Undecidability of the structure of the Solovay degrees of c.e. reals

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Abstract

We show that the elementary theory of the structure of the Solovay degrees of computably enumerable reals is undecidable.

1 Introduction

In this paper we work in Cantor space 2^ω, with basic clopen sets $[\sigma] = \{\sigma\alpha : \alpha \in 2^\omega\}$ having Lebesgue measure $\mu([\sigma]) = 2^{-|\sigma|}$. While this space is not homeomorphic to the real interval $(0,1)$, it is measure-theoretically isomorphic to it. We will identify a set A with its characteristic function χ_A, and hence with the real $0.\chi_A$. We write '$[s]$' after expressions to indicate that everything in the expression is taken with its value at stage s of the given construction.

We assume the basics of the theory of effective randomness, in particular the definitions of prefix-free Turing machine and prefix-free Kolmogorov complexity, which we denote by K. For definitions of these and related concepts, see for instance [6, 7, 11, 14].

Our basic objects of study will be the *computably enumerable* reals (which have also been called *left computable* and *left-c.e.*). The computably enumerable reals are those reals α such that the left cut $L(\alpha) = \{q \in \mathbb{Q} : q \leqslant \alpha\}$ forms a c.e. set of rationals. Equivalently, c.e. reals are those that are the limits of computable increasing sequences of rationals. These reals should not be confused with the *computable reals*, which are those that are the limits of a computable sequence of rationals for which the modulus of convergence is also a computable function. Nor should they be confused with the *strongly c.e. reals*, which are those that are of the form $0.\chi_A$ for some c.e. *set* A. Computably enumerable reals arise naturally as the measures of the domains of prefix-free Turing machines in the same way that in classical computability theory c.e. *sets* arise as the halting sets of Turing machines.

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A classic example of a c.e. real that is not strongly c.e. is Chaitin's halting probability [5]:

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix-free machine. This real is famously *1-random*, in the sense that there is a constant c such that for all n,

$$K(\Omega \upharpoonright n) > n - c,$$

where $\Omega \upharpoonright n$ denotes the first n bits of Ω. As is well-known, this initial segment definition coincides with Martin-Löf's definition [15] of randomness as avoiding all "effectively presented" statistical tests.

The context of the current paper is a program to try to understand the relative initial segment complexity of (c.e.) reals as articulated in the papers [6, 8, 9, 10]. For instance, the above definition of Ω *seems* to depend upon the choice of the relevant universal machine U. Perhaps for a different machine $\widehat{U}$, the real $\Omega_{\widehat{U}}$ would behave rather differently than Ω_U.

The situation is akin to that for "the" halting set $\emptyset' = \{e : \varphi_e(e)\downarrow\}$, where φ_e is the eth partial computable function. Here we might well argue that the definition actually depends on the choice of universal machine enumerating the partial computable functions. Of course, by Myhill's Theorem (see [17]), we know that all versions of the halting problem are creative sets, and are all the same up to computable permutations of the natural numbers.

The first person to address this situation for Ω was Solovay [18, 19], who introduced an analytic version of m-reducibility appropriate for c.e. reals.

Definition 1 (Solovay [18, 19]). Let α and β be reals. We say β *dominates* α, or, alternatively, that α is *Solovay reducible to* β, and write $\alpha \leqslant_{\mathrm{S}} \beta$, if there exist a partial computable function φ and a constant c such that for all rational $q < \beta$, we have $\varphi(q)\downarrow$ and

$$\alpha - \varphi(q) \leqslant c(\beta - q).$$

If A, $B \subseteq \mathbb{N}$, then $A \leqslant_{\mathrm{S}} B$ if and only if $0.\chi_A \leqslant_{\mathrm{S}} 0.\chi_B$.

Note that Solovay reducibility implies Turing reducibility. (See e.g. [8] for a proof.)

One reason that Solovay was interested in this reducibility is that if $\alpha \leqslant_{\mathrm{S}} \beta$ then there is a constant c such that for all n,

$$K(\beta \upharpoonright n) > K(\alpha \upharpoonright n) - c.$$

(See e.g. [9] for a proof.) This fact makes Solovay reducibility a possible measure of relative randomness. Solovay defined a class of c.e. reals, the Ω*-like* c.e. reals, as being those that dominate Ω. Clearly, all Ω-like reals are 1-random. In the wonderful unpublished notes [18], Solovay proved a number of very interesting results about the initial segment complexity of Ω-like reals. He remarked: "It seems strange that we are able to prove so much about the behavior of Ω-like reals when, a priori, the definition of Ω is thoroughly model dependent…”. Solovay's notes have not been published, save for a fragment in Solovay [19], but most of the material will appear in the forthcoming monograph of Downey and Hirschfeldt [7].

Solovay's intuition has been more recently confirmed by two groups of authors. First, Calude, Khoussainov, Hertling and Wang [4] used Kraft's inequality to show that if a c.e. real is Ω-like then it is the halting probability of some universal prefix-free machine. Thus it is a version of Ω. Second, Kučera and Slaman [13] proved that if a c.e. real is 1-random then it is Ω-like.

Thus we have the following very remarkable consequence. Fundamental work of Chaitin and Levin (see e.g. [11]) has shown that for all n there are strings σ of length n such that $K(\sigma) = n + K(n) + O(1)$. Thus it is possible for the Kolmogorov complexity of Ω to oscillate upwards above $n + \log n$, and indeed we can show that infinitely often we have $K(\Omega \upharpoonright n) > n + \log n$. On the other hand, the complexity of a 1-random real can oscillate down towards n infinitely often. The Kučera-Slaman Theorem shows that all 1-random c.e. reals have the *same* initial segment behavior, and all oscillate downwards and upwards at the *same* ns. Thus the situation for halting probabilities is in this respect similar to that for versions of the halting set.

This paper is part of the effort to understand the structure of the c.e. reals under Solovay reducibility. Note that for c.e. reals α and β, since there exist increasing computable functions of rationals $\alpha[s]$ and $\beta[s]$ such that $\lim_s \alpha[s] = \alpha$ and $\lim_s \beta[s] = \beta$, we have $\alpha \leqslant_{\mathrm{S}} \beta$ if and only if there exist a computable function φ and a constant c such that for all s,

$$\alpha - \alpha[\varphi(s)] < c \cdot (\beta - \beta[s]).$$

(This characterization was first given in [3]; see also [9] for a proof.) Since we deal exclusively with c.e. reals in what follows, we generally use this equivalent characterization without explicit comment. One easy consequence of it that we will use below is that if for all s,

$$\alpha[s+1] - \alpha[s] \leqslant \beta[s+1] - \beta[s],$$

then $\alpha \leqslant_{\mathrm{S}} \beta$. Another useful variation is given in the following lemma.

Lemma 1.1. *Let $\alpha[s]$ and $\beta[s]$ be increasing sequences of rationals such that $\lim_s \alpha[s] = \alpha$ and $\lim_s \beta[s] = \beta$. Then $\alpha \leqslant_S \beta$ if and only if there exist a computable function φ, a constant c, and a nonincreasing computable function $\psi : \mathbb{N} \to \mathbb{Q}$ such that $\lim_s \psi(s) = 0$ and for all s,*

$$\alpha - \alpha[\varphi(s)] < c \cdot (\beta - \beta[s]) + \psi(s).$$

Proof. The “only if” direction follows from the previous characterization of Solovay reducibility. For the “if” direction, suppose that the displayed inequality holds for all s. Given s, find a $t > s$ such that $c \cdot (\beta[t] - \beta[s]) > \psi(t)$, which is possible since ψ goes to 0. Define $\theta(s) = \varphi(t)$. Then

$$\alpha - \alpha[\theta(s)] < c \cdot (\beta - \beta[t]) + \psi(t) = c \cdot (\beta - \beta[s]) - c \cdot (\beta[t] - \beta[s]) + \psi(t) < c \cdot (\beta - \beta[s]).$$

So $\alpha \leqslant_{\rm S} \beta$ by the previous characterization of Solovay reducibility. □

We call the equivalence classes of c.e. reals induced by the Solovay reducibility pre-ordering *c.e. Solovay degrees*. We denote the degree of α by $[\alpha]_{\rm S}$. Thus $[\alpha]_{\rm S} = \{\beta : \beta \equiv_{\rm S} \alpha\}$. The first paper to study this structure in detail was Downey, Hirschfeldt, and Nies [9], where it was shown that the Solovay degrees of c.e. reals form a dense distributive upper-semilattice, with join induced by ordinary arithmetical addition, that is, $[\alpha]_{\rm S} \vee [\beta]_{\rm S} = [\alpha + \beta]_{\rm S}$ (though some of these facts were previously known). It was also shown in [9] that while every nonrandom degree splits over all lesser ones, remarkably, $[\Omega]_{\rm S}$ is qualitatively different in that if α and β are c.e. reals and $\alpha + \beta \equiv_{\rm S} \Omega$, then at least one of α or β is 1-random. (According to Kučera, the latter result had been proved earlier by Demuth.)

The goal of this paper is to add to our global understanding of the structure of the c.e. Solovay degrees. We prove that the structure of the c.e. Solovay degrees has an undecidable first order theory. The proof is, of course, a priority argument, in this case one employing a $\mathbf{0}'''$ tree of strategies.

Of relevance to this paper is another measure of relative randomness.

Definition 2 (Downey, Hirschfeldt, and LaForte [8])**.** Let A, $B \subseteq \mathbb{N}$. We say A is *strongly weak-truth-table reducible (sw-reducible) to* B, and write $A \leqslant_{\rm sw} B$, if there exist a computable functional Γ and a constant c such that $A = \Gamma^B$ and $\gamma(x) \leqslant x + c$ for all x, where γ is the use function of Γ, which is independent of B. If α and β are reals with $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, then $\alpha \leqslant_{\rm sw} \beta$ if and only if $A \leqslant_{\rm sw} B$.

Again it is relatively easy to show that if $A \leqslant_{\rm sw} B$ then $K(B \restriction n) > K(A \restriction n) - O(1)$. For c.e. reals in general, sw-reducibility and Solovay reducibility are incomparable measures, but for strongly c.e. reals they coincide. For these results and others concerning the spectrum of measures of relative randomness see Downey, Hirschfeldt, and LaForte [8], the survey articles Downey [6] and Downey, Hirschfeldt, Nies, and Terwijn [11], or the forthcoming monograph Downey and Hirschfeldt [7].

Most of our notation is standard, and follows that of Soare [17]. In particular, we write '$[s]$' after any expression to denote its value at stage s. A c.e. real can be identified with the characteristic function of a *nearly c.e. set*, that is, a set B with an approximation $B[s]$ such that for all s and x, if $B(x)[s+1] < B(x)[s]$ then there exists some $y < x$ such that $B(y)[s] < B(y)[s+1]$. In order to avoid possible confusion between c.e. *sets* and c.e. *reals*, in the following sections, we will talk only about c.e. sets and nearly c.e. sets. Given either c.e. or nearly c.e. sets Y and Z, we generally write $Y + Z$ and $Y - Z$ for the ordinary arithmetic sum and difference of the c.e. reals $0.\chi_Y$ and $0.\chi_Z$. This convention will never cause any confusion below, and simplifies our notation somewhat.

We show that the theory of the structure of the c.e. Solovay degrees is undecidable using the method of Nies [16], involving the notions of effectively dense boolean algebras and hereditarily undecidable theories. Suppose we have a structure $(\mathbb{N}, \preccurlyeq, \vee, \wedge)$ such that $\preccurlyeq$ is a Σ^0_k pre-ordering and $\vee$ and $\wedge$ are total computable binary functions, and let $\approx$ be the equivalence relation modulo which $\preccurlyeq$ becomes a partial order. (In other words, $m \approx n$ if and only if $m \preccurlyeq n$ and $n \preccurlyeq m$.) If the quotient structure $\mathcal{B} = (\mathbb{N}, \preccurlyeq, \vee, \wedge)/\approx$ is a boolean algebra, then we call $\mathcal{B}$ a Σ^0_k *boolean algebra*. (We also abuse terminology and call $\mathcal{B}$ a Σ^0_k boolean algebra if it is isomorphic to a Σ^0_k boolean algebra.) A boolean algebra $\mathcal{B}$ with least element 0 is *effectively dense* if there is a computable function F such that $0 \prec F(x) \prec x$ for all $x \neq 0$ in the domain of $\mathcal{B}$.

A theory T in a first-order language L is *hereditarily undecidable* if every set $X \subseteq T$ containing the valid L-sentences is undecidable. This notion is useful to us because of the following *transfer principle* (see e.g. [1, 2]): if A is an L_1-structure with a hereditarily undecidable theory, and A can be interpreted with parameters in an L_2-structure B, then the theory of B is hereditarily undecidable.

A Solovay degree b is *complemented below* a Solovay degree a if there is a Solovay degree c such that $b \wedge c = a$ and $b \vee c = [0]_{\mathrm{S}}$. In Nies [16] it is shown that the lattices of Σ^0_k ideals of effectively dense Σ^0_k boolean algebras have hereditarily undecidable theories. (Actually, Theorem 2.1 of [16] is proved only for the Σ^0_1 case, but it is clear how to relativize the proof for $k > 1$. See, for example, Nies and Downey [12, Theorem 2.1] for the case $k = 2$.) We use this result to show that the structure of the c.e. Solovay degrees is undecidable by finding a c.e. Solovay degree a such that the collection of c.e. Solovay degrees complemented below a (with join and meet) forms an effectively dense Σ^0_3 boolean algebra $\mathcal{B}(a)$ for which the lattice of Σ^0_3 ideals is definable in $\mathcal{B}(a)$.

2 The structure $\mathcal{B}(a)$

We produce the Solovay degree a mentioned above using two technical lemmas, Theorems 1 and 2 below, involving the following notion:

Definition 3. A c.e. set $A \subset \mathbb{N}$ is *weakly sparse via f* if

1. $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing, computable function,
2. $A \subseteq \{ f(k) : k \in \mathbb{N} \}$, and
3. $f(k) + k + 1 < f(k+1)$ for all k.

This notion is a significant weakening of the notion of super-sparseness used in various results characterizing polynomial-time degrees of computable sets. A simple example of a weakly sparse set is the set of positive squares $\{ n^2 : n > 0 \}$. In fact, for our main lemma, Theorem 2 below, we will choose a to be the Solovay degree of a subset of this set.

Let W_e denote the eth c.e. set, and let $B \sqcup C = A$ denote the fact that B and C are disjoint and their union is A. For a c.e. set A, let $\mathcal{B}(A)$ be the boolean algebra of c.e. subsets of A that have c.e. complements included in A, that is, $\{ W_e : \exists j\, (W_j \sqcup W_e = A) \}$, with union and intersection. If A is weakly sparse, we can prove the following exact pair theorem characterizing the ideals of $\mathcal{B}(A)$ that are closed downward under sw-reducibility (or equivalently, since we are dealing with c.e. sets, Solovay reducibility).

Theorem 1. *Let A be weakly sparse. Let $\mathcal{J}$ be a Σ^0_3 ideal of $\mathcal{B}(A)$ that is closed downward under sw-reducibility. Then there exist c.e. sets X and Y such that $W_e \in \mathcal{J}$ if and only if $W_e \leqslant_{sw} X, Y$.*

We prove this result in Section 3 below.

We can then produce a weakly sparse set A with certain useful properties. For sets B_1 and B_2, we write $B_1 \wedge B_2 \equiv_{\mathrm{S}} 0$ to denote the fact that the infimum of $[B_1]_{\mathrm{S}}$ and $[B_2]_{\mathrm{S}}$ exists and is $[0]_{\mathrm{S}}$.

Theorem 2. *There exists a c.e., non-computable, weakly sparse set A such that*

1. *for all c.e. splittings $A_1 \sqcup A_2 = A$, the infimum of the Turing degrees of A_1 and A_2 is $\mathbf{0}$; and*

2. *for all nearly c.e. sets B_1 and B_2 such that $B_1 + B_2 \equiv_S A$ and $B_1 \wedge B_2 \equiv_S 0$, there exist c.e. sets A_1 and A_2 with $A_1 \sqcup A_2 = A$, such that $B_1 \equiv_S A_1$ and $B_2 \equiv_S A_2$.*

We prove this result in Section 4 below. From now on, fix a set A as in Theorem 2 and let $a = [A]_{\mathrm{S}}$.

We will show that for all $B, C \in \mathcal{B}(A)$, we have $[B]_{\mathrm{S}} \wedge [C]_{\mathrm{S}} = [B \cap C]_{\mathrm{S}}$ and $[B]_{\mathrm{S}} \vee [C]_{\mathrm{S}} = [B \cup C]_{\mathrm{S}}$. Together with Theorem 2, this fact implies that the collection of c.e. Solovay degrees complemented below a (with join and meet) forms a boolean algebra $\mathcal{B}(a)$, which is equal to $\left\{ [B]_{\mathrm{S}} : B \in \mathcal{B}(A) \right\}$. We will also show that this boolean algebra is Σ^0_3 and effectively dense. Thus the theory of the lattice $\mathcal{I}(a)$ of Σ^0_3 ideals of $\mathcal{B}(a)$ is hereditarily undecidable. Since Theorem 1 will imply that there is an interpretation of the theory of $\mathcal{I}(a)$ in the theory of the Solovay degrees below a, this will show that the theory of the Solovay degrees of c.e. reals is undecidable, by the transfer principle mentioned in the introduction.

We now prove a series of lemmas showing that the set of c.e. splittings of A together with the relation of Solovay reducibility and the operations of intersection and union can be used to construct a Σ^0_3 boolean algebra. First we show that this algebra can be presented in a c.e. way, and that the density of the structure for sw-reducibility is effective. Since sw-reducibility is equivalent to Solovay reducibility on c.e. sets, this result will serve our purposes. Recall that the ordinary arithmetic sum gives the join in the Solovay degrees of c.e. reals.

Lemma 2.1. *There exist a computable enumeration $\left\{ A_e : e \in \mathbb{N} \right\}$ of c.e. sets and computable functions f, g, and h such that the following hold.*

(a) *For each i and j such that $W_i \sqcup W_j = A$, there exists an e such that $W_i = A_e$.*

(b) *For each e, there exists a k such that $A_e \sqcup A_k = A$.*

(c) *For all i and j, we have $A_i \cup A_j = A_{f(i,j)}$ and $A_i \cap A_j = A_{g(i,j)}$.*

(d) *For all e, if A_e is noncomputable, then $\emptyset <_{sw} A_{h(e)} <_{sw} A_e$, and hence $\emptyset <_S A_{h(e)} <_S A_e$.*

Proof. We first note that the collection of sets that are halves of c.e. splittings of A can be enumerated by the following procedure. Let $\langle W_e, V_e\rangle$ be an enumeration of all pairs of c.e. subsets of A. For each e, define a sequence of e-expansionary stages $s_0^e < s_1^e, \ldots$ recursively by letting $s_0^e = 0$ and letting s_{k+1}^e be the least stage $s > s_k^e$ such that $\forall t \leqslant s\, (W_e \cap V_e = \emptyset)$ and $A[s_k^e] \subseteq (W_e \cup V_e)[s]$. Note that this sequence of stages could be finite. Let $B_e = \bigcup_k W_e[s_k^e]$. Then $\{ B_e : e \in \mathbb{N} \}$ is an indexing of all the c.e. sets that can be used to split A. This collection is closed under union and intersection, and under c.e. splittings.

We construct the sequence $\{ A_e : e \in \mathbb{N} \}$ from $\{ B_e : e \in \mathbb{N} \}$ by first constructing a natural computable enumeration of all closed terms of the first-order language $\mathcal{L} = \{\wedge, \vee, h_D\} \cup \{ b_e : e \in \mathbb{N} \}$, where $\wedge$ and $\vee$ are two-place function symbols, h_D is a one-place function symbol, and each b_e is a constant symbol. Let $\{ t_e : e \in \mathbb{N} \}$ be this sequence. The construction familiar from the Sacks splitting theorem (see [17]) is uniform in the enumeration of the set being split, so that given any noncomputable c.e. set B, we can uniformly find a c.e. set $B_D \subset B$ that is half of a splitting of B and such that $\emptyset <_{\mathrm{T}} B_D <_{\mathrm{T}} B$. Since $+$ is the join in the sw-degrees of c.e. sets and B_D is half of a c.e. splitting of B, it follows that $B_D \leqslant_{\mathrm{sw}} B$. Combining this fact with the fact that sw-reducibility implies Turing reducibility, we see that $\emptyset <_{\mathrm{sw}} B_D <_{\mathrm{sw}} B$. Also, the operations of union and intersection are of course similarly uniform for c.e. sets. So we can define A_e to be the c.e. set with the obvious interpretation given by t_e, interpreting the symbol $\wedge$ as $\cap$, the symbol $\vee$ as $\cup$, the symbol h_D as the operation of taking the first member of the splitting given by the uniform version of the Sacks splitting theorem, and each symbol b_e as B_e. (Note that we can apply the construction in the proof of the uniform version of the Sacks splitting theorem even if B is computable, to obtain some c.e. set; in this case, we do not care what properties this set has, only that it exists, so that h_D is total.) For example, if $t_e = h_D((b_2 \wedge b_1) \vee b_3)$, then A_e is the first half of the Sacks splitting of $(B_2 \cap B_1) \cup B_3$. Clearly the sequence $\{ A_e : e \in \mathbb{N} \}$ is as required, since we can effectively find the indices of $t_i \vee t_j$, of $t_i \wedge t_j$, and of $h_D(t_i)$ from i and j, giving us the required functions f, g, and h. □

Fix an enumeration $\{ A_e : e \in \mathbb{N} \}$ as in the lemma.

As indicated in the introduction, we simplify notation by writing X for both a c.e. set and the real $0.\chi_X$, which means we often write '$-$' for both set complementation and ordinary arithmetic subtraction. When the sets involved in the expression $B - C$ are c.e. and not merely nearly c.e., and $C \subseteq B$, these two operations amount to the same thing, so no confusion should result. For a subset B of A, we write $\overline{B}$ for the complement of B in A, that is, $A - B$. Notice that $\overline{A}_e$ is a c.e. subset of A for every e. We now show that intersections and unions give infima and suprema, respectively, for the Solovay degrees of the sets A_e.

Lemma 2.2. *For all $i, j \in \mathbb{N}$, we have $A_i \cup A_j \equiv_S A_i + A_j$.*

Proof. First note that if n enters $A_i \cup A_j$ at stage $s+1$, then it contributes exactly 2^{-n} to the value of $(A_i \cup A_j)[s+1] - (A_i \cup A_j)[s]$, thought of as a real, and contributes at least 2^{-n} to the value of $(A_i + A_j)[s+1] - (A_i + A_j)[s]$. Thus

$$(A_i \cup A_j)[s+1] - (A_i \cup A_j)[s] \leqslant (A_i + A_j)[s+1] - (A_i + A_j)[s]$$

for all s. It follows that $A_i \cup A_j \leqslant_{\mathrm{S}} A_i + A_j$.

Showing that $A_i + A_j \leqslant_{\mathrm{S}} A_i \cup A_j$ is a little more involved. We can assume the usual convention that, at each stage, exactly one number enters one of A_i or A_j. (Clearly we can assume without loss of generality that $A_i \cup A_j$ is infinite.) Given $s \in \mathbb{N}$, let $f(s)$ be the least stage t at which for every $z \in (A_i \cup A_j)[s]$, we have $z \in (A_i \cap A_j)[t]$, or $z \in (A_i \cap \overline{A}_j)[t]$, or $z \in (\overline{A_i} \cap A_j)[t]$. Note that f is a computable function. Suppose $t \geqslant f(s)$ and $(A_i + A_j)[t+1] - (A_i + A_j)[t] = 2^{-k(t)}$. Then either $k(t) \in A_i[t+1] - A_i[t]$ or $k(t) \in A_j[t+1] - A_j[t]$. So $k(t) \notin (A_i \cup A_j)[s]$ by our choice of $f(s)$. Each k that enters $A_i \cup A_j$ at some t after stage $f(s)$ can contribute at most $2 \cdot 2^{-k} = 2^{-k+1}$ to the value $A_i + A_j$, since it can enter A_i at most once and A_j at most once. Thus

$$\begin{aligned} A_i + A_j - (A_i + A_j)[f(s)] &= \sum_{j=f(s)+1}^{\infty} 2^{-k(j)} \\ &\leqslant \sum_{k \in A_i \cup A_j - (A_i \cup A_j)[s]} 2 \cdot 2^{-k} \\ &= 2 \cdot (A_i \cup A_j - (A_i \cup A_j)[s]). \end{aligned}$$

It follows that $A_i + A_j \leqslant_{\mathrm{S}} A_i \cup A_j$. □

Since $\overline{A_i \cup A_j} = \overline{A}_i \cap \overline{A}_j$, and, by Theorem 2, every Solovay degree in $\mathcal{B}(a)$ contains a c.e. set that is half of a splitting of A, the previous lemma shows that $\mathcal{B}(a)$ is closed under suprema. We now show that infima also exist in $\mathcal{B}(a)$. One direction is almost immediate.

Lemma 2.3. *For all $i, k \in \mathbb{N}$, we have $A_i \cap A_k \leqslant_S A_k$.*

Proof. At stage s, we search for the least $t > s$ such that $A_k[s] \subset A_i[t] \cup \overline{A_i}[t]$, and set $f(s) = t$. If $k \in A_i \cap A_k - (A_i \cap A_k)[f(s)]$, then $k \notin A_k[s]$, so $k \in A_k - A_k[s]$. Hence $A_i \cap A_k - (A_i \cap A_k)[f(s)] \leqslant A_k - A_k[s]$ for all s, and so $A_i \cap A_k \leqslant_{\mathrm{S}} A_k$. □

For the other direction, we first prove a useful lemma.

Lemma 2.4. *For all $i, k \in \mathbb{N}$, we have $A_i \leqslant_S A_k$ if and only if $A_i \cap \overline{A_k}$ is computable.*

Proof. First, suppose $A_i \leqslant_{\mathrm{S}} A_k$. By Lemma 2.3, $A_i \cap \overline{A_k} \leqslant_{\mathrm{S}} \overline{A_k}$ and $A_i \cap \overline{A_k} \leqslant_{\mathrm{S}} A_i \leqslant_{\mathrm{S}} A_k$. By Theorem 2, the infimum of A_k and $\overline{A_k}$ in the Turing degrees is $\mathbf{0}$, so $A_i \cap \overline{A_k}$ is computable.

Now, suppose that $A_i \cap \overline{A_k}$ is computable. Then $A_i \cap \overline{A_k} \leqslant_{\mathrm{S}} A_k$. By Lemma 2.3, also $A_i \cap A_k \leqslant_{\mathrm{S}} A_k$. Since $A_i = (A_i \cap \overline{A_k}) + (A_i \cap A_k)$, it follows that $A_i \leqslant_{\mathrm{S}} A_k$. □

Lemma 2.5. *For all $i, j, k \in \mathbb{N}$, if $A_j \leqslant_S A_i$ and $A_j \leqslant_S A_k$, then $A_j \leqslant_S A_i \cap A_k$.*

Proof. By Lemma 2.4, $A_j \cap \overline{A_i}$ and $A_j \cap \overline{A_k}$ are both computable. But then $A_j \cap \overline{A_i \cap A_k}$ is computable. Since $A_i \cap A_k = A_l$ for some l, Lemma 2.4 implies that $A_j \leqslant_{\mathrm{S}} A_i \cap A_k$. □

The relation of Solovay reducibility on c.e. reals is certainly Σ^0_3, since $X \leqslant_{\mathrm{S}} Y$ if and only if

$$\exists e, c\, \forall s, t\, \exists u\, (\varphi_e(s)[u]\downarrow \wedge (X_t - X[\varphi_e(s)]) < c \cdot (Y_t - Y[s])).$$

Thus, we have shown that $\mathcal{B}(a)$ is an effectively dense Σ^0_3 boolean algebra. We can now easily prove our main result.

Theorem 3. *The structure of the Solovay degrees of c.e. reals is undecidable.*

Proof. By Theorem 2 and the lemmas above, we can represent $\mathcal{B}(a)$ by $\{ [A_e]_{\mathrm{S}} : e \in \mathbb{N} \}$, with the operations of union and intersection on these sets giving joins and meets on their Solovay degrees. (Hence $\mathcal{B}(a)$ actually is a boolean algebra.) Furthermore, this algebra is Σ^0_3, as explained above, and is effectively dense, by Lemma 2.1.(d), since $\leqslant_{\mathrm{sw}}$ and $\leqslant_{\mathrm{S}}$ coincide on c.e. sets. Let $\mathcal{I}(a)$ be the lattice of Σ^0_3 ideals of $\mathcal{B}(a)$. By Theorem 1, there is a (two-dimensional) interpretation of the theory of $\mathcal{I}(a)$ in the theory of the Solovay degrees below a, since the sets involved can all be taken to be c.e., so that Solovay reducibility and sw-reducibility coincide on them. Since the theory of $\mathcal{I}(a)$ is hereditarily undecidable, this fact implies that the theory of the structure of the Solovay degrees of c.e. reals is undecidable, by the transfer principle mentioned in the introduction. □

3 The proof of Theorem 1

In this section we prove the first of our two main technical lemmas:

Theorem 1. *Let A be weakly sparse. Let $\mathcal{J}$ be a Σ^0_3 ideal of $\mathcal{B}(A)$ that is closed downward under sw-reducibility. Then there exist c.e. sets X and Y such that $W_e \in \mathcal{J}$ if and only if $W_e \leqslant_{sw} X, Y$.*

Proof. To simplify the construction, we will consider all c.e. sets to be subsets of A. This convention just amounts to abusing standard notation by indexing only the c.e. subsets of A: in other words, W_e means for us what would ordinarily be written $W_e \cap A$. Let $\mathcal{J} = \{ W_e : \exists p\, \forall s\, R(e, p, s) \}$, where R is a Σ^0_1 relation.

When W and V are c.e. sets, the sw-degree of the union of W and V is not in general the degree of the join of their sw-degrees, even if such a join exists. (Joins in the sw-degrees of c.e. reals do not always exist; see [8].) Fortunately, it is easy to see that for elements W and V of $B(A)$, we have $W, V \leqslant_{\mathrm{sw}} W \cup V$, which is all that will be required in the construction below.

Lemma 3.1. *Let A be a c.e. set. If W and V are elements of $B(A) = \{ W_e : \exists j\, (W_j \sqcup W_e = A) \}$, then $W, V \leqslant_{sw} W \cup V$.*

Proof. The proof is almost immediate, since the complements of V and W in A, and hence in $W \cup V$, are c.e. sets. Given $n \in \mathbb{N}$, let $g(n)$ be the least stage s at which for every $z \in (W \cup V) \upharpoonright n$, one of the following holds: $z \in (W \cap V)[s]$, or $z \in (W \cap \overline{V})[s]$, or $z \in (\overline{W} \cap V)[s]$. Clearly $W \upharpoonright n = W \upharpoonright n[g(n)]$ and $V \upharpoonright n = V \upharpoonright n[g(n)]$. □

Let $V_{\langle e,p\rangle} = \bigcup \{ W_e[s] : \forall t < s\, R(e,p,s) \}$. To ensure that X and Y bound the ideal $\mathcal{J}$, we will satisfy two sequences of positive requirements:

$$\mathrm{P}^X_{e,p}: \quad \exists n\, \forall x > n\, (x \in V_{\langle e,p\rangle} \text{ if and only if } x + \langle e,p\rangle \in X)$$

and

$$\mathrm{P}^Y_{e,p}: \quad \exists n\, \forall x > n\, (x \in V_{\langle e,p\rangle} \text{ if and only if } x + \langle e,p\rangle \in Y).$$

These requirements clearly imply that

$$W_e \in \mathcal{J} \Longrightarrow W_e \leqslant_{\mathrm{sw}} X$$

and

$$W_e \in \mathcal{J} \Longrightarrow W_e \leqslant_{\mathrm{sw}} Y,$$

since if $W_e \in \mathcal{J}$, then there exists a p with $\forall s\, R(e,p,s)$, so that $W_e = V_{\langle e,p\rangle} \leqslant_{\mathrm{sw}} X, Y$.

To ensure exactness of the pair X, Y, we satisfy the sequence of negative requirements

$$\mathrm{N}_e: \quad \Phi^X_e = \Phi^Y_e = h \text{ total } \Longrightarrow h \leqslant_{\mathrm{sw}} \bigcup_{\langle j,p\rangle < e} V_{\langle j,p\rangle},$$

where $\langle \Phi_e : e \in \mathbb{N}\rangle$ is a sequence consisting of all partial sw-reductions (i.e., all partial computable functionals such that, for any oracle, the use function is bounded by $x + O(1)$). Note that we list our sw-reductions so that their use functions are independent of the oracle; we can assume that the use function of Φ_e is exactly $n + c$ for some c. These requirements suffice because they imply that if a set W is sw-reducible to both X and Y, then W is also sw-reducible to a union of elements of $\mathcal{J}$ (since every $V_{\langle j,p\rangle}$ is either equal to W_j, in which case $W_j \in \mathcal{J}$, or is finite), and hence $W \in \mathcal{J}$. (The fact that we can use Φ_e twice in the statement of N_e, rather than having one requirement for each pair of reductions, follows by the usual Posner trick; see [17].)

Our construction will be similar to that of a minimal pair of Turing degrees (see [17]). The coding involved in our positive requirements of course prevents us from building an actual minimal pair, but we will see that it is compatible with our negative requirements. Some care will need to be taken because the positive requirements are infinitary.

We employ a tree of strategies. There are two possible outcomes for a strategy α working for a negative requirement N_e, depending on whether or not the partial functions Φ^X_e and Φ^Y_e produce the same total function. The infinitary outcome will be coded by 0 and the finitary one by 1. If Φ^X_e and Φ^Y_e do produce the same total function, then the restraint involved in ensuring that this function is computable from $\bigcup_{\langle j,p\rangle<e} V_{\langle j,p\rangle}$ will tend to infinity in the limit, but this restraint will be imposed only on positive strategies to the right of $\alpha^\frown 0$ on the tree of strategies. The idea is that the action of positive strategies above α can be accounted for by the $\bigcup_{\langle j,p\rangle<e} V_{\langle j,p\rangle}$ term in the statement of N_e, while positive strategies at or below $\alpha^\frown 0$ will not be able to destroy both Φ_e^X and Φ_e^Y computations between successive expansionary stages (that is, stages at which the length of agreement between Φ_e^X and Φ_e^Y increases), as usual in variations of the minimal pair construction. (Of course, if α is on the true path, the action of strategies to the left of α will be finite.)

Thus positive requirements will inflict only finite injury, so our tree is not really needed for them. We put their strategies there anyway for the sake of uniformity.

Construction:
To control our construction, we use the tree of strategies $2^{<\omega}$, assigning requirement N_e to each node of length $3e$, requirement $\mathrm{P}_{e,p}^X$ to each node of length $3\langle e,p\rangle+1$, and requirement $\mathrm{P}_{e,p}^Y$ to each node of length $3\langle e,p\rangle+2$; and associating to each $\alpha \in 2^{<\omega}$ of length $3e$ a restraint function $r_\alpha[s]$. We use the usual priority ordering for nodes on the tree: $\beta \leqslant \alpha$ if and only if either $\beta \subseteq \alpha$ or $(\beta \cap \alpha)^\frown 0 \subseteq \beta$. If the latter case holds but the former does not (i.e., if β is to the left of α), we also write $\beta <_{\mathrm{L}} \alpha$.

At each stage s, there will be an *approximation to the true path* $g[s] \in 2^{<\omega}$ of length s, defined in the usual way. A stage s is an *α-stage* if either $\alpha \subseteq g[s]$ or $s=0$. As usual, we say that α is *initialized* whenever $g[s]$ is to the left of α.

Let f be a function witnessing the fact that A is weakly sparse. We say that a pair consisting of a node $\alpha = 3\langle e,p\rangle+1$ and a number x *requires attention* at stage s if

1. s is an α-stage,
2. $x \notin X[s]$,
3. $x \in V_{e,p}[s]$,
4. $x = f(n)$ for some $n > \langle e,p\rangle$, and
5. $x + \langle e,p\rangle > \max\{\, r_\beta[s] : \beta^\frown 0 <_{\mathrm{L}} \alpha \,\}$.

We also have the analogous definition for $\alpha = 3\langle e,p\rangle+2$, with the obvious changes.

Of all pairs requiring attention at stage s, let α and x be the one that has required it for the longest time, breaking ties by choosing the shortest α and then the least x. We say that α, x *receives attention* at this stage. (If there is no such pair, then no number enters X or Y at this stage.) If α has requirement $\mathrm{P}_{e,p}^X$ assigned to it, then put $x+\langle e,p\rangle$ into $X[s+1]$. If α has requirement $\mathrm{P}_{\langle e,p\rangle}^Y$ assigned to it, then put $x+\langle e,p\rangle$ into $Y[s+1]$. In either case, the outcome of α does not matter, so we arbitrarily let it be 0.

Now suppose a node β has requirement N_e attached to it and s is a β-stage. Let φ_e be the use function of Φ_e. For $Z = X$ or Y, a computation $\Phi_e^Z(y)$ is *β-correct* at stage s if for every $\langle j, p\rangle < e$, if s^- is the previous β-stage and $x + \langle j, p\rangle < \varphi_e(y)$, then $x + \langle j, p\rangle \in Z[s]$ if and only if $x + \langle j, p\rangle \in Z[s^-]$. Let

$$l_\beta[s] = \max\bigl\{\, y \;:\; \forall x < y\, (\Phi_e^X(x)[s]{\downarrow} = \Phi_e^Y(x)[s]{\downarrow}\ \ \beta\text{-correctly}) \,\bigr\}.$$

Let $r_\beta[s] = \max\bigl\{\, \varphi_e(x) \;:\; x < l_\beta[s] \,\bigr\}$. (For every t between s and the next β-stage, if any, we define $r_\beta[t] = r_\beta[s]$.) The stage s is *β-expansionary* if $l_\beta[t] < l_\beta[s]$ for every β-stage $t < s$. If s is β-expansionary, then α has outcome 0 at stage s; otherwise it has outcome 1 at stage s.

Verification:
Let $g = \liminf_s g[s]$ be the true path of the construction.

First, we verify that the positive requirements are satisfied. Let $\alpha = g \upharpoonright 3e + 1$. If $\beta^\frown 0 <_{\mathrm{L}} \alpha$ corresponds to a negative requirement, then $\beta^\frown 0$ is not on the true path, and hence there are only finitely many β-expansionary stages, which implies that $r_\beta[s]$ comes to a limit. So $\max\bigl\{\, r_\beta[s] \;:\; \beta^\frown 0 <_{\mathrm{L}} \alpha \,\bigr\}$ comes to a limit r. Suppose that $x = f(n)$ for some $n > \langle e, p\rangle$, and $x + \langle e, p\rangle > r$.

Since there are infinitely many α-stages, if $x \in V_{\langle e,p\rangle}$, then the pair α, x eventually requires attention, and thus it eventually receives attention, so that $x + \langle e, p\rangle \in X$. Next, suppose $x + \langle e, p\rangle \in X$. Then it must be the case that $x + \langle e, p\rangle = y + \langle e', p'\rangle$ for some y, e', and p' such that $y \in V_{\langle e',p'\rangle}$ and $y = f(m)$ for some $m > \langle e', p'\rangle$ (because every number put into X during the construction is of this form). We show that $y = x$. If $y > x$, then since f is increasing, $m > n > \langle e, p\rangle > \langle e', p'\rangle$. By the fact that f is a sparseness function, however, we then have $y = f(m) > f(n) + n + 1 > f(n) + \langle e, p\rangle = x + \langle e, p\rangle$, which contradicts the assumption that $y + \langle e', p'\rangle = x + \langle e, p\rangle$. Similarly, if $y < x$, we have the analogous contradiction with the roles of x and y reversed. Hence $x = y$ and so $e' = e$ and $p' = p$. Thus $x \in V_{\langle e,p\rangle}$.

The same argument works for $\alpha = g \upharpoonright 3e + 2$, with the obvious changes. Thus the positive requirements are satisfied.

Now suppose that $\Phi_e^X = \Phi_e^Y$, with use function $\varphi_e(x) = x + c$. Let $\alpha = g \upharpoonright 3e$ and let s be a stage such that $\alpha \leqslant g[t]$ for every $t > s$. Given n, let $s_n > s$ be the least α-expansionary stage such that for every $\langle a, p\rangle < e$, we have $V_{\langle a,p\rangle}[s_n] \upharpoonright (n+c) = V_{\langle a,p\rangle} \upharpoonright (n+c)$, and for every $m \leqslant n$, we have $\Phi_e^X(m)[s_n]{\downarrow} = \Phi_e^Y(m)[s]{\downarrow}$. By Lemma 3.1, the function $n \mapsto s_n$ can be computed from $\bigcup_{\langle a,p\rangle<e} V_{\langle a,p\rangle}$ via a procedure with use $n + O(1)$. We claim that for every α-expansionary stage $t \geqslant s_n$, we have $\Phi_e^X(n)[t] = \Phi_e^X(n)[s_n]$.

The proof is by induction. Suppose that $t \geqslant s_n$ is an α-expansionary stage such that $\Phi_e^X(n)[t] = \Phi_e^X(n)[s_n]$. By the definition of α-expansionary, we also have $\Phi_e^X(n)[t] = \Phi_e^Y(n)[t]$. Let $t^+ > t$ be the next α-expansionary stage after t. Let us consider what numbers can be enumerated into X or Y between stages t and t^+. No node $\beta <_{\mathrm{L}} \alpha$ can act, since $t > s$, and the nodes $\beta \subseteq \alpha$ never enumerate any numbers into either X or Y below $\varphi_e(n)$, since $t \geqslant s_n$. The nodes $\beta \supseteq \alpha^\frown 1$ must respect $r_\beta[t]$, so they do not enumerate any numbers below $\varphi_e(n)$. As for the nodes $\beta \supseteq \alpha^\frown 0$, at most one such node can act in this interval (since only one such node can act at stage t, and none can act until the next α-expansionary stage, namely t^+). This node may enumerate a number into X or into Y, but not both. Thus we see that in the interval between t and t^+, either no number enters X below $\varphi_e(n)$, or no number enters Y below $\varphi_e(n)$. Since $\Phi_e^X(n)[t^+] = \Phi_e^Y(n)[t^+]$, it follows that $\Phi_e^X(n)[t^+] = \Phi_e^X(n)[t] = \Phi_e^X(n)[s_n]$.

Thus, for every α-expansionary stage $t \geqslant s_n$, we have $\Phi_e^X(n)[t] = \Phi_e^X(n)[s_n]$, and hence $\Phi_e^X(n) = \Phi_e^X(n)[s_n]$. So Φ_e^X is sw-computable from $\bigcup_{\langle a,p\rangle<e} V_{\langle a,p\rangle}$. □

4 The proof of Theorem 2

In this section we establish our main technical result:

Theorem 2. *There exists a c.e., non-computable, weakly sparse set A such that*

1. *for all c.e. splittings $A_1 \sqcup A_2 = A$, the infimum of the Turing degrees of A_1 and A_2 is* $\mathbf{0}$*; and*
2. *for all nearly c.e. sets B_1 and B_2 such that $B_1 + B_2 \equiv_S A$ and $B_1 \wedge B_2 \equiv_S 0$, there exist c.e. sets A_1 and A_2 with $A_1 \sqcup A_2 = A$, such that $B_1 \equiv_S A_1$ and $B_2 \equiv_S A_2$.*

Proof. We make A weakly sparse by choosing all numbers enumerated into A to be from $\{\, n^2 : n > 0 \,\}$. We must satisfy three types of requirements. The simplest are the requirements for noncomputability: for each $e \in \mathbb{N}$,

$$\mathrm{P}_e : \overline{A} \neq W_e.$$

To ensure that condition 1 on all c.e. splittings of A holds, we satisfy

$$\mathrm{N}_e : (U_e \sqcup V_e = A \text{ and } \Phi_e^{U_e} = \Phi_e^{V_e} = h \text{ total}) \Longrightarrow h \leqslant_{\mathrm{T}} \emptyset,$$

where $\langle U_e, V_e, \Phi_e\rangle$ is an enumeration of all triples consisting of pairs of c.e. sets together with a partial computable functional.

The most complex requirements are those involving condition 2 on A. Say that $X \leqslant_{\mathrm{S}} Y$ *via* c, φ if $X - X[\varphi(s)] < c \cdot (Y - Y[s])$ for all s. Letting $\langle B_e, C_e, \varphi_e, \psi_e\rangle$ enumerate all pairs of nearly c.e. sets together with all pairs of partial computable functions, we must satisfy for each $c \in \mathbb{Q}$ and $e \in \mathbb{N}$ the requirement

$$\begin{aligned}\mathrm{R}_{\langle e,c\rangle} : (B_e + C_e \leqslant_{\mathrm{S}} A \text{ via } c, \psi_e \text{ and } A \leqslant_{\mathrm{S}} B_e + C_e \text{ via } c, \varphi_e) \Longrightarrow \\ \big(\exists \text{ c.e. } Q_e\,(Q_e \leqslant_{\mathrm{S}} B_e, C_e \wedge \forall i\,(\overline{Q_e} \neq W_i)) \vee \\ \exists \text{ c.e. } \widehat{B_e}, \widehat{C_e}\,(\widehat{B_e} \sqcup \widehat{C_e} = A \wedge \widehat{B_e} \equiv_{\mathrm{S}} B_e \wedge \widehat{C_e} \equiv_{\mathrm{S}} C_e)\big).\end{aligned}$$

The strategies for the first two classes of requirements are straightforward and familiar ones from the study of c.e. Turing degrees. For the requirements P_e, we pick some large $x \in \{\, n^2 : n > 0 \,\}$ and wait for a stage s so that $x \in W_e[s]$, at which point we add x to $A[s+1]$. This action ensures that the complement of A is not equal to W_e.

For the requirements N_e, we use a slightly modified version of the strategy familiar from the standard construction of a minimal pair of c.e. degrees. To avoid introducing some essentially irrelevant details involved in checking whether $U_e \sqcup V_e$ splits A, we actually work with $U_e^* = U_e \cap A \cap (U_e \searrow V_e)$ and $V_e^* = V_e \cap A \cap (V_e \searrow U_e)$. (Recall that if X and Y are c.e. sets, then $X \searrow Y$ denotes the set of numbers enumerated into X before being enumerated into Y.) In what follows, we omit the *, just writing U_e and V_e for the restricted versions of these sets. Define the length of agreement for N_e at stage s by

$$l_e[s] = \max\{\, x : \forall y < x\, (\Phi_e^{U_e}(y)[s]\downarrow = \Phi_e^{V_e}(y)[s]\downarrow) \,\}.$$

At each stage s we define a restraint $r[s]$ preventing lower-priority strategies from enumerating numbers into A. (In the full construction, this restraint function will be implicit in the way P-strategies choose their witnesses, so there will be no need to define it explicitly.) We recursively define a set of *expansionary* stages, with 0 being the first such, and let $r[0] = 0$. At each stage $s+1 > 0$, let s^- be the previous expansionary stage. If $A[s^- + 1] \nsubseteq (U_e \sqcup V_e)[s]$ or there is some $t \leqslant s$ such that $l_e[t] \geqslant l_e[s+1]$, then $r[s+1] = \max\{\, \varphi_e^{U_e}(y)[s^-], \varphi_e^{V_e}(y)[s^-] : y < l_e[s^-] \,\}$. Otherwise, we declare $s+1$ to be expansionary and let $r[s+1] = 0$.

Since we intend to allow at most one number n to enter A below the restraint between expansionary stages, this procedure will ensure the satisfaction of the requirement, in much the same way as in the proof of Theorem 1: the restraint will not be allowed to drop again until n has entered either U_e or V_e, and both computations' values have been restored as signaled by the increase in the length of agreement.

4.1 Strategies for the splitting requirements

The description of our strategy for satisfying a requirement

$$\begin{aligned} \mathrm{R} :\ & (B + C \leqslant_{\mathrm{S}} A \text{ via } c, \psi \text{ and } A \leqslant_{\mathrm{S}} B + C \text{ via } c, \varphi) \Longrightarrow \\ & \qquad \big(\exists \text{ c.e. } Q\, (Q \leqslant_{\mathrm{S}} B, C \,\wedge\, \forall i\, (\overline{Q} \neq W_i)) \,\vee \\ & \qquad\qquad \exists \text{ c.e. } \widehat{B}, \widehat{C}\, (\widehat{B} \sqcup \widehat{C} = A \,\wedge\, \widehat{B} \equiv_{\mathrm{S}} B \,\wedge\, \widehat{C} \equiv_{\mathrm{S}} C)\big) \end{aligned}$$

is much more involved.

Notice that we can assume that φ and ψ are strictly increasing functions, $\varphi(s) > s$ and $\psi(s) > s$ for all s, and $c \geqslant 1$. The strategy for satisfying R involves first approximating whether or not the condition

$$B + C \leqslant_{\mathrm{S}} A \text{ via } c, \psi \wedge\ A \leqslant_{\mathrm{S}} B + C \text{ via } c, \varphi$$

holds, by means of a length-of-correctness function that looks for the most recent stage below which the Solovay reductions appear to be correct. Let

$$l[s] = \mu t\big((B+C)[s] - (B+C)[\psi(t)] \geqslant c \cdot (A[s] - A[t]) \vee A[s] - A[\varphi(t)] \geqslant c \cdot ((B+C)[s] - (B+C)[t])\big).$$

We can assume that $s > l[s]$ for all s. Notice that if the above condition does hold, then $\lim_s l[s] = \infty$. (We will prove this fact formally as part of the proof of Lemma 4.6 below.)

As usual, we will call stages at which $l[s]$ appears to be approaching infinity as a limit *expansionary stages*. At such stages, we will attempt to construct a noncomputable c.e. set $Q \leqslant_{\mathrm{S}} B, C$ while satisfying the infinite sequence of subrequirements

$$\mathrm{S}_j: \ \overline{Q} \neq W_j.$$

(In the full construction, the jth subrequirement of $\mathrm{R}_{\langle e,c\rangle}$ will be called $\mathrm{S}_{\langle e,c,j\rangle}$.) We will arrange things so that the failure to satisfy any one of these subrequirements will allow us to construct c.e. sets $\widehat{B}$ and $\widehat{C}$ such that

$$\widehat{B} \sqcup \widehat{C} = A \wedge \widehat{B} \equiv_{\mathrm{S}} B \wedge \widehat{C} \equiv_{\mathrm{S}} C,$$

thus satisfying the full requirement R.

We will choose a sequence of witnesses targeted for $Q \cap W_j$, and restrain the set A so that we can exercise control over the approximations to B and C by using ψ. After each witness enters W_j, we will drop all restraint on A for exactly one stage, so that the positive requirements P_k will have a chance to be satisfied. After the length-of-correctness function rises enough for us to monitor what has occurred because of the dropping of the restraint, we will examine the effect on B and C. At this point we will either enumerate our witness into Q and satisfy the subrequirement, or, if this is impossible, split the total change in A since the witness was chosen between $\widehat{B}$ and $\widehat{C}$ in a way that records the changes in B and C, respectively.

As an aid to understanding, we will describe the procedure for this strategy in detail and prove that it works, before giving the full construction for A in section 4.2 below. The strategy is complicated by the fact that we must divide the interval on which the permission is being sought into four pieces as more and more information about A, B, and C is provided by φ and ψ.

At each stage s at which the length of correctness increases, we will choose a witness x that we hope to enumerate into $Q \cap W_j$, and at the same time restrain A (thought of as a real) on 2^{-x}. Then we will wait for an expansionary stage $s_0 > \psi(s) > s$ such that $x \in W_j$. Let $s_1 > s_0$ be the least subsequent expansionary stage such that $l[s_1] > \varphi(s_0)$. At stage s_1 we drop the restraint on A, and then immediately reimpose this restraint at stage $s_1 + 1$. Finally, we end this attempt at permission by letting $s_2 > s_1 + 1$ be the least subsequent expansionary stage such that $l[s_2] > \psi(s_1 + 1)$. This stage is the one at which we hope to have finally gained permission to enumerate x into Q. If $B[s_2] - B[s_0] \geqslant 2^{-x-1}$ and $C[s_2] - C[s_0] \geqslant 2^{-x-1}$, then we can put x into Q, and the subrequirement S_j is thus permanently satisfied. Otherwise there are two possibilities:

1. If $C[s_2] - C[s_0] < 2^{-x-1}$, then we let $\widehat{B}[s_2] - \widehat{B}[s] = A[s_2] - A[\varphi(s_0)]$ and $\widehat{C}[s_2] - \widehat{C}[s] = A[\varphi(s_0)] - A[s]$.

2. If $B[s_2] - B[s_0] < 2^{-x-1}$ (and $C[s_2] - C[s_0] \geqslant 2^{-x-1}$), then we let $\widehat{C}[s_2] - \widehat{C}[s] = A[s_2] - A[\varphi(s_0)]$ and $\widehat{B}[s_2] - \widehat{B}[s] = A[\varphi(s_0)] - A[s]$.

In either case, we choose a new witness x' greater any number yet mentioned in the construction, restrain A on $2^{-x'}$, and repeat the entire cycle again, starting at s_2.

The point of this procedure is that we put the amount by which A changed during the stage at which it was unrestrained into the hatted set associated to the set that changed significantly, and the controlled part of A into the hatted version of the set that did not change enough to allow the enumeration of x into Q. If we are never able to enumerate a witness into Q, this procedure will give rise to an infinite sequence of pairs of stages at which we make the right decisions about which part of A to put into the hatted versions of each set.

We sum up the important facts about this sequence in the following definition. Call a computable sequence $s_2(-1) < s_0(0) < s_1(j) < s_2(0) < s_0(1) < \cdots$ *good for R* if there exists a computable sequence $x(0) < x(1) < \cdots$ and c.e. sets $\widehat{B}$ and $\widehat{C}$ such that for every $j \geqslant 0$,

(1) $s_0(j)$, $s_1(j)$, and $s_2(j)$ are expansionary stages;

(2) $s_0(j) > l[s_0(j)] > \psi(s_2(j-1))$;

(3) $s_1(j) > l[s_1(j)] > \varphi(s_0(j))$;

(4) $s_2(j) > l[s_2(j)] > \psi(s_1(j)+1)$;

(5) $A[s_2(j)] - A[s_1(j)+1] < 2^{-x(j)}$;

(6) $A[s_1(j)] - A[s_2(j-1)] < 2^{-x(j)}$;

(7) $C[s_2(j)] - C[s_0(j)] < 2^{-x(j)-1}$ if and only if $\widehat{B}[s_2(j)] - \widehat{B}[s_2(j-1)] = A[s_2(j)] - A[\varphi(s_0(j))]$ and $\widehat{C}[s_2(j)] - \widehat{C}[s_2(j-1)] = A[\varphi(s_0(j))] - A[s_2(j-1)]$; and

(8) $B[s_2(j)] - B[s_0(j)] < 2^{-x(j)-1}$ and $C[s_2(j)] - C[s_0(j)] \geqslant 2^{-x(j)-1}$ if and only if $\widehat{C}[s_2(j)] - \widehat{C}[s_2(j-1)] = A[s_2(j)] - A[\varphi(s_0(j))]$ and $\widehat{B}[s_2(j)] - \widehat{B}[s_2(j-1)] = A[\varphi(s_0(j))] - A[s_2(j-1)]$.

As we will show in Lemma 4.7 below, our strategy will give rise to a good sequence if we infinitely often ask for and fail to receive permission to enumerate into $Q \cap W_j$. For now, we show that the existence of such a sequence is enough to satisfy R.

Lemma 4.1. *If there is a sequence that is good for R, then there is a c.e. splitting $A = \widehat{B} \sqcup \widehat{C}$ such that $B \equiv_S \widehat{B}$ and $C \equiv_S \widehat{C}$.*

Proof. Let $s_2(-1) < s_0(0) < s_1(j) < s_2(0) < s_0(1) < \cdots$ be a good sequence, and let $x(j)$, $\widehat{B}$, and $\widehat{C}$ be as in the definition of a good sequence. We can assume $A[s_2(-1)] - A[0] \subset \widehat{B}$ without loss of generality, so $A = \widehat{B} \sqcup \widehat{C}$. We show $B \equiv_{\mathrm{S}} \widehat{B}$. The proof that $C \equiv_{\mathrm{S}} \widehat{C}$ is very similar. We will justify the nontrivial steps in our calculations by citing properties (1)–(8) in the definition of good sequence, as well as two auxiliary properties that we now derive from them.

Note that $s_1(k-1)+1 < \psi(s_1(k-1)+1) < s_2(k-1) < \psi(s_2(k-1))$ (the second inequality following by (4)), so (2) implies that

$$(2') \qquad l[s_0(k)] > s_1(k-1)+1.$$

Similarly, $s_1(k-1)+1 < s_0(k) < s_1(k)+1 < \psi(s_1(k)+1)$, so (4) implies that

$$(4') \qquad l[s_2(k)] > s_0(k) > s_1(k-1)+1.$$

We call a number j such that $C[s_2(j)] - C[s_0(j)] < 2^{-x(j)-1}$ a j of *type 1*; a j such that $B[s_2(j)] - B[s_0(j)] < 2^{-x(j)-1}$ and $C[s_2(j)] - C[s_0(j)] \geqslant 2^{-x(j)-1}$ is a j of *type 2*.

We first show that $\widehat{B} \leqslant_{\mathrm{S}} B$. Given s, let j be such that $s_0(j) > s$ and let $\delta(s) = s_2(j)$. Then

$$\begin{aligned}
\widehat{B} - \widehat{B}[\delta(s)] &= \sum_{k=j+1}^{\infty} \widehat{B}[s_2(k)] - \widehat{B}[s_2(k-1)] \\
&= \sum_{\substack{k \text{ type } 1 \\ k>j}} A[s_2(k)] - A[\varphi(s_0(k))] + \\
&\qquad \sum_{\substack{k \text{ type } 2 \\ k>j}} A[\varphi(s_0(k))] - A[s_2(k-1)] && \text{by (7) and (8)} \\
&\leqslant \sum_{\substack{k \text{ type } 1 \\ k>j}} A[s_2(k)] - A[\varphi(s_0(k))] + \sum_{k>j} 2^{-x(k)} && \text{by (3) and (6)} \\
&\leqslant \sum_{\substack{k \text{ type } 1 \\ k>j}} A[s_2(k)] - A[\varphi(s_0(k))] + 2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1 \\ k>j}} c((B+C)[s_2(k)] - (B+C)[s_0(k)]) + 2^{-x(j)} && \text{by } (4') \\
&= \sum_{\substack{k \text{ type } 1 \\ k>j}} c(B[s_2(k)] - B[s_0(k)]) + \\
&\qquad \sum_{\substack{k \text{ type } 1 \\ k>j}} c(C[s_2(k)] - C[s_0(k)]) + 2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1 \\ k>j}} c(B[s_2(k)] - B[s_0(k)]) + \\
&\qquad \sum_{\substack{k \text{ type } 1 \\ k>j}} c2^{-x(k)-1} + 2^{-x(j)} && \text{by (7)}
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{\substack{k \text{ type } 1\\ k>j}} c(B[s_2(k)] - B[s_0(k)]) + (c+1)2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(B[s_0(k+1)] - B[s_0(k)]) + (c+1)2^{-x(j)} \\
&\leqslant c(B - B[s_0(j)]) + (c+1)2^{-x(j)} \\
&\leqslant c(B - B[s]) + (c+1)2^{-x(j)}.
\end{aligned}
$$

Thus, by Lemma 1.1, $\widehat{B} \leqslant_{\mathrm{S}} B$.

We now show that $B \leqslant_{\mathrm{S}} \widehat{B}$. Given s, let j be such that $s_2(j-1) > s$ and let $\xi(s) = s_2(j)$. Then

$$
\begin{aligned}
B - B[\xi(s)] &= \sum_{k=j+1}^{\infty} B[s_2(k)] - B[s_2(k-1)] \\
&\leqslant \sum_{k=j+1}^{\infty} B[s_2(k)] - B[\psi(s_1(k-1)+1)] && \text{by (4)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + \\
&\qquad\qquad \sum_{\substack{k \text{ type } 2\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + \\
&\quad \sum_{\substack{k \text{ type } 2\\ k>j}} B[s_2(k)] - B[s_0(k)] + \\
&\qquad \sum_{\substack{k \text{ type } 2\\ k>j}} B[s_0(k)] - B[\psi(s_1(k-1)+1)] \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + \sum_{k=j+1}^{\infty} 2^{-x(k)-1} + \\
&\qquad \sum_{\substack{k \text{ type } 2\\ k>j}} B[s_0(k)] - B[\psi(s_1(k-1)+1)] && \text{by (8)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + 2^{-(x(j)-1)} + \\
&\qquad \sum_{\substack{k \text{ type } 2\\ k>j}} c(A[s_0(k)] - A[s_1(k-1)+1]) && \text{by (2')}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + 2^{-x(j)+1} + \\
&\qquad\qquad \sum_{\substack{k \text{ type } 2\\ k>j}} c(A[s_0(k)] - A[s_2(k-1)] + \\
&\qquad\qquad\qquad\qquad A[s_2(k-1)] - A[s_1(k-1)+1]) \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + c2^{-x(j)+2} + \\
&\qquad\qquad\qquad \sum_{k=j+1}^{\infty} c(2^{-x(k)} + 2^{-x(k-1)}) \qquad \text{by (5) and (6)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + 2^{-x(j)+1} + \\
&\qquad\qquad\qquad\qquad\qquad c(2^{-x(j)} + 2^{-x(j)+1}) \\
&= \sum_{\substack{k \text{ type } 1\\ k>j}} B[s_2(k)] - B[\psi(s_1(k-1)+1)] + (3c+2)2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(A[s_2(k)] - A[s_1(k-1)+1]) + (3c+2)2^{-x(j)} \qquad \text{by } (4') \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(A[s_2(k)] - A[s_1(k)] + A[s_1(k)] - A[s_2(k-1)] \\
&\qquad\qquad + A[s_2(k-1)] - A[s_1(k-1)+1]) + (3c+2)2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(A[s_2(k)] - A[s_1(k)] + 2^{-x(k)} + 2^{-x(k-1)}) + \\
&\qquad\qquad\qquad\qquad (3c+2)2^{-x(j)} \qquad \text{by (5) and (6)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(A[s_2(k)] - A[s_1(k)]) + c(2^{-x(j)} + 2^{-x(j)+1}) + \\
&\qquad\qquad\qquad\qquad (3c+2)2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(A[s_2(k)] - A[s_1(k)]) + (6c+2)2^{-x(j)} \\
&\leqslant \sum_{\substack{k \text{ type } 1\\ k>j}} c(A[s_2(k)] - A[\varphi(s_0(k))]) + (6c+2)2^{-x(j)} \qquad \text{by (3)} \\
&= \sum_{\substack{k \text{ type } 1\\ k>j}} c(\widehat{B}[s_2(k)] - \widehat{B}[s_2(k-1)]) + (6c+2)2^{-x(j)} \qquad \text{by (7)}
\end{aligned}
$$

$$= c(\widehat{B} - \widehat{B}[s_2(j-1)]) + (6c+2)2^{-x(j)}$$
$$\leqslant c(\widehat{B} - \widehat{B}[s]) + (6c+2)2^{-x(j)}$$

Thus, by Lemma 1.1, $B \leqslant_{\mathrm{S}} \widehat{B}$. □

4.2 Details of the construction

Coordinating the activities of all our strategies, particularly the ones for satisfying R-requirements, is naturally done using the $\mathbf{0}'''$-priority method of Lachlan, implemented via the tree of strategies $\mathcal{T} = 2^{<\omega}$. For the P-strategies ensuring noncomputability of A, there is really no need to assign outcomes on $\mathcal{T}$, since their actions are finitary, but we place these strategies on $\mathcal{T}$ anyway for the sake of uniformity. We wish to arrange our requirements on $\mathcal{T}$ so that for every infinite path f, the following hold.

1. Each P_e, each N_e, and each $\mathrm{R}_{\langle e,c\rangle}$ is assigned to some node in f.
2. If $\mathrm{R}_{\langle e,c\rangle}$ is assigned to α and $\alpha^\frown 0$ is on f (0 being the infinitary outcome of $\mathrm{R}_{\langle e,c\rangle}$, which indicates that there are infinitely many expansionary stages), then each $\mathrm{S}_{\langle e,c,i\rangle}$ is assigned to some node in f *unless* some $\mathrm{S}_{\langle e,c,i\rangle}$ is assigned to α and $\alpha^\frown 0$ is in f (0 being the outcome of $\mathrm{S}_{\langle e,c,i\rangle}$ indicating that it succeeds in satisfying $\mathrm{R}_{\langle e,c\rangle}$), in which case no requirement $\mathrm{S}_{\langle e,c,j\rangle}$ is assigned to any proper extension of α.

Formally, requirements are assigned to strategies in the tree by using a list function L from $2^{<\omega} \times \omega$ to the set of all requirements. Each node $\sigma \in \mathcal{T}$ will have requirement $L(\sigma, 0)$ assigned to it. If σ has a requirement P_e assigned to it, we say σ has *type P*. If σ has a requirement N_e assigned to it, we say σ has *type N*. If σ has a requirement $\mathrm{R}_{\langle e,c\rangle}$ assigned to it, we say σ has *type R*. If σ has a subrequirement $\mathrm{S}_{\langle e,c,i\rangle}$ assigned to it we say σ has *type S*. In this case, if $\tau \subset \sigma$ is the longest node included in σ with requirement $\mathrm{R}_{\langle e,c\rangle}$ assigned to it, we say σ *works for* τ. We define $L(\sigma, k)$ recursively on $|\sigma|$. Let λ denote the empty sequence. Let $L(\lambda, 3e) = \mathrm{P}_e$, let $L(\lambda, 3e+1) = \mathrm{N}_e$, and let $L(\lambda, 3\langle e,c\rangle+2) = \mathrm{R}_{\langle e,c\rangle}$. Suppose $\sigma \neq \lambda$, and let $\sigma_0 = \sigma \restriction (|\sigma|-1)$. If σ_0 has type P or N, or σ_0 has type R and $\sigma = \sigma_0{}^\frown 1$, then for all k, let $L(\sigma, k) = L(\sigma_0, k+1)$. If σ_0 has type R and $\sigma = \sigma_0{}^\frown 0$, then let $L(\sigma, 2k) = L(\sigma_0, k+1)$, and let $L(\sigma, 2\langle c,i\rangle + 1) = \mathrm{S}_{\langle e,c,i\rangle}$. Finally, suppose σ_0 works for a strategy $\tau \subset \sigma_0$. If $\sigma = \sigma_0{}^\frown 1$, let $L(\sigma, k) = L(\sigma_0, k+1)$. Otherwise, $\sigma = \sigma_0{}^\frown 0$. In this case, τ's requirement is satisfied at σ, so we remove all of τ's subrequirements below $\sigma^\frown 0$. Let $n^{\sigma^\frown 0}(0) = 1$, and let $n^{\sigma^\frown 0}(k+1)$ be the least $n > n^{\sigma^\frown 0}(k)$ such that $\forall j\, (L(\sigma, n) \neq L(\tau, 2j+1))$. Then, for all k, let $L(\sigma^\frown 0, k) = L(\sigma, n^{\sigma^\frown 0}(k))$.

It is not hard to check that this assignment satisfies the conditions listed above, but we will do a formal verification of its relevant properties in Lemma 4.3 below.

A node is *initialized* by undefining all parameters and functionals assigned to it. At each stage s we define an approximation $g[s]$ to the true path g consisting of a sequence of nodes accessible at stage s. All nodes to the right of $g[s]$ are initialized at stage s. If $\alpha \subseteq g[s]$, we say s is an *α-stage*.

Action for a node of type P

Suppose α has requirement P_e assigned to it and $s+1$ is an α-stage. If $(W_e \cap A)[s] \neq \emptyset$, then do nothing for α and let $\alpha^\frown 0$ be accessible at stage $s+1$.

Otherwise, act as follows. If x^α is currently undefined, let x^α be the least element of $\{ n^2 : n \in \mathbb{N} \}$ greater than any number yet mentioned in the construction, initialize all $\beta \supset \alpha$, immediately end stage $s+1$, and go on to stage $s+2$.

If x^α is defined and $x^\alpha \notin W_e[s]$, then do nothing for α and let $\alpha^\frown 0$ be accessible at stage $s+1$.

If x^α is defined and $x^\alpha \in (W_e - A)[s]$, then enumerate x^α into A, initialize all $\beta \supset \alpha$, immediately end stage $s+1$, and go on to stage $s+2$.

Action for a node of type N

Suppose α has requirement N_e assigned to it. Recall that

$$l_e[s] = \max\left\{ x : \forall y < x \, (\Phi_e^{U_e}(y)[s]\downarrow = \Phi_e^{V_e}(y)[s]\downarrow) \right\}.$$

We use l^e to define a sequence of *α-expansionary* stages by recursion, and to describe the action of α at α-stages.

Let $s+1$ be an α-stage. If α has been initialized since the last α-stage, declare stage $s+1$ to be α-expansionary, and let $\alpha^\frown 0 = g[s]$, but immediately end stage $s+1$ and go on to stage $s+2$. Otherwise, let s^- be the last previous $\alpha^\frown 0$-stage, and s_0 be the stage at which α was last initialized. There are two possibilities. If $A[s^- + 1] \nsubseteq (U_e \sqcup V_e)$ or there exists some α-stage $t+1$ with $s_0 \leqslant t+1 \leqslant s^-$ such that $l_e[t] \geqslant l_e[s]$, then let $\alpha^\frown 1$ be accessible at stage $s+1$. Otherwise, declare s to be α-expansionary and let $\alpha^\frown 0$ be accessible at stage $s+1$.

Action for a node of type R

Suppose α has requirement $\mathrm{R}_{\langle e,c\rangle}$ assigned to it. Our length-of-correctness function $l^\alpha[s]$ looks for the most recent stage below which the relevant Solovay reductions appear to be correct. That is, let

$$\begin{aligned} l^\alpha[s] = \mu t\big((B_e + C_e)[s] - (B_e + C_e)[\psi_e(t)] \geqslant c \cdot (A[s] - A[t]) \vee \\ A[s] - A[\varphi_e(t)] \geqslant c \cdot ((B_e + C_e)[s] - (B_e + C_e)[t])\big). \end{aligned}$$

If $s+1$ is an α-stage, then we say that stage s is *α-expansionary* if for all α-stages $t+1 \leqslant s$ since α was last initialized, $l^\alpha[t] < l^\alpha[s]$.

If s is not α-expansionary, then let $\alpha^\frown 1$ be accessible at stage $s+1$. If s is α-expansionary and there is no link set with top α, let $\alpha^\frown 0$ be accessible at stage $s+1$. Any link set with top α will have an associated target value. Suppose s is α-expansionary and there is a link set with top α and target value k. If $l^\alpha[s] > \max\{\psi_e(k), \varphi_e(k)\}$, then let the $\beta \supset \alpha$ at the bottom of the link act and remove the link; otherwise, let $\alpha^\frown 1$ be accessible at stage $s+1$.

Action for a node of type S

Suppose α has requirement $\mathrm{S}_{\langle e,c,i\rangle}$ assigned to it, and works for a node $\beta \subset \alpha$ (with $\mathrm{R}_{\langle e,c\rangle}$ assigned to β). Let $s+1$ be an α-stage. Let s^- be the last previous α-stage, or the stage at which α was last initialized. The task of α is to make a series of attempts to satisfy $Q_\beta \neq \overline{W_i}$. If $s+1$ is the first α-stage since α was last initialized, let $x^\alpha(0)$ be the least number greater than any yet mentioned in the construction, and let $s_2^\alpha(-1) = s$. In this case, let $\alpha^\frown 1$ be accessible at stage $s+1$.

Otherwise, there are several cases.

Case 1: There exists a j such that $x^\alpha(j)$ is defined and $x^\alpha(j)$ has not yet been involved in an attack. If $x^\alpha(j) \notin W_i[s]$ or $l^\beta[s] \leqslant \psi(s_2(j-1))$, take no action for α, and let $\alpha^\frown 1$ be accessible at stage $s+1$. Otherwise, declare $x^\alpha(j)$ to be *involved in an attack*, set a link *of type* s_0 with *target value* $s+1$ to β, and let $s_0^\alpha(j) = s$. Immediately end stage $s+1$ and go on to stage $s+2$.

Case 2: There exists a j such that $x^\alpha(j)$ is involved in an attack, and α has been reached by traveling a link of type s_0 with top β. Set a link of type s_1 with target value $s+1$ to β, let $s_1^\alpha(j) = s$, and let $\alpha^\frown 0$ be accessible at stage $s+1$.

Case 3: There exists a j such that $x^\alpha(j)$ is involved in an attack, and α has been reached by traveling a link of type s_1 with top β. There are three possibilities:

(a) If $B_e[s] - B_e[s^-] \geqslant 2^{-x^\alpha(j)-1}$ and $C_e[s] - C_e[s^-] \geqslant 2^{-x^\alpha(j)-1}$, then let $x^\alpha(j) \in Q_\beta[s+1]$ and declare $x^\alpha(j)$ to have *succeeded*. Let $\alpha^\frown 1$ be accessible at stage $s+1$.

(b) If $C_e[s] - C_e[s^-] < 2^{-x^\alpha(j)-1}$, then let

$$\widehat{B_e^\alpha}[s+1] = \widehat{B_e^\alpha}[s_2^\alpha(j-1)] + (A[s+1] - A[\varphi_e(s_0^\alpha(j))]),$$

and let

$$\widehat{C_e^\alpha}[s+1] = \widehat{C_e^\alpha}[s_2^\alpha(j-1)] + (\varphi_e(A[s_0^\alpha(j)))] - A[s_2^\alpha(j-1)]).$$

(c) If $B_e[s] - B_e[s^-] < 2^{-x^\alpha(j)-1}$ and $C_e[s] - C_e[s^-] \geqslant 2^{-x^\alpha(j)-1}$, then let

$$\widehat{C_e^\alpha}[s+1] = \widehat{C_e^\alpha}[s_2^\alpha(j-1)] + (A[s+1] - A[\varphi_e(s_0^\alpha(j))]),$$

let

$$\widehat{B_e^\alpha}[s+1] = \widehat{B_e^\alpha}[s_2^\alpha(j-1)] + (A[\varphi_e(s_0^\alpha(j))] - A[s_2^\alpha(j-1)]).$$

If either (b) or (c) holds, declare $x^\alpha(j)$ to have *failed*, define $s_2^\alpha(j) = s$, and let $x^\alpha(j+1)$ be the least number greater than any yet mentioned in the construction. In this case, immediately end stage $s+1$ and go on to stage $s+2$.

4.3 Verification

Define the orderings $\leqslant$ and $<_{\mathrm{L}}$ on $\mathcal{T}$ as in the proof of Theorem 1. Define the true path g of the construction as the leftmost path of nodes that are included in $g[s]$ infinitely often. In other words, $\alpha \subset g$ if and only if $\exists t \forall s > t\, (\alpha \leqslant g[s])$ and $\forall t \exists s > t\, (\alpha \subset g[s])$.

We show that every requirement is satisfied by the actions of some node on the true path. It follows from a straightforward induction that the true path is infinite, since an examination of the actions taken for each node show that if $\alpha \subset g$ and $\alpha \subset g[s]$, then there exists some $t > s$ such that some proper extension of α is included in $g[t]$. It may not be so obvious that each node included in g is accessible (that is, gets an opportunity to act) infinitely often, because of the linking procedure, so we begin by proving that this is the case. We then show that the requirements are properly distributed along $\mathcal{T}$.

Lemma 4.2. *If $\sigma \subset g$, then there exist infinitely many stages s at which σ is accessible.*

Proof. Suppose $\tau \subset g$ is accessible infinitely often. If τ is of type P, N, or S, then τ can never be the top of a link, so either $\tau^\frown 0$ or $\tau^\frown 1$ is accessible infinitely often. Now suppose τ is of type R. If there are only finitely many τ-expansionary stages, then $\tau^\frown 1$ is accessible infinitely often. Otherwise, $\tau^\frown 0 \subset g[s]$ at infinitely many stages s. Suppose a link with top τ is set by some node $\sigma \supseteq \tau^\frown 0$ at stage t during an attack involving $x^\sigma(j)$. Then the only nodes extending $\tau^\frown 0$ that can act until after the link is removed at stage $s_1^\sigma(j)+1$ are nodes extending $\sigma^\frown 0$. But none of these can possibly work for a subrequirement of τ, since no such subrequirement is in the range of the function $L(\sigma, \cdot)$. Hence, no link with top τ can be set be set again until $\tau^\frown 0$ is accessible again after stage $s_2^\sigma(j)$, which will happen after the next τ-expansionary stage. □

Lemma 4.3. *For all $n \geqslant 1$ and all σ, if f is an infinite path with $\sigma \subset f$, then either there exists a τ with $\sigma \subseteq \tau \subset f$ and $L(\sigma, n) = L(\tau, 0)$; or $L(\sigma, n)$ is a subrequirement introduced by some $\sigma_0 \subset \sigma$, and there exists $\rho^\frown 0 \subset f$ such that ρ is a strategy working for σ_0.*

Proof. The proof is by induction. Note that $L(\sigma, 1) = L(\sigma^\frown 0, 0) = L(\sigma^\frown 1, 0)$. Suppose $\tau \supset \sigma$ has $L(\sigma, n) = L(\tau, 0)$. Then if $L(\sigma, n+1)$ is never removed at any extension of σ compatible with τ, there exists some k such that $L(\sigma, n+1) = L(\tau, k)$. For all positive $j < k$, the value of $L(\tau, j)$ must be a requirement of type S, since only this kind of requirement is newly introduced along any path. If τ is not of type S, or if $\tau^\frown 1 \subset f$, then $L(\sigma, n+1) = L(f \upharpoonright (|\tau| + k), 0)$; otherwise, $L(\sigma, n+1) = L(f \upharpoonright (|\tau| + 2k), 0)$. □

By Lemmas 4.2 and 4.3, every requirement in the range of the original list $L(\lambda, \cdot)$ is assigned to some node on the true path that is accessible infinitely often. This fact makes it relatively simple to show that the requirements of types P and N are satisfied. Notice that nodes on the true path are initialized only finitely often.

Lemma 4.4. *Suppose $\alpha \subset g$, and α has requirement P_e assigned to it. Then $A \neq \overline{W_e}$.*

Proof. Choose s_0 least such that for all $t \geqslant s_0$, the node α is not initialized at t and $\alpha \leqslant g[t]$. Let $t+1 > s$ be any α-stage such that $(W_e \cap A)[t] = \emptyset$. Then $x = x^\alpha$ is defined at stage $t+1$ and is never undefined thereafter. If $x \notin W_e$, then $x \in \overline{W_e} - A$, since P_e never puts x into A, and no other positive strategy ever has x as a witness. Otherwise, if $t^+ + 1 > t$ is the least α-stage such that $x \in W_e[t^+]$, then $x \in (W_e \cap A)[t^+ + 1]$. In either case $A \neq \overline{W_e}$. □

Lemma 4.5. *Suppose $\alpha \subset g$ has requirement N_e assigned to it and $U_e \sqcup V_e = A$. If $\Phi_e^{U_e} = \Phi_e^{V_e}$ is total, then $\Phi_e^{U_e}$ is computable.*

Proof. The hypotheses imply that there are infinitely many α-expansionary stages. Choose s_0 least such that for all $t \geqslant s_0$, the node α is not initialized at t and $\alpha \leqslant g[t]$. Given x, let $s > s_0$ be an α-expansionary stage such that $\Phi_e^{U_e}(x)[s]\downarrow = \Phi_e^{V_e}(x)[s]\downarrow$. Let s^+ be the next α-expansionary stage after s. If no $\beta \supset \alpha^\frown 0$ set any link with top $\gamma \subset \alpha$ at stage $s+1$, or if some $\beta \supset \alpha^\frown 0$ set a link of type s_1 with top $\gamma \subset \alpha$ at stage $s+1$, then only nodes β such that $\alpha^\frown 0 <_{\mathrm{L}} \beta$ can have acted at any stage t with $s < t \leqslant s^+$. Hence only one node extending $\alpha^\frown 0$ can have enumerated any number into A at any stage t with $s < t \leqslant s^+$. On the other hand, if some $\beta \supset \alpha^\frown 0$ set a link of type s_0 with top $\gamma \subset \alpha$ at stage $s+1$, then no node can have enumerated any element into A at stage $s+1$. Also, at most one node extending $\alpha^\frown 0$ can have enumerated an element into A at any stage t with $s < t \leqslant s^+$, namely, at the stage at which the link was removed. So, in this case as well, only one node extending $\alpha^\frown 0$ can have enumerated any number into A at any stage t with $s < t \leqslant s^+$. Since $s^+ + 1$ is α-expansionary, and all nodes to the right of $\alpha^\frown 0$ have witnesses greater than $\min\{\varphi_e^{U_e}(x)[s], \varphi_e^{V_e}(x)[s]\}$, it follows that $\Phi_e^{U_e}(x)[s^+] = \Phi_e^{U_e}(x)[s]$, since at least one side of the computation $\Phi_e^{U_e}(x)[s]\downarrow = \Phi_e^{V_e}(x)[s]\downarrow$ has been preserved. Thus, $\Phi_e^{U_e}(x)$ has the same value at every α-expansionary stage after s, and hence the function $\Phi_e^{U_e}$ is computable. □

Finally, we show in the following two lemmas that the requirements of type R are satisfied.

Lemma 4.6. *Suppose $\beta \subset g$ has requirement $R_{\langle e,c\rangle}$ assigned to it, $B_e + C_e \leqslant_S A$ via c, ψ_e, and $A \leqslant_S B_e + C_e$ via c, φ_e. Then $Q_\beta \leqslant_S B_e$ and $Q_\beta \leqslant_S C_e$.*

Proof. We show that $Q_\beta \leqslant_{\mathrm{S}} B_e$, the proof for C_e being analogous.

First we claim that $\lim_s l^\beta[s] = \infty$. Suppose not, so that there is a t such that $l^\beta[s] = t$ for infinitely many s. Then there are either infinitely many s such that

$$(B_e + C_e)[s] - (B_e + C_e)[\psi_e(t)] \geqslant c \cdot (A[s] - A[t])$$

or infinitely many s such that

$$A[s] - A[\varphi_e(t)] \geqslant c \cdot ((B_e + C_e)[s] - (B_e + C_e)[t]).$$

Suppose that the former case holds (the other case being analogous). Then

$$\begin{aligned}(B_e + C_e) - (B_e + C_e)[\psi_e(t)] &= \lim_s (B_e + C_e)[s] - (B_e + C_e)[\psi_e(t)] \\ &\geqslant \lim_s c \cdot (A[s] - A[t]) \\ &= c \cdot (A - A[t]),\end{aligned}$$

contradicting the hypothesis that $B_e + C_e \leqslant_{\mathrm{S}} A$ via c, ψ_e. Thus $\lim_s l^\beta[s] = \infty$, and hence there are infinitely many β-expansionary stages.

If there are only finitely many stages at which some number is enumerated into Q_β, then we are done. Otherwise, given s, let s^+ be the next β-stage after s. Let y be the first number enumerated into Q_β after stage s^+. This enumeration must happen at an β-stage $\xi(t)+1$ at which the action of β is dictated by Case 3(a) on page 22. Thus $B_e[\xi(t)] - B_e[s^+] \geqslant 2^{-y-1}$. Furthermore, every number enumerated into Q_β after stage $\xi(t)+1$ is bigger than y, so

$$Q_\beta - Q_\beta[\xi(t)] \leqslant 2^{-y+1} < 5 \cdot (B_e[\xi(t)] - B_e[s^+]) \leqslant 5 \cdot (B_e - B_e[s]).$$

Thus $Q_\beta \leqslant_{\mathrm{S}} B$ via $\xi, 5$. □

Lemma 4.7. *Suppose $\alpha \subset g$ is assigned the subrequirement $S_{\langle e,c,i\rangle}$ and $\beta \subset \alpha$ is assigned the corresponding requirement $R_{\langle e,c\rangle}$. Then*

(a) either $Q_\beta \neq \overline{W_i}$ or $\alpha^\frown 0 \subset g$; and

(b) if $\alpha^\frown 0 \subset g$, then there exists a good sequence for $R_{\langle e,c\rangle}$.

Proof. Let s be a stage such that for all $t \geqslant s$, the node α is not initialized at t and $\alpha \leqslant g[t]$. If $Q_\beta = \overline{W_i}$, then $\overline{Q_\beta} = W_i$, so there exists an infinite computable sequence of failures $x^\alpha(0), x^\alpha(1), \ldots$ at stages after s, which gives rise to a computable sequence of stages $s_2^\alpha(-1) < s_0^\alpha(0) < s_1(0) < s_2^\alpha(0) < s_0^\alpha(1) < \cdots$ as defined in the description of the action of $\mathrm{S}_{\langle e,c,i\rangle}$. We will show that this sequence is good for $\mathrm{R}_{\langle e,c\rangle}$. For convenience, we repeat here the eight conditions that we need to verify:

(1) $s_0^\alpha(j)$, $s_1^\alpha(j)$, and $s_2^\alpha(j)$ are expansionary stages;

(2) $s_0^\alpha(j) > l^\beta[s_0^\alpha(j)] > \psi_e(s_2^\alpha(j-1))$;

(3) $s_1^\alpha(j) > l^\beta[s_1^\alpha(j)] > \varphi_e(s_0^\alpha(j))$;

(4) $s_2^\alpha(j) > l^\beta[s_2^\alpha(j)] > \psi_e(s_1^\alpha(j)+1)$;

(5) $A[s_2^\alpha(j)] - A[s_1^\alpha(j)+1] < 2^{-x^\alpha(j)}$;

(6) $A[s_1^\alpha(j)] - A[s_2^\alpha(j-1)] < 2^{-x^\alpha(j)}$;

(7) $C_e[s_2^\alpha(j)] - C_e[s_0^\alpha(j)] < 2^{-x^\alpha(j)-1}$ if and only if $\widehat{B_e^\alpha}[s_2^\alpha(j)] - \widehat{B_e^\alpha}[s_2^\alpha(j-1)] = A[s_2^\alpha(j)] - A[\varphi_e(s_0^\alpha(j))]$ and $\widehat{C_e^\alpha}[s_2(j)] - \widehat{C_e^\alpha}[s_2^\alpha(j-1)] = A[\varphi_e(s_0^\alpha(j))] - A[s_2^\alpha(j-1)]$; and

(8) $B_e[s_2^\alpha(j)] - B[s_0^\alpha(j)] < 2^{-x^\alpha(j)-1}$ and $C_e[s_2^\alpha(j)] - C_e[s_0^\alpha(j)] \geqslant 2^{-x^\alpha(j)-1}$ if and only if $\widehat{C_e^\alpha}[s_2^\alpha(j)] - \widehat{C_e^\alpha}[s_2^\alpha(j-1)] = A[s_2^\alpha(j)] - A[\varphi(s_0^\alpha(j))]$ and $\widehat{B_e^\alpha}[s_2^\alpha(j)] - \widehat{B_e^\alpha}[s_2^\alpha(j-1)] = A[\varphi_e(s_0^\alpha(j))] - A[s_2^\alpha(j-1)]$.

Condition (1) is met by definition. Each $s_0^\alpha(j)$ is chosen (in Case 1 of the description of the action of $\mathrm{S}_{\langle e,c,i\rangle}$) so that condition (2) is met. By Case 2 of the description of the action of $\mathrm{S}_{\langle e,c,i\rangle}$, for each j, the node α is reached at stage $s_1^\alpha(j)+1$ by traveling a link of type s_0 with top β. This link was set at stage $s_0^\alpha(j)+1$, and hence has target value $s_0^\alpha(j)+1$. Thus, by the description of the action of $\mathrm{R}_{\langle e,c\rangle}$, condition (3) is met. Similarly, for each j, the node α is reached at stage $s_2^\alpha(j)+1$ by traveling a link of type s_1 with top β and target value $s_1^\alpha(j)+1$, so condition (4) is met.

Since $\alpha^\frown 0 \subset g[s_1(j)]$ for every j, we have $\alpha^\frown 0 \subset g$. Once $x^\alpha(j)$ is defined, which happens at stage $s_2^\alpha(j-1)$, nodes to the right of $\alpha^\frown 0$ can enumerate only elements greater than $x^\alpha(j)$ into A, and nodes extending $\alpha^\frown 0$ are accessible only at stages at which links of type s_1 are set, which are exactly the stages of the form $s_1(j)+1$ for some j. (Nodes to the left of $\alpha^\frown 0$ have stopped acting by the choice of s.) Thus the only stage after $s_2^\alpha(j-1)$ at which a number greater than or equal to $x^\alpha(j)$ can be enumerated into A is $s_1(j)+1$, which implies that conditions (5) and (6) are met.

Finally, the definitions of $\widehat{B_e^\alpha}$ and $\widehat{C_e^\alpha}$ in Cases 3(b)and 3(c) of the description of the action of $\mathrm{S}_{\langle e,c,i\rangle}$ show that conditions (7) and (8) are met. □

Fix a requirement $\mathrm{R}_{\langle e,c\rangle}$ and let β be a node on the true path that is assigned to this requirement. If there exists a good sequence for $\mathrm{R}_{\langle e,c\rangle}$, then the requirement is satisfied, by Lemma 4.1. Otherwise, by Lemma 4.7, no node α on the true path working for β can have outcome 0 on the true path. Since only this outcome can remove the requirements in $L(\beta,\cdot)$ from the true path, every subrequirement $S_{\langle e,c,i\rangle}$ is assigned to some node on the true path, by Lemma 4.3. But then Q_β is noncomputable by Lemma 4.7, so by Lemma 4.6, $\mathrm{R}_{\langle e,c\rangle}$ is satisfied.

Hence every R-requirement is satisfied. Lemmas 4.4 and 4.5 show that the same is true of every P-requirement and every N-requirement. Thus Theorem 2 holds. □

Our result is another in the series of results showing the undecidability of the theories of the structures associated with virtually every nontrivial reducibility. We conjecture that the degree of the theory here is as high as possible: namely that of true arithmetic, but so far we have no way of interpreting the natural numbers into the structure.

A related question is whether or not structures associated more directly with prefix-free complexity have undecidable theories. Given a fixed universal prefix-free oracle machine M and two strings σ and τ, the *prefix-free complexity of* σ *relative to* τ, $K(\sigma \mid \tau)$, is the length of the shortest μ such that $M^\tau(\mu) = \sigma$. Say $\alpha \leqslant_{\mathrm{rK}} \beta$ if and only if there is a constant c such that for all n, $K(\alpha \upharpoonright n \mid \beta \upharpoonright n) \leqslant c$. It is not hard to see that $\alpha \leqslant_{\mathrm{rK}} \beta$ if and only if there are a computable function

f and a constant k such for all n there exists $j \leqslant k$ for which $f(\alpha \upharpoonright n, j) = \beta \upharpoonright n$. It turns out that if $\alpha \leqslant_{\mathrm{S}} \beta$ or if $\alpha \leqslant_{\mathrm{sw}} \beta$, then $\alpha \leqslant_{\mathrm{rK}} \beta$, although the reverse is not necessarily true. Similarly, rK-reducibility implies Turing reducibility, although the reverse does not hold. Thus this relation provides an intermediate reducibility between the Solovay and Turing reducibilities. It seems natural to conjecture that the theory of the associated degree structure on the c.e. reals is also undecidable.

Another relation even more closely related to the relative randomness of one real to another is the measure $\leqslant_K$, where $\alpha \leqslant_K \beta$ if and only if there exists a c such that for all n, $K(\alpha \upharpoonright n) \leqslant K(\beta \upharpoonright n)+c$. Although this relation is merely a measure of complexity, not a true reducibility, we again conjecture that the associated degree structure on the c.e. reals has an undecidable theory.

References

[1] S. Burris and R. McKenzie, *Decidability and Boolean Representations*, Memoirs of the Amer. Math. Soc. 246, American Mathematical Society, Providence, Rhode Island, July 1981.

[2] S. Burris and H. P. Sankappanavar, *Lattice theoretic decision problems in universal algebra*, Algebra Universalis 5 (1975), 163–177.

[3] C. S. Calude, R. Coles, P. H. Hertling, and B. Khoussainov, *Degree-theoretic aspects of computably enumerable reals*, in *Models and Computability* (S. B. Cooper and J. K. Truss, eds.), London Math. Soc. Lecture Note Ser. 259, Cambridge University Press, 1999, 23–39.

[4] C. S. Calude, P. H. Hertling, B. Khoussainov, and Y. Wang, *Recursively enumerable reals and Chaitin Ω numbers*, Theoret. Comput. Sci. 255 (2001), 125–149 (extended abstract in STACS 98, Lect. Notes in Comput. Sci. 1373, Springer-Verlag, Berlin, 1998, 596–606).

[5] G. J. Chaitin, *A theory of program size formally identical to information theory*, J. Assoc. Comput. Mach. 22 (1975), 329–340.

[6] R. Downey, *Some computability-theoretic aspects of reals and randomness*, in *The Notre Dame Lectures* (P. Cholak, ed.), Lect. Notes Log. 18, Assoc. for Symbol. Logic, Urbana, IL, 2005, 97–148.

[7] R. Downey and D. R. Hirschfeldt, *Algorithmic Randomness and Complexity*, Springer-Verlag, in preparation.

[8] R. Downey, D. R. Hirschfeldt, and G. LaForte, *Randomness and reducibility*, J. Comput. System Sci. 68 (2004), 96–114 (extended abstract in *Mathematical Foundations of Computer Science 2001* (J. Sgall, A. Pultr, and P. Kolman, eds.), Lect. Notes in Comput. Sci. 2136, Springer-Verlag, 2001, 316–327).

[9] R. Downey, D. R. Hirschfeldt, and A. Nies, *Randomness, computability, and density*, SIAM J. Comput. 31 (2002), 1169–1183 (extended abstract in *STACS 2001 Proceedings* (A. Ferreira and H. Reichel, eds.), Lect. Notes in Comput. Sci. 2010, Springer-Verlag, 2001, 195–201).

[10] R. Downey, D. R. Hirschfeldt, A. Nies, and F. Stephan, *Trivial reals*, in *Proceedings of the 7th and 8th Asian Logic Conferences* (R. Downey, D. Decheng, T. S. Ping, Q. Y. Hui, and M. Yasugi, eds.), Singapore University Press and World Scientific, 2003, 103–131 (extended abstract in Electronic Notes in Theoretical Computer Science 66 (2002), no. 1).

[11] R. Downey, D. R. Hirschfeldt, A. Nies, and S. A. Terwijn, *Calibrating Randomness*, to appear in Bull. Symbolic Logic.

[12] R. Downey and A. Nies, *Undecidability results for low complexity time classes*, J. Comput. System Sci. 60 (2000), 465–479.

[13] A. Kučera and T. A. Slaman, *Randomness and recursive enumerability*, SIAM J. Comput. 31 (2001) 199–211.

[14] M. Li and P. Vitányi, *An Introduction to Kolmogorov Complexity and its Applications*, 2nd edition, Springer-Verlag, 1997.

[15] P. Martin-Löf, *The definition of random sequences*, Inform. and Control 9 (1966), 602–619.

[16] A. Nies, *Intervals of the lattice of c.e. sets and effective Boolean algebras*, Bull. Lond. Math. Soc. 29 (1997), 683–692.

[17] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, 1987.

[18] R. M. Solovay, *Draft of a paper (or series of papers) on Chaitin's work*, unpublished manuscript, IBM Thomas J. Watson Research Center, New York, May 1975, 215 pp.

[19] R. M. Solovay, *On random r.e. sets*, in *Non-Classical Logics, Model Theory, and Computability* (A. Arruda, N. Da Costa, and R. Chuaqui, eds.), Studies in Logic and the Foundations of Mathematics 89, North-Holland, Amsterdam, 1977, 283–307.