An uncountably categorical theory whose only computably presentable model is saturated

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Abstract

We build an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose only computably presentable model is the saturated one. As a tool, we introduce a notion related to limitwise monotonic functions.

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1 Introduction

An important theme in computable model theory is the study of computable models of complete first-order theories. More precisely, given a complete first-order theory T, one would like to know which models of T have computable copies and which do not. A special case of interest is when T is an $\aleph_1$-categorical theory. In this paper we are interested in computable models of $\aleph_1$-categorical theories, and we always assume that these theories are not $\aleph_0$-categorical. In addition, since we are interested in computable models, all the structures in this paper are countable.

We assume that all languages we consider are computable. A complete theory T in a language L is $\aleph_1$-*categorical* if any two models of T of power $\aleph_1$ are isomorphic. We say that a model $\mathcal{A}$ of T is *computable* if its domain and its atomic diagram are computable. A model $\mathcal{A}$ is *computably presentable* if it is isomorphic to a computable model, which is called a *computable presentation* of $\mathcal{A}$. The reader is referred to [2] for the basics of computable model theory and to [12] for the basics of computability theory.

In [1], Baldwin and Lachlan developed the theory of $\aleph_1$-categoricity in terms of strongly minimal sets. They showed that the countable models of an $\aleph_1$-categorical theory T can be listed in an $\omega + 1$ chain

$$\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega,$$

where the embeddings are elementary, $\mathcal{A}_0$ is the prime model of T, and $\mathcal{A}_\omega$ is the saturated model of T. Based on the theory developed by Baldwin and Lachlan, Harrington [4] and N. G. Khisamiev [5] proved that if an $\aleph_1$-categorical theory T is decidable then all the countable models of T have computable presentations. Thus, for decidable $\aleph_1$-categorical theories the question of which models of T have computable presentations is fully settled. However, the situation is far from clear when the theory T is not decidable. The following definition is given in [9]:

1.1 Definition. Let T be an $\aleph_1$-categorical theory and let $\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega$ be the countable models of T. The *spectrum of computable models* of T is the set $\{i : \mathcal{A}_i \text{ has a computable presentation}\}$.

If $X \subseteq \omega + 1$ is the spectrum of computable models of some $\aleph_1$-categorical theory, then we say that X is *realized as a spectrum*.

There has been some previous work on the possible spectra of computable models of (undecidable) $\aleph_1$-categorical theories. For example, Nies [11] gave an upper bound of $\Sigma^0_3(\emptyset^\omega)$ for the complexity of the sets realized as spectra. Interestingly, the following are the only subsets of $\omega+1$ known to be realizable as spectra: the empty set, $\omega+1$ itself ([4], [5]), the initial segments $\{0,\dots,n\}$, where $n \in \omega$ ([3], [10]), the sets $(\omega+1) \setminus \{0\}$ and ω ([9]), and the intervals $\{1,\dots,n\}$, where $n \in \omega$ ([11]). Our main result adds $\{\omega\}$ to this list by showing that there exists an $\aleph_1$-categorical theory whose only computably presentable model is the saturated one.

This paper is organized as follows. The next section contains the proof of a computability-theoretic result that will be used in constructing the desired theory. In section 3 we introduce the basic building blocks of the models of this theory, which are called cubes. Finally, the last section contains the proof our main result.

2 A Computability-Theoretic Result

Limitwise monotonic functions were introduced by N. G. Khisamiev [6, 7, 8] and have found a number of applications in computable model theory. In particular, Khoussainov, Nies, and Shore [9] used them to show that $(\omega+1) \setminus \{0\}$ is realized as a spectrum. We now introduce a related notion.

Let $[\omega]^{<\omega}$ denote the collection of all finite sets of natural numbers, and let ∞ be a special symbol. We define the class of S-limitwise monotonic functions from ω to $[\omega]^{<\omega} \cup \{\infty\}$, where S is an infinite set. This class captures the idea of a family $A_0, A_1, \dots$ of uniformly c.e. sets, each of which is either finite or equal to S (represented by the symbol ∞), such that we can enumerate the set of i for which $A_i = S$.

2.1 Definition. Let S be an infinite set of natural numbers. An *S-limitwise monotonic function* is a function $f : \omega \to [\omega]^{<\omega} \cup \{\infty\}$ for which there is a computable function $g : \omega \times \omega \to [\omega]^{<\omega} \cup \{\infty\}$ such that

1. $f(n) = \lim_s g(n,s)$ for all n, and
2. for all $n, s \in \omega$, the following properties hold:
   (a) if $g(n, s+1) \neq \infty$ then $g(n,s) \subseteq g(n,s+1)$,
   (b) if $g(n,s) = \infty$ then $g(n,s+1) = \infty$, and

(c) if $g(n,s) \neq \infty$ and $g(n,s+1) = \infty$ then $g(n,s) \subset S$.

We refer to g as a *witness* to f being S-limitwise monotonic.

Note that if f is an S-limitwise monotonic function then its witness g can be chosen to be primitive recursive.

2.2 Definition. A collection of finite sets is *S-monotonically approximable* if it is equal to $\{f(n) : f(n) \neq \infty\}$ for some S-limitwise monotonic function f.

The main result of this section is the following computability-theoretic proposition, which shows that there is an infinite set S and a family of sets that is not S-monotonically approximable and has certain properties that will allow us to code it into a model of an $\aleph_1$-categorical structure.

2.3 Proposition. *There exists an infinite c.e. set S and uniformly c.e. sets $A_0, A_1, \ldots$ with the following properties:*

1. *each A_i is either finite or equal to S,*
2. *if $x \in S$ then $x \in A_i$ for almost all i,*
3. *if $x \notin S$ then $x \in A_i$ for only finitely many i,*
4. *if A_i is finite then there is a $k \in A_i$ such that $k \notin A_j$ for all $j \neq i$, and*
5. *$\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable.*

Proof. Let $g_0, g_1, \ldots$ be an effective enumeration of all primitive recursive functions from $\omega \times \omega$ to $\omega^{<\omega} \cup \{\infty\}$ such that for all $n, s \in \omega$, if $g_e(n, s+1) \neq \infty$ then $g(n,s) \subseteq g(n,s+1)$, and if $g(n,s) = \infty$ then $g(n,s+1) = \infty$.

We want to build S and $A_0, A_1, \ldots$ to satisfy 1–3 and the requirements $\mathcal{R}_e$ stating that if g_e is a witness to some function f being S-limitwise monotonic, then $\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable via f.

For each e, we define a procedure for enumerating A_e. We think of the procedures as alternating their steps, with the eth procedure taking place at stages of the form $\langle e, k\rangle$, which we call *e-stages*. All procedures may enumerate elements into S. The eth procedure is designed to satisfy $\mathcal{R}_e$ by ensuring that if g_e is a witness to some function f being S-limitwise monotonic and every $f(n) \neq \infty$ is equal to some A_i, then A_e is finite and not equal to $f(n)$ for any n. The eth procedure works as follows.

Let $A_e[s]$ and $S[s]$ denote the set of all numbers enumerated into A_e and S, respectively, by the end of stage s.

The main idea is to find an appropriate number n_e such that if $\lim_s g_e(n, s) = A_e$ for some n then $n = n_e$, and let $A_e[s]$ always contain an element not in $g_e(n_e, s)$, thus ensuring that either A_e is finite but $\lim_s g_e(n_e, s) \neq A_e$ or $g_e(n_e, s)$ is eternally playing catch-up, and hence does not come to a limit.

At the first e-stage s, put $\langle e, 0\rangle$, $\langle e, 1\rangle$, and all elements of $S[s]$ into A_e. Let $m_{e,s} = 1$ and let n_e be undefined. (For each e-stage t, we will let $m_{e,t}$ be the largest m such that $\langle e, m\rangle \in A_e[t]$.)

At any other e-stage s, proceed as follows. Let t be the previous e-stage. If n_e is undefined and there is an $n \leqslant s$ such that $g_e(n, s) = A_e[t]$, then let $n_e = n$. If n_e is now defined and $g_e(n_e, s) = A_e[t]$ then put $\langle e, m_{e,t} - 1\rangle$ into S, put $\langle e, m_{e,t} + 1\rangle$ and all elements of $S[s]$ into A_e, and let $m_{e,s} = m_{e,t} + 1$. Otherwise, let $m_{e,s} = m_{e,t}$ and do nothing else.

This finishes the description of the eth procedure. Running all procedures concurrently, as described above, we build a uniformly c.e. collection of sets $A_0, A_1, \ldots$ and a c.e. set S. Now our goal is to show that these sets satisfy the properties in the statement of the proposition.

Since at every stage s at which we put numbers into A_e, we put $S[s]$ into A_e and the second largest element of $A_e[s-1]$ into S, every infinite A_e is equal to S. This shows that the first property in the proposition holds.

Since for each e we put $S[s]$ into A_e, where s is the first e-stage, every element of S is in cofinitely many A_e. This shows that the second property in the proposition holds.

Since the only way a number of the form $\langle e, k\rangle$ can enter A_i for $i \neq e$ is if it first enters S, every number that is in infinitely many A_i must be in S. This shows that the third property in the proposition holds.

If A_e is finite, then $m = \lim_s m_{e,s}$ exists, and $\langle e, m\rangle$ is in A_e but not in A_j for $j \neq e$. This shows that the fourth property in the proposition holds.

We now show that the last property in the proposition holds. Assume for a contradiction that $\{A_i : |A_i| < \omega\} = \{f(n) : f(n) \neq \infty\}$ for some S-limitwise monotonic function f witnessed by g_e. Then n_e must eventually be defined, since otherwise A_e is finite but not in the range of f.

First suppose that $f(n_e) \neq \infty$. At the e-stage s_0 at which n_e is defined, $g_e(n_e, s_0)$ contains $\langle e, 0\rangle$ and $\langle e, 1\rangle$. If there is no e-stage $s_1 > s_0$ at which $g_e(n_e, s_1) = A_e[s_0]$, then $f(n_e)$ cannot equal any of the A_i, since A_e is then the only one of our sets that contains $\langle e, 1\rangle$, and $\langle e, 1\rangle \in g_e(n_e, s_0)$. So there must be such an e-stage s_1. Note that $g_e(n_e, s_1)$ contains $\langle e, 2\rangle$. By the same argument, there must be an e-stage $s_2 > s_1$ such that $g_e(n_e, s_2) = A_e[s_1]$, and this set contains $\langle e, 3\rangle$. Proceeding in this way, we see that $g_e(n_e, s)$ never reaches a limit.

Now suppose that $f(n_e) = \infty$. Let s_0 be the least s such that $g_e(n_e, s) = \infty$, and let t be the largest e-stage less than s_0. It is easy to check that $\langle e, m_{e,t} - 1\rangle \in g(n_e, t)$ but $\langle e, m_{e,t} - 1\rangle \notin S[t]$. We never put $\langle e, m_{e,t} - 1\rangle$ into S after stage t, so in fact $\langle e, m_{e,t} - 1\rangle \notin S$. Since $g_e(n_e, t) \subseteq g_e(n_e, s_0 - 1)$, we have $g_e(n_e, s_0 - 1) \not\subseteq S$, contradicting the choice of g_e. $\square$

3 Cubes

In this section we introduce a special family of structures, which we call cubes. These will be used in the next section to build an $\aleph_1$-categorical theory. They generalize the n-cubes and ω-cubes used in [9].

We work in the language $\mathcal{L} = \{P_i : i \in \omega\}$, where each P_i is a binary predicate symbol. We will define structures for sublanguages $\mathcal{L}'$ of $\mathcal{L}$. Any such structure can be thought of as an $\mathcal{L}$-structure by interpreting the P_i not contained in $\mathcal{L}'$ by the empty set. We denote the domain of a structure denoted by a calligraphic letter such as $\mathcal{A}$ by the corresponding roman letter A.

We begin with the following inductive definition of the finite cubes.

3.1 Definition. *Base case.* For $n \in \omega$, an (n)-*cube* is a structure $\mathcal{A} = (\{a, b\}; P_n^{\mathcal{A}})$, where $P_n^{\mathcal{A}}(x, y)$ holds if and only if $x \neq y$.

Inductive Step. Now suppose we have defined σ-cubes for a non-repeating sequence $\sigma = (n_1, \dots, n_k)$, and let $n_{k+1} \notin \sigma$. An $(n_1, \dots, n_k, n_{k+1})$-*cube* is a structure $\mathcal{C}$ defined in the following way. Take two σ-cubes $\mathcal{A}$ and $\mathcal{B}$ such that $A \cap B = \emptyset$ and let $f : \mathcal{A} \to \mathcal{B}$ be an isomorphism. Let $\mathcal{C}$ be the structure

$$(A \cup B; P_{n_1}^{\mathcal{A}} \cup P_{n_1}^{\mathcal{B}}, \dots, P_{n_k}^{\mathcal{A}} \cup P_{n_k}^{\mathcal{B}}, P_{n_{k+1}}^{\mathcal{C}}),$$

where $P_{n_{k+1}}^{\mathcal{C}}(x, y)$ holds if and only if $f(x) = y$ or $f^{-1}(x) = y$.

3.2 Example. Let σ be a finite non-repeating sequence. Consider $A = \mathbb{Z}_2^{|\sigma|}$ as a vector space over $\mathbb{Z}_2$, with basis $b_1, \ldots, b_{|\sigma|}$. If we define the structure $\mathcal{A}$ with domain A by letting $P^{\mathcal{A}}_{\sigma(i)}(x, y)$ iff $x + b_i = y$, then $\mathcal{A}$ is a σ-cube.

The following property of finite cubes, which is easily checked by induction, shows that we could have taken Example 3.2 as the definition of a σ-cube.

3.3 Lemma. *Let σ be a finite non-repeating sequence. Any two σ-cubes are isomorphic.*

Furthermore, we have the following stronger property.

3.4 Lemma. *If σ is a finite non-repeating sequence and τ is a permutation of σ, then every τ-cube is isomorphic to every σ-cube.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be a σ-cube and a τ-cube respectively. By Lemma 3.3, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are constructed as in Example 3.2. Since τ is a permutation of σ, there is a bijection f such that $\sigma(i) = \tau(f(i))$. Let φ be the vector space isomorphism induced by taking b_i to $b_{f(i)}$. We then have

$$\begin{aligned} &P^{\mathcal{A}}_{\sigma(i)}(x, y) \text{ iff } x + b_i = y \text{ iff } \varphi(x) + \varphi(b_i) = \varphi(y) \\ &\qquad \text{iff } \varphi(x) + b_{f(i)} = \varphi(y) \text{ iff } P^{\mathcal{B}}_{\tau(f(i))}(\varphi(x), \varphi(y)) \text{ iff } P^{\mathcal{B}}_{\sigma(i)}(\varphi(x), \varphi(y)). \end{aligned}$$

Thus φ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. $\square$

So instead of "σ-cube", where $\sigma = (n_1, \ldots, n_k)$, we will write "$A$-cube", where $A = \{n_1, \ldots, n_k\}$. (This notation matches that of [9], if we make the usual set-theoretic identification of n with $\{0, \ldots, n-1\}$.)

We now define infinite cubes.

3.5 Definition. Let $\alpha = (n_0, n_1, \ldots)$ be an infinite non-repeating sequence of natural numbers. An *α-cube* is a structure of the form $\bigcup_{i\in\omega} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an $\{n_0, \ldots, n_i\}$-cube, and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$.

As with finite sequences, the order of an infinite sequence α does not affect the isomorphism type of α-cubes, so we can talk about S-cubes, where S is an infinite set. To show that this is the case, we will use the following fact, which is easy to check. Suppose that $A \subset B \subset C$ are finite, $\mathcal{Z}$ is a C-cube, and $\mathcal{X} \subset \mathcal{Z}$ is an A-cube. Then there exists a B-cube $\mathcal{Y}$ such that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$.

3.6 Lemma. *If σ is an infinite non-repeating sequence and τ is a permutation of σ, then every τ-cube is isomorphic to every σ-cube.*

Proof. Let $\sigma = (m_0, m_1, \ldots)$ be an infinite non-repeating sequence, and let $\tau = (n_0, n_1, \ldots)$ be a permutation of σ. Let $s_i = \{m_0, \ldots, m_i\}$ and $t_i = \{n_0, \ldots, n_i\}$.

Let $\mathcal{A}$ be a σ-cube and let $\mathcal{B}$ be a τ-cube. Then $\mathcal{A} = \bigcup_{i\in\omega} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an s_i-cube, and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$. Similarly, $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$, where each $\mathcal{B}_i$ is a t_i-cube, and $\mathcal{B}_i \subset \mathcal{B}_{i+1}$.

We build a sequence of finite partial isomorphisms $\varphi_0 \subseteq \varphi_1 \subseteq \cdots$ such that $A_i \subseteq \operatorname{dom} \varphi_{2i+1}$ and $B_i \subseteq \operatorname{rng} \varphi_{2i+2}$. We begin with $\varphi_0 = \emptyset$.

Given φ_{2i}, let $k \geqslant i$ be such that $A_k \supseteq \operatorname{dom} \varphi_{2i}$, and let l be such that $B_l \supseteq \operatorname{rng} \varphi_{2i}$ and $s_k \subseteq t_l$. Then there is an s_k-cube $\mathcal{C} \subseteq \mathcal{B}_l$ such that $\operatorname{rng} \varphi_{2i} \subseteq C$. Extend φ_{2i} to an isomorphism $\varphi_{2i+1} : \mathcal{A}_k \to \mathcal{C}$.

Given φ_{2i+1}, proceed in an analogous fashion to define a finite partial isomorphism φ_{2i+2} including B_i in its range.

Now $\varphi = \bigcup_{i\in\omega} \varphi_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

4 The Main Theorem

In this section we prove the main result of this paper.

4.1 Theorem. *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose only computably presentable model is the saturated one.*

Proof. Let $\{A_i\}_{i\in\omega}$ and S be as in Proposition 2.3. Fix an enumeration of $\{A_i\}_{i\in\omega}$ such that at each stage exactly one element is enumerated into some A_i. (For instance, we can take the enumeration given in the proof of Proposition 2.3.) Construct a computable model $\mathcal{M}_\omega = \bigcup_{n\in\omega} \mathcal{M}_\omega^n$ as follows. Begin with $\mathcal{M}_\omega^n[0] = \emptyset$ for all n. At stage $s+1$, if $A_n[s+1] \neq A_n[s]$ then extend $\mathcal{M}_\omega^n[s]$ to an $A_n[s+1]$-cube using fresh large numbers.

It is clear that this procedure can be carried out effectively, so that $\mathcal{M}_\omega$ is computable. Furthermore, $\mathcal{M}_\omega$ is the disjoint union of one A_n-cube for each $n \in \omega$. In particular, every infinite cube in $\mathcal{M}_\omega$ is an S-cube.

Now let $T = \mathrm{Th}(\mathcal{M}_\omega)$ be the first-order theory of $\mathcal{M}_\omega$. We show that T is $\aleph_1$-categorical but not $\aleph_0$-categorical, $\mathcal{M}_\omega$ is saturated, and the only computably presentable model of T (up to isomorphism) is $\mathcal{M}_\omega$.

We begin by showing that that T is $\aleph_1$-categorical. Since T includes sentences saying that for each n and x there is at most one y such that $P_n(x, y)$, we are free to use functional notation and write $P_n(x) = y$ instead of $P_n(x, y)$. For $n \in S$, let $k(n)$ be the number of elements $x \in M_\omega$ for which $P_n^{\mathcal{M}_\omega}(x)$ is not defined. For $n \notin S$, let $k(n)$ be the number of elements $x \in M_\omega$ for which $P_n^{\mathcal{M}_\omega}(x)$ is defined. Note that $k(n)$ is finite for all n.

It is easy to see that $\mathcal{M}_\omega$ satisfies the following list of statements, which can be written as an infinite set $\Sigma \subset T$ of first-order sentences:

1. For each n, the relation P_n is a partial one-to-one function and $P_n(x) = y \rightarrow P_n(y) = x$.

2. For all $n \neq m$ and all x, we have $P_n(x) \neq P_m(x)$ and $P_n(x) \neq x$.

3. For all $n \neq m$ and all x, if $P_n(x)$ and $P_m P_n(x)$ are defined, then $P_m(x)$ and $P_n P_m(x)$ are defined, and $P_n P_m(x) = P_m P_n(x)$.

4. For all k, all $n > n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, and all x, we have $P_{n_1} \ldots P_{n_k}(x) \neq P_n(x)$.

5. For each $n \in S$ there are exactly $k(n)$ many elements x for which $P_n(x)$ is not defined.

6. For each $n \notin S$ there are exactly $k(n)$ many elements x for which $P_n(x)$ is defined.

7. Let A_i be finite, and let $m \in A_i$ be such that $m \notin A_j$ for all $j \neq i$. Then there exists a finite A_i-cube $\mathcal{C}_i$ such that $\forall x$ ($P_m(x)$ is defined $\rightarrow x \in \mathcal{C}_i$). (Note that $m \notin S$ and $\mathcal{C}_i$ has $k(m)$ many elements, so together with Statements 3 and 6, this statement implies that $\mathcal{C}_i$ is not contained in a larger cube.)

4.2 Remark. Note that Statements 1 and 3 imply the following statement: for all $n \neq m$ and all u, if $P_n(u)$ and $P_m(u)$ are defined then $P_m P_n(u)$ and $P_n P_m(u)$ are defined and equal. To prove this let $v = P_n(u)$, which, by Statement 1, implies that $P_n(v) = u$. Since $P_m P_n(v) = P_m(u)$ is defined, applying Statement 3 with $x = v$, we have that $P_m(v)$ and $P_n P_m(v)$ are defined, and $P_n P_m(v) = P_m P_n(v)$. If we let $w = P_m(v)$ then $P_m P_n(u) = w$. Since $P_n(w) = P_n P_m(v) = P_m P_n(v) = P_m(u)$, Statement 1 implies that $P_n P_m(u) = P_n P_n(w) = w$. Thus $P_m P_n(u) = P_n P_m(u)$.

Now suppose that $\mathcal{M}$ is a model of Σ. Let $A \subseteq \omega$ and $x \in M$. Using the statements above, it is easy to check that $\forall n \in A$ ($P_n^{\mathcal{M}}(x)$ is defined) if and only if x belongs to an A-cube. It is also clear that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are A-cubes in $\mathcal{M}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, then $\mathcal{C}_1 = \mathcal{C}_2$.

It now follows that $\mathcal{M}$ is the disjoint union of components $\mathcal{M}_0$ and $\mathcal{M}_1$, where $\mathcal{M}_0$ is the disjoint union of exactly one A_i-cube for each finite A_i. Let $x \in M_1$. If $n \in S$ then there are $k(n)$ elements in M_0 on which $P_n^{\mathcal{M}}$ is not defined. Statement 5 says that there are exactly $k(n)$ such elements in M. Hence $P_n^{\mathcal{M}}(x)$ is defined. Similarly, Statement 6 implies that if $n \notin S$ then $P_n^{\mathcal{M}}(x)$ is not defined. Therefore, x belongs to an S-cube. Thus, $\mathcal{M}_1$ is a disjoint union of S-cubes.

Let $\mathfrak{C}$ be the class of all structures that are the disjoint union of exactly one A_i-cube for each finite A_i and some finite or infinite number of S-cubes. Clearly, any structure in $\mathfrak{C}$ is a model of Σ, and we have shown that any model of Σ is in $\mathfrak{C}$. Let $\mathfrak{M}$ be a model of Σ. Each of the S-cubes in $\mathcal{M}$ is countable, so if $|M| = \aleph_1$, then there must be $\aleph_1$ many such S-cubes. Therefore, any two models of Σ of size $\aleph_1$ are isomorphic, and hence Σ is uncountably categorical. It now follows by the Łoś-Vaught Test that any model of Σ is a model of T. Thus T is uncountably categorical and, since $\mathfrak{C}$ contains infinitely many nonisomorphic countable structures, T is not countably categorical.

4.3 Lemma. *Let $\mathcal{M}$ be a computable model of T. Then $\mathcal{M}$ contains infinitely many S-cubes.*

Proof. Assume for a contradiction that $\mathcal{M}$ contains a finite number r of S-cubes (which may be 0). We can assume without loss of generality that the domain of $\mathcal{M}$ is ω. Let $\mathcal{M}_s$ be the structure obtained by restricting the domain of $\mathcal{M}$ to $\{0, \dots, s\}$ and the language to $P_0, \dots, P_s$. Choose one element from each S-cube, say $c_1, \dots, c_r$. Define a computable function $g : \omega \times \omega \to [\omega]^{<\omega} \cup \{\infty\}$ as follows.

If $x > s$ then $g(x, s) = \emptyset$. If x is connected to some c_i in $\mathcal{M}_s$ then $g(x, s) = \infty$. Otherwise, $g(x, s)$ is the set of all $k \leqslant s$ for which there is a $y \leqslant s$ such that $P_k^{\mathcal{M}}(x, y)$.

Clearly, $g(x, s)$ is computable. Also, if x belongs to some A_i-cube in $\mathcal{M}$ then $g(x, s) \subseteq A_i$, and if $g(x, s) = \infty$ then x must belong to an S-cube. It is now easy to check that $f(x) = \lim_s g(x, s)$ is S-limitwise monotonic and $\{f(x) : f(x) \neq \infty\} = \{A_i : |A_i| < \omega\}$. But this contradicts the fact that $\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable. $\square$

Since $\mathcal{M}_\omega$ is computable, it contains infinitely many S-cubes, and therefore is saturated. Other countable models of T have only finitely many S-cubes, and hence do not have computable presentations. □

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