Five papers on reverse mathematics and Ramsey-theoretic principles

C. T. Chong, Theodore A. Slaman, and Yue Yang. *The metamathematics of Stable Ramsey's Theorem for Pairs.* ***Journal of the American Mathematical Society***, vol. 27 no. 3 (2014), pp. 863–892.

Manuel Lerman, Reed Solomon, and Henry Towsner. *Separating principles below Ramsey's Theorem for Pairs.* ***Journal of Mathematical Logic***, vol. 13 no. 2 (2013), 1350007, 44 pp.

Jiayi Liu. RT^2_2 *does not imply* WKL_0. ***Journal of Symbolic Logic***, vol. 77 no. 2 (2012), pp. 609–620.

Lu Liu. *Cone avoiding closed sets.* ***Transactions of the American Mathematical Society***, vol. 367 no. 3 (2015), pp. 1609–1630.

Wei Wang. *Some logically weak Ramseyan theorems.* ***Advances in Mathematics***, vol. 261 (2014), pp. 1–25.

These five papers are all major advances in the reverse-mathematical and computability-theoretic analysis of combinatorial principles related to Ramsey's Theorem. Reverse mathematics seeks to calibrate the strength of theorems provable in the theory Z_2 of second-order arithmetic. Typically, given such a theorem T, one endeavors to find a subsystem S of Z_2 that is equivalent to T, in the sense that T is provable in S, but also each axiom of S is provable from T over a weak base system. This base system is usually RCA_0, which roughly corresponds to the practice of computable mathematics (and hence lends the area a distinctively computability-theoretic flavor). A celebrated phenomenon is that there are a few such subsystems that suffice to classify many theorems across mathematics, most famously the "big five" systems RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. Furthermore, these systems are linearly ordered by strength.

This is not the full picture, however. Combinatorics has proved to be a particularly rich source of theorems that fall outside the big five systems. An important example is Ramsey's Theorem for Pairs (and two colors), RT^2_2, which states that for any coloring of the unordered pairs of natural numbers into two colors, there is an infinite set H such that all pairs of elements of H have the same color. (Such a set is said to be homogeneous. The choice of number of colors does not matter in the context of this review.) The analogs of this principle for triples and larger tuples are all equivalent to ACA_0, which roughly corresponds to the existence of arithmetic sets, but RT^2_2 itself was shown by Seetapun (in a 1995 paper by Seetapun and Slaman) to be strictly weaker. (Here and below, all implications and nonimplications are over RCA_0.) On the other hand, Hirst (1987) used a computability-theoretic result of Jockusch (1972), combined with the low basis theorem, to show that RT^2_2 is not provable in WKL_0.

Thus it was natural to ask whether RT^2_2 implies WKL_0. There are several factors that made this an important question. The computability-theoretic study of versions of Ramsey's Theorem has been a fertile field since the work of Specker and especially Jockusch in the early 1970's (and in a sense even earlier, in the study of cohesive sets). Weak König's Lemma, the axiom added to RCA_0 to obtain WKL_0, states that every infinite binary tree has an infinite path. It is a way of expressing the compactness of spaces like the unit interval, and hence turns out to be equivalent to a host of theorems that make essential use of compactness arguments. RT^2_2 does not have many known equivalents, but there are several theorems not provable in RCA_0, or even in WKL_0, that follow from it, some of which will be discussed below. So establishing the precise connection between the world of WKL_0 and the world below RT^2_2 was a major task for reverse mathematics. While the results mentioned above showed that RT^2_2 does not fit neatly into the big-five picture, they left open the possibility that it might still fit into the linearly ordered collection of major systems that includes not only the big five but systems such as WWKL_0, which will be discussed below. Furthermore, Seetapun's result required a significant methodological development in the area, as did further analysis of the strength of RT^2_2 by Cholak, Jockusch, and Slaman (2001) and several others. The general feeling seemed to be that RT^2_2 does not imply WKL_0, but it soon became clear that strengthening Seetapun's result from ACA_0 to WKL_0 in this way would likely require another technical breakthrough.

This breakthrough comes in Liu's 2012 paper, which is likely the best work in computability theory by an undergraduate since the Friedberg-Muchnik Theorem in the 1950's. Liu's method is indeed computability-theoretic. It makes use of the separation of RT^2_2 into the principles COH and SRT^2_2 (which we will discuss further below) by Cholak, Jockusch, and Slaman. RT^2_2 is equivalent to the conjunction of these two principles, so it is often possible to obtain results about it by dealing with each of them separately. Typically, the SRT^2_2 case is the more difficult one. In computability theory, we can think of SRT^2_2 as the following problem: Given a Δ^0_2 set A, find an infinite subset of either A or its complement. (See below for more on theorems as problems.) The PA degrees (i.e., the Turing degrees of completions of Peano Arithmetic) are exactly the ones that can compute infinite paths on every computable infinite binary tree. Liu shows that if X does not have PA degree and A is any set (hence in particular any Δ^0_2 set), then there is an infinite set G contained in either A or its complement such that $G \oplus X$ still does not have PA degree. The proof requires an intricate forcing construction, using a considerable elaboration on the notion of Mathias forcing.

An ω-structure in the language of second-order arithmetic (which is really a two-sorted first-order language) is one whose first-order part is just the standard natural numbers, and hence is determined by which subsets of $\mathbb{N}$ it contains. Such a structure is an ω-model of RCA_0 if and only if it is a Turing ideal, i.e., is closed under joins and Turing reducibility. By iterating the relativized form of his computability-theoretic result and combining it with an analog for COH due to Cholak, Jockusch, and Slaman, Liu constructs an ω-model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ that is not a model of WKL_0. Building on this work, in his 2015 paper Liu improves his result to yield an ω-model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ that is not a model of WWKL_0. The latter system, intermediate in strength between RCA_0 and WKL_0, is important in the reverse-mathematical analysis of measure theory and related areas, and can be thought of as corresponding to the existence of Martin-Löf random sets. Indeed, Liu's 2015 paper also contains applications of his methods to the theory of algorithmic randomness.

The existence of these ω-models of course means that RT^2_2 does not imply WKL_0, or even WWKL_0, but it is in fact a stronger result. This point is highlighted by considering the relationship between RT^2_2 and SRT^2_2. The latter principle is the restriction of RT^2_2 to stable colorings of pairs, i.e., colorings c such that $\lim_y c(\{x,y\})$ exists for all x, and as mentioned above can also be thought of in terms of subsets of Δ^0_2 sets or their complements. There are several ways to show that the Cohesive Set Principle COH is strictly weaker than RT^2_2, but the question of whether SRT^2_2 implies RT^2_2 remained open since it was stated by Cholak, Jockusch, and Slaman. This question attracted the attention of many researchers, and led to several papers with partial and related results. One of its intriguing aspects is that there is a computability-theoretic sense in which SRT^2_2 is clearly weaker than RT^2_2: It is easy to show that every computable stable coloring of pairs has a Δ^0_2 infinite homogeneous set, but Jockusch showed that this is not always the case for non-stable colorings. This fact does not suffice to separate the two principles in the context of reverse mathematics, however, because a proof of RT^2_2 from SRT^2_2 could use multiple applications of the latter principle. As in the case of the previous question, it became clear that solving it would likely require a particularly fine understanding of the combinatorial subtleties of these deceptively simple-sounding principles.

The solution comes in Chong, Slaman, and Yang's paper. An early idea for separating RT^2_2 and SRT^2_2 was to try to show that SRT^2_2 always has low solutions, i.e., that every Δ^0_2 set has a low infinite subset of either it or its complement. Combining the relativization of this fact with Jockusch's result that there are computable 2-colorings of pairs with no Σ^0_2 infinite homogeneous sets would then lead to the construction of an ω-model of $RCA_0 + SRT^2_2$ that is not a model of RT^2_2. However, Downey, Hirschfeldt, Lempp, and Solomon (2001) built a Δ^0_2 set with no low subset of either it or its complement. This might have seemed like the end of this approach, but reverse mathematics does not always reduce to computability theory on the standard natural numbers. RCA_0 does not have the full induction axiom scheme, but only its restriction to Σ^0_1 formulas (indeed, its first-order part is Σ_1-PA, Peano Arithmetic with induction restricted to the Σ_1 formulas of first-order arithmetic), so it has models with nonstandard first-order parts. Chong, Slaman, and Yang define a carefully constructed model M of Σ_1-PA (and the principle of Σ_2-bounding, which does not hold in Σ_1-PA but is weaker than Σ_2-induction) and show that, in the sense of M, every Δ^0_2 set does indeed have a low infinite subset in either it or its complement. (Computability theory over models with restricted induction is of course different from standard computability theory, but a small amount of induction suffices to have many of the basic definitions and results available.)

This result allows Chong, Slaman, and Yang to build a model $\mathcal{M}$ of $RCA_0 + SRT^2_2$ with first order part M consisting entirely of sets that are low in the sense of M. Cholak, Jockusch, and Slaman showed that SRT^2_2 implies the Σ^0_2-bounding principle $B\Sigma^0_2$, and Jockusch's result on Σ^0_2 homogeneous sets is provable in $RCA_0 + B\Sigma^0_2$, so $\mathcal{M}$ is not a model of RT^2_2. It follows that $RCA_0 + SRT^2_2 \nvdash RT^2_2$. With some further work, Chong, Slaman, and Yang show that in fact $WKL_0 + SRT^2_2 \nvdash RT^2_2$. Whether RT^2_2 and SRT^2_2 can be separated in the context of ω-models remains an open question, and continues to generate significant partial results.

Another important feature of $\mathcal{M}$ is that it does not satisfy the Σ^0_2-induction principle $I\Sigma^0_2$, and hence SRT^2_2 does not imply $I\Sigma^0_2$. Chong, Slaman, and Yang (to appear) have since improved this result to show that RT^2_2 does not imply $I\Sigma^0_2$, and Patey and Yokoyama (to appear) have gone even further by showing that RT^2_2 is a Π^0_3-conservative extension of $I\Sigma^0_1$.

As mentioned above, there are many principles not provable in RCA_0 that follow from RT^2_2 (often with quite natural proofs). In addition to SRT^2_2 and COH, examples include the Chain / Antichain Principle CAC, which states that every infinite partial order has an infinite chain or antichain; the Ascending / Descending Sequence Principle ADS, which states the every infinite linear order has an infinite ascending or descending sequence; and the Erdős-Moser Principle EM, which states that every infinite tournament has an infinite transitive subtournament, i.e. that if T is an irreflexive binary relation on $\mathbb{N}$ such that if $x \neq y$ then exactly one of $T(x,y)$ and $T(y,x)$ holds, then there is an infinite set S such that the restriction of T to S is transitive.

Principles like these often have complex computability-theoretic and reverse-mathematical relationships, which are in some cases quite difficult to establish. For example, while it is not difficult to show that CAC implies ADS and that RT^2_2 implies EM, it was not known whether these implications reverse. Existing methods seemed insufficient to settle these questions, particularly in the latter case.

Lerman, Solomon, and Towsner's paper answers both questions by producing ω-models that separate CAC from ADS and RT^2_2 from EM. These models are obtained in a remarkable way, as we now discuss. All of the principles discussed here have the form $\forall X(\Phi(X) \to \exists Y \Psi(X,Y))$, where Φ and Ψ are arithmetic properties. We can think of such a principle P as a problem, where an instance is an X such that $\Phi(X)$ holds, and a solution to this instance is a Y such that $\Psi(X, Y)$ holds. Let P and Q be two such principles. Typically, to build an ω-model of $\mathrm{RCA}_0 + P$ that is not a model of Q, one begins with a computable instance X of Q with no computable solutions. Usually X is chosen so that its solutions are particularly far from computable in some appropriate sense. One then shows that each computable instance of P has a solution that does not compute any solutions to X, often via a forcing argument. Relativizing this result allows one to build a Turing ideal I in a step-by-step manner, adding to I a solution to each instance of P in I, without at any point adding a solution to X. This ideal determines an ω-model of $\mathrm{RCA}_0 + P$ that is not a model of Q.

This is the procedure followed in Liu's proof that RT^2_2 does not imply WKL_0, for example, with X taken to be a computable infinite binary tree each of whose paths has PA degree. In both of their proofs, however, Lerman, Solomon and Towsner need considerably more from the "bad instance" X than just having difficult solutions. In each case, for their iterated forcing construction to work, they first need to build X carefully (and noncomputably) using a separate forcing construction. The methods developed in this paper are likely to continue to be useful, as already demonstrated in the work of Patey.

The key computability-theoretic result behind Seetapun's proof that RT^2_2 does not imply ACA_0 is that RT^2_2 is cone-avoiding, in the sense that if X is an A-computable instance of RT^2_2 and $B \nleq_{\mathrm{T}} A$, then X has a solution Y such that $B \nleq_{\mathrm{T}} A \oplus Y$. Taking $B = \emptyset'$ and applying the iterative model-building procedure discussed above results in an ω-model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ that does not contain $\emptyset'$, and hence cannot be a model of ACA_0. As mentioned above, the versions RT^n_2 of Ramsey's Theorem for n-tuples with $n > 2$ are all equivalent to ACA_0, and hence are not cone-avoiding. One might expect similar behavior from other Ramsey-theoretic principles. For example, the Free Set Theorem for n-tuples, FS^n, states that for every coloring c of the unordered n-tuples of natural numbers into infinitely many colors, there is an infinite set F such that if $x_0, \dots, x_{n-1}$ are distinct elements of F, then $c(\{x_0, \dots, x_{n-1}\}) \notin F \setminus \{x_0, \dots, x_{n-1}\}$. FS^n follows from RT^n_2, so FS^2 does not imply ACA_0, but the situation for higher exponents was not clear.

Wang's paper solves this problem, which was left open in the work of Cholak, Giusto, Hirst, and Jockusch (2005), by showing that FS^n is cone-avoiding for every n, and hence full $\mathrm{FS} \equiv \forall n\, \mathrm{FS}^n$ does not imply ACA_0. Indeed, Wang shows that FS is strongly cone-avoiding, in the sense that if $B \nleq_{\mathrm{T}} A$ and X is any instance of FS, then X has a solution Y such that $B \nleq_{\mathrm{T}} A \oplus Y$. The point here is that there is no requirement that X be A-computable, and hence the cone-avoidance works even for instances X that compute B. Wang obtains the same results for the Thin Set Theorem, which was known to follow from from FS, and for the Rainbow Ramsey Theorem, which he shows also follows FS; as well as for the Achromatic Ramsey Theorem of Erdős, Hajnal, and Rado. Like FS, each of these principles involves colorings of tuples of a fixed size. Thus the phenomenon highlighted by Wang—the existence of natural Ramsey-theoretic principles that are weak in the sense that, unlike Ramsey's Theorem itself, their full versions for arbitrarily large tuples fail to imply ACA_0—seems quite widespread. Wang's arguments use cohesive/stable decompositions like the one for RT^2_2 discussed above, and Mathias forcing with the complexity of generic sets controlled by Π^0_1 classes. In addition to settling several open questions in a compelling way, this paper suggests new computability-theoretic connections between various Ramsey-theoretic principles that seem well worth exploring.

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