcal{M}$ by examining the images in these copies of a particular relation on the domain of $\mathcal{M}$, but concepts and techniques developed for this purpose have also been instrumental in resolving several other types of questions about computable structures. (For more on this theme, see for instance [15], [18], or [21].) Before we proceed, let us recall the basic definitions we will need below.

1.1. **Definition.** A structure $\mathcal{A}$ is *computable* if both its domain $|\mathcal{A}|$ and the atomic diagram of $\langle \mathcal{A}, a\rangle_{a\in|\mathcal{A}|}$ are computable. An isomorphism from a structure $\mathcal{M}$ to a computable structure is called a *computable presentation* of $\mathcal{M}$. (We often abuse terminology and refer to the image of a computable presentation as a computable presentation.)

If $\mathcal{M}$ has a computable presentation, then it is *computably presentable*.

Ash and Nerode were concerned with relations that maintain some degree of effectiveness in different computable presentations of a structure.

1.2. **Definition.** Let U be a relation on the domain of a computable structure $\mathcal{M}$ and let $\mathfrak{C}$ be a class of relations. U is *intrinsically* $\mathfrak{C}$ on $\mathcal{M}$ if the image of U in any computable presentation of $\mathcal{M}$ is in $\mathfrak{C}$.

Another approach to the study of relations on computable structures, which began with the work of Harizanov [12], is to fix a reducibility (most often Turing reducibility) and look at the collection of degrees of the images of a relation in different computable presentations of a structure.

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1.3. **Definition.** Let r be a reducibility, such as many-one reducibility (m-reducibility) or Turing reducibility. Let U be a relation on the domain of a computable structure $\mathcal{M}$. The *r-degree spectrum* of U on $\mathcal{M}$, $\mathrm{DgSp}^{\mathrm{r}}_{\mathcal{M}}(U)$, is the set of r-degrees of the images of U in all computable presentations of $\mathcal{M}$.

For simplicity in the statement of results below, we will call an r-degree *nontrivial* if it contains an infinite and coinfinite set. Note that if r and s are reducibilities such that r is stronger than s, and U is a relation on a computable structure $\mathcal{M}$, then $\mathrm{DgSp}^{\mathrm{s}}_{\mathcal{M}}(U)$ is equal to the set of s-degrees that contain at least one r-degree in $\mathrm{DgSp}^{\mathrm{r}}_{\mathcal{M}}(U)$.

There has been a large amount of research on degree spectra of relations, both in the general case and with certain restrictions imposed on the structure or the relation. Much of this work has been devoted to exploring syntactic conditions on a relation that guarantee that its degree spectrum has certain properties. Another rich vein has been the study of "pathological" examples, such as degree spectra that are finite but not singletons, with work focused both on constructing such examples and on giving conditions that imply that they cannot occur.

Less work has been devoted to giving examples of relations whose degree spectra form natural classes of degrees. Despite the fact that there are no known nontrivial restrictions on which sets of degrees can be realized as degree spectra of relations, such examples are still important to our understanding of degree spectra of relations, particularly if they are in some sense natural. The purpose of this paper is to present a family of relations on computable structures whose degree spectra (with respect to any reducibility stronger than or equal to m-reducibility) coincide with levels of the hyperarithmetic hierarchy.

Of course, naturalness is very much in the eye of the beholder, but we believe the examples in this paper qualify because they are built up from trees that explicitly code the alternations of quantifiers that define the levels of the hyperarithmetic hierarchy. Furthermore, by the results of [17], our results automatically imply that such examples also exist within well-known classes of algebraic structures, such as integral domains and 2-step nilpotent groups.

There are two further reasons for our interest in the family of examples constructed in this paper. First, the trees that form their basic building blocks are useful in establishing a wide range of results in the classification of the complexity of index sets of computable structures with given model-theoretic properties (see [25] for details).

Second, realizing levels of the hyperarithmetic hierarchy as degree spectra of relations gives an illustration of the difference between the general case, in which we are trying to realize certain sets of degrees as degree spectra of relations on computable structures with no additional restrictions, and cases in which we impose extra conditions on one or more aspects of this realization. This is due to the following result, proved independently by Ash, Cholak, and Knight [2] and Harizanov [14].

1.4. **Theorem** (Ash, Cholak, and Knight; Harizanov)**.** *Let U be a relation on the domain of a computable structure $\mathcal{A}$. Suppose that for each Δ_3 set C there is an isomorphism f from $\mathcal{A}$ to a computable structure $\mathcal{B}$ such that $f \leqslant_T C \leqslant_T f(U)$. Then for each set C there is an isomorphism f from $\mathcal{A}$ to a computable structure $\mathcal{B}$ such that $f \leqslant_T C \leqslant_T f(U)$. In particular, $\mathrm{DgSp}^T_{\mathcal{A}}(U)$ contains every degree.*

We should stress that, although our result and the specific trees used to prove it appear to be new, our approach is hardly novel. Indeed, when analyzing hyperarithmetic structures and relations, it is quite natural to consider structures similar to our trees, and to analyze their complexity in terms of recursively defined infinitary formulae as we do. Examples of this approach to various questions in computable model theory are the following papers of Ash and Knight [1, 3, 5, 6, 7, 8, 19]; see also their book [4]. (A general abstract framework for these kinds of results would be useful; Soskov and Baleva [23, 24] have introduced such a framework, but from a quite different perspective.)

In [13], Harizanov gave a syntactical condition on a computable structure $\mathcal{A}$ and a relation U on $\mathcal{A}$ which, with certain additional effectiveness conditions, is equivalent to U being intrinsically c.e. with degree spectrum consisting of all c.e. Turing degrees. A reasonable extension of this condition from Σ_1 to Σ_α would of course imply our result (at least for Turing degrees), but in [7] it is shown that such an extension is unlikely to exist. Indeed, it appears that the natural classes of degrees captured by such syntactical conditions are not the Σ_α degrees but the Σ_α degrees possessing an *α-table* (see [7] for a definition), or the Σ_α degrees over Δ^0_α, as in [8].

In [3], conditions on a pair of computable structures $\mathcal{A}$ and $\mathcal{B}$ and a computable ordinal α are given which ensure that for any Σ_α set S there is a uniformly computable family of structures $\mathcal{C}_0, \mathcal{C}_1, \ldots$ such that $\mathcal{C}_n \cong \mathcal{A}$ if $n \in S$ and $\mathcal{C}_n \cong \mathcal{B}$ if $n \notin S$. We could use these conditions in proving Proposition 3.2 below, but the direct proof we give is simpler. Indeed, because we are just producing an example, rather than giving general conditions as in the papers mentioned above, our analysis is a particularly simple example of this line of research, so we prefer to give direct proofs rather than attempt to employ some of the abstract theorems available in the literature.

In the next section, we define relevant concepts relating to the hyperarithmetic hierarchy and ordinal notations. In Section 3, we define our basic building blocks, the *back-and-forth trees*. Finally, in Section 4, we define our relations and structures and establish their relevant properties.

2. The Hyperarithmetic Hierarchy and Ordinal Notations

Throughout this paper, we will use the standard notions of the arithmetic and analytic hierarchies for predicates $\mathcal{R}(f, n)$ on $\omega^\omega \times \omega$, as in Sacks [20]. However, our definition of the hyperarithmetic hierarchy will follow the less standard terminology of Ash and Knight [9], since this is more in line with several important concepts from model theory.

2.1. **Definition.** A system of *notations* for ordinals consists of a set $O \subset \omega$ and a function $|\ |_O$ taking each element of O to an ordinal. The function $|\ |_O$ defines a natural partial ordering on O given by

$$a \leqslant_O b \quad \Leftrightarrow \quad |a|_O \leqslant |b|_O$$

Our notations will follow certain standard conventions.

(1) $1 \in O$ is the notation for 0. That is, $|1|_O = 0$.
(2) If $a \in O$ is a notation for α, then $2^a \in O$ is a notation for $\alpha + 1$.
(3) Suppose that the e^{th} partial computable function $\{e\}$ determines a *fundamental sequence* for a limit ordinal γ. In other words, $\{e\}$ is total, $\{e\}(n) \leqslant_O \{e\}(n+1)$ for all n, and γ is the least upper bound for the ordinals $|\{e\}(n)|_O$. Then $3 \cdot 5^e \in O$ is a notation for γ.

As there are only countably many notations, not every ordinal will have a notation. However, these standard conventions ensure that every computable ordinal does have a notation. The computable ordinals are an initial sequence of all ordinals bounded above by ω_1^{ck}, the first noncomputable ordinal.

It is clear from the final convention that ordinal notations need not be unique, since any limit ordinal may have multiple fundamental sequences. In this paper we will restrict ourselves to a subset $O_1 \subseteq O$ of unique notations so that every limit ordinal has a unique fundamental sequence. For any limit ordinal γ, we define

$$\gamma_n = |\{e\}(n)|_O$$

where $\{e\}$ determines our fixed fundamental sequence for γ. Furthermore, we require that this fixed fundamental sequence contain only successor ordinals and that $\gamma_0 = 1$. We do this simply as a matter of convenience; none of the results in this paper depend on our choice of fundamental sequences.

Given these notations for ordinals, we define the hyperarithmetic hierarchy in terms of computable infinitary formulae.

2.2. **Definition.** A Σ_0 (Π_0) *index* for a computable predicate $\mathcal{R}(f,n)$ is a triple $\langle \Sigma, 0, e\rangle$ ($\langle \Pi, 0, e\rangle$) where e is an index for the predicate $\mathcal{R}$. For any computable ordinal α, a Σ_α (Π_α) *index* for a predicate $\mathcal{R}(f,n)$ is a triple $\langle \Sigma, a, e\rangle$ ($\langle \Pi, a, e\rangle$), where a is a notation for α and e is an index for a c.e. set of Π_{β_k} (Σ_{β_k}) indices for predicates $\mathcal{Q}_k(f,n,x)$, such that $\beta_k < \alpha$ for all $k \in \omega$ and

$$\mathcal{R}(f,n) \Leftrightarrow \bigvee_{k\in\omega} \exists x\, \mathcal{Q}_k(f,n,x) \qquad \Big(\mathcal{R}(f,n) \Leftrightarrow \bigwedge_{k\in\omega} \forall x\, \mathcal{Q}_k(f,n,x)\Big)$$

We say that a predicate is Σ_α (Π_α) if it has a Σ_α (Π_α) index. We say that a predicate is Δ_α if it is both Σ_α and Π_α.

It is straightforward to check that this definition of the hyperarithmetic hierarchy is equivalent to other common definitions, with one important exception. In the case of infinite ordinals, the levels of our hierarchy may be indexed by the appropriate successor ordinal in other definitions of the hierarchy. For example, in Soare [22, p. 259], a predicate is $\Sigma_{\omega+1}$ if it is $\Sigma_1^{\emptyset^\omega}$. In our definition above, these are exactly the Σ_ω predicates. We choose this definition because it gives a more natural correspondence between computable ordinals and infinitary formulae. Furthermore, this choice does not omit any interesting levels of complexity in the hyperarithmetic hierarchy.

3. Back-and-Forth Trees

Our starting point is the following very simple but highly suggestive example. Let C_0 be the directed graph consisting of a single node and no edges, and let C_1 be the directed graph consisting of two nodes x and y with an edge from x to y. Consider the directed graph $\mathcal{G} = \langle |\mathcal{G}|, E\rangle$ that is the disjoint union of infinitely many copies of each of C_0 and C_1. Let U be the unary relation on the domain of $\mathcal{G}$ that holds of x if and only if there is a y such that $E(x,y)$. Since U is defined by an existential formula in the language of directed graphs, U is intrinsically c.e. Furthermore, it is not hard to check that $\mathrm{DgSp}^{\mathrm{m}}_{\mathcal{G}}(U)$ contains all nontrivial c.e. m-degrees.

For any $n \in \omega$, it is possible to modify this example in a natural way to realize the set of all n-c.e. degrees as the degree spectrum of an intrinsically n-c.e. relation on the domain of a computable structure. In fact, this is true even with n replaced by any computable ordinal α (for the definition of α-c.e. sets and degrees, see [11]). We will not do this here since a stronger result appears in [16]. Instead, we generalize the above example in a different direction. We begin by defining building blocks that generalize the graphs C_0 and C_1 above.

3.1. **Definition.** We define the *back-and-forth trees* by induction on α.

- $\mathcal{A}_1$ consists of a single node.
- $\mathcal{E}_1$ consists of a root node, to which are attached infinitely many nodes with no children. This tree is illustrated in Figure 3.1.

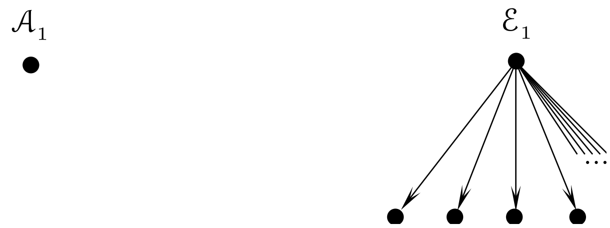

Figure 3.1. The base back-and-forth trees

- For any successor ordinal $\alpha+1$, $\mathcal{A}_{\alpha+2}$ consists of a root node, with infinitely many copies of $\mathcal{E}_{\alpha+1}$ attached to this root. This tree is illustrated in Figure 3.2.
- For any successor ordinal $\alpha+1$, $\mathcal{E}_{\alpha+2}$ consists of a root node, with infinitely many copies of $\mathcal{A}_{\alpha+1}$ and infinitely many copies of $\mathcal{E}_{\alpha+1}$ attached to this root. This tree is also illustrated in Figure 3.2.
- Let γ be a limit ordinal with fundamental sequence $\{\gamma_k\}_{k\in\omega}$. For any k, $\mathcal{L}_k^\gamma$ consists of a root node with exactly one copy of $\mathcal{A}_{\gamma_n}$ attached to this root for each $n \leqslant k$, and exactly one copy of $\mathcal{E}_{\gamma_n}$ attached to this root for each $n > k$. This tree is well-defined, since our fundamental sequences consist only of successor ordinals. It is illustrated in Figure 3.3.
- For any limit ordinal γ, $\mathcal{L}_\infty^\gamma$ consists of a root node with exactly one copy of $\mathcal{A}_{\gamma_n}$ attached to this root for each $n \in \omega$. This tree is also illustrated in Figure 3.3.
- For any limit ordinal γ, $\mathcal{A}_{\gamma+1}$ consists of a root node with infinitely many copies of $\mathcal{L}_n^\gamma$ attached to this root for each $n \in \omega$. This tree is illustrated in Figure 3.4.
- For any limit ordinal γ, $\mathcal{E}_{\gamma+1}$ consists of a root node with infinitely many copies of $\mathcal{L}_n^\gamma$ attached to this root for each $n \in \omega \cup \{\infty\}$. This tree is also illustrated in Figure 3.4.

We say that the back-and-forth trees $\mathcal{L}_n^\gamma$ for $n \in \omega \cup \{\infty\}$ have *rank* γ, while the back-and-forth trees $\mathcal{E}_\alpha$ and $\mathcal{A}_\alpha$ each have rank α.

Note that all of these trees are computably presentable, by effective transfinite recursion on the computable ordinals. In the case of infinite ordinals, the isomorphism structure of these

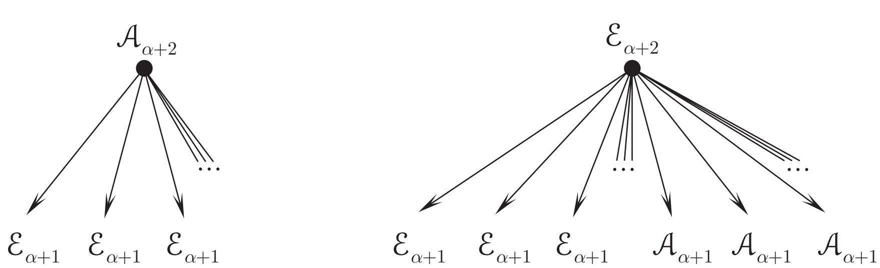

Figure 3.2. The successor back-and-forth trees

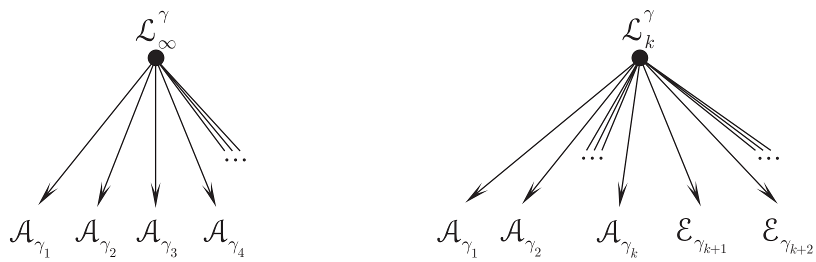

FIGURE 3.3. The limit back-and-forth trees

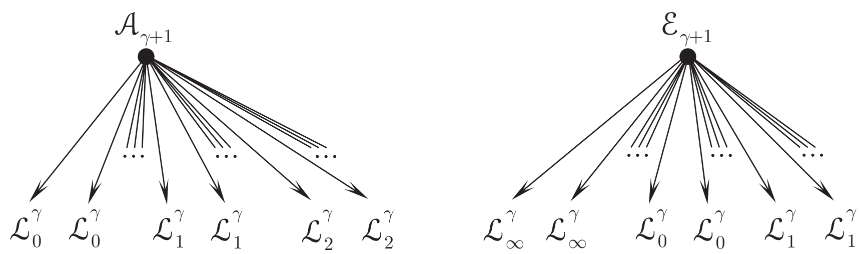

FIGURE 3.4. The transition back-and-forth trees

trees does depend on our choice of fundamental sequences for the limit ordinals. However, the following basic properties of back-and-forth trees are independent of our choice of fundamental sequences.

3.2. **Proposition.** *Let $\mathcal{P}(n)$ be a Σ_α predicate.*

(1) *If α is a successor ordinal, there is sequence of trees $\mathcal{T}_n$, uniformly computable from a Σ_α index for $\mathcal{P}$, such that for all n,*

$$\mathcal{T}_n \cong \begin{cases} \mathcal{E}_\alpha & \text{if } \mathcal{P}(n) \\ \mathcal{A}_\alpha & \text{otherwise} \end{cases}$$

(2) *If α is a limit ordinal, there is sequence of trees $\mathcal{T}_n$, uniformly computable from a Σ_α index for $\mathcal{P}$, such that for all n,*

$$\mathcal{T}_n \cong \begin{cases} \mathcal{L}^\alpha_\infty & \text{if } \neg\mathcal{P}(n) \\ \mathcal{L}^\alpha_k \text{ for some } k & \text{otherwise} \end{cases}$$

Proof. We proceed by effective transfinite recursion.

Case $\alpha = 1$: Since we can code disjunction over a c.e. set of indices as an existential quantifier, it follows from Definition 2.2 that $\mathcal{P}(n) = \exists x\, \mathcal{Q}(x, n)$ for some computable predicate $\mathcal{Q}$. We enumerate a root element into $\mathcal{T}_n$. So long as no witness appears for $\mathcal{Q}(x, n)$, we do not add any children. Once a witness appears, we add infinitely many children to $\mathcal{T}_n$. We can do this while keeping $\mathcal{T}_n$ computable by ensuring that the children are all elements greater than our witness for $\mathcal{Q}(x, n)$.

Case $\alpha = \beta+1$, with β a successor ordinal: Again $\mathcal{P}(n) = \exists x\, \mathcal{Q}(x, n)$ for some Π_β predicate $\mathcal{Q}(x, n)$. Since we can go uniformly from a Σ_α index for $\mathcal{P}$ to a Σ_β index for $\mathcal{Q}$, we apply our induction hypothesis.

As β is a successor ordinal, we have a computable sequence of trees $\mathcal{U}_{x,n}$ satisfying 1. For each n, let $\mathcal{T}_n$ be a tree whose root has infinitely many copies of $\mathcal{U}_{x,n}$ for each $x \in \omega$, and infinitely many copies of $\mathcal{E}_\beta$. As each $\mathcal{U}_{x,n}$ is either $\mathcal{E}_\beta$ or $\mathcal{A}_\beta$, $\mathcal{T}_n$ is either $\mathcal{E}_\alpha$ or $\mathcal{A}_\alpha$. Furthermore, $\mathcal{T}_n$ is $\mathcal{E}_\alpha$ only when at least one of the $\mathcal{U}_{x,n}$ is $\mathcal{A}_\beta$, or equivalently, when $\exists x\, \mathcal{Q}(x, n)$.

Case α is a limit ordinal: By padding, we can assume without loss of generality that $\mathcal{P}(n)$ is a computable disjunction of $\mathcal{Q}_k(n) \in \Sigma_{\alpha_k}$ with $\{\, \alpha_k \,\}_{k\in\omega}$ the fundamental sequence for α. Furthermore, we can go uniformly from a Σ_α index for $\mathcal{P}$ to Σ_{α_k} indices for the $\mathcal{Q}_k$. By induction, we have a computable sequence $\mathcal{U}_{m,n}$ satisfying 1 for the predicate

$$\mathcal{C}(m, n) =_{\mathtt{def}} \bigvee_{k\leqslant m} \mathcal{Q}_k(n)$$

Let $\mathcal{T}_n$ be the tree whose root has exactly one copy of $\mathcal{U}_{m,n}$ for each $m \in \omega$. Then $\mathcal{T}_n$ is $\mathcal{L}^\alpha_n$ for some $n \in \omega \cup \{\infty\}$. Moreover, $\mathcal{T}_n$ is $\mathcal{L}^\alpha_\infty$ exactly when $\mathcal{P}(n)$ fails. So the trees $\mathcal{T}_n$ satisfy 2.

Case $\alpha = \beta + 1$, with β a limit ordinal: Again we write $\mathcal{P}(n) = \exists x\, \mathcal{Q}(x, n)$ for some Π_β predicate $\mathcal{Q}(x, n)$. By induction, there is a computable sequence of trees $\mathcal{S}_{x,n}$ satisfying 2 for $\mathcal{Q}$. Let $\mathcal{T}_n$ be a tree whose root has infinitely many copies of $\mathcal{L}^\alpha_m$ for each $m \in \omega$, and infinitely many copies of $\mathcal{S}_{x,n}$ for each $x \in \omega$. These trees satisfy 1. □

By appropriately padding the indices of our computable structures, this result allows us several 1-reductions from predicates to our collection of back-and-forth trees. Each back-and-forth tree has a natural level of complexity such that we can go from predicates of that complexity to the corresponding back-and-forth tree. The following results show that we can also go in the reverse direction, from computable presentations of back-and-forth trees to hyperarithmetic predicates.

3.3. **Definition.** Let $\mathcal{T}$ be a tree with edge relation E. A tree $\mathcal{S}$ is a *limb* of $\mathcal{T}$ if $\mathcal{S} \subseteq \mathcal{T}$ and

$$\forall x \in \mathcal{S}\, \forall y \in \mathcal{T}\, (E(x, y) \Rightarrow y \in \mathcal{S})$$

If $a \in \mathcal{T}$ is the parent of the root of $\mathcal{S}$, we say that $\mathcal{S}$ is a *limb attached to a in* $\mathcal{T}$. For convenience, we often say that $\mathcal{S}$ is a limb attached to a if it is isomorphic to a limb attached to a.

3.4. **Lemma.** *Let $\mathcal{T}$ be any tree. For each computable ordinal α, there is an infinitary formula $\chi_\alpha(x) \in \mathcal{L}_{\omega_1,\omega}$ such that for any back-and-forth limb $\mathcal{S}$ of $\mathcal{T}$ with root $a \in \mathcal{T}$,*

$$\mathcal{T} \vDash \chi_\alpha(a) \Leftrightarrow \operatorname{rank}(\mathcal{S}) = \alpha \tag{1}$$

Furthermore, $\mathcal{T} \vDash \chi_\alpha(a)$ is a Π_α condition for computable structures $\mathcal{T}$.

Proof. We proceed by transfinite induction on the complexity on α.

Case $\alpha = 1$: We define the formula

$$\chi_1(x) =_{\mathtt{def}} \forall y, z\, \big(E(x, y) \Rightarrow \neg E(y, z)\big)$$

This formula is universal, and so $\mathcal{T} \vDash \chi_1(a)$ is Π_1 for computable structures $\mathcal{T}$.

Note that $\mathcal{T} \vDash \chi_1(a)$ if and only if every element of $\mathcal{S}$ has depth at most 1. Since $\mathcal{S}$ is a back-and-forth tree, this can only be the case when $\mathcal{S}$ is either $\mathcal{E}_1$ or $\mathcal{A}_1$.

Case $\alpha = \beta + 1$: We define the formula

$$\chi_\alpha(x) =_{\mathtt{def}} \exists y\, E(x,y) \wedge \forall z\, \big(E(x,z) \Rightarrow \chi_\beta(z)\big)$$

Since $\beta \geqslant 1$, $\mathcal{T} \vDash \chi_\alpha(a)$ is Π_α for computable structures $\mathcal{T}$.

Because $\mathcal{S}$ is a back-and-forth tree, all of the limbs attached to the root of $\mathcal{S}$ are also back-and-forth trees. As limbs of $\mathcal{S}$ are also limbs of $\mathcal{T}$, $\mathcal{T} \vDash \chi_\alpha(a)$ if and only if all of the limbs attached to a have rank β and there is at least one attached limb with rank β. It follows from Definition 3.1 that this is true exactly when $\mathcal{S}$ has rank α.

Case α **is a limit ordinal**: Let $\{\, \alpha_k \,\}_{k\in\omega}$ be the fundamental sequence for α. We define the formula

$$\begin{aligned}\chi_\alpha(x) =_{\mathtt{def}} \bigwedge_{n\in\omega} & \exists^{\geqslant n} y\, E(x,y) \wedge \\ & \forall z \left(E(x,z) \Rightarrow \bigvee_{k\in\omega} \chi_{\alpha_k}(z) \right) \wedge \\ & \bigwedge_{n\in\omega} \forall u,v\, \Big(\big(E(x,u) \wedge E(x,v) \wedge \chi_{\alpha_n}(u) \wedge \chi_{\alpha_n}(v)\big) \Rightarrow u = v \Big)\end{aligned}$$

Since $\alpha \geqslant \omega$, $\mathcal{T} \vDash \chi_\alpha(a)$ is Π_α for computable structures $\mathcal{T}$.

Because $\mathcal{S}$ is a back-and-forth tree, $\mathcal{T} \vDash \chi_\alpha(a)$ if and only $\mathcal{S}$ has exactly one limb of rank α_k for each $k \in \omega$. It follows from Definition 3.1 that this is true exactly when $\mathcal{S}$ has rank α. □

3.5. **Lemma.** *Let $\mathcal{T}$ be a tree and let $\mathcal{B}$ be any back-and-forth tree. Then there is an infinitary formula $\phi_{\mathcal{B}}(x) \in \mathcal{L}_{\omega_1,\omega}$ such that for any back-and-forth limb $\mathcal{S}$ of $\mathcal{T}$ which has root $a \in \mathcal{T}$ and is of the same rank as $\mathcal{B}$,*

$$\mathcal{T} \vDash \begin{cases} \phi_{\mathcal{B}}(a) & \text{if } \mathcal{S} \cong \mathcal{B} \\ \neg\phi_{\mathcal{B}}(a) & \text{otherwise} \end{cases} \tag{2}$$

Furthermore, for computable $\mathcal{T}$, the complexity of $\mathcal{T} \vDash \phi_{\mathcal{B}}(a)$ is the natural complexity of $\mathcal{B}$.

Proof. We proceed by transfinite induction on the complexity of $\mathcal{B}$.

Case $\mathcal{B} \cong \mathcal{A}_1$: We define the formula

$$\phi_{\mathcal{A}_1}(x) =_{\mathtt{def}} \forall y\, \neg E(x,y)$$

This formula is universal, and so $\mathcal{T} \vDash \phi_{\mathcal{A}_1}(a)$ is Π_1 for computable structures $\mathcal{T}$.

Note that $\mathcal{T} \vDash \phi_{\mathcal{A}_1}(a)$ if and only if a has no children in $\mathcal{T}$. Since any child of a in $\mathcal{T}$ is also a child of the root in $\mathcal{S}$, this is the case exactly when $\mathcal{S} \cong \mathcal{A}_1$.

Case $\mathcal{B} \cong \mathcal{E}_1$: We define the formula

$$\phi_{\mathcal{E}_1}(x) =_{\mathtt{def}} \exists y\, E(x,y)$$

This formula is existential, and so $\mathcal{T} \vDash \phi_{\mathcal{E}_1}(a)$ is Σ_1 for computable structures $\mathcal{T}$.

Since $\mathcal{S}$ is a back-and-forth tree of the same rank as $\mathcal{B}$, it is isomorphic to either $\mathcal{E}_1$ or $\mathcal{A}_1$. $\mathcal{T} \vDash \phi_{\mathcal{E}_1}(a)$ if and only if a has a child in $\mathcal{T}$, and hence in $\mathcal{S}$. This is the case exactly when $\mathcal{S} \cong \mathcal{E}_1$.

Case $\mathcal{B} \cong \mathcal{A}_{\alpha+1}$**,** α **a successor ordinal**: We define the formula

$$\phi_{\mathcal{A}_{\alpha+1}}(x) =_{\mathtt{def}} \exists y\, E(x,y) \wedge \forall z\, \big(E(x,z) \Rightarrow \phi_{\mathcal{E}_\alpha}(z)\big)$$

Since $\alpha \geqslant 1$, $\mathcal{T} \vDash \phi_{\mathcal{A}_{\alpha+1}}(a)$ is Π_α for computable structures $\mathcal{T}$.

Note that $\mathcal{T} \vDash \phi_{\mathcal{A}_{\alpha+1}}(a)$ if and only if all of the limbs attached to the root of $\mathcal{S}$ are isomorphic to $\mathcal{E}_\alpha$. Since $\mathcal{S}$ has rank $\alpha+1$, this is the case exactly when $\mathcal{S} \cong \mathcal{A}_{\alpha+1}$.

Case $\mathcal{B} \cong \mathcal{E}_{\alpha+1}$, α a successor ordinal: We define the formula

$$\phi_{\mathcal{E}_{\alpha+1}}(x) =_{\texttt{def}} \exists y \left(E(x,y) \wedge \phi_{\mathcal{A}_\alpha}(y) \right)$$

The argument that $\phi_{\mathcal{E}_{\alpha+1}}$ works is as before.

Case $\mathcal{B} \cong \mathcal{L}^\gamma_\infty$: Let $\{ \gamma_k \}_{k\in\omega}$ be the fundamental sequence for α. We define the formula

$$\phi_{\mathcal{L}^\gamma_\infty}(x) =_{\texttt{def}} \forall z \left(E(x,z) \Rightarrow \bigvee_{n\in\omega} \phi_{\mathcal{A}_{\gamma_n}}(z) \right)$$

Note that $\phi_{\mathcal{L}^\gamma_\infty}(x) \in \mathcal{L}_{\omega_1,\omega}$, since it involves a countable disjunction of formulae. Furthermore, $\mathcal{T} \vDash \phi_{\mathcal{L}^\gamma_\infty}(a)$ is Π_γ for computable structures $\mathcal{T}$.

Since $\mathcal{S}$ is a back-and-forth tree of the same rank γ as $\mathcal{B}$, $\mathcal{S} \cong \mathcal{L}^\gamma_k$ for some $k \in \omega \cup \{ \infty \}$. By induction, $\mathcal{T} \vDash \phi_{\mathcal{L}^\gamma_\infty}(a)$ if and only if, for each $n \in \omega$, any limb of rank α_n is isomorphic to $\mathcal{A}_{\gamma_n}$. This is the case exactly when $\mathcal{S} \cong \mathcal{L}^\gamma_\infty$.

Case $\mathcal{B} \cong \mathcal{L}^\gamma_n$, $n \in \omega$: We define the formula

$$\phi_{\mathcal{L}^\gamma_n}(x) =_{\texttt{def}} \exists y,z \left(E(x,y) \wedge E(x,z) \wedge \chi_{\gamma_n}(y) \wedge \phi_{\mathcal{A}_{\gamma_n}}(y) \wedge \chi_{\gamma_{n+1}}(z) \wedge \phi_{\mathcal{E}_{\gamma_{n+1}}}(z) \right)$$

This is a formula of $\mathcal{L}_{\omega_1,\omega}$, and $\mathcal{T} \vDash \phi_{\mathcal{L}^\gamma_n}(a)$ is Σ_γ for computable structures $\mathcal{T}$.

Again, $\mathcal{S} \cong \mathcal{L}^\gamma_k$ for some $k \in \omega \cup \{ \infty \}$. By Lemma 3.4, $\mathcal{T} \vDash \phi_{\mathcal{L}^\gamma_n}(a)$ if and only if $\mathcal{S}$ has a limb of rank γ_n isomorphic to $\mathcal{A}_{\gamma_n}$ and another of rank γ_{n+1} isomorphic to $\mathcal{E}_{\gamma_{n+1}}$. By Definition 3.1, this is true exactly when $\mathcal{S} \cong \mathcal{L}^\gamma_n$.

Case $\mathcal{B} \cong \mathcal{A}_{\gamma+1}$, γ a limit ordinal: We define the formula

$$\phi_{\mathcal{A}_{\gamma+1}}(x) =_{\texttt{def}} \forall y \left(E(x,y) \Rightarrow \bigvee_{k\in\omega} \phi_{\mathcal{L}^\gamma_k}(y) \right)$$

The argument that $\phi_{\mathcal{A}_{\gamma+1}}$ works is as before.

Case $\mathcal{B} \cong \mathcal{E}_{\gamma+1}$, γ a limit ordinal: We define the formula

$$\phi_{\mathcal{E}_{\gamma+1}}(x) =_{\texttt{def}} \exists y \left(E(x,y) \wedge \phi_{\mathcal{L}^\gamma_\infty}(y) \right)$$

The argument that $\phi_{\mathcal{E}_{\gamma+1}}$ works is as before. □

4. Main Results

We are now ready to use the trees defined in the previous section to obtain new examples of possible degree spectra of relations. A relation on the domain of a structure is *invariant* if it is mapped to itself by every automorphism of the structure.

4.1. **Theorem.** *Let α be a computable ordinal and let s be any reducibility stronger than or equal to m-reducibility. There exists an intrinsically Σ_α invariant relation U on a computably presentable structure $\mathcal{M}$ such that $\mathrm{DgSp}^s_{\mathcal{M}}(U)$ consists of all nontrivial Σ_α s-degrees.*

Proof. Let $\mathcal{M}$ be a copy of $\mathcal{E}_{\alpha+1}$, and let r be the root of $\mathcal{M}$.

Case α is a successor ordinal: Let U be the unary relation consisting of the children of r that are the roots of $\mathcal{E}_\alpha$ limbs. Every limb attached to r is a copy of either $\mathcal{A}_\alpha$ or $\mathcal{E}_\alpha$, and hence all such limbs are of the same rank. By Lemma 3.5,

$$a \in U \Leftrightarrow \mathcal{M} \models E(r,a) \wedge \phi_{\mathcal{E}_\alpha}(a) \tag{3}$$

Therefore, U is intrinsically Σ_α.

The parameter r in (3) is first-order definable by the formula $\forall y \neg E(y,x)$. Therefore, U is definable by a parameter-free formula of $\mathcal{L}_{\omega_1,\omega}$. Thus U is invariant.

Given an infinite and coinfinite Σ_α relation $\mathcal{P}(n)$ in Σ_α, let $\mathcal{T}_n$ be the computable sequence from part 1 of Proposition 3.2. It is straightforward to define a computable presentation M of $\mathcal{M}$ consisting of a copy of $\mathcal{E}_\alpha$ with root node $\langle 0,n\rangle$ for each $n \in \omega$, a copy of $\mathcal{A}_\alpha$ with root node $\langle 1,n\rangle$ for each $n \in \omega$, and a copy of $\mathcal{T}_n$ with root node $\langle 2,n\rangle$ for each $n \in \omega$. Now $x \in U$ if and only if either $x = \langle 0,n\rangle$ for some $n \in \omega$ or $x = \langle 2,n\rangle$ for some n such that $\mathcal{P}(n)$. Thus $U^M \equiv_{\mathrm{m}} \mathcal{P}$.

Case α is a limit ordinal: Let U be the unary relation consisting of the children of r that are the roots of copies of $\mathcal{L}_k^\alpha$ for $k \in \omega$. By Lemma 3.5,

$$a \in U \Leftrightarrow \mathcal{M} \models E(r,a) \wedge \bigvee_{k\in\omega} \phi_{\mathcal{L}_k^\alpha}(a)$$

This is a computable disjunction of Σ_α predicates, and so U is intrinsically Σ_α. Furthermore, U is invariant since r is first-order definable.

Given an infinite and coinfinite relation $\mathcal{P}(n)$ in Σ_α, let $\mathcal{T}_n$ be the computable sequence from part 2 of Proposition 3.2. It is straightforward to define a computable presentation M of $\mathcal{M}$ consisting of a copy of $\mathcal{L}_k^\alpha$ with root node $\langle 0,k,n\rangle$ for each n, $k \in \omega$, a copy of $\mathcal{L}_\infty^\alpha$ with root node $\langle 1,n\rangle$ for each $n \in \omega$, and a copy of $\mathcal{T}_n$ with root node $\langle 2,n\rangle$ for each $n \in \omega$. Now $x \in U$ if and only if either $x = \langle 0,k,n\rangle$ for some n, $k \in \omega$ or $x = \langle 2,n\rangle$ for some n such that $\mathcal{P}(n)$. Thus $U^M \equiv_{\mathrm{m}} \mathcal{P}$. ☐

By replacing U with its complement, we can replace Σ_α with Π_α in the statement of Theorem 4.1. The next result shows that we can also replace Σ_α with Δ_α.

4.2. **Theorem.** *Let α be a computable ordinal and let s be any reducibility stronger than or equal to m-reducibility. There exists an intrinsically Δ_α invariant relation V on a computably presentable structure $\mathcal{N}$ such that $\mathrm{DgSp}^s_{\mathcal{N}}(V)$ consists of all nontrivial Δ_α s-degrees.*

Proof. Let $\mathcal{M}$ and U be as in the proof of Theorem 4.1, and let $\mathcal{M}_0$ and $\mathcal{M}_1$ be copies of $\mathcal{M}$. Let S_i be the set of children of the root of $\mathcal{M}_i$, let U_i be the copy of U in $\mathcal{M}_i$, and let $\widehat{U}_i = S_i - U_i$. Note that U_i is intrinsically Σ_α and $\widehat{U}_i$ is intrinsically Π_α. Let $a_0^i, a_1^i, \ldots$ and $b_0^i, b_1^i, \ldots$ be the elements of U_i and $\widehat{U}_i$, respectively.

In addition to the edge relation, the language of $\mathcal{N}$ has two unary relations D_0 and D_1 and a binary relation F. To define $\mathcal{N}$, we begin with $\mathcal{M}_0$ and $\mathcal{M}_1$, add new elements $c_0, c_1, \ldots$ and $d_0, d_1, \ldots$, and let

$$\begin{aligned} F^{\mathcal{N}}(x,y) \Leftrightarrow & (x = c_k \wedge (y = a_k^0 \vee y = b_k^1)) \vee \\ & (x = d_k \wedge (y = b_k^0 \vee y = a_k^1)) \end{aligned}$$

and

$$D_i^{\mathcal{N}}(x) \Leftrightarrow x \in \mathcal{M}_i$$

This completes the definition of $\mathcal{N}$. Let $V = \{ c_0, c_1, \dots \}$.

Since V can be defined both as

$$\{ x \in \mathcal{N} \mid \exists y \in U_0 \, (F(x,y)) \}$$

and as

$$\left\{ x \in \mathcal{N} \mid \forall y \, (F(x,y) \Rightarrow y \in \widehat{U}_1) \right\}$$

V is intrinsically Δ_α.

Given an infinite and coinfinite relation $\mathcal{P}(n)$ in Δ_α, we can use the construction in the proof of Theorem 4.1 to build computable presentations M_0 and M_1 of $\mathcal{M}$ satisfying the following conditions.

(1) $|M_0| \cap |M_1| = \emptyset$.
(2) $|M_0| \cup |M_1|$ is coinfinite.
(3) For some computable list $y^0_0, y^0_1, \dots$ of the elements of S_0, $U^{M_0}(y^0_k) \Leftrightarrow \mathcal{P}(k)$.
(4) For some computable list $y^1_0, y^1_1, \dots$ of the elements of S_1, $U^{M_1}(y^1_k) \Leftrightarrow \neg\mathcal{P}(k)$.

Now build a computable presentation N of $\mathcal{N}$ as follows. Begin with M_0 and M_1. Let $x_0 < x_1 < \cdots$ be the elements of $\omega - (|M_0| \cup |M_1|)$. Let $D_i^N = |M_i|$, and let $F^N = \left\{ (x_k, y^i_k) \mid i < 1, \ k \in \omega \right\}$.

It is easy to check that N is a computable presentation of $\mathcal{N}$. Furthermore, if $x \notin \{ x_0, x_1, \dots \}$ then $\neg V^N(x)$, while $V^N(x_k) \Leftrightarrow U^N_0(y^0_k) \Leftrightarrow \mathcal{P}(k)$, and hence $V^N \equiv_{\mathrm{m}} \mathcal{P}$. □

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