Randomness and Reducibility*

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1 Introduction

How random is a real? Given two reals, which is more random? If we partition reals into equivalence classes of reals of the "same degrees of randomness", what does the resulting structure look like? The goal of this paper is to look at questions like these, specifically by studying the properties of reducibilities that act as measures of relative randomness, as embodied in the concept of initial-segment complexity.

The initial segment complexity of a real is a natural measure of its relative randomness, and has been implicitly studied by many authors. For instance, by the work of Schnorr we know that a real α is Martin-Löf random if and only if its initial segment complexity is roughly speaking as big as it can be. (See below for the relevant definitions.) That is, if we denote prefix-free Kolmogorov complexity by H, then α is Martin-Löf random if and only if there is a constant c such that $H(\alpha \upharpoonright n) \geqslant n - c$ for all n, where $\alpha \upharpoonright n$ denotes the initial segment of α of length n. Furthermore, the work of Barzdins [3] shows that if a set is computably enumerable then its plain Kolmogorov complexity is bounded by $2 \log n$, and this bound can be sharp, as shown by Kummer [30]. Finally, recent work of Levin, Lutz, Mayordomo, Staiger, and others (e.g., [38, 52, 36, 34]) proves that effective Hausdorff dimension is essentially intertwined with initial segment complexity.

We look at reducibilities $\leqslant_{\mathrm{R}}$ which have the property that if $\alpha \leqslant_{\mathrm{R}} \beta$ then the prefix-free initial segment complexity of α is no greater than that of β (up to an additive constant), and hence act as measures of relative randomness. One such reducibility, called domination or Solovay reducibility, was introduced by Solovay [50], and has been studied by Calude, Hertling, Khoussainov, and Wang [8], Calude [4], Kučera and Slaman [29], and Downey, Hirschfeldt, and Nies [18], among others. Solovay reducibility has proved to be a powerful tool in the study of randomness of effectively presented reals. Motivated by certain shortcomings of Solovay reducibility, which we will discuss below, we introduce two new reducibilities and study, among other things, the relationships between these various measures of relative randomness.

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We work in Cantor space 2^ω with basic clopen sets $[\sigma] = \{\sigma\alpha : \alpha \in 2^\omega\}$ for strings $\sigma \in 2^{<\omega}$. The Lebesgue measure of a clopen set $[\sigma]$ is $2^{-|\sigma|}$. This space is measure-theoretically identical with the interval of reals $(0, 1)$, though the two spaces are not homeomorphic. We identify a real with its binary expansion, which we may think of as an element of 2^ω, and hence with the set of natural numbers whose characteristic function is the same as that expansion. (Some reals have two binary expansions; for such a real, which is always rational, we choose the nonterminating expansion.) We also identify finite binary strings with rationals. Our computability-theoretic notation follows the standard of Soare [45].

Our main concern will be reals that are limits of computable increasing sequences of rationals. We call such reals *computably enumerable* (c.e.), though they have also been called *recursively enumerable*, *left computable* (by Ambos-Spies, Weihrauch, and Zheng [2]), *left semicomputable*, and *lower semicomputable*. If, in addition to the existence of a computable increasing sequence $q_0, q_1, \ldots$ of rationals with limit α, there is a total computable function f such that $\alpha - q_{f(n)} < 2^{-n}$ for all n, then α is called *computable*. These and related concepts have been widely studied. In addition to the papers and books mentioned elsewhere in this introduction, we may cite, among others, early work of Rice [41], Lachlan [31], Soare [43], and Ceĭtin [10], and more recent papers by Ko [24, 25], Calude, Coles, Hertling, and Khoussainov [7], Ho [23], and Downey and LaForte [20]. Several of the results mentioned below provide strong evidence that computably enumerable reals are natural objects in the study of effective randomness in the same way that computably enumerable sets are natural objects in classical computability theory.

An alternate definition of c.e. reals can be given as follows.

Definition 1.1. *A set $A \subseteq \mathbb{N}$ is* nearly computably enumerable *if there is a computable approximation $\{A_s\}_{s\in\omega}$ such that $A(x) = \lim_s A_s(x)$ for all x and $A_s(x) > A_{s+1}(x) \Rightarrow \exists y < x(A_s(y) < A_{s+1}(y))$.*

As shown by Calude, Coles, Hertling, and Khoussainov [7], a real $0.\chi_A$ is c.e. if and only if A is nearly c.e.. An interesting subclass of the class of c.e. reals is the class of strongly c.e. reals. A real $0.\chi_A$ is said to be *strongly c.e.* if A is c.e.. Soare [44] noted that there are c.e. reals that are not strongly c.e..

A computer M is *self-delimiting* if, for all finite binary strings σ and $\tau \subsetneq \tau'$, we have $M^\sigma(\tau)\downarrow \Rightarrow M^\sigma(\tau')\uparrow$, where $M^\sigma(\tau)\downarrow$ means that the computation of M on input τ and using oracle σ converges, and $M^\sigma(\tau)\uparrow$ means that this computation diverges.

It is not difficult to see that a real is c.e. if and only if it is the measure of the domain of a self-delimiting machine. This fact is analogous to the statement that a set is c.e. if and only if it is the domain of a function on $\mathbb{N}$ computed by a Turing machine.

The self-delimiting computer M is *universal* if for each self-delimiting computer N there is a constant c such that, for all binary strings σ and τ, if $N^\sigma(\tau)\downarrow$ then $M^\sigma(\mu)\downarrow = N^\sigma(\tau)$ for some μ with $|\mu| \leqslant |\tau| + c$. We call c the *coding constant* of N.

Fix a self-delimiting universal computer M. We can define Chaitin's number $\Omega = \Omega_M$ via

$$\Omega = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|},$$

which is the *halting probability* of the computer M. The properties of Ω relevant to this paper are independent of the choice of M. A c.e. real is an Ω-*number* if it is Ω_M for some self-delimiting universal computer M.

The c.e. real Ω is random in the canonical Martin-Löf sense [37] of c.e. randomness. There are many equivalent formulations of c.e. randomness. The one that is most relevant to us here is based on prefix-free complexity, which we define below. (The history of effective randomness is quite rich and involved; references include van Lambalgen [53], Calude [5], Li and Vitanyi [35], and Ambos-Spies and Kučera [1].)

Recall that the prefix-free complexity $H(\tau)$ of a binary string τ is the length of the shortest binary string σ such that $M(\sigma)\downarrow = \tau$. (Often $K(\tau)$ is used instead of $H(\tau)$. The choice of self-delimiting universal computer M does not affect the prefix-free complexity, up to a constant additive factor.) For $n \in \mathbb{N}$, we write $H(n)$ for $H(1^n)$. Most of the statements about $H(\tau)$ made below also hold for the plain Kolmogorov complexity $C(\tau)$. For more on the definitions and basic properties of $H(\tau)$ and $C(\tau)$, see Chaitin [14], Calude [5], Li and Vitanyi [35], and Fortnow [21]. Among the many works dealing with these and related topics, and in addition to those mentioned elsewhere in this paper, we may cite Solomonoff [47–49], Kolmogorov [26–28], Levin [32–34], Zvonkin and Levin [56], Gács [22], Schnorr [42], and Chaitin [11].

A real α is *random*, or more precisely, 1-random, if there is a constant c such that $H(\alpha \upharpoonright n) \geqslant n - c$ for all n. As mentioned earlier, Schnorr showed that this definition is equivalent to the earlier, measure-theoretical one due to Martin-Löf [37]. (An earlier article by Levin [34] studied monotone complexity and proved a similar characterization of 1-randomness.)

Many authors have studied Ω and its properties, notably Chaitin [12–14] and Martin-Löf [37]. In the very long and widely circulated manuscript [50] (a fragment of which appeared in [51]), Solovay carefully investigated relationships between prefix-free complexity, Kolmogorov complexity, and properties of random languages and reals. See Chaitin [12] for an account of some of the results in this manuscript.

Solovay discovered that several important properties of Ω (whose definition is model-dependent) are shared by another class of reals he called Ω-like, whose definition is model-independent. The point here is that when we look at classical computability we always talk about *the* halting problem rather than *a* halting problem, even though the actual definition depends on the relevant enumeration of the partial computable functions. The reason we can do this is that we can show that all versions of the halting problem are "the same" by showing that they all have the same m-degree. Solovay's idea was to define an appropriate type of reduction to show that all the versions of Ω are "the same". Indeed, the reduction below is a kind of analytic m-reducibility.

Definition 1.2. *Let α and β be c.e. reals. We say that α* dominates *β and that β is* Solovay reducible *(S-reducible) to α, and write $\beta \leqslant_{\mathrm{S}} \alpha$, if there are a constant c and a partial computable function $\varphi : \mathbb{Q} \to \mathbb{Q}$ such that for each rational $q < \alpha$ we have $\varphi(q)\downarrow < \beta$ and*

$$\beta - \varphi(q) \leqslant c(\alpha - q).$$

We write $\alpha \equiv_{\mathrm{S}} \beta$ if $\alpha \leqslant_{\mathrm{S}} \beta$ and $\beta \leqslant_{\mathrm{S}} \alpha$.

The idea is that, given an approximation to α, we can generate one converging to β just as fast. Solovay reducibility is reflexive and transitive, and hence $\equiv_{\mathrm{S}}$ is an equivalence relation on the c.e. reals. Thus we can define the *Solovay degree* $\deg_{\mathrm{S}}(\alpha)$ of a c.e. real α to be its $\equiv_{\mathrm{S}}$ equivalence class.

Solovay reducibility is naturally associated with randomness due to the following fact.

Theorem 1.3 (Solovay [50]). *Let $\beta \leqslant_{\mathrm{S}} \alpha$ be c.e. reals. There is a constant c such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + c$ for all n.*

It is this property of Solovay reducibility (which we will call the *Solovay property*), which makes it a measure of relative randomness. This is in contrast with Turing reducibility, for example, which does not have the Solovay property, since the complete c.e. Turing degree contains both random and nonrandom reals.

Solovay observed that Ω dominates all c.e. reals, and Theorem 1.3 implies that if a c.e. real dominates all c.e. reals then it must be random. This led him to define a c.e. real to be *Ω-like* if it dominates all c.e. reals (that is, if it is S-complete). The point is that the definition of Ω-like seems quite model-independent (in the sense that it does not require a choice of self-delimiting universal computer), as opposed to the model-dependent definition of Ω. However, Calude, Hertling, Khoussainov, and Wang [8] showed that the two notions coincide, by showing that if a c.e. real is Ω-like, then it is the halting probability of some universal machine. This circle of ideas was completed recently by Kučera and Slaman [29], who showed that all random c.e. reals are Ω-like.

This collection of results gives great insight into the structure of random c.e. reals and their initial segment complexity. We know that there is a c such that $H(\sigma) \leqslant |\sigma| + H(|\sigma|) + c$ for all $\sigma \in 2^{<\omega}$, and that this upper bound cannot be improved. It would therefore seem reasonable that there would be many possible initial segment complexities of random c.e. reals, since to be random a real need only have its prefix-free initial segment complexity be above n. However, the results above show that for c.e. reals this is not so. There is essentially only one c.e. random real, namely Ω, and all versions of it oscillate in prefix-free initial segment complexity at essentially the same rate.

For more on c.e. reals and S-reducibility, see for instance Chaitin [12–14], Calude, Hertling, Khoussainov, and Wang [8], Calude and Nies [9], Calude [4], Kučera and Slaman [29], and Downey, Hirschfeldt, and Nies [18]. For instance, in [18], Downey, Hirschfeldt, and Nies proved that the S-degrees of c.e. reals form a dense distributive uppersemilattice, where the natural join operation is induced by arithmetical addition. They also showed that if $\alpha + \beta$ is random for c.e. reals α and β, then one of α and β must also be random, but, on the other hand, for any non-random c.e. real γ, there are c.e. reals α and β such that $\alpha + \beta = \gamma$ and $\alpha, \beta <_{\mathrm{S}} \gamma$. These facts demonstrate a qualitative difference between random and non-random c.e. reals, as reflected in the structure of the S-degrees.

Solovay reducibility is an excellent tool in the study of the relative randomness of reals, but it has several shortcomings. One such shortcoming is that S-reducibility is quite ill-behaved outside the c.e. reals. It is not very hard to construct a noncomputable real that is not S-above the computable reals (in fact, this real can be chosen to be d.c.e., that is, of the form $\alpha - \beta$ where α and β are c.e.). This and similar facts show that S-reducibility is very unnatural when applied to non-c.e. reals. Another problem with S-reducibility is that it is uniform in a way that relative initial-segment complexity is not. This makes it too strong, in a sense, and appears to preclude its having a natural characterization in terms of initial-segment complexity. In particular, Calude and Coles [6] answered a question of Solovay by showing that the converse of Theorem 1.3 does not hold (see below for an easy proof of this fact). Thus, if our goal is to study relative initial segment complexity of reals, it behooves us to look beyond S-reducibility.

In this paper, we introduce two new measures of relative randomness that provide additional tools for the study of the relative randomness of reals, and investigate their properties and the relationships between them and S-reducibility. In the same way that m-reducibility is a very refined reducibility in classical computability theory, and is extended by other reducibilities such as Turing reducibility, so is S-reducibility extended by the new ones we introduce. Our hope is that these new measures of relative randomness will reveal insights like those discussed above into the very nature of initial segment complexity. Thus the purpose of this paper is to establish some of their basic properties and relationships, and hence provide a solid basis for further work in this area. (We will mention below some of the work that has already been done since the original writing of this paper.)

We begin with sw-reducibility, which has some nice features but also some shortcomings. It is related to a reducibility recently studied by Soare [46] and Csima [15] in connection with computability-theoretic notions arising from the work of Nabutovsky and Weinberger [39] in differential geometry. Informally, sw-reducibility says that there is a natural way, with little compression, to produce the bits of one real from another. It agrees with Solovay reducibility on *strongly* c.e. reals but is in general different. Recently, Yu and Ding [54] have proven a number of interesting results about sw-reducibility, one of which is that there is no maximum sw-degree of c.e. reals, meaning that while Solovay completeness captures 1-randomness, there is no such characterization based on uniform reductions between initial segments of similar lengths.

We then move on to the very interesting rH-reducibility, which shares many of the best features of S-reducibility, while not being restricted to the c.e. reals, and can also be seen as a less uniform version of sw-reducibility. This nonuniformity allows us to circumvent the problems with sw-reducibility, such as the lack of a maximum degree among c.e. reals. Furthermore, rH-reducibility has a very nice characterization, in terms of relative initial-segment complexity, which can be seen as a partial converse to the Solovay property. (Indeed, rH stands for "relative H".) More specifically, we prove that $\alpha \leqslant_{\mathrm{rH}} \beta$ if and only if there is a constant c such that, for all n, the initial segment complexity of $\alpha \upharpoonright n$ given $\beta \upharpoonright n$ is less than or equal to c. We also prove that $\leqslant_{\mathrm{rH}}$ is well-behaved on the c.e. reals. For instance, we show that the rH-degrees of c.e. reals form a dense uppersemilattice with top degree that of Ω and join induced by addition.

We remark that, since the original writing of this paper, there have been quite a number of subsequent investigations into the notion of relative randomness via initial segment complexities. These have yielded significant insights into the nature of randomness and have seen nice applications in other arenas. For instance, it has recently been shown that constructing a c.e. set A for which there is a c with $H(A \upharpoonright n) \leqslant H(n) + c$ for all n gives a simple priority-free solution to Post's problem (see [17, 19, 40]). There appears to be a strong potential in this area for the development of a complex theory paralleling and interacting with the theory of measures of relative complexity studied in classical computability theory.

2 Strong Weak Truth Table Reducibility

Solovay reducibility has many attractive features, but it is not the only interesting measure of relative randomness. In this section, we introduce another such measure, sw-reducibility, which is more explicitly derived from the idea of initial segment complexity, and which is in some ways nicer than S-reducibility. In particular, sw-reducibility is much better adapted to dealing with non-c.e. reals. Furthermore, sw-reducibility is also helpful in the study of S-reducibility, as we will indicate below, and provides a motivation for the definition of rH-reducibility in the next section, since rH-reducibility is a kind of "sw-reducibility with advice".

Recall that a Turing reduction $\Gamma^A = B$ is called a weak truth table (wtt) reduction if there is a computable function φ such that the use function $\gamma(x)$ is bounded by $\varphi(x)$.

Definition 2.1. *Let $A, B \subseteq \mathbb{N}$. We say that B is* strongly weak truth table reducible *(sw-reducible) to A, and write $B \leqslant_{\mathrm{sw}} A$, if there are a constant c and a wtt reduction Γ such that $B = \Gamma^A$ and $\forall x(\gamma(x) \leqslant x + c)$.*

For reals $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, we say that β is sw-reducible to α, and write $\beta \leqslant_{\mathrm{sw}} \alpha$, if $B \leqslant_{\mathrm{sw}} A$.

Since sw-reducibility is reflexive and transitive, we can define the *sw-degree* $\deg_{\mathrm{sw}}(\alpha)$ of a real α to be its sw-equivalence class.

Solovay [50] noted that for each k there is a constant c such that for all $n \geqslant 1$ and all binary strings σ, τ of length n, if $|0.\sigma - 0.\tau| < k2^{-n}$ then $|H(\tau) - H(\sigma)| \leqslant$

c. Using this result, it is easy to check that sw-reducibility has the Solovay property.

Proposition 2.2. *Let $\beta \leqslant_{\mathrm{sw}} \alpha$ be c.e. reals. There is a constant c such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + c$ for all $n \in \omega$.*

Theorem 2.5 below shows that the converse of Proposition 2.2 does not hold even for c.e. reals.

We now explore the relationship between S-reducibility and sw-reducibility on the c.e. and strongly c.e. reals. We begin by noting the following lemma, implicit in Solovay [50].

Lemma 2.3. *Let α and β be c.e. reals, and let $\alpha_0, \alpha_1, \dots$ and $\beta_0, \beta_1, \dots$ be computable increasing sequences of rationals converging to α and β, respectively. Then $\alpha \leqslant_{\mathrm{S}} \beta$ if and only if there are a constant c and a total computable function f such that for all $n \in \omega$ we have $\alpha - \alpha_{f(n)} \leqslant c(\beta - \beta_n)$.*

Proof. First suppose that $\alpha \leqslant_{\mathrm{S}} \beta$ and let c and φ be as in Definition 1.2. For each n let $f(n)$ be the least s such that $\alpha_s \geqslant \varphi(\beta_n)$. Then $\alpha - \alpha_{f(n)} \leqslant \alpha - \varphi(\beta_n) \leqslant c(\beta - \beta_s)$.

For the converse, suppose that c and f are as above. For each rational q, if there is a stage s_q such that $\beta_{s_q} \geqslant q$ then let $\varphi(q) = \alpha_{f(s_q)}$, and otherwise let $\varphi(q)\uparrow$. Then φ is defined on all rationals less than β, and for any such rational q we have $\alpha - \varphi(q) = \alpha - \alpha_{f(s_q)} \leqslant c(\beta - \beta_{s_q}) \leqslant c(\beta - q)$. Thus $\alpha \leqslant_{\mathrm{S}} \beta$. □

Whenever we mention a c.e. real α below, we assume that we have chosen a computable increasing sequence $\alpha_0, \alpha_1, \dots$ converging to α. The previous lemma guarantees that, in determining whether one c.e. real dominates another, the particular choice of such sequences is irrelevant.

In general, neither of the reducibilities under consideration implies the other.

Theorem 2.4. *There exist c.e. reals $\alpha \leqslant_{\mathrm{sw}} \beta$ such that $\alpha \nleqslant_{\mathrm{S}} \beta$. Moreover, α can be chosen to be strongly c.e..*

Proof. We must build α and β so that $\alpha \leqslant_{\mathrm{sw}} \beta$ and α is strongly c.e., while satisfying the following requirements for each $e, c \in \omega$.

$$\mathcal{R}_{e,c} : \exists q \in \mathbb{Q}(c(\beta - q) \ngeqslant \alpha - \Phi_e(q)),$$

where Φ_e is the eth partial computable function. We do this with a straightforward finite injury argument.

We discuss the strategy for a single requirement $\mathcal{R}_{e,c}$. Let k be such that $c \leqslant 2^k$. We must make the difference between β and some rational q quite small while making the difference between α and $\Phi_e(q)$ relatively large. At a stage t we pick a new big number d. For the sake of $\mathcal{R}_{e,c}$, we will control the first $d + k + 3$ places of (the binary expansion of) β_s and α_s for $s \geqslant t$. We set $\beta_t(x) = 1$ for all x with $d \leqslant x \leqslant d + k + 2$, while at the same time keeping $\alpha_s(x) = 0$ for all such x. We let $q = \beta_t$. Note that, since we are restraining the first $d + k + 3$ places of

β_s, we know that, unless this restraint is lifted, β_s can only change on positions greater than or equal to $d+k+3$, and hence $\beta - q \leqslant 2^{-(d+k+3)}$. This means that, unless we lift the restraint, $c(\beta - q) \leqslant 2^k 2^{-(d+k+3)} = 2^{-(d+3)}$.

We now need do nothing until we come to a stage $s \geqslant t$ such that $\Phi_{e,s}(q)\downarrow$ *and* $0 < \alpha_s - \Phi_{e,s}(q) \leqslant 2^{-(d+3)}$. Our action then is the following. First we add $2^{-(d+k+2)}$ to β_s. Then we restrain β_u for $u > s+1$ on its first $d+k+3$ places. Assuming that this restraint is successful, it follows that $c(\beta - q) \leqslant 2^{-(d+3)} + 2^{-(d+2)} < 2^{-(d+1)}$.

Finally we win by our second action, which is to add 2^{-d} to α_{s+1}. Then $\alpha - \alpha_s \geqslant 2^{-d}$, so $\alpha - \Phi_e(q) \geqslant 2^{-d} > c(\beta - q)$, as required.

The theorem now follows by a simple application of the finite injury priority method.

It is easy to see that $\alpha \leqslant_{\mathrm{sw}} \beta$. When we add $2^{-(d+k+2)}$ to β_s, since $\beta_t(x) = 1$ for all x with $d \leqslant x \leqslant d+k+2$, the effect is to make position $d-1$ of β change from 0 to 1. On the α side, the only change is that position $d-1$ changes from 0 to 1. Hence we keep $A \leqslant_{\mathrm{sw}} B$ (with constant 0). It is also clear that α is strongly c.e.. □

We note that, since sw-reducibility has the Solovay property, the previous result gives a quick proof of the theorem, due to Calude and Coles [6], that the converse of Theorem 1.3 does not hold. This is one example of the usefulness of sw-reducibility in the study of S-reducibility.

Theorem 2.5. *There exist c.e. reals $\alpha \leqslant_{\mathrm{S}} \beta$ such that $\alpha \nleqslant_{\mathrm{sw}} \beta$ (in fact, even $\alpha \nleqslant_{\mathrm{wtt}} \beta$). Moreover, β can be chosen to be strongly c.e..*

Proof. The proof is a straightforward diagonalization argument, similar to the previous proof, but even easier. The strategy is described below. We build sets A and B and let $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$. We must meet the following requirements.

$$\mathcal{R}_{e,c} : \text{If } \Gamma_e \text{ has use } x+c \text{ then } \Gamma_e^B \neq A.$$

The idea is quite simple. We need only make B "sparse" and A "sometimes thick". That is, for the sake of $\mathcal{R}_{e,c}$, we set aside a block of $c+2$ positions of the binary expansion of β, say $n, n+1, \ldots, n+c+1$. Initially we have *none* of these numbers in B, but we put *all* of $n+1, \ldots, n+c+1$ into A. If we ever see a stage s where $\Gamma_{e,s}^{B_s}(n)\downarrow = 0$ with use $n+c$, we can satisfy the requirement by adding $2^{-(n+c+1)}$ to both α_s and β_s, the effect being that $B_s(n+c+1)$ changes from 0 to 1, $A_s(n+i)$ for $1 \leqslant i \leqslant c+1$ changes from 1 to 0, and $A_s(n)$ changes from 0 to 1.

It is easy to check that $\alpha \leqslant_{\mathrm{S}} \beta$ and that β is strongly c.e.. □

The counterexamples above can be jazzed up with relatively standard degree control techniques to prove the following result.

Theorem 2.6. *Let* **a** *be a nonzero c.e. Turing degree. There exist c.e. reals α and β of degree* **a** *such that α is strongly c.e., $\alpha \leqslant_{\mathrm{sw}} \beta$, and $\alpha \nleqslant_{\mathrm{S}} \beta$. There also exist c.e. reals γ and δ of degree* **a** *such that δ is strongly c.e., $\gamma \leqslant_{\mathrm{S}} \delta$, and $\gamma \nleqslant_{\mathrm{sw}} \delta$.*

On the strongly c.e. reals, however, S-reducibility and sw-reducibility coincide. Since sw-reducibility is sometimes easier to deal with than S-reducibility, this fact makes sw-reducibility a useful tool in the study of S-reducibility on strongly c.e. reals. An example of this phenomenon is Theorem 2.10 below, which is most easily proved using sw-reducibility, as the proof included below illustrates.

Theorem 2.7. *If β is strongly c.e. and α is c.e. then $\alpha \leqslant_{\mathrm{sw}} \beta$ implies $\alpha \leqslant_{\mathrm{S}} \beta$.*

Proof. Let A and B be such that $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, and suppose that $\Gamma^B = A$ with use $x + c$. We may assume that we have the approximations of A and B sped up so that every stage is expansionary. That is, for all stages s and all $z \leqslant s$, we have $\Gamma_s^{B_s}(z) = A_s(z)$. We may also assume that if z enters A at stage s then $s \geqslant z$. Now if z enters A at stage s then some number less than or equal to $z + c$ must enter B at stage s. Since B is c.e., this means that $\beta_s - \beta_{s-1} \geqslant 2^{-(z+c)}$. But z entering A corresponds to a change of at most 2^{-z} in the value of α, so $\beta_s - \beta_{s-1} \geqslant 2^{-c}(\alpha_s - \alpha_{s-1})$. Thus for all s we have $\alpha - \alpha_s \leqslant 2^c(\beta - \beta_s)$, and hence, by Lemma 2.3, $\alpha \leqslant_{\mathrm{S}} \beta$. □

Theorem 2.8. *If α is strongly c.e. and β is c.e. then $\alpha \leqslant_{\mathrm{S}} \beta$ implies $\alpha \leqslant_{\mathrm{sw}} \beta$.*

Proof. Let A and B be such that $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$. Note that, since α is strongly c.e., for all k and s we have $A \upharpoonright k = A_s \upharpoonright k$ if and only if $\alpha - \alpha_s \leqslant 2^{-(k+1)}$. Let f and c be as in Lemma 2.3 and let k be such that $c \leqslant 2^{k-2}$. To decide whether $x \in A$ using the first $x + k$ bits of B, find the least stage s such that $B_s \upharpoonright x + k = B \upharpoonright x + k$. We claim that $x \in A$ if and only if $x \in A_{f(s)}$. To verify this claim, first note that $\beta - \beta_s < 2^{-(x+k)}$, since otherwise β_s would have to change on one of its first $x + k$ places after stage s. Thus $\alpha - \alpha_{f(s)} \leqslant 2^{k-2}2^{-(x+k)} = 2^{-(x+2)}$, and hence, as noted above, A has stopped changing on the numbers $0, \dots, x$ by stage $f(x)$. □

Corollary 2.9. *If α and β are strongly c.e. then $\alpha \leqslant_{\mathrm{S}} \beta$ if and only if $\alpha \leqslant_{\mathrm{sw}} \beta$.*

There is a greatest S-degree of c.e. reals, namely that of Ω, but the situation is different for strongly c.e. reals.

Theorem 2.10. *Let α be strongly c.e.. There is a strongly c.e. real that is not sw-below α, and hence not S-below α.*

Proof. The argument is nonuniform, but is still finite injury. Since sw-reducibility and S-reducibility coincide for strongly c.e. reals, it is enough to build a strongly c.e. real that is not sw-below α. Let A be such that $\alpha = 0.\chi_A$. We build c.e. sets B and C to satisfy the following requirements.

$$\mathcal{R}_{e,i} : \Gamma_e^A \neq B \vee \Gamma_i^A \neq C,$$

where Γ_e is the eth wtt reduction with use less than $x + e$. It will then follow that either $0.\chi_B \nleqslant_{\mathrm{sw}} \alpha$ or $0.\chi_C \nleqslant_{\mathrm{sw}} \alpha$.

The idea for satisfying a single requirement $\mathcal{R}_{e,i}$ is simple. Let $l(e,i,s) = \max\{x : \forall y \leqslant x(\Gamma_{e,s}^{A_s}(y) = B_s(y) \wedge \Gamma_{i,s}^{A_s} = C_s(y))\}$. Pick a large number $k >> e, i$ and let $\mathcal{R}_{e,i}$ assert control over the interval $[k, 3k]$ in both B and C, waiting until a stage s such that $l(e,i,s) > 3k$.

First work with C. Put $3k$ into C, and wait for the next stage s' where $l(e,i,s') > 3k$. Note that some number must enter $A_{s'} - A_s$ below $3k+i$. Now repeat with $3k-1$, then $3k-2, \ldots, k$. In this way, $2k$ numbers are made to enter A below $3k+i$. Now we can win using B, by repeating the process and noticing that, by the choice of the parameter k, A cannot respond another $2k$ times below $3k+e$.

The theorem now follows by a standard application of the finite injury method. □

Some structural properties are much easier to prove for sw-reducibility than for S-reducibility. One example is the fact that there are no minimal sw-degrees of c.e. reals, that is, that for any noncomputable c.e. real α there is a c.e. real strictly sw-between α and the computable reals. The analogous property for S-reducibility was proved by Downey, Hirschfeldt, and Nies [18] with a fairly involved priority argument.

Definition 2.11. *Let A be a nearly c.e. set. The* sw-canonical c.e. set A^* *associated with A is defined as follows. Begin with $A_0^* = \emptyset$. For all x and s, if either $x \notin A_s$ and $x \in A_{s+1}$, or $x \in A_s$ and $x \notin A_{s+1}$, then for the least j with $\langle x, j\rangle \notin A_s^*$, put $\langle x, j\rangle$ into A_{s+1}^*.*

Lemma 2.12. *$A^* \leqslant_{\mathrm{sw}} A$ and $A \leqslant_{\mathrm{tt}} A^*$.*

Proof. Since A is nearly c.e., $\langle x, j\rangle$ enters A^* at a given stage only if some $y \leqslant x$ enters A at that stage. Such a y will also be below $\langle x, j\rangle$. Hence $A^* \leqslant_{\mathrm{sw}} A$ with use x. Clearly, $x \in A$ if and only if A^* has an odd number of entries in row x, and furthermore, since A is nearly c.e., the number of entries in this row is bounded by x. Hence $A \leqslant_{\mathrm{tt}} A^*$. □

Corollary 2.13. *If A is nearly c.e. and noncomputable then there is a noncomputable c.e. set $A^* \leqslant_{\mathrm{sw}} A$.*

Corollary 2.14. *There are no minimal sw-degrees of c.e. reals.*

Proof. Let A be nearly c.e. and noncomputable. Then $A^* \leqslant_{\mathrm{sw}} A$ is noncomputable, and we can c.e. Sacks split A^* into two disjoint c.e. sets A_1^* and A_2^* of incomparable Turing degree. Note that $A_i^* \leqslant_{sw} A^*$. (To decide whether $x \in A_i^*$, ask whether $x \in A^*$ and, if the answer is yes, then run the enumerations of A_1^* and A_2^* to see which set x enters.) So $\emptyset <_{\mathrm{sw}} A_1^* <_{\mathrm{sw}} A^* \leqslant_{\mathrm{sw}} A$. □

Actually, while the above proof yields more than just nonminimality, there is an easier proof that the sw-degrees of c.e. reals have no minimal members. Given a c.e. real $A = 0.a_1a_2\ldots$, consider the c.e. real $B = 0.a_1 0 a_2 00 a_3 000 a_4 \ldots$. It is easy to prove that if A is noncomputable then so is B. But it is also easy to see that $B \leqslant_{\text{sw}} A$, and that if it were the case that $A \leqslant_{\text{sw}} B$ then A would be computable. Hence $\emptyset <_{\text{sw}} B <_{\text{sw}} A$.

One thing we can get out of the proof of Corollary 2.14 is that every c.e. real has a noncomputable strongly c.e. real sw-below it. The same is not true for S-reducibility.

Theorem 2.15. *There is a noncomputable c.e. real α such that all strongly c.e. reals dominated by α are computable.*

Proof. We begin by noting the following lemma, proved in [18].

Lemma 2.16. *Let $\beta \leqslant_{\text{S}} \alpha$ be c.e. reals. There are a c.e. real γ and a positive $c \in \mathbb{Q}$ such that $\alpha = c\beta + \gamma$.*

A c.e. set $A \subseteq \{0,1\}^*$ *presents* a c.e. real α if A is prefix-free and

$$\alpha = \sum_{\sigma \in A} 2^{-|\sigma|}.$$

In [20], Downey and LaForte constructed a noncomputable c.e. real α such that if A presents α then A is computable. We claim that, for this α, if $\beta \leqslant_{\text{S}} \alpha$ is strongly c.e. then β is computable.

To verify this claim, let $\beta \leqslant_{\text{S}} \alpha$ be strongly c.e.. By Lemma 2.16, there is a positive $c \in \mathbb{Q}$ such that $\alpha = c\beta + \gamma$. Let $k \in \omega$ be such that $2^{-k} \leqslant c$ and let $\delta = \gamma + (c - 2^{-k})\beta$. Then δ is a c.e. real such that $\alpha = 2^{-k}\beta + \delta$.

It is easy to see that there exist computable sequences of natural numbers $b_0, b_1, \dots$ and $d_0, d_1, \dots$ such that $2^{-k}\beta = \sum_{i\in\omega} 2^{-b_i}$ and $\delta = \sum_{i\in\omega} 2^{-d_i}$. Furthermore, since β is strongly c.e., so is $2^{-k}\beta$, and hence we can choose $b_0, b_1, \dots$ to be pairwise distinct, so that the nth bit of the binary expansion of $2^{-k}\beta$ is 1 if and only if $n = b_i$ for some i.

Since $\sum_{i\in\omega} 2^{-b_i} + \sum_{i\in\omega} 2^{-d_i} = 2^{-k}\beta + \delta = \alpha < 1$, Kraft's inequality tells us that there is a prefix-free c.e. set $A = \{\sigma_0, \sigma_1, \dots\}$ such that $|\sigma_0| = b_0$, $|\sigma_1| = d_0$, $|\sigma_2| = b_1$, $|\sigma_3| = d_1$, etc.. Now $\sum_{\sigma\in A} 2^{-|\sigma|} = \sum_{i\in\omega} 2^{-b_i} + \sum_{i\in\omega} 2^{-d_i} = \alpha$, and thus A presents α.

By our choice of α, this means that A is computable. But now we can compute the binary expansion of $2^{-k}\beta$ as follows. Given n, compute the number m of strings of length n in A. If $m = 0$ then $b_i \neq n$ for all i, and hence the nth bit of binary expansion of $2^{-k}\beta$ is 0. Otherwise, run through the b_i and d_i until either $b_i = n$ for some i or $d_{j_1} = \cdots = d_{j_m} = n$ for some $j_1 < \cdots < j_m$. By the definition of A, one of the two cases must happen. In the first case, the nth bit of the binary expansion of $2^{-k}\beta$ is 1. In the second case, $b_i \neq n$ for all i, and hence the nth bit of the binary expansion of $2^{-k}\beta$ is 0. Thus $2^{-k}\beta$ is computable, and hence so is β. ∎

As we have seen, in some ways the sw-degrees are nicer than the S-degrees. Unfortunately, the theorem below shows that this is not always the case. There is a simple join operator, arithmetic addition, which induces a join operation on the S-degrees. No such operation exists for the sw-degrees.

Theorem 2.17. *There exist nearly c.e. sets A and B such that for all nearly c.e. $W \geqslant_{\mathrm{sw}} A, B$ there is a nearly c.e. Q with $A, B \leqslant_{\mathrm{sw}} Q$ but $W \nleqslant_{\mathrm{sw}} Q$. Thus the sw-degrees of c.e. reals do not form an uppersemilattice.*

Proof. We build A, B, and W in stages, to meet the following requirements.

$$\mathcal{R}_e : (\Gamma_e^{W_e} = A \wedge \Delta_e^{W_e} = B) \Rightarrow \exists Q_e(A, B \leqslant_{\mathrm{sw}} Q_e \wedge W_e \nleqslant_{\mathrm{sw}} Q_e).$$

Here we assume that each Γ_e and Δ_e is an sw procedure with use bounded by $x+e$, and that the triples $\langle \Gamma_e, \Delta_e, W_e \rangle$ run through all triples consisting of a pair of such procedures together with a nearly c.e. set W_e. The above requirements are broken into subrequirements

$$\mathcal{R}_{e,i} : (\Gamma_e^{W_e} = A \wedge \Delta_e^{W_e} = B) \Rightarrow \exists Q_e(A, B \leqslant_{\mathrm{sw}} Q_e \wedge \Phi_i^{Q_e} \neq W_e),$$

where each Φ_i is an sw procedure with use bounded by $x+i$ and the Φ_i run over all such procedures.

Actually, the argument is *nonuniform*. We *really* construct sets Q_e together with backup sets $Q_{e,i}$ and meet the requirements

$$\mathcal{R}_{e,i} : (\Gamma_e^{W_e} = A \wedge \Delta_e^{W_e} = B) \Rightarrow$$
$$(A, B \leqslant_{\mathrm{sw}} Q_e \wedge A, B \leqslant_{\mathrm{sw}} Q_{e,i} \wedge (\Phi_i^{Q_e} = W_e \Rightarrow \Phi_j^{Q_{e,i}} \neq W_e)).$$

These naturally have subrequirements $\mathcal{R}_{e,i,j}$ trying to make $\Phi_i^{Q_e} \neq W_e$ or $\Phi_j^{Q_{e,i}} \neq W_e$.

The argument is a finite injury one, and hence it suffices to give the strategy for a single $\mathcal{R}_{e,i,j}$. The idea is the following. For a single $\mathcal{R}_{e,i,j}$, one picks a killing point n, which is large and fresh. If this happens at stage s then choosing $n = s$ would suffice with the standard use conventions. We may assume that $e, i, j << n$ and $e < i < j$.

Now the idea is that $\mathcal{R}_{e,i,j}$ will control the region $[n, (2j+1)n^2]$ of both A and B. We assume by priorities that the regions below n have ceased changing.

The key observation is the following. Suppose that we wish to kill $\Phi_i^{Q_e} = W_e$ or $\Phi_j^{Q_{e,i}} = W_e$. We need to have a situation where, through our changing A or B, we cause W_e to have to change on some m, while Q_e or $Q_{e,i}$ changes only on $k > m+i$ or $k > m+j$, respectively. However, W_e is not really under our control. But suppose that using *only B changes* we can get to a situation where W_e has a block of $2j+1$ consecutive 1's. That is, at stage s, we have $(\Phi_i^{Q_e}(z) = W_e(z))[s]$ and $(\Phi_j^{Q_{e,i}}(z) = W_e(z))[s]$ for all $z \leqslant m+j+1$, where $[m-j, m+j+1] \subseteq W_{e,s}$. (Here, m is the central number in the interval.) Further assume that the stage is e-expansionary, that is, $l(e,s) > \max\{l(e,t) : t < s\}$ and $l(e,s) > m+j+1$, where

$$l(e,s) = \max\{z : \forall y \leqslant z((\Gamma_e^{W_e}(y) = A(y) \wedge \Delta_e^{W_e}(y) = B(y))[s]\}.$$

Then we can win as follows.

Step 1. First we put some small number $p << m - i$ into $Q_e[s+1]$ and take all the numbers bigger than p (including, in particular, the interval $[m-i, m+i+1]$) out of $Q_e[s+1]$. We do *not*, however, change $Q_{e,i}$.

Step 2. Then we wait for the length of agreement to recover. That is, we wait for an e-expansionary stage $t > s$ such that $(\Phi_i^{Q_e}(z) = W_e(z))[t]$ and $(\Phi_j^{Q_{e,i}}(z) = W_e(z))[t]$ for all $z \leqslant m + j + 1$. Since we have not changed $Q_{e,i}$ between stages s and t, we have $W_e[s] \upharpoonright m + j + 1 = W_e[t] \upharpoonright m + j + 1$.

We can now win by putting m into A, Q_e, and $Q_{e,i}$. Since W_e is supposedly above both A and B via Γ_e and Δ_e, respectively, W_e must change below $m+e < m+j$. Because W_e is nearly c.e. and contains the whole interval $[m-j, m+j+1]$, such a change can only occur *below* $m - j$. Thus some $p < m - j$ must enter W_e. But supposedly $\Phi^{Q_e}(p) = W_e(p)$. Therefore Q_e should have changed in the region below $p + j$, which it did not.

The conclusion is that one of the equalities is wrong.

Thus if we ever see a situation where, at some e, i, j expansionary stage, W_e contains a full interval $[j - m, m + j + 1]$ with the end points between n and $(2j + 1)n^2$ then we are done.

We must now deal with the case in which such a good block never occurs. We think of the argument to follow as an *entropy* one. The idea is that if W_e never contains a block of the appropriate size then it cannot change as often as we can change B, and hence we can ensure that W_e is not sw-above B.

We cycle through B configurations as follows, using the B changes to induce changes in W_e. At an e-expansionary stage s, we put $b_1 = (2j + 1)n^2 - j$ into B. We wait until the next e-expansionary stage $s_1 > s$. Note that W_e must have changed between stages s and s_1, and indeed a number must have entered W_e below $(2j + 1)n^2 - j + e$, and hence below $(2j + 1)n^2$. Now we can repeat. We put $b_1 - 1$ into B, take b_1 out of B, and wait for the next e-expansionary stage $s_2 > s_1$, at which point there will have been another change in W_e below $(2j + 1)n^2$. We keep repeating this: we next put b_1 into B again; at the next e-expansionary stage, we put $b_1 - 2$ into B and take out $b_1 - 1$ and b_1. We continue until we have put the whole block $[n + j, (2j + 1)n^2 - j]$ into B. Our assumption is that, throughout this entire procedure, we never get a large block of consecutive 1's in W_e.

To keep $A, B \leqslant_{\mathrm{sw}} Q_e, Q_{e,i}$, we copy what we do to B into Q_e and $Q_{e,i}$. These will be the only changes to these sets below $(2j+1)n^2$, unless we see the desired block of 1's in W_e. Notice also that W_e will not change below n throughout this procedure, since otherwise the e, i, j computations could not recover. (Any $p < n$ entering W_e would require a change in the Q sets below $p + j < n + j$.)

The above procedure allows us to make $2^{(2j+1)n^2-n-2j}$ changes to B between $n + j$ and $(2j + 1)n^2 - j$. If W_e is sw-above B then it must change in response to each of these changes. We compute an upper bound on how many times W_e can change in the interval $[n, (2j+1)n^2]$, assuming that it has no block of $2j+1$ many 1's in that interval.

We can split $[n, (2j+1)n^2]$ into less than n^2 consecutive blocks of size $2j+1$. For each W_e configuration at an e-expansionary stage, each of these intervals must contain at least one 0. For each such interval, it follows that there are only 2^{2j} possible configurations of that interval that can be realized. This gives W_e a maximum of $(2^{2j})^{n^2} = 2^{2n^2 j}$ possible configurations in the interval $[n, (2j+1)n^2]$. But since $n >> j$, which implies that $n^2 > n - 2j$, we have $2n^2 j < (2j+1)n^2 - n - 2j$. This means that W_e cannot change as often in the interval $[n, (2j+1)n^2]$ as we can change B in the interval $[n+j, (2j+1)n^2 - j]$, and hence we can force it to be the case that $B \nleqslant_{\mathrm{sw}} W_e$.

A standard application of the finite injury priority method completes the proof. □

The lack of a join operation leads to difficulties in exploring the structure of the sw-degrees beyond what is done here, and is one of the motivations for the introduction of rH-reducibility in the following section.

3 Relative *H* Reducibility

Both S-reducibility and sw-reducibility are uniform in a way that relative initial-segment complexity is not. This makes them too strong, in a sense, and it is natural to wish to investigate nonuniform versions of these reducibilities. Motivated by this consideration, as well as by the problems with sw-reducibility, we introduce another measure of relative randomness, called relative H reducibility, which can be seen as a nonuniform version of both S-reducibility and sw-reducibility, and which combines many of the best features of these reducibilities. Its name derives from a characterization, discussed below, which shows that there is a very natural sense in which it is an *exact* measure of relative randomness.

Definition 3.1. *Let α and β be reals. We say that β is* relative H reducible *(rH-reducible) to α, and write $\beta \leqslant_{\mathrm{rH}} \alpha$, if there are a constant k and a partial computable binary function f such that for each n there is a $j \leqslant k$ for which $f(\alpha \upharpoonright n, j)\downarrow = \beta \upharpoonright n$.*

Since rH-reducibility is reflexive and transitive, we can define the *rH-degree* $\deg_{\mathrm{rH}}(\alpha)$ of a real α to be its rH-equivalence class.

There are several characterizations of rH-reducibility, each revealing a different facet of the concept. We mention three, beginning with a "relative entropy" characterization whose proof is quite straightforward. For a c.e. real β and a fixed computable approximation $\beta_0, \beta_1, \dots$ of β, we will let the mind-change function $m(\beta, n, s, t)$ be the cardinality of $\{u \in [s,t] : \beta_u \upharpoonright n \neq \beta_{u+1} \upharpoonright n\}$.

Proposition 3.2. *Let α and β be c.e. reals. The following condition holds if and only if $\beta \leqslant_{\mathrm{rH}} \alpha$. There are a constant k and computable approximations $\alpha_0, \alpha_1, \dots$ and $\beta_0, \beta_1, \dots$ of α and β, respectively, such that for all n and $t > s$, if $\alpha_t \upharpoonright n = \alpha_s \upharpoonright n$ then $m(\beta, n, s, t) \leqslant k$.*

The following is a more analytic characterization of rH-reducibility, which clarifies its nature as a nonuniform version of both S-reducibility and sw-reducibility.

Proposition 3.3. *For any reals α and β, the following condition holds if and only if $\beta \leqslant_{\mathrm{rH}} \alpha$. There are a constant c and a partial computable function φ such that for each n there is a τ of length $n+c$ with $|\alpha - \tau| \leqslant 2^{-n}$ for which $\varphi(\tau)\downarrow$ and $|\beta - \varphi(\tau)| \leqslant 2^{-n}$.*

Proof. First suppose that $\beta \leqslant_{\mathrm{rH}} \alpha$ and let f and k be as in Definition 3.1. Let c be such that $2^c \geqslant k$ and define the partial computable function φ as follows. Given a string σ of length n, whenever $f(\sigma, j)\downarrow$ for some new $j \leqslant k$, choose a new $\tau \supseteq \sigma$ of length $n+c$ and define $\varphi(\tau) = f(\sigma, j)$. Then for each n there is a $\tau \supseteq \alpha \upharpoonright n$ such that $\varphi(\tau)\downarrow= \beta \upharpoonright n$. Since $|\alpha - \tau| \leqslant |\alpha - \alpha \upharpoonright n| \leqslant 2^{-n}$ and $|\beta - \beta \upharpoonright n| \leqslant 2^{-n}$, the condition holds.

Now suppose that the condition holds. For a string σ of length n, let S_σ be the set of all μ for which there is a τ of length $n+c$ with $|\sigma - \tau| \leqslant 2^{-n+1}$ and $|\mu - \varphi(\tau)| \leqslant 2^{-n+1}$. It is easy to check that there is a k such that $|S_\sigma| \leqslant k$ for all σ. So there is a partial computable binary function f such that for each σ and each $\mu \in S_\sigma$ there is a $j \leqslant k$ with $f(\sigma, j)\downarrow= \mu$. But, since for any real γ and any n we have $|\gamma - \gamma \upharpoonright n| \leqslant 2^{-n}$, it follows that for each n we have $\beta \upharpoonright n \in S_{\alpha \upharpoonright n}$. Thus f and k witness the fact that $\beta \leqslant_{\mathrm{rH}} \alpha$. □

The most interesting characterization of rH-reducibility (and the reason for its name) is given by the following result, which shows that there is a very natural sense in which rH-reducibility is an exact measure of relative randomness. Recall that the prefix-free complexity $H(\tau \mid \sigma)$ of τ relative to σ is the length of the shortest string μ such that $M^\sigma(\mu)\downarrow= \tau$, where M is a fixed self-delimiting universal computer.

Theorem 3.4. *Let α and β be reals. Then $\beta \leqslant_{\mathrm{rH}} \alpha$ if and only if there is a constant c such that $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant c$ for all n.*

Proof. First suppose that $\beta \leqslant_{\mathrm{rH}} \alpha$ and let f and k be as in Definition 3.1. Let m be such that $2^m \geqslant k$ and let $\tau_0, \dots, \tau_{2^m-1}$ be the strings of length m. Define the prefix-free machine N to act as follows with σ as an oracle. For all strings μ of length not equal to m, let $N^\sigma(\mu)\uparrow$. For each $i < 2^m$, if $f(\sigma, i)\downarrow$ then let $N^\sigma(\tau_i)\downarrow= f(\sigma, i)$, and otherwise let $N^\sigma(\tau_i)\uparrow$. Let e be the coding constant of N and let $c = e + m$. Given n, there exists a $j \leqslant k$ for which $f(\alpha \upharpoonright n, j)\downarrow= \beta \upharpoonright n$. For this j we have $N^{\alpha \upharpoonright n}(\tau_j)\downarrow= \beta \upharpoonright n$, which implies that $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant |\tau_j| + e \leqslant c$.

Now suppose that $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant c$ for all n. Let $\tau_0, \dots, \tau_k$ be a list of all strings of length less than or equal to c and define f as follows. For a string σ and a $j \leqslant k$, if $M^\sigma(\tau_j)\downarrow$ then $f(\sigma, j)\downarrow= M^\sigma(\tau_j)$, and otherwise $f(\sigma, j)\uparrow$. Given n, since $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant c$, it must be the case that $M^{\alpha \upharpoonright n}(\tau_j)\downarrow= \beta \upharpoonright n$ for some $j \leqslant k$. For this j we have $f(\alpha \upharpoonright n, j)\downarrow= \beta \upharpoonright n$. Thus $\beta \leqslant_{\mathrm{rH}} \alpha$. □

An immediate consequence of this result is that rH-reducibility satisfies the Solovay property.

Corollary 3.5. *If $\beta \leqslant_{\mathrm{rH}} \alpha$ then there is a constant c such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + c$ for all n.*

On the other hand, the converse of this corollary is not true even for strongly c.e. reals. This follows from Theorem 3.10 below and a result of Zambella [55], who showed, using a technique due to Solovay [50], that there is a noncomputable strongly c.e. real β such that for some c we have $H(\beta \upharpoonright n) \leqslant H(n) + c$ for all n.

The next two results, which show that rH-reducibility is a common weakening of S-reducibility and sw-reducibility, follow easily from Proposition 3.3.

Proposition 3.6. *Let α and β be c.e. reals. If $\beta \leqslant_{\mathrm{S}} \alpha$ then $\beta \leqslant_{\mathrm{rH}} \alpha$.*

Corollary 3.7. *A c.e. real α is rH-complete if and only if it is random.*

Proposition 3.8. *If $\beta \leqslant_{\mathrm{sw}} \alpha$ then $\beta \leqslant_{\mathrm{rH}} \alpha$.*

Theorems 2.4 and 2.5 show that the converses of Propositions 3.6 and 3.8 do not hold, but even among strongly c.e. reals, where S-reducibility and sw-reducibility agree, rH-reducibility is not equivalent to its stronger counterparts.

Theorem 3.9. *There exist strongly c.e. reals α and β such that $\beta \leqslant_{\mathrm{rH}} \alpha$ but $\beta \nleqslant_{\mathrm{sw}} \alpha$ (equivalently, $\beta \nleqslant_{\mathrm{S}} \alpha$).*

Proof. We build c.e. sets A and B to satisfy the following requirements.

$$\mathcal{R}_e : \Gamma_e^A \neq B,$$

where Γ_e is the eth wtt reduction with use less than $x + e$. We think of α and β as $0.\chi_A$ and $0.\chi_B$, respectively, and we build A and B in such as way as to enable us to apply Proposition 3.2 to conclude that $\beta \leqslant_{\mathrm{rH}} \alpha$.

The construction is a standard finite injury argument. We discuss the satisfaction of a single requirement $\mathcal{R}_e$. For the sake of this requirement, we choose a large n, restrain n from entering B, and restrain $n+e+1$ from entering A. If we find a stage s such that $\Gamma_{e,s}^{A_s}(n)\downarrow = 0$ then we put n into B, put $n+e+1$ into A, and restrain the initial segment of A of length $n+e$. Unless a higher priority strategy acts at a later stage, this guarantees that $\Gamma_e^A(n) \neq B(n)$.

Furthermore, it is not hard to check that, because of the numbers that we put into A, for each n and $t > s$, if $\alpha_t \upharpoonright n = \alpha_s \upharpoonright n$ then $m(\beta, n, s, t) \leqslant 2$ (where $m(\beta, n, s, t)$ is as defined before Proposition 3.2). Thus, by Proposition 3.2, $\beta \leqslant_{\mathrm{rH}} \alpha$. □

It is interesting to note that, despite the nonuniform nature of its definition, rH-reducibility implies Turing reducibility. Since any computable real is obviously rH-reducible to any other real, this implies that the computable reals form the least rH-degree.

Theorem 3.10. *If $\beta \leqslant_{\mathrm{rH}} \alpha$ then $\beta \leqslant_{\mathrm{T}} \alpha$.*

Proof. Let k be the least number for which there exists a partial computable binary function f such that for each n there is a $j \leqslant k$ with $f(\alpha \upharpoonright n, j)\downarrow = \beta \upharpoonright n$. There must be infinitely many n for which $f(\alpha \upharpoonright n, j)\downarrow$ for all $j \leqslant k$, since otherwise we could change finitely much of f to contradict the minimality of

k. Let $n_0 < n_1 < \cdots$ be an α-computable sequence of such n. Let T be the α-computable subtree of 2^ω obtained by pruning, for each i, all the strings of length n_i except for the values of $f(\alpha \upharpoonright n_i, j)$ for $j \leqslant k$.

If γ is a path through T then for all i there is a $j \leqslant k$ such that γ extends $f(\alpha \upharpoonright n_i, j)$. Thus there are at most k many paths through T, and hence each path through T is α-computable. But β is a path through T, so $\beta \leqslant_{\mathrm{T}} \alpha$. □

On the other hand, by Theorem 2.5, S-reducibility does not imply wtt-reducibility, even among c.e. reals, and hence rH-reducibility does not imply wtt-reducibility.

Structurally, the rH-degrees of c.e. reals are nicer than the sw-degrees of c.e. reals.

Theorem 3.11. *The rH-degrees of c.e. reals form an uppersemilattice with least degree that of the computable sets and highest degree that of Ω. The join of the rH-degrees of the c.e. reals α and β is the rH-degree of $\alpha + \beta$.*

Proof. All that is left to show is that addition is a join. Since $\alpha, \beta \leqslant_{\mathrm{S}} \alpha + \beta$, it follows that $\alpha, \beta \leqslant_{\mathrm{rH}} \alpha + \beta$. Let γ be a c.e. real such that $\alpha, \beta \leqslant_{\mathrm{rH}} \gamma$. Then Proposition 3.2 implies that $\alpha + \beta \leqslant_{\mathrm{rH}} \gamma$, since for any n and $s < t$ we have $m(\alpha + \beta, n, s, t) \leqslant 2(m(\alpha, n, s, t) + m(\beta, n, s, t)) + 1$. □

In [18], Downey, Hirschfeldt, and Nies studied the structure of the S-degrees of c.e. reals. As mentioned in the introduction, they showed that the S-degrees of c.e. reals are dense. They also showed that every incomplete S-degree splits over any lesser degree, while the complete S-degree does not split at all. The methods of that paper can easily be adapted to prove the analogous results for rH-degrees of c.e. reals.

Theorem 3.12. *For any rH-degrees $\mathbf{a} < \mathbf{b}$ of c.e. reals there is an rH-degree $\mathbf{c}$ of c.e. reals such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.*

Theorem 3.13. *For any rH-degrees $\mathbf{a} < \mathbf{b} < \deg_{\mathrm{rH}}(\Omega)$ of c.e. reals, there are rH-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of c.e. reals such that $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ and $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.*

Theorem 3.14. *For any rH-degrees $\mathbf{a}, \mathbf{b} < \deg_{\mathrm{rH}}(\Omega)$ of c.e. reals, $\mathbf{a} \vee \mathbf{b} < \deg_{\mathrm{rH}}(\Omega)$.*

Thus we see that rH-reducibility shares many of the nice structural properties of S-reducibility on the c.e. reals, while still being a reasonable reducibility on non-c.e. reals. Together with its various characterizations, especially the one in terms of relative H-complexity of initial segments, this makes rH-reducibility a tool with great potential in the study of the relative randomness of reals.

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