ON NOTIONS OF COMPUTABILITY-THEORETIC REDUCTION BETWEEN Π^1_2 PRINCIPLES

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Abstract. Several notions of computability-theoretic reducibility between Π^1_2 principles have been studied. This paper contributes to the program of analyzing the behavior of versions of Ramsey's Theorem and related principles under these notions. Among other results, we show that for each $n \geqslant 3$, there is an instance of RT^n_2 all of whose solutions have PA degree over $\emptyset^{(n-2)}$, and use this to show that König's Lemma lies strictly between RT^2_2 and RT^3_2 under one of these notions. We also answer two questions raised by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [2016] on comparing versions of Ramsey's Theorem and of the Thin Set Theorem with the same exponent but different numbers of colors. Still on the topic of the effect of the number of colors on the computable aspects of Ramsey-theoretic properties, we show that for each $m \geqslant 2$, there is an $(m+1)$-coloring c of $\mathbb{N}$ such that every m-coloring of $\mathbb{N}$ has an infinite homogeneous set that does not compute any infinite homogeneous set for c, and connect this result with the notion of infinite information reducibility introduced by Dzhafarov and Igusa [ta]. Next, we introduce and study a new notion that provides a uniform version of the idea of implication with respect to ω-models of RCA_0, and related notions that allow us to count how many applications of a principle P are needed to reduce another principle to P. Finally, we fill in a gap in the proof of Theorem 12.2 in Cholak, Jockusch, and Slaman [2001].

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Contents

1. Introduction

Many of the mathematical principles that have been analyzed in the setting of reverse mathematics have the form

$$\forall X\,[\Theta(X) \rightarrow \exists Y\,\Psi(X,Y)],$$

where X and Y range over sets of natural numbers and Θ and Ψ are arithmetic. We think of a true sentence of this form as a *problem*. An *instance* of this problem is an X such that $\Theta(X)$ holds and a *solution* to this instance is a Y such that $\Psi(X,Y)$ holds. From a computability-theoretic point of view, several ways of comparing the relative strength of two such problems have been studied. These are different ways to formalize the idea of reducing a problem P to a problem Q, and each of them can give us interesting information within the broad program of analyzing the computability-theoretic content of mathematics.

Definition 1.1. Let P and Q be problems.

(1) Recall that a *Turing ideal* is a nonempty collection of sets closed under join and closed downward under Turing reducibility. A problem R *holds* in a Turing ideal I if for every instance $X \in I$ of R, there is a solution $Y \in I$ to X. We write $P \leqslant_\omega Q$ to mean that for every Turing ideal I, if Q holds in I then so does P. In the language of reverse mathematics, this definition says that every ω-model of $\mathrm{RCA}_0 + Q$ is a model of P.

(2) We say that P is *computably reducible* to Q, and write $P \leqslant_{\mathrm{c}} Q$, if for every instance X of P, there is an X-computable instance $\widehat{X}$ of Q such that, for every solution $\widehat{Y}$ to $\widehat{X}$, there is an $X \oplus \widehat{Y}$-computable solution to X.

(3) We say that P is *strongly computably reducible* to Q, and write $P \leqslant_{\mathrm{sc}} Q$, if for every instance X of P, there is an X-computable instance $\widehat{X}$ of Q such that, for every solution $\widehat{Y}$ to $\widehat{X}$, there is a $\widehat{Y}$-computable solution to X.

(4) We say that P is *Weihrauch reducible* to Q, and write $P \leqslant_{\mathrm{W}} Q$, if there are Turing functionals Φ and Ψ such that, for every instance X of P, the set $\widehat{X} = \Phi^X$ is an instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$, the set $Y = \Psi^{X \oplus \widehat{Y}}$ is a solution to X.

(5) We say that P is *strongly Weihrauch reducible* to Q, and write $P \leqslant_{\mathrm{sW}} Q$, if there are Turing functionals Φ and Ψ such that, for every instance X of P, the set $\widehat{X} = \Phi^X$ is an instance of Q, and for every solution $\widehat{Y}$ to $\widehat{X}$, the set $Y = \Psi^{\widehat{Y}}$ is a solution to X.

Note that all of these notions are transitive.

Remark 1.2. The reducibility $\leqslant_\omega$ is equivalent to a special case of the notion of computable entailment discussed by Shore [52], which is defined as follows: if Φ, Ψ are sentences of second order arithmetic, then Ψ *computably entails* Φ if every ω-model of $\mathrm{RCA}_0 + \Psi$ is a model of Φ.

Weihrauch reducibility has also been called uniform reducibility in this context. It is a more general notion, first introduced by Weihrauch [59, 60] in the context of computable analysis, and widely studied since. It was used for the kind of computability-theoretic comparison of mathematical principles we are engaged here by Gherardi and Marcone [22], Brattka and Gherardi [2, 3], and several other researchers since then. Dzhafarov [15] introduced the notion of computable reducibility. Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12] then uniformized this notion, yielding the definition of Weihrauch reducibility given above. This definition is different from the one given in the aforementioned papers coming from the computable analysis tradition, but it was proved to be equivalent to (a special case) of the latter in Appendix A of [12].

If P and Q are the trivial problems $\forall X\, \exists Y\, [Y = X]$ and $\forall X\, \exists Y\, [Y = \emptyset]$, respectively, then $P \leqslant_{\mathrm{W}} Q$ but $P \nleqslant_{\mathrm{sc}} Q$. Intuitively, we would probably want to think of P as reducible to any problem, so $\leqslant_{\mathrm{sc}}$ and $\leqslant_{\mathrm{sW}}$ may seem a bit suspect. Nevertheless, they can still give us useful information. To give an example, and for future use, we define some well-known versions of Ramsey's Theorem and related concepts.

Definition 1.3. For a set X, let $[X]^n$ be the collection of n-element subsets of X. We refer to an element of $[X]^n$ as an *n-tuple*, although these elements are unordered. A *k-coloring* of $[X]^n$ is a map $c : [X]^n \to k$. We write $c(x_1, \dots, x_n)$ for $c(\{x_1, \dots, x_n\})$. A set $H \subseteq X$ is *homogeneous for c* if there is an $i < k$ such that $c(s) = i$ for all $s \in [H]^n$. We also say that H is *homogeneous to i*. A set $A \subseteq X$ is *prehomogeneous for c* if the color of an element of $[A]^n$ depends only on its least $n-1$ many elements.

Ramsey's Theorem for n-tuples and k colors RT^n_k is the statement that every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set. $\mathrm{RT}^n_{<\infty}$ is the statement $\forall k\, \mathrm{RT}^n_k$. RT is the statement $\forall n\, \forall k\, \mathrm{RT}^n_k$.

A coloring $c : [\mathbb{N}]^2 \to 2$ is *stable* if $\lim_y c(x, y)$ exists for all x (in other words, for each x, there is an $i < 2$ such that $c(x, y) = i$ for all sufficiently large y). *Stable Ramsey's Theorem for Pairs* SRT^2_2 is the statement that every stable 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

Let D^2_2 be the statement that for every stable 2-coloring of pairs c there is an infinite set H that is *limit-homogeneous* in the sense that there is an $i < 2$ such that for all $x \in H$, if y is a sufficiently large element of H then $c(x, y) = i$. The principles SRT^2_2 and D^2_2 are closely related. Indeed, D^2_2 captures the way computability theorists generally think of SRT^2_2, by considering the task of computing an infinite homogeneous set given a computable stable 2-coloring of pairs to be equivalent to that of computing a subset of A or its complement $\overline{A}$ given a Δ^0_2 set A. SRT^2_2 trivially implies D^2_2 (indeed, $\mathrm{D}^2_2 \leqslant_{\mathrm{sW}} \mathrm{SRT}^2_2$), and it was shown by Chong, Lempp, and Yang [9, Theorem 1.4] that $\mathrm{RCA}_0 \vdash \mathrm{D}^2_2 \to \mathrm{SRT}^2_2$. (Their proof involved showing that D^2_2 implies Σ^0_2-bounding over RCA_0, thus justifying a hidden use of Σ^0_2-bounding in the proof of Lemma 7.10 in [7].) It is also easy to see that $\mathrm{SRT}^2_2 \leqslant_{\mathrm{c}} \mathrm{D}^2_2$. However, Dzhafarov [16] has shown that $\mathrm{SRT}^2_2 \nleqslant_{\mathrm{W}} \mathrm{D}^2_2$ (see also [5, Corollary 6.12]) and $\mathrm{SRT}^2_2 \nleqslant_{\mathrm{sc}} \mathrm{D}^2_2$, so we regard SRT^2_2 and D^2_2 as separate principles. We can of course also define SRT^2_k, $\mathrm{SRT}^2_{<\infty}$, D^2_k, and $D^2_{<\infty}$ in the obvious ways.

A set C is *cohesive* for a collection of sets $R_0, R_1, \dots$ if C is infinite and for each i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$ (where $X \subseteq^* Y$ means that $X \cap \overline{Y}$ is finite). The *Cohesive Set Principle* COH is the statement that every countable collection of sets has a cohesive set.

Remark 1.4. In reverse mathematics, we work in the setting of second order arithmetic, where the only first order objects are natural numbers, and the only second order objects are sets of natural numbers. Thus we must encode other first order objects, such as tuples of natural numbers and strings, as natural numbers; and other second order objects, such as colorings and trees, as sets of natural numbers. For the kinds of objects we consider in this paper, though, these codings are straightforward. Furthermore, in the computability-theoretic setting, we may argue informally, and use the fact that it is well understood what is meant by saying, for example, that one coloring computes another. Thus, we regard instances and solutions of a problem P as the actual objects mentioned in the (informal) statement of P. For example, an instance of RT^2_2 is a 2-coloring of $[\mathbb{N}]^2$. However, it will also be useful at times to be able to think of the instances and solutions of our problems as subsets of $\mathbb{N}$, so we identify all second order objects with subsets of $\mathbb{N}$ (or, equivalently, elements of 2^ω) via some reasonable choice of coding (such as the ones in [54]). The exact choice of coding does not affect any of our results.

Remark 1.5. In almost all cases we consider, it is clear from the informal description of a problem what its instances and solutions are. The only exceptions are $\mathrm{RT}^n_{<\infty}$, $\mathrm{SRT}^2_{<\infty}$, $\mathrm{D}^2_{<\infty}$, and RT. One way to formalize $\mathrm{RT}^n_{<\infty}$ is to have an instance be a function $f : [\mathbb{N}]^n \to \mathbb{N}$ such that $\mathrm{rng}(f)$ is bounded (and a solution to this instance be an infinite set H such that $\mathrm{rng}(f \upharpoonright [H]^n)$ is a singleton). Another is to have an instance consist of a number k together with a function $f : [\mathbb{N}]^n \to k$. As the second notion seems closer to the intent of the principle, let us call it $\mathrm{RT}^n_{<\infty}$, and let us call the first notion $(\mathrm{RT}^n_{<\infty})'$. In the nonuniform setting, the two principles are equivalent, but we do have $(\mathrm{RT}^n_{<\infty})' \nleqslant_{\mathrm{W}} \mathrm{RT}^n_{<\infty}$, with a proof similar to that of Theorem 3.3 below. However, none of the results in this paper depend on which of these formalizations we adopt, so we will not treat $(\mathrm{RT}^n_{<\infty})'$ as a separate principle. The same considerations apply to the cases of $\mathrm{SRT}^2_{<\infty}$, $\mathrm{D}^2_{<\infty}$, and RT. (See also Brattka and Rakotoniaina [5], where $\mathrm{RT}^n_{<\infty}$ and $(\mathrm{RT}^n_{<\infty})'$ are denoted by $\mathrm{RT}_{n,+}$ and $\mathrm{RT}_{n,\mathbb{N}}$, respectively.)

Dzhafarov [15] showed that $\mathrm{COH} \nleqslant_{\mathrm{sc}} \mathrm{D}^2_2$, and has recently improved this result in [16] to $\mathrm{COH} \nleqslant_{\mathrm{sc}} \mathrm{SRT}^2_2$. It is still an open question whether $\mathrm{COH} \leqslant_{\mathrm{c}} \mathrm{SRT}^2_2$, or, equivalently, whether $\mathrm{COH} \leqslant_{\mathrm{c}} \mathrm{D}^2_2$. Since RT^2_2 is equivalent to $\mathrm{SRT}^2_2 + \mathrm{COH}$ over RCA_0 (as shown by Mileti [41] and Jockusch and Lempp [unpublished], see [8, item 2]), answering this question might be a step toward answering the question of whether $\mathrm{RT}^2_2 \leqslant_\omega \mathrm{SRT}^2_2$. The latter question has become particularly interesting in light of the proof by Chong, Slaman, and Yang [10] that $\mathrm{RCA}_0 + \mathrm{SRT}^2_2 \nvdash \mathrm{RT}^2_2$. Note that Dzhafarov [16] has also shown that $\mathrm{COH} \nleqslant_{\mathrm{W}} \mathrm{SRT}^2_2$.

For any two problems P and Q, we clearly have

$$\begin{tikzcd}
 & & P \leqslant_{\mathrm{W}} Q \arrow[dl, Rightarrow] & \\
P \leqslant_\omega Q & P \leqslant_{\mathrm{c}} Q \arrow[l, Rightarrow] & & P \leqslant_{\mathrm{sW}} Q. \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
 & & P \leqslant_{\mathrm{sc}} Q \arrow[ul, Rightarrow] &
\end{tikzcd}$$

No implications between these notions hold in general other than the ones shown or implied by transitivity of implication. We have already seen that $\leqslant_{\mathrm{W}}$ does not imply $\leqslant_{\mathrm{sc}}$. For the other cases, we can give examples using versions of Ramsey's Theorem.

Here and below, we adopt standard computability-theoretic notation. In particular, we use $\downarrow$ to indicate that a computation converges, and for an object X being computably approximated, we write $X[s]$ for the stage s approximation to X; thus, for a Turing functional Φ, we write $\Phi^X(n)[s]$ for the value of the computation of Φ with oracle X on input n at stage s (which might of course be undefined if this computation has not yet converged).

If P is RT^1_2 and Q is $\forall X\,\exists Y\,[Y = X]$ then clearly $P \leqslant_{\mathrm{sc}} Q$. However, $P \nleqslant_{\mathrm{W}} Q$. To see that this is the case, assume that $P \leqslant_{\mathrm{W}} Q$ as witnessed by the functionals Φ and Ψ. Define $c : \mathbb{N} \to 2$ as follows. Start defining $c(s) = 0$ at each stage s until we find $n \leqslant s$ such that $\Psi^{c\oplus\Phi^c}(n)[s]\downarrow = 1$ (adopting the usual convention that the use of this computation must be at most s). We must eventually find such n and s, since Φ^c is a solution to itself as an instance of Q. Then define $c(t) = 1$ for all $t > s$. Any solution to P must consist entirely of numbers greater than s, so $\Psi^{c\oplus\Phi^c}$ is not a solution to P. We will see nontrivial examples of this kind of phenomenon in Section 4.

Jockusch [30, Theorems 5.1 and 5.5] showed that for each $n, k \geqslant 2$, every computable k-coloring of $[\mathbb{N}]^n$ has an infinite Π^0_n homogeneous set, but there is a computable k-coloring of $[\mathbb{N}]^n$ with no infinite Σ^0_n homogeneous set. It follows that $\mathrm{RT}^{n+1}_k \nleqslant_{\mathrm{c}} \mathrm{RT}^n_k$. On the other hand, using results from Jockusch [30, Section 5], Simpson [53, Theorem III.7.6] showed that if $n \geqslant 3$ and $k \geqslant 2$, then RT^n_k is equivalent to ACA_0 over RCA_0, so for such n we have $\mathrm{RT}^{n+1}_k \leqslant_\omega \mathrm{RT}^n_k$. ($\mathrm{RT}^2_k$, on the other hand, is weaker, not just reverse-mathematically but also in the sense of ω-models, as shown by Seetapun in Seetapun and Slaman [51, Theorem 3.1].)

It is also worth remarking on the connections between these notions of reducibility and implication over RCA_0. If $\mathrm{RCA}_0 + Q \vdash P$ then $P \leqslant_\omega Q$, but the example of RT^{n+1}_k and RT^n_k, where $n \geqslant 3$, shows that it is not necessarily the case that $P \leqslant_{\mathrm{c}} Q$. Conversely, even having $P \leqslant_{\mathrm{sW}} Q$ is not enough to ensure that $\mathrm{RCA}_0 + Q \vdash P$, although it is somewhat more difficult to find a counterexample. To do so, we consider the principle $\Pi^0_1\mathrm{GA}$, introduced by Hirschfeldt, Lange, and Shore [27], as a "miniaturization" of the principle $\Pi^0_1\mathrm{G}$ studied by Hirschfeldt, Shore, and Slaman [29].

Definition 1.6. Let D be a property of binary strings. We write $\sigma \in D$ to mean that D holds of σ. We say that D is *dense* if for every σ there is a $\tau \succcurlyeq \sigma$ such that $\tau \in D$.

$\Pi^0_1\mathrm{G}$ is the principle stating that for any X and any uniformly $\Pi^{0,X}_1$ collection $D_0, D_1, \ldots$ of dense properties of binary strings, there is a G such that

$$\forall i\,\exists m\,[G \upharpoonright m \in D_i].$$

Π^0_1GA is the principle stating that for any X and any uniformly $\Pi^{0,X}_1$ collection $D_0, D_1, \dots$ of dense properties of binary strings, there is a sequence $g_0, g_1, \dots$ of sets such that

$$\forall i\, \exists m\, \exists t\, \forall u > t\, [g_u \upharpoonright m = g_t \upharpoonright m \in D_i].$$

We can think of Π^0_1GA as a problem as follows. An instance of Π^0_1GA is a sequence $A_0, A_1, \dots \subseteq \mathbb{N} \times 2^{<\omega}$ such that for every σ and i, there is a $\tau \succcurlyeq \sigma$ for which $\forall k\, [\langle k, \tau\rangle \in A_i]$ holds. A solution to this instance is a sequence $g_0, g_1, \dots$ of sets such that

$$\forall i\, \exists m\, \exists t\, \forall u > t\, \forall k\, [g_u \upharpoonright m = g_t \upharpoonright m \,\wedge\, \langle k, g_t \upharpoonright m\rangle \in A_i].$$

It is easy to check that Π^0_1GA is uniformly computably true, i.e., that there is a uniform procedure to obtain $g_0, g_1, \dots$ as above from $A_0, A_1, \dots$. In particular, $\Pi^0_1\mathrm{GA} \leqslant_{\mathrm{W}} \mathrm{WKL}$. We will see in Proposition 4.9 that this fact implies that $\Pi^0_1\mathrm{GA} \leqslant_{\mathrm{sW}} \mathrm{WKL}$. On the other hand, it is shown in [27, Theorem 3.3] that Π^0_1GA implies Σ^0_2-induction over RCA_0 together with Σ^0_2-bounding. By Hájek [24, Corollary 3.14], WKL is conservative over Σ^0_2-bounding for arithmetic statements, so $\mathrm{RCA}_0 + \mathrm{WKL} \nvdash \Pi^0_1\mathrm{GA}$.

In this paper, we consider several issues connected with the above notions of reducibility. In Section 2, motivated both by a question about computable reducibility and by one raised by Liu's proof in [40] that RT^2_2 does not imply Weak König's Lemma (WKL) over RCA_0, we show that for every $n \geqslant 3$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that any infinite homogeneous set has PA degree over $\emptyset^{(n-2)}$. Using this result we show that full König's Lemma (KL) lies strictly between RT^2_2 and RT^3_2 under computable reducibility. We also analyze the limits of codability into solutions of (not necessarily computable) instances of Ramsey's Theorem and use this analysis to show that WKL is not strongly computably reducible to RT, and prove several other results on the relationships between versions of Ramsey's Theorem and König's Lemma with respect to the notions of reducibility defined above.

In Section 3, we answer two questions raised by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12] on Weihrauch reducibility between versions of Ramsey's Theorem and of the Thin Set Theorem with the same exponent but different numbers of colors. We also show that for each $m \geqslant 2$, there is an $(m+1)$-coloring c of $\mathbb{N}$ such that every m-coloring d of $\mathbb{N}$ has an infinite homogeneous set that does not compute any infinite homogeneous set for c, which strengthens both a theorem of Dzhafarov [15], who established this result under the assumption that d is hyperarithmetic in c (and hence showed that $\mathrm{RT}^1_3 \nleqslant_{\mathrm{sc}} \mathrm{RT}^1_2$), and one of Dzhafarov and Igusa [17] on the notion of infinite information reducibility introduced by them. Our result demonstrates that, while colorings of $\mathbb{N}$ might at first seem uninteresting, once we look at noncomputable colorings, surprising phenomena can emerge.

The difference between $P \leqslant_{\mathrm{c}} Q$ and $P \leqslant_{\omega} Q$ is that the former captures the idea of solving an instance of P by reducing it to a single instance of Q, while the latter captures the idea of solving an instance of P by reducing it to multiple instances of Q (taken in parallel, in series, or in a combination of both modes). In Section 4, we make this interpretation of $\leqslant_{\omega}$ more precise, in a way that allows us to combine this general form of reduction with the idea of uniformity, to arrive at a uniform version of $\leqslant_{\omega}$, whose basic properties we study. We also look at ways to count the number of instances of Q used in reducing P to Q (uniformly or not), and to extend our notions to non-ω-models.

Section 5 is a summary of some of our results, their context, and some remaining open questions in the form of diagrams.

In the course of our work on this paper, we found a gap in the proof of the following result in Cholak, Jockusch, and Slaman [7, Theorem 12.2]: Let $n, k \geqslant 2$ and let $C_0, C_1, \dots$ be such that $C_i \nleqslant_{\mathrm{T}} \emptyset^{(n-2)}$ for all i. Then each computable k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set H such that $H' \ngeqslant_{\mathrm{T}} \emptyset^{(n)}$ and $C_i \nleqslant_{\mathrm{T}} H$ for all i. In the appendix, we show how to fill in this gap.

We assume familiarity with the basic terminology and notation of computability theory and reverse mathematics, as found for instance in Soare [55] and Simpson [54], respectively. In Section 3.2, we will also need some basic facts about the hyperjump (see for instance Sacks [49]). We denote the eth $\{0,1\}$-valued Turing functional in a fixed effective listing of such functionals by Φ_e. A *tree* is a subset of $\mathbb{N}^{<\omega}$ that is closed downward under extension. A tree T is *finitely bounded* if for each n there is a k such that $\sigma(n) < k$ for every $\sigma \in T$ of length $n+1$. A *binary tree* is a tree that is a subset of $2^{<\omega}$.

Recall that König's Lemma KL is the statement that every infinite, finitely bounded tree has an infinite path, and Weak König's Lemma WKL is the statement that every infinite binary tree has an infinite path. Recall also that a set has PA degree if it can compute a solution to each computable instance of WKL. More generally, we say that X has PA degree over Y, and write $X \gg Y$, if X can compute a solution to each Y-computable instance of WKL. It is well known that $X \gg Y$ if and only if X computes a completion of the partial function $e \mapsto \Phi_e^Y(e)$.

Weak Weak König's Lemma WWKL, which comes up often in the reverse mathematics of measure theory, states that if T is a binary tree such that

$$\liminf_n \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n} > 0,$$

then T has an infinite path.

2. Versions of Ramsey's Theorem and König's Lemma

In this section we compare various versions of Ramsey's Theorem (i.e., RT^n_k, SRT^2_k, D^2_k, $\mathrm{RT}^n_{<\infty}$, $\mathrm{SRT}^2_{<\infty}$, $\mathrm{D}^2_{<\infty}$, RT, and COH) and König's Lemma (i.e., KL, WKL, and WWKL). We leave a discussion of the case of RT^n_j versus RT^n_k for $j \neq k$ to the following section. For a summary of our results and how they fit into a more general picture, see the diagrams in Section 5. The key new technical result needed here is that there is a computable 2-coloring of triples such that every infinite homogeneous set has PA degree over $\emptyset'$.

We begin with a reminder of results for $\leqslant_\omega$, summarized in Figure 2.1. Since the number of colors does not matter in this case, the subscript k here stands for any number greater than 1 or for $<\infty$. No other implications than the ones shown (or implied by transitivity) hold. For justifications of these implications and nonimplications, see [25]. There are two arrows with questions marks, but they represent the same open question, since RT^2_2 is equivalent to $\mathrm{SRT}^2_2 + \mathrm{COH}$ over RCA_0.

$$\begin{tikzcd}
 & & \mathrm{RT} = \mathrm{RT}^3_k = \mathrm{KL} \arrow[dl] \arrow[dr] & \\
 & \mathrm{RT}^2_k \arrow[dl] \arrow[d, shift left] & & \mathrm{WKL} \arrow[ddl] \\
\mathrm{COH} \arrow[drr] & \mathrm{SRT}^2_k = \mathrm{D}^2_k \arrow[l, "?"'] \arrow[u, shift left, "?"] \arrow[dr] & & \\
 & & \mathrm{RT}^1_k &
\end{tikzcd}$$

Figure 2.1. Versions of RT and KL under $\leqslant_\omega$

Liu [40] showed that RT^2_2 does not imply WKL over RCA_0. In fact he showed in his Corollary 1.6 that there is an ω-model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ that does not contain any set of PA degree. It follows immediately that $\mathrm{WKL} \nleqslant_\omega \mathrm{RT}^2_2$. It also follows immediately that for every computable 2-coloring of $[\mathbb{N}]^2$ there is an infinite homogeneous set whose degree is not PA. For colorings of exponent $n \geqslant 3$ it is natural to work over $\emptyset^{(n-2)}$, since it was shown in Jockusch [30, Lemma 5.9] that there is a computable 2-coloring of $[\mathbb{N}]^n$ with every infinite homogeneous set computing $\emptyset^{(n-2)}$. In particular, one might ask whether Liu's result above could be extended by showing that for every $n \geqslant 2$ and every computable coloring of $[\mathbb{N}]^n$, there is an infinite homogeneous set whose degree is not PA over $\emptyset^{(n-2)}$.

A closely related question comes from considering the relationship between RT^3_2 and KL. These principles are both equivalent to ACA_0 over RCA_0. On the other hand, it is not difficult to see that every computable instance of KL has a Δ^0_3 solution, while by Jockusch [30, Theorem 5.1], there are computable instances of RT^3_2 with no Σ^0_3 solution, so $\mathrm{RT}^3_2 \not\leqslant_{\mathrm{c}} \mathrm{KL}$. We might then ask whether $\mathrm{KL} \leqslant_{\mathrm{c}} \mathrm{RT}^3_2$. Suppose for the moment that this is so. By Theorem 5 of Jockusch, Lewis, and Remmel [31] (and the existence of a nonempty $\Pi^{0,\emptyset'}_1$ set with all elements of PA degree over $\emptyset'$) there is a computable instance of KL with all solutions of PA degree over $\emptyset'$. Hence, by our supposition, there must be a computable instance of RT^3_2 with all solutions of PA degree over $\emptyset'$. Our next result shows that this is the case. We then use this result in relativized form to show in Corollary 2.4 that $\mathrm{KL} \leqslant_{\mathrm{c}} \mathrm{RT}^3_2$ and in fact that $\mathrm{KL} \leqslant_{\mathrm{W}} \mathrm{RT}^3_2$, a result which was obtained independently by Brattka and Rakotoniaina [5].

Theorem 2.1. *There is a computable 2-coloring of $[\mathbb{N}]^3$ such that any infinite prehomogeneous set has PA degree over $\emptyset'$.*

Proof. Let $m < s < t$. If for all $e < m$ we have $\Phi^{\emptyset'[s]}_e(e)[s] = \Phi^{\emptyset'[t]}_e(e)[t]$ (which includes the possibility that both sides diverge) then let $c(m,s,t) = 1$. Otherwise, let $c(m,s,t) = 0$.

Suppose that A is prehomogeneous for c. Define a function $g \leqslant_{\mathrm{T}} A$ as follows. Given x, let $m \in A$ be such that $x < m$. Search for $s, t \in A$ such that $m < s < t$ and $\Phi^{\emptyset'[s]}_e(e)[s] = \Phi^{\emptyset'[t]}_e(e)[t]$ for all $e < m$. Such numbers must exist since there are only finitely many possibilities for the outcomes of these computations. Let $g(x) = \Phi^{\emptyset'[s]}_x(x)[s]$ if the latter is defined, and otherwise let $g(x) = 0$. By prehomogeneity, $\Phi^{\emptyset'[s]}_x(x)[s] = \Phi^{\emptyset'[u]}_x(x)[u]$ for all $u > s$ in A, so if $\Phi^{\emptyset'}_x(x)\downarrow$ then $\Phi^{\emptyset'}_x(x) = \Phi^{\emptyset'[s]}_x(x)[s] = g(x)$. Thus g is an A-computable completion of the partial function $e \mapsto \Phi^{\emptyset'}_e(e)$, and hence $A \gg \emptyset'$. ☐

In particular, there is a computable 2-coloring of $[\mathbb{N}]^3$ such that any infinite homogeneous set has PA degree over $\emptyset'$. We can extend this result to higher exponents as follows. As in [30], a coloring c is called *unbalanced* if all infinite homogeneous sets are homogeneous to the same color. Note that the coloring constructed in the proof of Theorem 2.1 is unbalanced.

Corollary 2.2. *Let $n \geqslant 3$. There is an unbalanced computable 2-coloring of $[\mathbb{N}]^n$ such that any infinite homogeneous set has PA degree over $\emptyset^{(n-2)}$.*

Proof. We prove a strong relativized version by induction on n, specifically the following: For each $n \geqslant 3$ and each set X there is an unbalanced X-computable 2-coloring of $[\mathbb{N}]^n$ such that for any infinite homogeneous set H, the set $X \oplus H$ has PA degree over $X^{(n-2)}$. The base case $n = 3$ is obtained by relativizing Theorem 2.1 to X. Now assume the result for n, where $n \geqslant 3$, and let X be given. By the inductive hypothesis (for X')

there is an unbalanced X'-computable 2-coloring c of $[\mathbb{N}]^n$ such that for any infinite homogeneous set H, the set $X' \oplus H$ has PA degree over $X^{(n-1)}$. By [30, Lemma 5.2], relative to X, there is an X-computable 2-coloring d of $[\mathbb{N}]^{n+1}$ such that every infinite set homogeneous for d is homogeneous for c. Furthermore, since c is unbalanced, the proof of Lemma 5.2 of [30] shows that we can take d to be unbalanced. Next, by [30, Theorem 5.7], relative to X, there is an X-computable unbalanced 2-coloring e of $[\mathbb{N}]^{n+1}$ such that, for every infinite homogeneous set H, the set $X \oplus H$ computes $X^{(n-2)}$ and hence computes X', since $n \geqslant 3$. By [30, Lemma 5.10], relative to X, there is an unbalanced X-computable 2-coloring f of $[\mathbb{N}]^{n+1}$ such that the infinite sets homogeneous for f are precisely the infinite sets homogeneous for both d and e. To complete the induction, it suffices to show that every infinite homogeneous set H for f is such that $X \oplus H$ has PA degree over $X^{(n-1)}$. So assume that H is homogeneous for f. It follows that H is homogeneous for d and hence for c, so $X' \oplus H$ has PA degree over $X^{(n-1)}$. Since H is homogeneous for e, the set $X \oplus H$ computes X'. Hence $X \oplus H \geqslant_{\mathrm{T}} X' \oplus H$, so $X \oplus H$ has PA degree over $X^{(n-1)}$ as needed. ☐

Thus we see that the analog of Liu's Theorem for higher exponents fails for each exponent $n > 2$.

We now consider the relation between König's Lemma and various versions of Ramsey's Theorem. Our first result is a corollary to Theorem 2.1.

Corollary 2.3 (due independently to Brattka and Rakotoniaina [5]). $\mathrm{KL} \leqslant_{\mathrm{W}} \mathrm{RT}^3_2$.

Proof. We show first that $\mathrm{KL} \leqslant_{\mathrm{c}} \mathrm{RT}^3_2$. Let X be any instance of KL, so X is an infinite, finitely branching tree. By relativizing Theorem 2.1 to X, we obtain an X-computable 2-coloring c of $[\mathbb{N}]^3$ such that for any infinite homogeneous set P, the set $X \oplus P$ has PA degree over X'. It now suffices to show that, for every such P, the set $X \oplus P$ computes a solution to X, i.e. a path through X. This fact is clear because the paths through X are bounded by a function $b \leqslant_{\mathrm{T}} X'$, and thus X is essentially a computable binary tree relative to X'. In more detail, let $b(n)$ be the greatest value of $\sigma(n)$ over all strings $\sigma \in X$ of length $n+1$. (There are only finitely many such strings because X is finitely branching, and at least one such string because X is infinite.) Clearly $b \leqslant_{\mathrm{T}} X'$. Thus X is an infinite tree that is X'-computable and X'-computably bounded. As remarked in [35, page 606], for every computable, computably bounded tree T, there is a computable tree $U \subseteq 2^{<\omega}$ such that the degrees of the paths through T coincide with the degrees of the paths through U. Relativizing this remark to X' yields the fact that every set C of PA degree over X' is such that $C \oplus X'$ computes a path through X. In particular, if P is an infinite homogeneous set for c, then $X \oplus P$ is of PA degree over X', so $X \oplus P \oplus X'$ computes a path through X. But $X \oplus P \geqslant_{\mathrm{T}} X'$ since $X \oplus P$ is of PA degree over X', so $X \oplus P$ computes a path through X, as needed to complete the proof that $\mathrm{KL} \leqslant_{\mathrm{c}} \mathrm{RT}^3_2$.

To show that $\mathrm{KL} \leqslant_{\mathrm{W}} \mathrm{RT}^3_2$, observe that the coloring c obtained from the proof of (the relativized version of) Theorem 2.1 is such that there is a uniform procedure taking infinite homogeneous sets for c to completions of the partial function $e \mapsto \Phi_e^{X'}(e)$. From such a completion we can uniformly obtain X', and hence uniformly obtain a path through X. ☐

The following corollary compares the computability-theoretic strength of König's Lemma with that of Ramsey's Theorem for pairs and triples. It shows that the strength of König's Lemma should be, fancifully speaking, roughly that of $\mathrm{RT}^{2.5}_2$. Of course, $P <_{\mathrm{c}} Q$ means that $P \leqslant_{\mathrm{c}} Q$ and $Q \nleqslant_{\mathrm{c}} P$.

Corollary 2.4. $\mathrm{RT}^2_2 <_{\mathrm{c}} \mathrm{KL} <_{\mathrm{c}} \mathrm{RT}^3_2$.

Proof. The corollary we have just proved shows that $\mathrm{KL} \leqslant_{\mathrm{c}} \mathrm{RT}^3_2$, and we have already remarked that $\mathrm{RT}^3_2 \nleqslant_{\mathrm{c}} \mathrm{KL}$. To see that $\mathrm{RT}^2_2 \leqslant_{\mathrm{c}} \mathrm{KL}$, note that in [30, Proposition 4.6], it is shown that for any computable 2-coloring of pairs there is a finitely branching computable tree T such that every path through T computes an infinite homogeneous set. (The fact that T is finitely branching is clear from the proof.) Relativizing the proof of this result shows that $\mathrm{RT}^2_2 \leqslant_{\mathrm{c}} \mathrm{KL}$. Since there is a computable instance of KL with all solutions computing $\emptyset'$ (by [31, Corollary 5.1], for example) but no computable instance of RT^2_2 with all solutions computing $\emptyset'$ by Seetapun's Theorem ([51, Theorem 2.1]), we have that $\mathrm{KL} \nleqslant_{\mathrm{c}} \mathrm{RT}^2_2$. ☐

The same proof also shows that $\mathrm{RT}^2_{<\infty} <_{\mathrm{c}} \mathrm{KL}$. We can improve this result by applying the following fact.

Proposition 2.5. *For any problem P, if $P \leqslant_{\mathrm{c}} \mathrm{KL}$ then $P \leqslant_{\mathrm{sc}} \mathrm{KL}$.*

Proof. Let X be an instance of P and let T be an X-computable instance of KL such that for any solution Y to T, there is an $(X \oplus Y)$-computable solution to X. Let $\widehat{T}$ consist of all σ for which there is a $\tau \in T$ with $\sigma(2n) = \tau(n)$ for all $2n < |\sigma|$ and $\sigma(2n+1) = X(n)$ for all $2n+1 < |\sigma|$. Then $\widehat{T}$ is also an X-computable instance of KL, and every solution $\widehat{Y}$ to $\widehat{T}$ computes $X \oplus Y$ for some solution Y to T, and hence computes a solution to X. ☐

Corollary 2.6. $\mathrm{RT}^2_{<\infty} \leqslant_{\mathrm{sc}} \mathrm{KL}$.

On the other hand, $\mathrm{RT}^2_2 \nleqslant_{\mathrm{W}} \mathrm{KL}$. We will prove a stronger version of this result in part (2) of Theorem 2.10 below.

For WKL we have a considerably stronger version of Proposition 2.5.

Proposition 2.7. *For any problem P, if $P \leqslant_{\omega} \mathrm{WKL}$ then $P \leqslant_{\mathrm{sc}} \mathrm{WKL}$.*

Proof. Suppose that $P \leqslant_\omega$ WKL and let X be an instance of P. Let T be an X-computable infinite binary tree such that every infinite path on T has PA degree over X. If Z is an infinite path on T then, by the work of Scott [50], there is a Turing ideal I in which WKL holds such that $X \in I$ and I consists entirely of Z-computable sets. Since $P \leqslant_\omega$ WKL, X has a solution in I and hence a Z-computable solution. $\square$

We will prove a uniform version of Proposition 2.7 in Proposition 4.9.

The next result will be used to show that KL $\nleqslant_{\mathrm{sc}}$ RT^3_2, and indeed WKL $\nleqslant_{\mathrm{sc}}$ RT.

Theorem 2.8. *The following are equivalent for $n, k \geqslant 2$ and $X \subseteq \mathbb{N}$.*

(i) *There is a k-coloring c of $[\mathbb{N}]^n$ (not necessarily computable) such that every infinite homogeneous set computes X.*

(ii) *X is hyperarithmetic.*

Proof. To show that (i) implies (ii), assume (i). Given an infinite set A, the restriction of c to A has an infinite homogeneous set $H \subseteq A$. Since H is also homogeneous for c, it computes X. Thus every infinite set has a subset that computes X, i.e. X is *encodable*. It follows from Solovay [56, Theorem 2.3] that X is hyperarithmetic.

The converse follows from the remark on page 278 of Jockusch [30] that there is a 2-coloring of $[\mathbb{N}]^2$ such that all infinite homogeneous sets compute all hyperarithmetic sets. We give a proof here for the convenience of the reader. By Jockusch and McLaughlin [32, Theorem 4.13], there is an increasing function f such that $X \leqslant_{\mathrm{T}} f$ and $f \leqslant_{\mathrm{T}} g$ for every function g that dominates f. Now for $x_1 < x_2 < \cdots < x_n$, let $c(x_1, \ldots, x_n) = 1$ if $x_2 \geqslant f(x_1)$ and otherwise let $c(x_1, \ldots, x_n) = 0$. Then if H is any infinite homogeneous set for c, it is easily seen that H is homogeneous to 1 and that p_H dominates f, where p_H is the function that enumerates H in increasing order. It follows that $X \leqslant_{\mathrm{T}} f \leqslant_{\mathrm{T}} p_H \leqslant_{\mathrm{T}} H$. $\square$

Note that the above result fails for $n = 1$. Indeed, Dzhafarov and Jockusch showed in [18, Lemma 5.2(i)] that if (i) above holds for $n = 1$ and $k = 2$, then X is computable.

Corollary 2.9. WKL $\nleqslant_{\mathrm{sc}}$ RT.

Proof. Let X be any set that is not hyperarithmetic. Then WKL has an instance whose only solution is X, but there is no instance of RT such that all solutions compute X. $\square$

Monin and Patey [42] have recently extended this result from WKL to WWKL, answering a question stated in an earlier version of this paper. (The weaker version with $\leqslant_{\mathrm{sW}}$ in place of $\leqslant_{\mathrm{sc}}$, which had also been a question in an earlier version of this paper, was answered independently by Brattka and Rakotoniaina [5].)

We finish this section with various facts about versions of Ramsey's Theorem and of König's Lemma. Part (4) of the following theorem was proved independently by Brattka and Rakotoniaina [5].

Theorem 2.10.

(1) $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}} \mathrm{WKL}$.

(2) $\mathrm{D}^2_2 \nleqslant_{\mathrm{W}} \mathrm{KL}$ *(and hence* $\mathrm{SRT}^2_2 \nleqslant_{\mathrm{W}} \mathrm{KL}$*).*

(3) $\mathrm{RT}^1_k \leqslant_{\mathrm{sW}} \mathrm{D}^2_k$ *(and hence* $\mathrm{RT}^1_k \leqslant_{\mathrm{sW}} \mathrm{SRT}^2_k$*) for all* $k \geqslant 2$. *Similarly,* $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{sW}} \mathrm{D}^2_{<\infty}$.

(4) $\mathrm{RT}^1_{k+1} \nleqslant_{\mathrm{W}} \mathrm{SRT}^2_k$ *(and hence* $\mathrm{RT}^1_{k+1} \nleqslant_{\mathrm{W}} \mathrm{D}^2_k$*) for all* $k \geqslant 2$.

(5) $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}} \mathrm{COH}$.

(6) $\mathrm{COH} \leqslant_{\mathrm{sW}} \mathrm{KL}$.

(7) $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{sW}} \mathrm{KL}$.

(8) $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{sW}} \mathrm{RT}^2_2$.

(9) $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{sc}} \mathrm{COH}$.

(10) $\mathrm{RT}^1_2 \nleqslant_{\mathrm{sc}} \mathrm{WWKL}$.

Proof. (1) Suppose $\mathrm{RT}^1_2 \leqslant_{\mathrm{W}} \mathrm{WKL}$ as witnessed by Φ and Ψ. Begin building $c : \mathbb{N} \to 2$ by letting $c(s) = 0$ at stage s, and let T^c be the infinite binary tree coded by Φ^c. Wait until we find an n and an s such that $T^c(\sigma)[s]\downarrow$ for all $\sigma \in 2^n$, and for each such σ for which $\sigma \in T^c$, there is an $m < s$ such that $m \in \Psi^{c\oplus\sigma}[s]$. Such n and s must eventually be found, as otherwise either T^c is not total or, by compactness, it has an infinite path X for which $\Psi^{c\oplus X}$ is not an infinite set. When they are found, let $c(t) = 1$ for all $t \geqslant s$. Let X be an infinite path on T^c. Since X extends some $\sigma \in T^c \cap 2^n$, we thus ensure that $\Psi^{c\oplus X}$ is not an infinite homogeneous set for c.

(2) Suppose $\mathrm{D}^2_2 \leqslant_{\mathrm{W}} \mathrm{KL}$ as witnessed by Φ and Ψ. The proof is similar to that of part (1), but "one jump up". Instead of building a stable coloring of pairs directly, we build an $\emptyset'$-computable 2-coloring c of singletons. By the recursion theorem and the limit lemma, we may assume that we know an index for the reduction from $\emptyset'$ to c, and hence know an index for a computable stable coloring d of pairs such that $c(x) = \lim_s d(x,s)$. Note that if an infinite set is limit-homogeneous for d then it is homogeneous for c. For a function $f : \mathbb{N} \to \mathbb{N}$, let f^n be the set of all $\sigma \in \mathbb{N}^n$ such that $\sigma(i) < f(i)$ for all $i < n$, and let $f^{<\omega}$ be the finitely branching tree $\bigcup_n f^n$.

Begin building c by letting $c(s) = 0$ at stage s, and let T^d be the infinite, finitely branching tree coded by Φ^d. Since we have access to $\emptyset'$, we also have access to a function f such that $T^d \subseteq f^{<\omega}$. Wait until we find an n and an s such that $T^d(\sigma)[s]\downarrow$ for all $\sigma \in f^n$, and for each such σ for which $\sigma \in T^d$, there is an $m < s$ such that $m \in \Psi^{d \oplus \sigma}[s]$. As in part (1), such n and s must eventually be found. When they are found, let $c(t) = 1$ for all $t \geqslant s$. Let X be an infinite path on T^d. Since X extends some $\sigma \in T^d \cap f^n$, we thus ensure that $\Psi^{d \oplus X}$ is not an infinite homogeneous set for c, and hence is not an infinite limit-homogeneous set for d.

(3) Given $c : \mathbb{N} \to k$, let $d(x, y) = c(x)$ for all $x < y$; then d is stable, and any set that is limit-homogeneous for d is homogeneous for c.

(4) Suppose that $\mathrm{RT}^1_{k+1} \leqslant_{\mathrm{W}} \mathrm{SRT}^2_k$ as witnessed by Φ and Ψ. We build $c : \mathbb{N} \to (k+1)$ in stages. Let d^c be the coloring of $[\mathbb{N}]^2$ coded by Φ^c, and for $x < s$ let $e(x, s) = d^c(x, u)[s]$ for the largest u such that $d^c(x, u)[s]\downarrow$ (or undefined if there is no such u). At each stage s, we have parameters i_j, F_j, and n_j (which we call the j-parameters) for $j < k$, all of which are initially undefined. Proceed as follows at stage s for each $j < k$.

If the j-parameters are undefined then look for a finite set F with $\max F < s$ such that $d^c(x, y)\downarrow = j$ for all $x, y \in F$ with $x < y$ and $e(x, s) = j$ for all $x \in F$, and an $n < s$ such that $n \in \Psi^{c \oplus F}[s]$. If such F and n exist then let $F_j = F$, let $n_j = n$, and let $i_j = c(n)$.

If the j-parameters are defined and $e(x, s) \neq j$ for some $x \in F_j$, then undefine the j-parameters.

At the end of stage s, define $c(s)$ to be different from i_j for each i_j that is currently defined.

Now let H be an infinite homogeneous set for d^c and let j be the color to which it is homogeneous. Then $\Psi^{c \oplus H}$ is an infinite homogeneous set for c. There are a finite $E \subset H$ and an n such that $n \in \Psi^{c \oplus E}$, so a final F_j and n_j will be found. This F_j is homogeneous to j for d^c, and there is a y such that $d^c(x, z) = j$ for all $x \in F_j$ and $z > y$, and $\Psi^{c \oplus F_j}(n_j)\downarrow = 1$ with use at most y. Then $I = F_j \cup (H \upharpoonright {>y})$ is an infinite homogeneous set for d^c, and $n_j \in \Psi^{c \oplus I}$. However, $c(s) \neq i_j = c(n_j)$ for all sufficiently large s, so $\Psi^{c \oplus I}$ is not an infinite homogeneous set for c.

(5) Suppose $\mathrm{RT}^1_2 \leqslant_{\mathrm{W}} \mathrm{COH}$ as witnessed by Φ and Ψ. Begin building $c : \mathbb{N} \to 2$ by letting $c(s) = 0$ at stage s, until we find a finite set F and an n such that $n \in \Psi^{c \oplus F}$. Then ensure that $c(n) = 0$ and $c(t) = 1$ for all sufficiently large t. Now, Φ^c is an instance of COH. No matter what this instance is, there is a solution S to it such that $F \subset S$ and $\min S \setminus F$ is larger than the use of $\Psi^{c \oplus F}(n)$. Then $n \in \Psi^{c \oplus S}$, so $\Psi^{c \oplus S}$ is not an infinite homogeneous set for c.

This result can also be obtained from Proposition 4.3 in [4], since, in the terminology of that paper, COH is densely realized, while RT^1_2 is indiscriminative. We will strengthen it in Section 4.3.

(6) Let $\mathcal{A} = (A_0, A_1, \ldots)$ be a sequence of sets. Let $A^1_i = A_i$ and $A^0_i = \overline{A_i}$. Let T be the binary tree consisting of all σ such that $|\bigcap_{i<|\sigma|} A^{\sigma(i)}_i| \geqslant$

$|\sigma|$. It is easy to see that T is infinite and $\mathcal{A}'$-computable. By the relativized form of Theorem 5 of Jockusch, Lewis, and Remmel [31], there is an $\mathcal{A}$-computable finitely branching tree U such that the infinite paths on U are in an effective, degree-preserving one-to-one correspondence with those of T. The definition of U in that proof is uniform, so U can be obtained uniformly from $\mathcal{A}$, and from an infinite path on U, we can uniformly obtain an infinite path P on T. We have $|\bigcap_{i<n} A_i^{P(i)}| = \omega$ for all n, so from $\mathcal{A} \oplus P$ we can uniformly obtain a cohesive set $C = \{c_0, c_1, \ldots\}$ for $\mathcal{A}$ by letting $c_0 = 0$ and letting c_{n+1} be the least number greater than c_n in $\bigcap_{i<n} A_i^{P(i)}$.

(7) Given a function $f : \mathbb{N} \to \mathbb{N}$, let T consist of all sequences of the form $(\langle i, n_0\rangle, \langle i, n_1\rangle, \ldots, \langle i, n_{k-1}\rangle)$ where $n_0 < \cdots < n_{k-1}$ are the first k many numbers n such that $f(n) = i$. Then T is uniformly f-computable. If $\mathrm{rng}(f)$ is bounded then T is finitely branching and from any infinite path on T we can compute an infinite homogeneous set for f in a uniform way.

(8) Given a function $f : \mathbb{N} \to \mathbb{N}$, let $c : [\mathbb{N}]^2 \to 2$ be defined by letting $c(x, y) = 1$ if $f(x) = f(y)$ and $c(x, y) = 0$ otherwise. Suppose H is homogeneous for c. If f has bounded range, then H cannot be homogeneous to 0, so it is homogeneous to 1 and hence also homogeneous for f.

(9) Given a function $f : \mathbb{N} \to \mathbb{N}$, let $A_i = \{n : f(n) = i\}$, and let C be cohesive for $A_0, A_1, \ldots$. The A_i are pairwise disjoint, and if f has bounded range, then only finitely many A_i are nonempty, so C must be almost entirely contained in some A_i. Then C computes $C \cap A_i$, which is an infinite homogeneous set for f.

(10) Every instance of WWKL has positive measure many solutions, so it is enough to show that there is a 2-coloring c of $\mathbb{N}$ such that the class of sets that can compute an infinite homogeneous set for c has measure 0. The existence of such a c follows at once from the following result of Mileti, which is Theorem 5.2.6 of his dissertation [41]. We include a proof here for the convenience of the reader. Recall that a set A is *hyperimmune* if there is no computable function g such that A has at least n many elements less than $g(n)$ for all n.

Lemma 2.11 (Mileti [41])**.** *If A is hyperimmune, then the family $\mathcal{C}$ of sets that compute infinite subsets of A has measure* 0.

Proof. We suppose that $\mu(\mathcal{C}) > 0$ and show that A is not hyperimmune. Since $\mu(\mathcal{C}) > 0$ there exists an e such that $\mu(\mathcal{C}_e) > 0$, where $\mathcal{C}_e$ is the set of X such that Φ_e^X is an infinite subset of A. By the Lebesgue Density Theorem, there is a string $\sigma \in 2^{<\omega}$ such that $\frac{\mu(\mathcal{C}_e \cap [\sigma])}{\mu([\sigma])} > \frac{1}{2}$, where $[\sigma] = \{X : \sigma \prec X\}$. For notational convenience we assume that σ is the empty string, i.e., $\mu(\mathcal{C}_e) > \frac{1}{2}$. We claim now that there is a computable function

f such that $[n, f(n)) \cap A \neq \emptyset$ for all n. The claim suffices to prove the lemma, since if f is as in the claim, then it follows by induction on n that A has at least n many elements less than $f^{(n)}(0)$ for each n, where $f^{(n)}$ is the nth iterate of f.

We now prove the claim. Given n, to calculate $f(n)$ search for a number k such that $\mu(\{X : \Phi^X_e \cap [n, k) \neq \emptyset\}) > \frac{1}{2}$ and let $f(n)$ be the first such k that is found. (In this definition, we identify Φ^X_e with $\{j : \Phi^X_e(j) = 1\}$ without assuming that Φ^X_e is total.) Such a number k exists since

$$\begin{aligned}\frac{1}{2} < \mu(\mathcal{C}_e) \leqslant \mu(\{X : \Phi^X_e \text{ is infinite}\}) \leqslant \mu(\{X : \exists j\, [j \in \Phi^X_e \wedge j \geqslant n]\}) \\ = \lim_z \mu(\{X : \exists j \leqslant z\, [j \in \Phi^X_e \wedge j \geqslant n]\}),\end{aligned}$$

where the final equation follows easily from the countable additivity of μ. Thus all sufficiently large numbers z meet the criterion to be chosen as k. Furthermore, k can be found by effective search since the condition $\mu(\{X : \Phi^X_e \cap [n, k) \neq \emptyset\}) > \frac{1}{2}$ is a Σ^0_1 predicate of n and k. Hence f is a (total) computable function. It remains to be shown that the interval $[n, f(n))$ always contains at least one element of A. Since $\mathcal{C}_e$ and $\{X : \Phi^X_e \cap [n, k] \neq \emptyset\}$ each have measure greater than $\frac{1}{2}$, these two sets must have a nonempty intersection. Fix a set X that belongs to both sets. Then Φ^X_e is a subset of A that intersects the interval $[n, f(n))$, so A intersects this interval. □

Now to complete the proof of (10), let A be any set such that both A and $\overline{A}$ are hyperimmune, and let the coloring c be the characteristic function of A. For A we may choose any weakly 1-generic set, i.e., any set that meets every dense c.e. set of binary strings. By Kurtz [38, Corollaries 2.10 and 2.11], every hyperimmune degree contains a weakly 1-generic set, and every such set is hyperimmune and has a hyperimmune complement. □

3. The number of colors

3.1. **Weihrauch reducibility and the number of colors.** In Theorem 3.1 of [12], Dorais, Dzhafarov, Hirst, Mileti, and Shafer showed that $\mathrm{RT}^n_k \nleqslant_{\mathrm{sW}} \mathrm{RT}^n_j$ for all $n \geqslant 1$ and $k > j \geqslant 2$. They asked whether this result remains true for Weihrauch reducibility. We show here that it does. This result was obtained independently by Brattka and Rakotoniaina [5, 48]. During the final stages of preparation of this paper, we were informed that Patey [46] has proved the stronger result that $\mathrm{RT}^n_k \nleqslant_{\mathrm{c}} \mathrm{RT}^n_j$ for all $n \geqslant 2$ and $k > j \geqslant 2$ (with a proof considerably more involved than the one we give for our weaker result).

To motivate our proof, let us first consider the $n = 1$ case. Here the proof is easy: Assume for a contradiction that there is a Weihrauch reduction of RT^1_k to RT^1_j given by the functionals Φ and Ψ. We build a k-coloring c of $\mathbb{N}$. For each $i < j$, we search for a finite set F_i such that every element of F_i has Φ^c-color i and $\Psi^{c \oplus F_i}(m_i)\downarrow = 1$, say with use u_i, for some m_i. Then we ensure that $c(m) \neq c(m_i)$ for all sufficiently large m. Since there are at most j values of i such that m_i is defined, and we have k colors available with $k > i$, this is easy to arrange. Let i be such that infinitely many numbers have Φ^c-color i. Then F_i must eventually be found, since otherwise there is an infinite homogeneous set H for Φ^c such that $\Psi^{c \oplus H}$ is not infinite. There is an infinite homogeneous set I for Φ^c such that $F_i \subset I$ and $\min(I \setminus F_i) > u_i$. But $m_i \in \Psi^{c \oplus I}$, and $c(m) \neq c(m_i)$ for all sufficiently large m, so $\Psi^{c \oplus I}$ is not an infinite homogeneous set for c, giving us the desired contradiction.

To lift this construction to higher exponents, we proceed as in the proof of part (2) of Theorem 2.10: Instead of constructing a computable coloring of n-tuples directly, we construct an $\emptyset^{(n-1)}$-computable coloring c of singletons. By the recursion theorem and the iterated form of the limit lemma, we may assume that we know an index for the reduction from $\emptyset^{(n-1)}$ to c, and hence know an index for a computable coloring d of n-tuples such that $c(x) = \lim_{s_1} \cdots \lim_{s_{n-1}} d(x, s_1, \ldots, s_{n-1})$. Note that if an infinite set is homogeneous for d then it is homogeneous for c.

Of course, we need to deal with the fact that Φ^d is a coloring of n-tuples. But as we will see, we have access to an infinite $\Pi_1^{0,\emptyset^{(n-1)}}$ set P such that the Φ^d-color of a tuple in $[P]^n$ depends only on its least element. Let e be the induced coloring of P (i.e., $e(x)$ is the Φ^d-color of a tuple in $[P]^n$ with least element x). Then, as long as we stay inside P, we can work with the coloring of singletons e in place of Φ^d, and proceed much as above, to build c so that, for some infinite homogeneous set H for e, and hence for Φ^d, the set $\Psi^{d \oplus H}$ is not an infinite homogeneous set for c, and hence is also not an infinite homogeneous set for d.

There is a slight complication in that we have to approximate both e and P, and hence might have to make several attempts at defining our m_i and F_i before we hit on suitable ones, but this fact will not affect the construction much.

Note that by Jockusch [30, Theorem 5.5], each computable 2-coloring d of n-tuples has an infinite Π_n^0 homogeneous set P. However, this fact does not serve our purposes here because this homogeneous set P is obtained by a non-uniform construction. By weakening the requirement of homogeneity, we can obtain uniformity using proofs from [30], as we now show.

Lemma 3.1. *Let $n \geqslant 1$ and $k \geqslant 2$ and let $X \subseteq \mathbb{N}$.*

(1) *Let $Y \gg X'$. There is an effective procedure that, from the index of an X-computable k-coloring c of $[\mathbb{N}]^n$, produces an index for a Y-computable infinite prehomogeneous set for c. (Note that we are not claiming that the definition of this procedure is uniform with respect to Y.)*

(2) *There is an effective procedure that, from the index of an X-computable k-coloring c of $[\mathbb{N}]^n$, produces an index for a $\Pi^{0,X'}_1$ infinite prehomogeneous set for c.*

Proof. For part (1), use the proof of Lemma 5.4 of [30], together with the fact we have already noted that if $Y \gg X'$ then each infinite X-computable finitely branching tree has a Y-computable path.

For part (2), if $n = 2$ use the proof of Theorem 4.2 of [30]. The set M constructed there is Π^0_2, uniformly obtained, and prehomogeneous. An elaboration of the same argument gives the full result. However, we omit the proof since we need the result only for $n = 2$ in the proof of the following lemma. □

Lemma 3.2. *Let $n, k \geqslant 2$. Then there is an effective procedure that, from the index of a computable k-coloring of $[\mathbb{N}]^n$, produces an index for an infinite $\Pi^{0,\emptyset^{(n-1)}}_1$ set P such that the color of a tuple in $[P]^n$ depends only on its least element.*

Proof. For each $i \leqslant n-2$, let $X_i \gg \emptyset^{(i)}$ be low over $\emptyset^{(i)}$. Such sets exist by the low basis theorem. We first claim that, for each $i \leqslant n-2$, there is an effective procedure that, from an index of a computable k-coloring of $[\mathbb{N}]^n$, produces an index for an X_i-computable infinite set P_i such that the color of a tuple in $[P_i]^n$ depends only on its least $n-i$ many elements.

We proceed by induction. The $i = 0$ case is trivial. Now assume the claim holds for $i-1 \leqslant n-3$. Applying the first part of Lemma 3.1 to the induced coloring of $[P_{i-1}]^{n-i+1}$ (with $X = X_{i-1}$, noting that $X'_{i-1} \equiv_{\mathrm{T}} \emptyset^{(i)}$) yields the desired procedure for obtaining P_i.

Now applying the second part of Lemma 3.1 to the induced coloring of $[P_{n-2}]^2$ (with $X = X_{n-2}$, noting that $X'_{n-2} \equiv_{\mathrm{T}} \emptyset^{(n-1)}$) yields the desired procedure for obtaining P. □

Theorem 3.3 (due independently to Patey [46] and Brattka and Rakotoniaina [5, 48])**.** *Let $n \geqslant 1$ and $k > j \geqslant 2$. Then $\mathrm{RT}^n_k \nleqslant_{\mathrm{W}} \mathrm{RT}^n_j$.*

Proof. Assume for a contradiction that there is a Weihrauch reduction from RT^n_j to RT^n_k given by the functionals Φ and Ψ. We build an $\emptyset^{(n-1)}$-computable k-coloring of singletons $c : \mathbb{N} \to k$. As mentioned above, we may assume that we know an index for the reduction from $\emptyset^{(n-1)}$ to c, and hence know an index for a computable coloring $d : [\mathbb{N}]^n \to k$ such that $c(x) = \lim_{s_1} \cdots \lim_{s_{n-1}} d(x, s_1, \ldots, s_{n-1})$. (If $n = 1$ then we just take $d = c$.) Note that if an infinite set is homogeneous for d then it is homogeneous for c. By Lemma 3.2, we have access to an infinite $\Pi^{0,\emptyset^{(n-1)}}_1$ set P such that the Φ^d-color of a tuple in $[P]^n$ depends only on its least element. Let e be the induced coloring of P. Let P_s be stage s of an $\emptyset^{(n-1)}$-computable approximation to P. For $x \in P_s$, let $e_s(x)$ be the Φ^d-color of the least element of $[P_s]^n$ with first element x (in some fixed ordering of

$[\mathbb{N}]^n$). We think of e_s as a coloring of P_s. Note that $e(x) = \lim_s e_s(x)$ for all $x \in P$.

As discussed above, the idea is to build c as in the $n = 1$ case described above, but via a $\emptyset^{(n-1)}$-computable construction instead of a computable one, with Φ^c replaced by e, and working inside P. Our task is to build c so that Φ^d has an infinite homogeneous set H such that $\Psi^{d \oplus H}$ is not an infinite homogeneous set for d.

We have parameters $m_0, \ldots, m_{j-1} \in \mathbb{N}$ and $F_0, \ldots, F_{j-1} \subset \mathbb{N}$, with the latter being finite sets when defined. All parameters are initially undefined. Whenever m_i is defined, $c(m_i)$ will also be defined. Let D_i be the ith finite set in a standard listing of finite sets.

At stage s, we act as follows. There are three steps to this stage.

Step 1. For each $i < j$, if F_i is defined and either $F_i \not\subseteq P_s$ or $e_s(x) \neq e_{s-1}(x)$ for some $x \in F_i$, then undefine m_i and F_i.

Step 2. Search for the least pair (i, l) with $i < j$ and $l < s$, if any, such that F_i is undefined and for $F = D_l$,

1. $F \subset P_s$,
2. $e_s(x) = i$ for all $x \in F$, and
3. $\Psi^{d \oplus F}(m)[s]\downarrow = 1$ for some m such that $c(m)$ is defined.

If there is no such pair, then proceed to Step 3. Otherwise, let $F_i = F$ and let $m_i = m$.

Step 3. Define $c(s)$ to be the least number different from $c(m_i)$ for each m_i that is defined.

There is an $i < j$ such that $e(x) = i$ for infinitely many $x \in P$. It is then easy to see that there is a stage s at which F_i becomes defined and is never later undefined. Furthermore, $F_i \subset H$ for some infinite $H \subseteq P$ such that $\min(H \setminus F_i)$ exceeds the use of the computation $\Psi^{d \oplus F_i}(m_i)$ and H is homogeneous for e, and hence for Φ^d. Then $m_i \in \Psi^{d \oplus H}$, but for every sufficiently large m, we have $c(m) \neq c(m_i)$, so $\Psi^{d \oplus H}$ is not an infinite homogeneous set for c, and hence is not an infinite homogeneous set for d. $\square$

Note that in the $n = 2$ case of the above proof, d is stable, and $\Psi^{d \oplus H}$ is in fact not an infinite limit-homogeneous set for d. Thus we have the following result.

Theorem 3.4 (due independently to Patey [46])**.** *If $k > j \geqslant 2$ then $\mathrm{D}^2_k \nleq_{\mathrm{W}} \mathrm{RT}^2_j$, and hence $\mathrm{SRT}^2_k \nleq_{\mathrm{W}} \mathrm{RT}^2_j$.*

We can use our exponent-lifting technique to answer another question raised by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12]. In that paper, they studied the following Ramsey-theoretic principles, related to the Thin Set Theorem, whose reverse-mathematical analysis was introduced by Friedman [21] and further developed by Cholak, Giusto, Hirst, and Jockusch [6] and Wang [58], among others.

Definition 3.5. A set S is *thin* for a k-coloring c of $[\mathbb{N}]^n$ if there is an $i < k$ such that $f(s) \neq i$ for all $s \in [S]^n$. Let TS^n_k be the statement that every k-coloring of $[\mathbb{N}]^n$ has an infinite thin set.

Note that TS^n_2 is the same as RT^n_2. Indeed, let $\mathrm{RT}^n_{k,j}$ be the statement that for every k-coloring c of $[\mathbb{N}]^n$, there is an infinite set S such that the image of $[S]^n$ under c consists of at most j many colors. Then RT^n_k becomes $\mathrm{RT}^n_{k,1}$, and TS^n_k becomes $\mathrm{RT}^n_{k,k-1}$. The general reverse-mathematical study of this family of principles was suggested by J. Miller (see Montalbán [43, Section 2.2.3]) and has been pursued by Lempp, Miller and Ng (see [43]), Wang [58], and Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12].

Among several other results, Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12, Proposition 5.2 and Theorem 5.27] showed that if $k > j \geqslant 2$ and $n \geqslant 1$ then $\mathrm{TS}^1_j \nleqslant_{\mathrm{W}} \mathrm{TS}^1_k$ and $\mathrm{TS}^n_j \nleqslant_{\mathrm{sW}} \mathrm{TS}^n_k$. They left open whether these results can be extended to prove the analog of the former result for higher exponents. As we will see, we can use the same exponent-lifting strategy as above to show that this is indeed the case. Patey [personal communication] has announced the stronger result that $\mathrm{TS}^n_j \nleqslant_{\mathrm{c}} \mathrm{TS}^n_k$ for $n \geqslant 2$ and $k > j \geqslant 2$.

To motivate our proof, let us review the proof in [12, Theorem 5.27] that $\mathrm{TS}^1_j \nleqslant_{\mathrm{W}} \mathrm{TS}^1_k$ for $k > j \geqslant 2$. Assume for a contradiction that there is a Weihrauch reduction of TS^1_j to TS^1_k given by the functionals Φ and Ψ. To obtain a contradiction, we define a computable k-coloring of singletons c such that Φ^c (which is a j-coloring of singletons) has an infinite thin set I such that $\Psi^{c \oplus I}$ is not a thin set for c. A key observation is that if d is a k-coloring of $\mathbb{N}$ and F is a finite set whose image under d consists of at most $i \leqslant k-2$ many colors, then F is contained in an infinite set I whose image under d consists of at most $i+1$ many colors, and hence is thin for d. For instance, let $I = F \cup H$, where H is an infinite homogeneous set for d.

Our construction now proceeds as follows. We start building a k-coloring of singletons c (say by setting $c(x) = 0$ for $x = 0, 1, \dots$) while looking for a finite set F_1 such that the image of F_1 under Φ^c consists of only 1 color and $\Psi^{c \oplus F_1}(m_1)\downarrow = 1$ for some m_1, say with use u_1. (In the foregoing, we require that any oracle information about c uses only the part of c already defined when the first such set F_1 is found.) We must eventually find such an F_1, since otherwise c is total and hence there is an infinite homogeneous set I for Φ^c. (Then $\Psi^{c \oplus I}$ is infinite, so we can take m_1 to be any element of $\Psi^{c \oplus I}$ and F_1 to be a finite subset of I with $\Psi^{c \oplus F_1}(m_1) = 1$.) We then continue defining c and ensure that $c(m) \neq c(m_1)$ for all sufficiently large m. Now, assuming $j > 2$, we look for a finite set $F_2 \supseteq F_1$ such that $\min(F_2 \setminus F_1) > u_1$, the image of F_2 under Φ^c consists of at most

2 colors, and $\Psi^{c\oplus F_2}(m_2)\downarrow = 1$, say with use u_2, for some m_2 such that $c(m_2) \neq c(m_1)$. Again, we must eventually find such an F_2 because, as noted above, F_1 is contained in an infinite set I with $\min(I - F_1) > u_1$ such that image of I under Φ^c consists of at most 2 colors, and $\Psi^{c\oplus I}$ must be infinite. We then ensure that $c(m) \neq c(m_2)$ for all sufficiently large m. Note that $\Psi^{c\oplus F_2}(m_1) = 1$ via the same computation as $\Psi^{c\oplus F_1}(m_1) = 1$, since F_2 and F_1 agree below u_1.

We continue in this way to define F_i and m_i for all $i < j$, in such a way that the m_i all have different c-colors and $\Psi^{c\oplus F_i}(m_i) = 1$ with use u_i, where $\min(F_{i+1} \setminus F_i) > u_i$. The image of F_{j-1} under Φ^c consists of at most $j-1 \leqslant k-2$ many colors, so it is contained in an infinite thin set I for Φ^c. We may assume that $\min(I \setminus F_{j-1}) > u_{j_1}$. Then $\Psi^{c\oplus I}(m_i) = \Psi^{c\oplus F_i}(m_i) = 1$ for all $0 < i < j$. But $\Psi^{c\oplus I}$ must be infinite, so $\Psi^{c\oplus I}(m) = 1$ for some m such that $c(m) \neq c(m_i)$ for all $0 < i < j$. Then the image of $\Psi^{c\oplus I}$ under c consists of j many colors, so $\Psi^{c\oplus I}$ is not a thin set for c, which is a contradiction.

We are now ready to prove the following theorem.

Theorem 3.6. *Let $n \geqslant 1$ and $k > j \geqslant 2$. Then $\mathrm{TS}^n_j \nleqslant_{\mathrm{W}} \mathrm{TS}^n_k$.*

Proof. Assume for a contradiction that there is a Weihrauch reduction from TS^n_k to TS^n_j given by the functionals Φ and Ψ. We build an $\emptyset^{(n-1)}$-computable k-coloring of singletons $c : \mathbb{N} \to k$. Let d, P, e, P_s, and e_s be as in the proof of Theorem 3.3. Note that if an infinite set is thin for d then it is thin for c. As in the proof of Theorem 3.3, the idea is to build c as in the $n = 1$ case described above, but via an $\emptyset^{(n-1)}$-computable construction instead of a computable one, with Φ^c replaced by e, and working inside P. Our task is to build c so that Φ^d has an infinite thin set I such that $\Psi^{d\oplus I}$ is not an infinite thin set for d.

We have parameters $m_1, \dots, m_{j-1}, u_1, \dots, u_{j-1} \in \mathbb{N}$ and $F_0, \dots, F_{j-1} \subset \mathbb{N}$, with the latter being finite sets when defined. We will always have $F_0 = \emptyset$. All other parameters are initially undefined. Whenever m_i is defined, $c(m_i)$ will also be defined. Let D_i be the ith finite set in a standard listing of finite sets.

At stage s, we act as follows. There are three steps to this stage.

Step 1. If there exists i such that F_i is defined and either $F_i \nsubseteq P_s$ or $e_s(x) \neq e_{s-1}(x)$ for some $x \in F_i$, then, for the least such i and all $l \geqslant i$, undefine m_l, u_l, and F_l. If there is no such i, do nothing.

Step 2. Let i be largest such that F_i is defined. If $i = j-1$ then proceed to Step 3. Otherwise, search for the least $l < s$, if any, such that, letting $F = D_l$,

1. $F_i \subseteq F \subset P_s$,
2. $\min(F \setminus F_i) > u_i$,
3. the image of F under e_s consists of at most $i + 1$ many colors, and

4. $\Psi^{d\oplus F}(m)[s]\downarrow = 1$, say with use u, for an m such that $c(m)$ is defined and is different from $c(m_i)$ for each m_i that is defined.

If there is no such pair then proceed to Step 3. Otherwise, let $F_{i+1} = F$, let $m_{i+1} = m$, and let $u_{i+1} = u$.

Step 3. Define $c(s)$ to be the least number different from $c(m_i)$ for each m_i that is defined.

We claim there is a stage s_0 by which all F_i and m_i are defined, and are never later undefined. Suppose not, and let i be largest such that there is a stage s by which F_i and m_i are defined and never later undefined (or let $i = s = 0$ if there is no such $i > 0$). Then $F_i \subset P$ and $e(x) = e_t(x)$ for all $x \in F_i$ and $t \geqslant s$, so the image of F_i under e consists of at most $i \leqslant k-2$ many colors. As noted above, it follows that F_i is contained in some infinite subset I of P that is thin for e, and hence thin for Φ^d. Then $\Psi^{d\oplus I}$ must be an infinite set that is thin for d. Thus there must be an F such that $F_i \subseteq F \subset P$ and $\min(F \setminus F_i) > u_i$, the image of F under e consists of only $i+1$ many colors, and $\Psi^{d\oplus F}(m)\downarrow = 1$ for some $m \geqslant s$. Note that $c(m)$ is different from $c(m_j)$ for all $j \in (0, i]$. It is now easy to see that there is a stage $t \geqslant s$ at which F_{i+1} and m_{i+1} are defined and never later undefined.

Let s_0 be as above. As before, $F_{j-1} \subset P$ and $e(x) = e_t(x)$ for all $x \in F_{j-1}$ and $t \geqslant s_0$, so the image of F_{j-1} under e consists of at most $j-1 \leqslant k-2$ many colors. Thus F_{j-1} is contained in some infinite subset I of P that is thin for e, and hence thin for Φ^d. Then $\Psi^{d\oplus I}$ must be an infinite set that is thin for d. In particular, it must contain some $m \geqslant s$. But then $c(m)$ is different from $c(m_i)$ for all $i \in (0, j]$, so $\Psi^{d\oplus I}$ contains elements with j many different c-colors, and hence is not thin for c. Thus $\Psi^{d\oplus I}$ is not thin for d, giving us our desired contradiction. □

3.2. **Strong computable reducibility, nonuniform infinite information reducibility, and the number of colors.** In his proof that COH $\nleqslant_{\mathrm{sc}}$ D^2_2, Dzhafarov [15] showed something stronger, namely that if $n \geqslant 2$ and $m < 2^n$, there is a sequence $R = (R_0, \dots, R_{n-1})$ of sets such that for any partition $A_0, \dots, A_{m-1}$ of $\mathbb{N}$ hyperarithmetical in R, there is an infinite subset of some A_j that does not compute any cohesive set for R. There are 2^n many intersections of the form $\bigcap_{i<n} S_i$ with each S_i equal to either R_i or $\overline{R_i}$. As Dzhafarov [personal communication] has remarked, it is easy to adapt the proof in [15] to ensure that all but $m+1$ many of these intersections are empty. (For those familiar with this proof, all that needs to be done is the following: Choose a set $I \subseteq 2^n$ of size $m+1$. It is easy to restrict the set of conditions in the proof to ensure that for every n, there is a $\rho \in I$ such that $R_i(n) = \rho(i)$ for all $i < n$. Lemma 2.3 in [15] remains true when we add the condition that $u_0 \dots u_{n-1} \in I$. In the proof of Lemma 2.4 in [15], we again restrict ourselves to such $u_0, \dots, u_{n-1}$, and may still conclude in the last paragraph that there are two different such n-tuples for which we can pick the same $j < m$, since $|I| > m$.) Thus we can obtain an $(m+1)$-coloring c of $\mathbb{N}$ from R such that every infinite homogeneous set for c is cohesive for R. Of course, a partition $A_0, \dots, A_{m-1}$ of $\mathbb{N}$ yields an m-coloring of $\mathbb{N}$. Thus the proof in [15] yields the following result.

Theorem 3.7 (Dzhafarov [15]). *Let $2 \leqslant j < k$. Then there is a $c : \mathbb{N} \to k$ such that for every c-hyperarithmetic $d : \mathbb{N} \to j$, there is an infinite homogeneous set for d that does not compute any infinite homogeneous set for c. In particular, $\mathrm{RT}^1_k \nleqslant_{\mathrm{sc}} \mathrm{RT}^1_j$.*

In fact, since we can view a view a stable coloring of pairs c as a c'-computable coloring d of $\mathbb{N}$, with limit-homogeneous sets for c corresponding to homogeneous sets for d, Dzhafarov's proof shows that if $2 \leqslant j < k$ then $\mathrm{RT}^1_k \nleqslant_{\mathrm{sc}} \mathrm{D}^2_j$. More recently, Dzhafarov, Patey, Solomon, and Westrick [19] have shown that, in fact, if $2 \leqslant j < k$ then $\mathrm{RT}^1_k \nleqslant_{\mathrm{sc}} \mathrm{SRT}^2_j$, answering a question stated in an earlier version of this paper.

Montalbán [personal communication] asked whether the first statement in Theorem 3.7 might hold without any computability assumptions on d. We will give a positive answer to this question in Theorem 3.9. (This theorem has been independently obtained by Patey [46], whose proof remarkably shows that c can be chosen to be low.) This result demonstrates a perhaps surprising difference in the degrees of homogeneous sets resulting from partitions of $\mathbb{N}$ into j parts and into k parts for $j \neq k$.

Our result is also connected with one of the several reducibilities Dzhafarov and Igusa [17] call "notions of robust information coding". These include the generic-case and coarse reducibilities defined by Jockusch and Schupp [33], as well as several ones introduced in [17]. One of these is infinite information reducibility, denoted by $\leqslant_{\mathrm{ii}}$. A *partial oracle* for a set A is a set (A) such that if there is an s with $\langle n, 0, s\rangle \in (A)$ then $n \notin A$, and if there is an s with $\langle n, 1, s\rangle \in (A)$ then $n \in A$. The domain of (A) is the set of n such that $\langle n, i, s\rangle \in (A)$ for some s and some $i < 2$. We say that $B \leqslant_{\mathrm{ii}} A$ if there is a Turing functional Γ such that for every partial oracle (A) for A with infinite domain, $\Gamma^{(A)}$ has infinite domain and is equal to B where defined.

As noted in [17], infinite information reducibility acts in the opposite way to what one might at first expect. For instance, as shown in [17, Proposition 3.6], if $A \leqslant_1 B$ then $B \leqslant_{\mathrm{ii}} A$, and $A \oplus B$ is the infimum of A and B under $\leqslant_{\mathrm{ii}}$. Another interesting feature of ii-reducibility is the existence of maximal pairs.

Theorem 3.8 (Dzhafarov and Igusa [17, Theorem 5.1]). *There are sets B_0 and B_1 for which there is no set A such that $B_0 \leqslant_{\mathrm{ii}} A$ and $B_1 \leqslant_{\mathrm{ii}} A$.*

One of the motivations of Dzhafarov and Igusa in introducing ii-reducibility was its potential connections with the computability-theoretic analysis of versions of Ramsey's Theorem. In this regard, it is particularly interesting to examine the nonuniform analog to ii-reducibility. We say that B is *nonuniformly infinite information reducible* to A, and write $B \leqslant_{\text{nii}} A$, if each partial oracle (A) for A with infinite domain computes a partial function that has infinite domain and is equal to B where defined. The connection with Ramsey's Theorem is particularly apparent when we consider the following equivalent definition: $B \leqslant_{\text{nii}} A$ if every infinite subset of A or $\overline{A}$ computes an infinite subset of B or $\overline{B}$.

To see that these definitions are indeed equivalent, first suppose that the first definition holds of A and B and let S be an infinite subset of A or $\overline{A}$. Let $i = 1$ if $S \subseteq A$ and let $i = 0$ otherwise. Then $\{\langle n, i, 0\rangle \mid n \in S\}$ is an S-computable partial oracle for A, so it computes a partial function f that has infinite domain and is equal to B where defined. Given f we can compute an infinite subset of B or $\overline{B}$. Now suppose that the second definition holds of A and B and let (A) be a partial oracle for A with infinite domain. From (A) we can compute an infinite subset S of A or $\overline{A}$. From S we can compute an infinite subset of B or $\overline{B}$, and from such a subset we can compute a partial function with infinite domain that is equal to B where defined.

Using ideas from its proof, we can extend Theorem 3.8 to nii-reducibility. Given a maximal pair B_0, B_1 for nii-reducibility, let $c : \mathbb{N} \to 4$ be defined by $c(n) = 2B_0(n) + B_1(n)$. Then c is a 4-coloring of $\mathbb{N}$ such that for every 2-coloring d of $\mathbb{N}$, there is an infinite homogeneous set for d that does not compute any infinite homogeneous set for c. By working directly with colorings (but still using ideas from the proof of Theorem 3.8), we can make c a 3-coloring, and generalize this result as follows.

Theorem 3.9 (due independently to Patey [46])**.** *Let $m \geqslant 2$. There is a $c : \mathbb{N} \to (m+1)$ such that for every $d : \mathbb{N} \to m$, there is an infinite homogeneous set for d that does not compute any infinite homogeneous set for c.*

Proof. We will show that any sufficiently Cohen generic $c : \mathbb{N} \to (m+1)$ has this property. To determine a sufficient level of genericity, we begin with a few definitions based on the notion of α-deduction for ordinals α in [17, Definition 5.5]. A *Mathias condition* is a pair (F, I) such that F is a finite set, I is an infinite set, and $\max F < \min I$. We write $(F, I) \leqslant (G, J)$ if $I \subseteq J$ and $G \subseteq F \subset G \cup J$. Given a coloring $d : \mathbb{N} \to m$, the construction of an infinite homogeneous set for d that does not compute any infinite homogeneous set for c will use Mathias conditions. Thus we will need to adapt the notion of α-deduction to work relative to a given infinite set I.

Let $\Phi_0, \Phi_1, \ldots$ be an effective listing of the $[0, m]$-valued Turing functionals. For a finite set F, we write $\Phi^F_e(n)\Downarrow$ to mean that $\Phi^F_e(n)\downarrow$ with use no larger than $\max F + 1$ (where we take this max to be -1 for $F = \emptyset$). A set X is *compatible* with the Mathias condition (F, I) if $F \subseteq X \subseteq F \cup I$. Note that if $\Phi_e^F(n)\Downarrow$ and X is compatible with (F, I) then $\Phi_e^X(n)\downarrow = \Phi_e^F(n)$.

Definition 3.10. Let I be an infinite set and let F be a finite set. We say that F *α-deduces relative to I that* $\Phi_e(n) = i$ if either $\Phi_e^F(n)\Downarrow = i$ or there are infinitely many $k \in I$ such that $F \cup \{k\}$ β-deduces relative to I that $\Phi_e(n) = i$ for some $\beta < \alpha$. We say that F *deduces relative to I that* $\Phi_e(n) = i$ if it α-deduces relative to I that $\Phi_e(n) = i$ for some α.

Note that it is possible for F to deduce relative to I both that $\Phi_e(n) = i$ and that $\Phi_e(n) = j$ for some $i \neq j$. Note also that for each I there is a countable ordinal γ_I such that if F deduces relative to I that $\Phi_e(n) = i$, then F α-deduces relative to I that $\Phi_e(n) = i$ for some $\alpha < \gamma_I$.

For each number e, finite set F, and infinite set I, define a partial function $f_{e,F,I}$ by letting $f_{e,F,I}(n) = i$ if and only if F deduces relative to I that $\Phi_e(n) = i$ but does not deduce relative to I that $\Phi_e(n) = j$ for any $j \neq i$.

Remark 3.11. It is easy to show by transfinite induction on α that if F α-deduces relative to I that $\Phi_e(n) = j$, then for every k there is a finite set $G \subset I$ with $\min G > k$ such that $\Phi_e^{F\cup G}(n)\downarrow = j$. If F 0-deduces relative to I that $\Phi_e(n) = i$ then $\Phi_e^{F\cup G}(n)\downarrow = i$ for all $G \subset I$ such that $\min G > \max F$, so F cannot deduce relative to I that $\Phi_e(n) = j$ for $j \neq i$, and hence $f_{e,F,I}(n) = i$.

The basic reason for choosing our coloring c to be generic is to ensure that, thinking of c as a partition of $\mathbb{N}$, each part "knows nothing about" the other parts. In fact, we will need it to be the case that if F and G are disjoint subsets of $[0, m]$ such that $F \cup G \neq [0, m]$, then $c^{-1}(F)$ "knows nothing about" $c^{-1}(G)$, in the sense that (roughly speaking) we cannot use $c^{-1}(F)$ to compute an infinite subset of $c^{-1}(G)$, even with the help of some additional information, represented by a Turing ideal that we now define. (We will be more precise about the properties we need c to satisfy below. Note that the requirement that $F \cup G \neq [0, m]$ is necessary, since $c^{-1}(F)$ can compute $c^{-1}([0, m] \setminus F)$.)

Definition 3.12. Let $\mathcal{I}$ be a countable Turing ideal such that for each $X \in \mathcal{I}$ there is a $Y \in \mathcal{I}$ with the following properties.

1. For any X-computable infinite set I and any ordinal $\alpha \leqslant \gamma_I$, given numbers e, n, and i and a finite set F, we can use Y to compute whether F β-deduces relative to I that $\Phi_e(n) = i$ for some $\beta < \alpha$. (In particular, taking $\alpha = \gamma_I$, given numbers e, n, and i and a finite set F, we can use Y to compute whether F deduces relative to I that $\Phi_e(n) = i$, and hence we can use Y to compute whether $f_{e,F,I}(n) = i$.)
2. For any X-computable infinite set I, given a number e and finite sets F and G, we can use Y to compute whether $\operatorname{dom} f_{e,F,I} \subseteq \operatorname{dom} f_{e,G,I}$.

3. For any nonempty $\Pi^{0,X}_1$ subclass $\mathcal{C}$ of ω^ω, we can use Y to compute the range of some element of $\mathcal{C}$.

Though we do not need this fact here, we note that we can take $\mathcal{I}$ to be the smallest hyperjump-ideal, i.e., the smallest Turing ideal closed under the hyperjump.

For sets A and B, we say that A is $(B \oplus \mathcal{I})$-computable if it is $(B \oplus X)$-computable for some $X \in \mathcal{I}$, that A is $\mathcal{O}^{B\oplus\mathcal{I}}$-computable if it is $\mathcal{O}^{B\oplus X}$-computable for some $X \in \mathcal{I}$ (where $\mathcal{O}^Y$ is the hyperjump of Y), and that a subclass of ω^ω is a $\Pi^{0,\mathcal{I}}_1$ class if it is a $\Pi^{0,X}_1$ class for some $X \in \mathcal{I}$.

Let $c : \mathbb{N} \to (m+1)$ be sufficiently generic so that the following conditions hold. (The first condition was already mentioned above; the second will be needed in the proof of Lemma 3.17 below.)

(C1) Let $F, G \subset [0,m]$ be such that $F \cap G = \emptyset$ and $F \cup G \neq [0,m]$. Then $\mathcal{O}^{c^{-1}(F)\oplus\mathcal{I}}$ does not compute any infinite subset of $c^{-1}(G)$.

(C2) Let $D_0, D_1, \ldots$ be an $\mathcal{I}$-computable sequence of finite sets such that $\min D_n > n$ for all n. Then for each $i \leqslant m$ there are infinitely many n such that $c(x) = i$ for all $x \in D_n$.

Note that, in particular, there is no infinite homogeneous set for c in $\mathcal{I}$.

Remark 3.13. One way to see that such a c exists is the following: Let Z be a set computing all elements of $\mathcal{I}$. Then 1-genericity relative to Z suffices to ensure that (C2) holds. Let F and G be as in (C1). Hyperarithmetic genericity relative to $\mathcal{O}^Z$ suffices to ensure that there is no infinite subset of $c^{-1}(G)$ hyperarithmetic in $c^{-1}(F) \oplus \mathcal{O}^Z$. A slight variation on Exercise IV.3.13 of Sacks [49] shows that if c is sufficiently generic relative to Z then $\mathcal{O}^{c^{-1}(F)\oplus Z}$ is hyperarithmetically reducible to $c^{-1}(F) \oplus \mathcal{O}^Z$. Thus, if c is sufficiently generic then there is no infinite subset of $c^{-1}(G)$ computable (or even hyperarithmetic) in $\mathcal{O}^{c^{-1}(F)\oplus Z}$.

Fix $d_0 : \mathbb{N} \to m$. We must show that there is an infinite homogeneous set for d_0 that does not compute any infinite homogeneous set for c. It will be convenient to "work inside" a certain infinite set $I \in \mathcal{I}$. The property we desire for I is that for each $i < m$ either $d_0^{-1}(i) \cap I$ is empty or there is no infinite subset of $I \setminus d_0^{-1}(i)$ in $\mathcal{I}$. This property can be ensured by requiring that $|d(I)|$ (the number of colors assigned by d to elements of I) be as small as possible as I ranges over infinite sets in $\mathcal{I}$. Fix such a "minimal" I. Suppose that there are l many values of $i < m$ such that $d_0^{-1}(i) \cap I$ is nonempty. By permuting the numbers less than m we may assume that these l many values of i are exactly $0, 1, \ldots, l-1$, so that $d_0 \restriction I$ is an l-coloring of I. Let $d = d_0 \restriction I$. Then d has the property that for each $i < l$ there is no infinite subset of $I \setminus d^{-1}(i)$ in $\mathcal{I}$. We may assume without loss of generality that $l \geqslant 2$ since clearly $l > 0$, while if $l = 1$ then I is an infinite homogeneous set for d_0, and I cannot compute an infinite homogeneous set for c, as there are no infinite homogeneous sets for c in $\mathcal{I}$. Our goal now is to construct an infinite set $S \subset I$ that is homogeneous for d and computes no infinite homogeneous set for c, which of course suffices to prove the theorem since S is also homogeneous for d_0.

To give some intuition for our construction, let us consider the case where $l = 2$. We may assume that every infinite subset of $d^{-1}(1)$ computes an infinite homogeneous set for c, as otherwise we are done. It is enough to build an infinite set $S \subseteq d^{-1}(0)$ to satisfy the requirements

$$R_e : \ |\operatorname{dom} \Phi_e^S| = \omega \ \Rightarrow \ \exists n\, [\Phi_e^S(n)\downarrow \neq c(n)],$$

as it then clearly follows that S cannot compute an infinite homogeneous set for c. The idea is to build a sequence of Mathias conditions $(F_0, I_0) > (F_1, I_1) > \cdots$ beginning with $F_0 = \emptyset$ and $I_0 = \operatorname{dom} d = I$, using (F_{e+1}, I_{e+1}) to satisfy R_e, and to let $S = \bigcup_e F_e$. (We can ensure that S is infinite by also making $F_{e+1} \supsetneq F_e$ for all e.) As in many Mathias forcing constructions, the easiest way to satisfy R_e is to have a finite set $F \subset d^{-1}(0)$ compatible with (F_e, I_e) and an n such that $\Phi_e^F(n)\Downarrow \neq c(n)$. We then set $F_{e+1} = F$ and let I_{e+1} be the set of all elements of I_e greater than $\max F$. We call this outcome a "finite win". If there is no such finite set F, then we want to define $(F_{e+1}, I_{e+1}) < (F_e, I_e)$ so that for any X compatible with (F_{e+1}, I_{e+1}), the domain of Φ_e^X is finite. (Notice that S is such an X.)

As we will show below in Lemma 3.14 of our more general setting, the assumption that we are not in the "finite win" case allows us to prove the following Correctness Lemma by transfinite induction: If the finite set $H \subset d^{-1}(0)$ is compatible with (F_e, I_e), then $f_{e,H,I_e}(n) = c(n)$ when defined.

Let $\mathcal{C}$ be the $\Pi_1^{0,\mathcal{I}}$ class of all $g \in \omega^\omega$ such that g is strictly increasing, $\operatorname{rng} g \subseteq I_e$, and $\operatorname{dom} f_{e,F_e \cup H,I_e} \subseteq \operatorname{dom} f_{e,F_e,I_e}$ for all finite $H \subset \operatorname{rng} g$. If $\mathcal{C}$ is nonempty, then it has an element g such that $Y = \operatorname{rng} g$ is in $\mathcal{I}$. We claim that we can take $I_{e+1} = Y$ and satisfy R_e. Indeed, suppose that $F_e \subset X \subseteq F_e \cup Y$. It is easy to see using Remark 3.11 that if $\Phi_e^X(n)\downarrow$ then $n \in \operatorname{dom} f_{e,F_e,I_e}$. However, the graph of f_{e,F_e,I_e} is in $\mathcal{I}$, and by the Correctness Lemma, $f_{e,F_e,I_e}(n) = c(n)$ when defined, so $\operatorname{dom} f_{e,F_e,I_e}$ cannot be infinite, as otherwise there would be an infinite homogeneous set for c in $\mathcal{I}$. Thus $\operatorname{dom} \Phi_e^X$ is finite, as desired.

So we are left with arguing that $\mathcal{C}$ is nonempty. Assume for a contradiction that $\mathcal{C} = \emptyset$. As we will show below in Lemma 3.15, it follows that there is a finite set $G \subset d^{-1}(0)$ compatible with (F_e, I_e) such that $\operatorname{dom} f_{e,G\cup\{k\},I_e} \nsubseteq \operatorname{dom} f_{e,G,I_e}$ for infinitely many $k \in I_e$. Let E be the set of such k. Note that $E \in \mathcal{I}$. For each $k \in E$, let n_k be the least element of $\operatorname{dom} f_{e,G\cup\{k\},I_e} \setminus \operatorname{dom} f_{e,G,I_e}$. We claim that there is no n such that $n_k = n$ for infinitely many $k \in E$. Assume for a contradiction that there is such an n. By the definition of deducibility, the fact that $n \in \operatorname{dom} f_{e,G\cup\{k\},I_e}$ for infinitely many $k \in I_e$ implies that there is an i such that G deduces relative to I_e that $\Phi_e(n) = i$. By the Correctness Lemma, the fact that G is a subset of $d^{-1}(0)$ and is compatible with (F_e, I_e) implies that G cannot deduce relative to I_e that $\Phi_e(n) = i$ unless $i = c(n)$. It follows that $n \in \operatorname{dom} f_{e,G,I_e}$, which is a contradiction and hence establishes our claim.

It is now easy to see that there are an $i \leqslant m$ and an $\mathcal{I}$-computable sequence $k_0, k_1, \ldots \in E$ such that $n_{k_p} > p$ and $f_{e,G\cup\{k_p\},I_e}(n_{k_p}) = i$ for all p. Let $j \leqslant m$ be such that $j \neq i$. The sequence $n_{k_0}, n_{k_1}, \ldots$ is also $\mathcal{I}$-computable, so condition (C2) above, applied to the singletons $\{n_{k_0}\}, \{n_{k_1}\}, \ldots$, implies that there are infinitely many p such that $c(n_{k_p}) = j$. Thus

$$R = \{k \in E : f_{e,G\cup\{k\},I_e}(n_k)\downarrow = i \wedge c(n_k) = j\}$$

is infinite. (In this case, we can argue that R is infinite for some choice of $j \neq i$ without appealing to condition (C2), but that condition will be crucial in obtaining an analogous set in the full proof.)

If $k \in R$ then $G \cup \{k\}$ is compatible with (F_e, I_e) but does not satisfy the conclusion of the Correctness Lemma, so it must be the case that $G\cup\{k\} \nsubseteq d^{-1}(0)$. Since $G \subset d^{-1}(0)$ and $l = 2$, it follows that $k \in d^{-1}(1)$. So R is an infinite subset of $d^{-1}(1)$, and hence computes an infinite homogeneous set H for c. If H is not homogeneous to j then we have our desired contradiction, since R is $(c^{-1}(j) \oplus \mathcal{I})$-computable, but we cannot be sure that this is the case. Instead, we use the fact that R cannot compute an infinite subset of $c^{-1}(i)$ (which follows from condition (C1) above and the fact that R is $(c^{-1}(j) \oplus \mathcal{I})$-computable) to analyze the following superset of R:

$$Q = \{k \in E : f_{e,G\cup\{k\},I_e}(n_k)\downarrow = i \neq c(n_k)\}.$$

The set Q is $(c^{-1}(i) \oplus \mathcal{I})$-computable. Furthermore, Q has an infinite subset that does not compute any infinite subset of $c^{-1}(i)$, namely R. The class of all such subsets of Q is arithmetic relative to $c^{-1}(i) \oplus \mathcal{I}$, so it contains an $\mathcal{O}^{c^{-1}(i)\oplus\mathcal{I}}$-computable member P. By the same argument as for R, we have that P is an infinite subset of $d^{-1}(1)$, and hence computes an infinite set H that is homogeneous for c. Since P does not compute any infinite subset of $c^{-1}(i)$, the set H must be contained in $c^{-1}(i')$ for some $i' \neq i$. Thus there is an $\mathcal{O}^{c^{-1}(i)\oplus\mathcal{I}}$-computable infinite subset of $c^{-1}(i')$, contradicting condition (C1).

Having given an outline of our proof in the case where $l = 2$, let us now turn to our full proof. Now we assume that every infinite subset of $d^{-1}(l-1)$ computes an infinite homogeneous set for c, and want to make S an infinite subset of $d^{-1}(i)$ for some $i \leqslant l-2$. As candidates for S we will build infinite sets $S_0, \ldots, S_{l-2}$ with $S_i \subseteq d^{-1}(i)$. For each $i \leqslant l-2$, we seek to satisfy the requirements

$$R_{i,e} : \ |\operatorname{dom} \Phi_e^{S_i}| = \omega \ \Rightarrow \ \exists n\, \Phi_e^{S_i}(n)\!\downarrow \neq c(n).$$

Fix a bijection $\langle \cdot, \dots, \cdot \rangle$ between $\mathbb{N}^{l-1}$ and $\mathbb{N}$. At stage $\langle e_0, \dots, e_{l-2} \rangle + 1$ of our construction, we will ensure the satisfaction of $R_{0,e_0} \vee \cdots \vee R_{l-2,e_{l-2}}$. We claim that it follows that there is an $i \leqslant l-2$ such that every $R_{i,e}$ is satisfied. If $l = 2$ then the claim is clear. Otherwise, suppose that for each $i < l-2$ there is an e_i such that R_{i,e_i} is never satisfied. Then at stage $\langle e_0, \dots, e_{l-3}, e \rangle$, we necessarily satisfy $R_{l-2,e}$. Thus our claim holds. Furthermore, it is easy to see that if every $R_{i,e}$ is satisfied then S_i, which is an infinite homogeneous set for d, cannot compute an infinite homogeneous set for c.

We define sequences of Mathias conditions $(F_0^i, I_0) > (F_1^i, I_1) > \cdots$ for $i \leqslant l-2$ by induction. (Notice that the finite parts of the conditions depend on i, but the infinite parts do not.) We will ensure that $F_s^i \subset d^{-1}(i)$, that $|F_{s+1}^i| > |F_s^i|$, and that $I_s \in \mathcal{I}$. Our desired infinite subset S_i of $d^{-1}(i)$ will be $\bigcup_s F_s^i$. We will begin by setting $I_0 = \operatorname{dom} d = I$, which ensures that for each s, we have $I_s \subseteq \operatorname{dom} d$, and hence $|I_s \cap d^{-1}(i)| = \omega$ for all $i \leqslant l-2$, by our assumption that there is no infinite subset of $\operatorname{dom} d \setminus d^{-1}(i)$ in $\mathcal{I}$.

Let $F_0^i = \emptyset$ and $I_0 = \operatorname{dom} d$. Suppose we have defined $F_s^i \subset d^{-1}(i)$ and $I_s \in \mathcal{I}$. Let $e_0, \dots, e_{l-2}$ be such that $s = \langle e_0, \dots, e_{l-2} \rangle$. We first ask whether there are an $i \leqslant l-2$, an n, and a finite set $G \subset d^{-1}(i)$ compatible with (F_s^i, I_s) such that $\Phi_{e_i}^G(n)\!\Downarrow \neq c(n)$. If so then, by adding an element to G if needed, we may assume that $G \supsetneq F_s^i$. In this case, proceed as follows. Let $F_{s+1}^i = G$. For each $j \leqslant l-2$ such that $j \neq i$, let k be the least element of $I_s \cap d^{-1}(j)$ and let $F_{s+1}^j = F_s^j \cup \{k\}$. Let I_{s+1} be the restriction of I_s to numbers greater than $\max(\bigcup_{j \leqslant l-2} F_{s+1}^j)$. Notice that $I_{s+1} \in \mathcal{I}$, and for each $j \leqslant l-2$ (including i), $F_{s+1}^j \subset d^{-1}(j)$ and $|F_{s+1}^j| > |F_s^j|$, as required. We say that the conditions (F_{s+1}^j, I_{s+1}) are defined by finite extension. Note that, in this case, R_{i,e_i} is satisfied.

If we are not in the above case then proceed as follows. We first prove a general version of the Correctness Lemma discussed in our outline of the case where $l = 2$ above.

Lemma 3.14. *For each $i \leqslant l-2$, if a finite set $G \subset d^{-1}(i)$ compatible with (F_s^i, I_s) deduces relative to I_s that $\Phi_{e_i}(n) = j$, then $c(n) = j$.*

Proof. We prove by induction on $\alpha < \gamma_{I_s}$ that if such a G α-deduces relative to I_s that $\Phi_{e_i}(n) = j$, then $c(n) = j$. The statement holds for $\alpha = 0$ because we have not defined the conditions (F_{s+1}^i, I_{s+1}) by finite extension. Now suppose it holds for all $\beta < \alpha$, and let $G \subset d^{-1}(i)$ be a finite set compatible with (F_s^i, I_s) that α-deduces relative to I_s that $\Phi_{e_i}(n) = j$. Assume for a contradiction that $c(n) \neq j$. Since we do not have $\Phi_{e_i}^G(n)\!\Downarrow = j$, there are infinitely many $k \in I_s$ such that $G \cup \{k\}$

β-deduces relative to I_s that $\Phi_{e_i}(n) = j$ for some $\beta < \alpha$. By the inductive hypothesis, none of these k's can be in $d^{-1}(i)$, and the set of all such k's is in $\mathcal{I}$. So there is an infinite subset of $\operatorname{dom} d \setminus d^{-1}(i)$ in $\mathcal{I}$, contradicting our assumption on d. □

Now let $\mathcal{C}^i_s$ be the $\Pi^{0,\mathcal{I}}_1$ class of all $g \in \omega^\omega$ such that g is strictly increasing, $\operatorname{rng} g \subseteq I_s$, and $\operatorname{dom} f_{e_i,F^i_s \cup H,I_s} \subseteq \operatorname{dom} f_{e_i,F^i_s,I_s}$ for all finite $H \subset \operatorname{rng} g$. Note that if $F^i_s \subseteq S \subseteq F^i_s \cup \operatorname{rng} g$ for some element g of $\mathcal{C}^i_s$ and $\Phi^S_{e_i}(n){\downarrow} = j$, then $\Phi^{F^i_s \cup H}_{e_i}(n){\Downarrow} = j$ for some finite $H \subset \operatorname{rng} g$, which means that $F^i_s \cup H$ 0-deduces relative to I_s that $\Phi_{e_i}(n) = j$. It follows by Remark 3.11 that $n \in \operatorname{dom} f_{e_i,F^i_s \cup H,I_s}$, and hence that $n \in \operatorname{dom} f_{e_i,F^i_s,I_s}$.

We now show that there is at least one $i \leqslant l-2$ for which $\mathcal{C}^i_s$ is nonempty. We will then be able to take (a finite modification of) the range of an element of $\mathcal{C}^i_s$ as I_{s+1}, which, as in our outline of the case where $l = 2$, will allow us to satisfy $R^i_{e_i}$ because the graph of f_{e_i,F^i_s,I_s} is in $\mathcal{I}$, and hence cannot compute an infinite homogeneous set for c. We begin with two auxiliary lemmas, which allow us to use the assumption that each $\mathcal{C}^i_s$ is empty to choose finite sets $G_0, \dots, G_{l-2}$ with properties analogous to the finite set G in the outline of the case where $l = 2$. We will also want a set E analogous to the one in that outline. It will be important that there is a single such E that works for all G_i.

Lemma 3.15. *Let $i \leqslant l-2$ and let $D \subseteq I_s \cap d^{-1}(i)$ be infinite. Suppose that for every finite set $G \subset d^{-1}(i)$ compatible with (F^i_s, I_s), we have that $\operatorname{dom} f_{e_i,G \cup \{k\},I_s} \subseteq \operatorname{dom} f_{e_i,G,I_s}$ for all but finitely many $k \in D$. Then $\mathcal{C}^i_s$ is nonempty.*

Proof. We define a function $g \in \mathcal{C}^i_s$ by recursion. Suppose that $g(i)$ is defined for all $i < n$ and let $Y_n = \{g(i) : i < n\}$. Assume by induction that if $H \subseteq Y_n$ then $f_{e_i,F^i_s \cup H,I_s} \subseteq f_{e_i,F^i_s,I_s}$. By assumption, for each such H we have $\operatorname{dom} f_{e_i,F^i_s \cup H \cup \{k\},I_s} \subseteq \operatorname{dom} f_{e_i,F^i_s \cup H,I_s}$ for almost all $k \in D$. So there is a $k \in D$ such that $\operatorname{dom} f_{e_i,F^i_s \cup H \cup \{k\},I_s} \subseteq \operatorname{dom} f_{e_i,F^i_s \cup H,I_s}$ for all $H \subseteq Y_n$ and $k > g(i)$ for all $i < n$. Let $g(n)$ be the least such k. Then $g \in \mathcal{C}^i_s$ by construction. □

Lemma 3.16. *Suppose $\mathcal{C}^i_s = \emptyset$ for all $i \leqslant l-2$. Then there are an infinite set $E \subseteq I_s$ and finite sets $G_i \subset d^{-1}(i)$ for $i \leqslant l-2$ such that $\operatorname{dom} f_{e_i,G_i \cup \{k\},I_s} \nsubseteq \operatorname{dom} f_{e_i,G_i,I_s}$ for every $k \in E$ and $i \leqslant l-2$.*

Proof. By Lemma 3.15, there is a finite set $G_0 \subset d^{-1}(0)$ compatible with (F^0_s, I_s) such that $\operatorname{dom} f_{e_0,G_0 \cup \{k\},I_s} \nsubseteq \operatorname{dom} f_{e_0,G_0,I_s}$ for infinitely many $k \in I_s \cap d^{-1}(0)$. Then there is an infinite set $E_0 \subseteq I_s$ in $\mathcal{I}$ such that $\operatorname{dom} f_{e_0,G_0 \cup \{k\},I_s} \nsubseteq \operatorname{dom} f_{e_0,G_0,I_s}$ for all $k \in E_0$. Now, $E_0 \cap d^{-1}(1)$ must be infinite, so again by Lemma 3.15, there is a finite set $G_1 \subset d^{-1}(1)$ compatible with (F^1_s, I_s) such that $\operatorname{dom} f_{e_1,G_1 \cup \{k\},I_s} \nsubseteq \operatorname{dom} f_{e_1,G_1,I_s}$ for infinitely many $k \in E_0 \cap d^{-1}(1)$. Again, there is an infinite set $E_1 \subseteq E_0$ in $\mathcal{I}$ such that $\operatorname{dom} f_{e_1,G_1\cup\{k\},I_s} \nsubseteq \operatorname{dom} f_{e_1,G_1,I_s}$ for all $k \in E_1$. We can continue in this manner to define, for each $i \leqslant l-2$, a finite set $G_i \subset d^{-1}(i)$ compatible with (F_s^i, I_s) and an infinite set $E_i \in \mathcal{I}$ with $E_i \subseteq E_{i-1}$ such that $\operatorname{dom} f_{e_i,G_i\cup\{k\},I_s} \nsubseteq \operatorname{dom} f_{e_i,G_i,I_s}$ for all $k \in E_i$. We have $E_0 \supseteq E_1 \supseteq \cdots \supseteq E_{l-2}$, so the lemma follows by taking $E = E_{l-2}$. □

Lemma 3.17. *There is an $i \leqslant l-2$ such that $\mathcal{C}_s^i$ is nonempty.*

Proof. Assume otherwise, and let E and $G_0, \dots, G_{l-2}$ be as in Lemma 3.16. Our argument will be similar to that in our outline of the case where $l = 2$. We will define sets R, Q, and P analogous to the ones in that outline. The main difference is that the single color i in the definitions of R and Q will now be replaced by a set F of up to $l-1$ many colors.

For $k \in E$ and $i \leqslant l-2$, let n_k^i be the least element of $\operatorname{dom} f_{e_i,G_i\cup\{k\},I_s} \setminus \operatorname{dom} f_{e_i,G_i,I_s}$. Fix $i \leqslant l-2$. We claim that there is no n such that $n_k^i = n$ for infinitely many $k \in E$. Assume for a contradiction that there is such an n. By the definition of deducibility, the fact that $n \in \operatorname{dom} f_{e_i,G_i\cup\{k\},I_s}$ for infinitely many $k \in I_s$ implies that there is a j such that G_i deduces relative to I_s that $\Phi_{e_i}(n) = j$. By Lemma 3.14, G_i cannot deduce relative to I_s that $\Phi_{e_i}(n) = j$ unless $j = c(n)$. It follows that $n \in \operatorname{dom} f_{e_i,G_i,I_s}$, which is a contradiction and hence establishes our claim.

It is now easy to see that there are $j_0, \dots, j_{l-2} \leqslant m$ and an $\mathcal{I}$-computable sequence $k_0, k_1, \dots \in E$ such that $n_{k_p}^i > p$ and $f_{e_i,G_i\cup\{k_p\},I_s}(n_{k_p}^i) = j_i$ for all p and $i \leqslant l-2$. Let $F = \{j_0, \dots, j_{l-2}\}$ and let $j \leqslant m$ be such that $j \notin F$. Applying condition (C2) to the sets $\{n_{k_p}^i : i \leqslant l-2\}$, we see that there are infinitely many p such that $c(n_{k_p}^i) = j$ for all $i \leqslant l-2$. Thus

$$R = \{k \in E : (\forall i \leqslant l-2)\,[f_{e_i,G_i\cup\{k\},I_s}(n_k^i)\downarrow \in F \,\wedge\, c(n_k^i) = j]\}$$

is infinite. Since R is $(c^{-1}(j) \oplus \mathcal{I})$-computable, $|F| \leqslant l-1 \leqslant m-1$, and $j \notin F$, condition (C1) implies that R cannot compute an infinite subset of $c^{-1}(F)$.

Now let

$$Q = \{k \in E : (\forall i \leqslant l-2)\,[f_{e_i,G_i\cup\{k\},I_s}(n_k^i)\downarrow \in F \,\wedge\, c(n_k^i) \notin F]\}.$$

Then Q is $(c^{-1}(F) \oplus \mathcal{I})$-computable. Furthermore, Q has an infinite subset that does not compute any infinite subset of $c^{-1}(F)$, namely R. The class of all such subsets of Q is arithmetic relative to $c^{-1}(F) \oplus \mathcal{I}$, so it contains an $\mathcal{O}^{c^{-1}(F)\oplus\mathcal{I}}$-computable member P. Let $k \in P$. For each $i \leqslant l-2$, we have that $f_{e_i,G_i\cup\{k\},I_s}(n_k^i) \neq c(n_k^i)$, which implies, by Lemma 3.14, that $G_i \cup \{k\} \nsubseteq d^{-1}(i)$, and hence that $k \notin d^{-1}(i)$. Thus $k \in d^{-1}(l-1)$. So P is an infinite subset of $d^{-1}(l-1)$, and hence computes an infinite homogeneous set H for c. Since P does not compute any infinite subset of $c^{-1}(F)$, the set H must be contained in $c^{-1}(j')$ for some $j' \notin F$. So there is an $\mathcal{O}^{c^{-1}(F)\oplus\mathcal{I}}$-computable infinite subset of $c^{-1}(j')$, which contradicts condition (C1). □

Let $i \leqslant l-2$ be such that $\mathcal{C}^i_s$ is nonempty, and let $Y \in \mathcal{I}$ be the range of an element of $\mathcal{C}^i_s$. Note that $Y \cap d^{-1}(j)$ is infinite for all $j \leqslant l-2$. For each $j \leqslant l-2$, let k be the least element of $Y \cap d^{-1}(j)$ and let $F^j_{s+1} = F^j_s \cup \{k\}$. Let I_{s+1} be the restriction of Y to numbers greater than $\max(\bigcup_{j \leqslant l-2} F^j_{s+1})$. Notice that $I_{s+1} \in \mathcal{I}$, and for each $j \leqslant l-2$, we have $F^j_{s+1} \subset d^{-1}(j)$ and $|F^j_{s+1}| > |F^j_s|$, as required.

We have completed the definition of our Mathias conditions. Now, for each $i \leqslant l-2$, let $S_i = \bigcup_s F^i_s$. We have ensured that $|F^i_{s+1}| > |F^i_s|$ and that $F^i_s \subset d^{-1}(i)$ for all s, so each S_i is an infinite subset of $d^{-1}(i)$. As argued above, it is enough to show that for each $s = \langle e_0, \ldots, e_{l-2} \rangle$, we satisfy at least one requirement $R^i_{e_i}$. Assume for a contradiction that we do not. Then it cannot be the case that the conditions (F^i_{s+1}, I_{s+1}) were defined by finite extension, so there is an $i \leqslant l-2$ such that $F^i_s \subset S_i \subseteq F^i_s \cup \operatorname{rng} g$ for some element g of $\mathcal{C}^i_s$, which, as noted above, implies that if $\Phi^{S_i}_{e_i}(n)\downarrow$ then $n \in \operatorname{dom} f_{e_i, F^i_s, I_s}$. Since we are assuming that $R^i_{e_i}$ is not satisfied, there are infinitely many such n, so $\operatorname{dom} f_{e_i, F^i_s, I_s}$ is infinite. Since $f_{e_i, F^i_s, I_s}(n) = c(n)$ when defined and the graph of f_{e_i, F^i_s, I_s} is in $\mathcal{I}$, there is an infinite homogeneous set for c in $\mathcal{I}$, contradicting the choice of c. □

Letting c be as in the theorem with $m = 2$ and defining $B_0 = c^{-1}(\{0,1\})$ and $B_1 = c^{-1}(\{0,2\})$ yields the following strengthening of Theorem 3.8.

Corollary 3.18. *There exist sets B_0 and B_1 for which there is no set A such that $B_0 \leqslant_{\text{nii}} A$ and $B_1 \leqslant_{\text{nii}} A$.*

The sets B_0 and B_1 constructed in the proof of Theorem 3.8 are quite complicated, and the ones in the above corollary even more so. As mentioned above, Patey [46] has shown that, remarkably, the coloring c in Theorem 3.9 can in fact be taken to be low, and hence so can the sets B_0 and B_1 in Corollary 3.18. We do not have any nontrivial lower bounds on the complexity of maximal pairs in either the ii or the nii case. Indeed, Dzhafarov and Igusa [17, Question 5.9] asked whether being mutually 1-random or mutually 1-generic is enough to ensure that a pair of sets is maximal with respect to ii-reducibility, and we may ask the same question for nii-reducibility. Even this level of complexity might conceivably not be necessary.

4. Reductions to multiple instances of a problem

4.1. **Reduction games.** As mentioned in the introduction, $P \leqslant_\omega Q$ can be thought of as meaning that solutions to P can be obtained from solutions to multiple instances of Q. We can make this intuition precise by thinking of reductions between problems as games.

Definition 4.1. For problems P and Q, the *reduction game* $G(Q \to P)$ is a two-player game that proceeds as follows.

On the first move, Player 1 plays an instance X_0 of P, and Player 2 either plays an X_0-computable solution to X_0 and declares victory, in which case the game ends, or responds with an X_0-computable instance Y_1 of Q. If Player 2 cannot move (which might happen if there is no X_0-computable instance of Q), then Player 1 wins, and the game ends.

For $n > 1$, on the nth move (if the game has not yet ended), Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q. Then Player 2 either plays a $(\bigoplus_{i<n} X_i)$-computable solution to X_0 and declares victory, in which case again the game ends, or plays a $(\bigoplus_{i<n} X_i)$-computable instance Y_n of Q.

Player 2 wins this play of the game if it ever declares victory. Otherwise, Player 1 wins.

Proposition 4.2. *If $P \leqslant_\omega Q$ then Player 2 has a winning strategy for $G(Q \to P)$. Otherwise, Player 1 has a winning strategy for $G(Q \to P)$. (Note that these strategies do not have to be effective in any way.)*

Proof. Suppose that $P \leqslant_\omega Q$. Player 2 then plays according to the following strategy. At the nth move, if there is a legal winning move, Player 2 makes it. Otherwise, it lets $Y_{n,0}, Y_{n,1}, \ldots$ be all instances of Q computable in the join of the moves of the game so far. For the least pair $\langle m, i\rangle$ with $m \leqslant n$ for which Player 2 has not yet acted, it then acts by playing $Y_{m,i}$ (to which Player 1 must reply with a solution to $Y_{m,i}$). Suppose Player 2 never has a winning move, and Player 1 never fails to have a legal move. Then the Turing ideal generated by the moves in the game is an ω-model of Q, and hence is a model of P. But then there must be some finite number of moves whose join computes a solution to Player 1's first move, which gives Player 2 a winning move. Thus Player 2 must eventually win the game no matter what Player 1 does.

Now suppose that $P \nleqslant_\omega Q$ and let S be an ω-model of $\mathrm{RCA}_0 + Q$ that is not a model of P. Since S is a Turing ideal, as long as Player 1's moves stay inside S, so must Player 2's moves. Furthermore, the fact that S is a model of Q implies that, as long as Player 2's moves stay inside S, Player 1 will always be able to reply with moves that stay inside S. So Player 1 can simply begin by playing an instance $X_0 \in S$ of P that has no solution in S, and then keep playing elements of S, which ensures that the game never ends (unless there is no X_0-computable instance of Q, in which case Player 2 loses on the first move). □

4.2. **Generalized Weihrauch reducibility.** We can now define a generalized notion of Weihrauch reduction by considering *computable* strategies. We assume that we have defined the join operation for finitely many sets so that we can determine n from $\bigoplus_{i\leqslant n} X_i$, say by letting $\bigoplus_{i\leqslant n} X_i = \{n\} \oplus \{\langle i, k\rangle : i \leqslant n \wedge k \in X_i\}$. In general, when we write $X_0 \oplus X_1$ we mean $\{2n : n \in X_0\} \cup \{2n+1 : n \in X_1\}$ as usual. However, we will sometimes write $X_0 \oplus X_1$ for $\bigoplus_{i \leqslant 1} X_i$, when it is clear that this is what we mean. Similarly, we will write simply X_0 instead of $\bigoplus_{i \leqslant 0} X_i$.

Definition 4.3. A *computable strategy* for Player 2 in a reduction game is a Turing functional that, given the join of Player 1's first n moves as an oracle, outputs Player 2's nth move. More precisely, the strategy is a functional Φ such that, if Z is the join of Player 1's first n moves, then $\Phi^Z = V \oplus Y$, where Y is Player 2's nth move, and V is $\{0\}$ if Player 2 declares victory at this move and $\emptyset$ otherwise. We write $\widehat{\Phi}^X$ for $\{n : 2n+1 \in \Phi^X\}$ (so if Z is the join of Player 1's first n moves, then $\widehat{\Phi}^Z$ is Player 2's nth move). This strategy is *winning* if it enables Player 2 to win no matter what Player 1 does.

We say that P is *Weihrauch* (or *uniformly*) *reducible to Q in the generalized sense*, and write $P \leqslant_{\mathrm{gW}} Q$, if Player 2 has a computable winning strategy in $G(Q \to P)$.

For example, consider RT^n_3 and RT^n_2. While $\mathrm{RT}^n_3 \leqslant_\omega \mathrm{RT}^n_2$, we have seen in Theorem 3.3 that $\mathrm{RT}^n_3 \nleqslant_{\mathrm{W}} \mathrm{RT}^n_2$. However, the procedure for obtaining solutions to instances of RT^n_3 using RT^n_2 is clearly uniform, though it does require two applications of RT^n_2. Indeed, we do have $\mathrm{RT}^n_3 \leqslant_{\mathrm{gW}} \mathrm{RT}^n_2$. Here is Player 2's strategy in $G(\mathrm{RT}^n_2 \to \mathrm{RT}^n_3)$: Player 1 plays a coloring $c : [\mathbb{N}]^n \to 3$. Then Player 2 plays the c-computable coloring $d : [\mathbb{N}]^n \to 2$ defined by $d(s) = 0$ if $c(s) = 2$ and $d(s) = 1$ otherwise. Next, Player 1 plays an infinite homogeneous set H for d. From $c \oplus H$, Player 2 can computably determine to which color H is homogeneous. If that color is 0, then H is also homogeneous for c, so Player 2 plays H and declares victory. If the color is 1, then Player 2 plays the 2-coloring $d \upharpoonright [H]^n$ (encoded as a coloring of $[\mathbb{N}]^n$). Player 1 must then play a homogeneous set I for this coloring. This set is also homogeneous for c, so Player 2 plays I and declares victory. Of course, this argument can easily be adapted to show that $\mathrm{RT}^n_k \leqslant_{\mathrm{gW}} \mathrm{RT}^n_j$ for any $k > j \geqslant 2$.

Proposition 4.4. *The relation $\leqslant_{\mathrm{gW}}$ is transitive.*

Proof. Let Φ and Ψ be computable winning strategies for Player 2 in $G(Q \to P)$ and $G(R \to Q)$, respectively. Then Player 2 has the following computable winning strategy in $G(R \to P)$. Player 1 plays the instance X_0 of P. If $\Phi^{X_0}(0) = 1$ then $\widehat{\Phi}^{X_0}$ is a solution to X_0, so player 2 plays it and wins. Otherwise, $\widehat{\Phi}^{X_0}$ is an instance Y_1 of Q. Player 2 can then begin a play of $G(R \to Q)$ by pretending that Player 1 has played Y_1 on its first move, and play this game according to Ψ until it obtains a solution X_1 to Y, which must happen no matter how player 1 replies. If $\Phi^{X_0 \oplus X_1}(0) = 1$ then $\widehat{\Phi}^{X_0 \oplus X_1}$ is a solution to X_0, so player 2 plays it and wins. Otherwise, $\widehat{\Phi}^{X_0 \oplus X_1}$ is an instance Y_2 of Q. Again, Player 2 can play $G(R \to Q)$ according to Ψ pretending that Player 1 has played Y_2 on its first move. Continuing in this way, Player 2 must eventually find an n such that $\Phi^{\bigoplus_{i \leqslant n} X_0}(0) = 1$. □

Clearly, if $P \leqslant_{\mathrm{W}} Q$ then $P \leqslant_{\mathrm{gW}} Q$, and if $P \leqslant_{\mathrm{gW}} Q$ then $P \leqslant_{\omega} Q$, so we have the following picture.

$$\begin{tikzcd}
& P \leqslant_{\mathrm{gW}} Q \arrow[dl, Rightarrow] & P \leqslant_{\mathrm{W}} Q \arrow[l, Rightarrow] \arrow[dl, Rightarrow] & \\
P \leqslant_{\omega} Q & P \leqslant_{\mathrm{c}} Q \arrow[l, Rightarrow] & & P \leqslant_{\mathrm{sW}} Q. \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
& & P \leqslant_{\mathrm{sc}} Q \arrow[ul, Rightarrow] &
\end{tikzcd}$$

Versions of Ramsey's Theorem for different exponents provide interesting witnesses to the fact that $\leqslant_{\mathrm{gW}}$ does not imply $\leqslant_{\mathrm{c}}$. (We will see an example showing that $\leqslant_{\mathrm{sc}}$ does not imply $\leqslant_{\mathrm{gW}}$ in Corollary 4.10 below.) Consider for instance RT^4_2 and RT^3_2. As discussed in the introduction, $\mathrm{RT}^4_2 \nleqslant_{\mathrm{c}} \mathrm{RT}^3_2$. On the other hand, let PRE be the principle stating that for every $k, n \geqslant 2$, every k-coloring of $[\mathbb{N}]^n$ has an infinite prehomogeneous set. The construction in Jockusch [30, proof of Lemma 5.4] that gives us the first part of Lemma 3.1 also shows that PRE $\leqslant_{\mathrm{W}}$ KL. We have already seen that KL $\leqslant_{\mathrm{W}} \mathrm{RT}^3_2$, so given an instance c of RT^4_2, we can use one application of RT^3_2 to obtain an infinite prehomogeneous set A for c, then another to obtain an infinite homogeneous set for the induced 2-coloring of $[A]^3$, which will also be homogeneous for c. This procedure is uniform, so $\mathrm{RT}^4_2 \leqslant_{\mathrm{gW}} \mathrm{RT}^3_2$. A similar argument establishes the following fact.

Proposition 4.5. RT $\leqslant_{\mathrm{gW}} \mathrm{RT}^n_k$ *for all* $n \geqslant 3$ *and* $k \geqslant 2$.

Similarly, given an instance c of RT^3_2, we can use one application of PRE to obtain an infinite prehomogeneous set A for c, then another to obtain an infinite prehomogeneous set B for the induced 2-coloring on $[A]^2$, then an application of RT^1_2 to obtain an infinite homogeneous set for the induced 2-coloring on B, which will also be homogeneous for c. Since $\mathrm{RT}^1_2 \leqslant_{\mathrm{W}}$ KL, we see that $\mathrm{RT}^3_2 \leqslant_{\mathrm{gW}}$ KL. By Proposition 4.5, we have the following result.

Proposition 4.6. RT $\leqslant_{\mathrm{gW}}$ KL.

Thus we see that the equivalences represented by the top line of Figure 2.1 also hold for $\leqslant_{\mathrm{gW}}$. The same is true of the bottom line of that figure, a fact worth noting in light of part (4) of Theorem 2.10, and of the similar equivalences for exponent 2.

Proposition 4.7. (1) $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{gW}} \mathrm{RT}^1_2$.
(2) $\mathrm{(S)RT}^2_{<\infty} \leqslant_{\mathrm{gW}} \mathrm{(S)RT}^2_2$ *and* $\mathrm{D}^2_{<\infty} \leqslant_{\mathrm{gW}} \mathrm{D}^2_2$.

Proof. For (1), let $f : \mathbb{N} \to \mathbb{N}$ have bounded range. Let $c_0 : \mathbb{N} \to 2$ be defined by letting $c_0(n) = 0$ if and only if $f(n) = 0$, and let H_0 be an infinite homogeneous set for c_0. If H_0 is homogeneous to 0 then it is homogeneous for f. Otherwise, let $c_1 : H_0 \to 2$ be defined by letting $c_0(n) = 0$ if and only if $f(n) = 1$, and let H_1 be an infinite homogeneous set for c_1. If H_1 is homogeneous to 0 then it is homogeneous for f. Otherwise, continue in this way to define $c_2, c_3, \ldots$ and $H_2, H_3, \ldots$. Eventually we must reach an n such that H_n is homogeneous to 0, and hence homogeneous for f.

The argument for (2) is essentially the same. □

In light of Dzhafarov's result [16] that $\mathrm{SRT}^2_2 \nleqslant_{\mathrm{W}} \mathrm{D}^2_2$, the following fact is also of interest.

Proposition 4.8. $\mathrm{SRT}^2_2 \leqslant_{\mathrm{gW}} \mathrm{D}^2_2$.

Proof. Let $c : [\mathbb{N}]^2 \to 2$ be stable. With one application of D^2_2, we can obtain an infinite limit-homogeneous set H for c. Let $a \in H$ and let $d(n) = c(a, a+n+1)$. With one application of RT^1_2, we can obtain an infinite homogeneous set I for d. Let $n \in I$ and let $i = c(a, a+n+1)$. Then $\lim_s c(a,s) = i$, so $\lim_s c(x,s) = i$ for all $x \in H$, and hence we can define an infinite homogeneous set $b_0, b_1, \ldots$ for c by letting b_0 be the least element of H and letting b_{k+1} be the least element of H greater than b_k such that $c(b_j, b_{k+1}) = i$ for all $j \leqslant k$. This construction is uniform, and $\mathrm{RT}^1_2 \leqslant_{\mathrm{W}} \mathrm{D}^2_2$, so $\mathrm{SRT}^2_2 \leqslant_{\mathrm{gW}} \mathrm{D}^2_2$. □

We will summarize the relationships between versions of Ramsey's Theorem and König's Lemma under $\leqslant_{\mathrm{gW}}$ in Section 5.

The following notions play an important role in the work of Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12]. For problems P_0 and P_1, let (P_0, P_1) be the problem whose instances are pairs (X_0, X_1) where each X_i is an instance of P_i, and such that a solution to such an instance is a pair (Y_0, Y_1) where each Y_i is a solution to X_i. Let $P^2 = (P, P)$. We can similarly define P^n for any n, and P^ω, which is also sometimes denoted by $\mathrm{Seq}P$ (the *sequential version* of P, also known as the *parallelization* of P) to avoid too many superscripts. We also have $P^{<\omega}$, where an instance is any instance of P^n for $n \in \omega$. Lemma 3.3 of [12] illustrates the fact that $P_0 \leqslant_{\mathrm{W}} Q$ and $P_1 \leqslant_{\mathrm{W}} Q$ does not necessarily imply that $(P_0, P_1) \leqslant_{\mathrm{W}} Q$; indeed, it is even possible to have $P^2 \nleqslant_{\mathrm{W}} P$. On the other hand, if $P_0 \leqslant_{\mathrm{gW}} Q$ and $P_1 \leqslant_{\mathrm{gW}} Q$ then clearly $(P_0, P_1) \leqslant_{\mathrm{gW}} Q$. In particular, we always have $P^n \leqslant_{\mathrm{gW}} P$, and even $P^{<\omega} \leqslant_{\mathrm{gW}} P$. It is not necessarily the case that $P^\omega \leqslant_{\mathrm{gW}} P$, however. In fact, we can even have $P^\omega \nleqslant_\omega P$. For example, as shown in [12, Lemma 3.2], there is a computable instance of SeqRT^2_2 such that every solution computes $\emptyset''$, but as shown by Seetapun in [51, Theorem 3.1], there is an ω-model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ that does not contain $\emptyset'$.

It is more difficult to give examples of principles P and Q such that $P \leqslant_\omega Q$ but $P \nleqslant_{\mathrm{gW}} Q$. We will discuss one class of examples in the following subsection. A different kind of example is given by the following proposition, which is a uniform version of Proposition 2.7.

Proposition 4.9. *Let P be a problem. If $P \leqslant_{\mathrm{gW}} \mathrm{WKL}$ then $P \leqslant_{\mathrm{sW}} \mathrm{WKL}$.*

Proof. For a set Z, let $Z^{[i]} = \{n : \langle i,n\rangle \in Z\}$. It is straightforward to define computable functions f, g, and h, and a functional T, such that T^X is a binary tree whose infinite paths are exactly the sets Z with the following properties.

1. $Z^{[0]} = X$.
2. $Z^{[f(i,j)]} = Z^{[i]} \oplus Z^{[j]}$ for all i, j.
3. If $\Phi_e^{Z^{[i]}}$ is total then $Z^{[g(e,i)]} = \Phi_e^{Z^{[i]}}$.
4. If $Z^{[i]}$ codes an infinite binary tree then $Z^{[h(i)]}$ is an infinite path on that tree.

Now suppose that $P \leqslant_{\mathrm{gW}} \mathrm{WKL}$ and let Φ_e be a winning strategy for Player 2 in $G(\mathrm{WKL} \to P)$. Given an instance X of P, let Z be an infinite path on T^X. Proceed as follows to obtain a solution to X uniformly from Z. Let $i_0 = 0$. Suppose we have defined $i_0, \ldots, i_n$. Using f and g, we can compute a j such that $Z_j = \bigoplus_{k \leqslant n} Z^{[i_k]}$. Wait for $\Phi_e^{Z^{[j]}}(0)$ to converge. When it does, if it converges to 1 then output $\widehat{\Phi}_e^{Z^{[j]}}$. (Recall the hat notation from Definition 4.3.) Otherwise, let $i_{n+1} = h(g(e_1, j))$, so that $Z^{[i_{n+1}]}$ is a solution to the instance $\widehat{\Phi}_e^{Z^{[j]}}$ of WKL. It is easy to see that this procedure eventually ends, producing a solution to X. □

Combining the above proposition with part (1) of Theorem 2.10 yields the following result.

Corollary 4.10. $\mathrm{RT}^1_2 \nleqslant_{\mathrm{gW}} \mathrm{WKL}$.

Since clearly $\mathrm{RT}^1_2 \leqslant_{\mathrm{sc}} \mathrm{WKL}$, we see that $\leqslant_{\mathrm{sc}}$ does not imply $\leqslant_{\mathrm{gW}}$.

4.3. **Diagonalizability.** In most cases, when one shows that $P \leqslant_\omega Q$, one does so via a generalized Weihrauch reduction. However, several examples where this is not the case fit a general pattern we now describe. (The definitions we give below are not the absolutely most general ones we could give, but they suffice for our examples.) For $\sigma_0, \ldots, \sigma_n \in 2^{<\omega}$ and a functional Φ, we write $\Phi^{\bigoplus_{i \leqslant n} \sigma_i}(k)\downarrow = j$ to mean that $\Phi^{\{n\} \oplus \{\langle i,k\rangle : i \leqslant n \wedge \sigma_i(k)=1\}}(k)\downarrow = j$ via a computation that does not query the oracle on $2\langle i,k\rangle + 1$ for any $i \leqslant n$ and $k \geqslant |\sigma_i|$. The point is that then, for any $X_0 \succ \sigma_0, \ldots, X_n \succ \sigma_n$, we have $\Phi^{\bigoplus_{i \leqslant n} X_i}(k)\downarrow = j$.

Let us begin with an example. We saw in Theorem 2.10(5) that $\mathrm{RT}^1_2 \nleqslant_{\mathrm{W}}$ COH. The proof of this result uses two properties of these problems: the ability to use instances of RT^1_2 to diagonalize against a supposed infinite homogeneous set, and the fact that for every instance X of COH, every string in $2^{<\omega}$ can be extended to a solution to X. It turns out that these properties suffice to show that, in fact, $\mathrm{RT}^1_2 \nleqslant_{\mathrm{gW}}$ COH, as we now discuss.

Suppose that Player 2 has a computable winning strategy Φ for the game $G(\mathrm{COH} \to \mathrm{RT}^1_2)$, and build a coloring $c : \mathbb{N} \to 2$ as follows. (As mentioned in Remark 1.4, we identify c with a subset of $\mathbb{N}$ coding it. We say that a binary string σ is an initial segment of c, and write $\sigma \prec c$, if σ

is an initial segment of this set.) Start by letting $c(s) = 0$ at stage s until we find $\sigma_0, \dots, \sigma_n \in 2^{<\omega}$ such that

1. $\sigma_0 \prec c$,
2. $\Phi^{\bigoplus_{i \leqslant j} \sigma_i}(0)\downarrow = 0$ for all $j < n$,
3. $\Phi^{\bigoplus_{i \leqslant n} \sigma_i}(0)\downarrow = 1$, and
4. $\widehat{\Phi}^{\bigoplus_{i \leqslant n} \sigma_i}(k)\downarrow = 1$ for some k.

If such strings are found, then ensure that $c(k) = 0$ and $c(t) = 1$ for all sufficiently large t.

Consider a play of $G(\mathrm{COH} \to \mathrm{RT}^1_2)$ where Player 1 plays c, then Player 2 plays the instance Y_0 of COH determined by Φ, then Player 1 plays a solution X_1 to Y_0, and so on, until Player 2 plays an infinite homogeneous set Y_n for c (still determined by Φ). For simplicity of notation, let $X_0 = c$. Then $\Phi^{\bigoplus_{i \leqslant n} X_i}(0)\downarrow = 1$ and $\Phi^{\bigoplus_{i \leqslant j} X_i}(0)\downarrow = 0$ for all $j < n$. Furthermore, there is a k such that $\widehat{\Phi}^{\bigoplus_{i \leqslant n} X_i}(k)\downarrow = 1$. Thus $\sigma_0, \dots, \sigma_n$ as above must eventually be found.

The crucial point now is that there are $X_0 \succ \sigma_0, \dots, X_n \succ \sigma_n$ such that $X_0 = c$ and if $\widehat{\Phi}^{\bigoplus_{i \leqslant j} X_i}$ is an instance of COH then X_{j+1} is a solution to this instance. This is the case simply because for every instance Z of COH, every string can be extended to a solution to Z. Now $X_0, \dots, X_n$ are valid moves by Player 1 in a game in which Player 2 plays according to Φ, and by the choice of $\sigma_0, \dots, \sigma_n$,

1. $\Phi^{\bigoplus_{i \leqslant j} X_i}(0)\downarrow = 0$ for all $j < n$,
2. $\Phi^{\bigoplus_{i \leqslant n} X_i}(0)\downarrow = 1$, and
3. $\widehat{\Phi}^{\bigoplus_{i \leqslant n} X_i}(k)\downarrow = 1$, where k is as above.

Then $\widehat{\Phi}^{\bigoplus_{i \leqslant n} X_i}$ must be an infinite homogeneous set for c. But $k \in \widehat{\Phi}^{\bigoplus_{i \leqslant n} X_i}$ and $c(k) = 0$, while $c(t) = 1$ for all sufficiently large t, so we have a contradiction. Thus $\mathrm{RT}^1_2 \nleqslant_{\mathrm{gW}} \mathrm{COH}$.

We did not use any properties of COH in this argument other than the fact that for every instance X of COH, every string can be extended to a solution to X. (In the terminology of [4], COH is densely realized.) We could also adapt the argument to the following more general situation (where we recall from Remark 1.4 the fact that we identify instances and solutions to problems with subsets of $\mathbb{N}$).

Definition 4.11. A problem P is *undiagonalizable* if there is a uniform procedure that, given an instance X of P and a string σ, decides whether σ can be extended to a solution to X.

Let us now discuss the essential property of RT^1_2 used in the above argument. We think of $\sigma_0, \dots, \sigma_n$ as forming a finite approximation to (Player 1's moves in) a possible run of $G(\mathrm{COH} \to \mathrm{RT}^1_2)$ where Player 1 begins by playing the coloring c we are building and Player 2 plays according to Φ. Then k is an element that is necessarily included in Player

2's final move in that run. We get a contradiction by ensuring that there is no infinite homogeneous set for c containing k. Suppose we were working with TS^1_3 instead of RT^1_2. Then finding a single k would not be enough. We would need a two-step process instead: First start defining $c(s) = 0$ at stage s and find a finite approximation F_0 to a possible run of $G(\mathrm{COH} \to \mathrm{TS}^1_3)$ where Player 1 begins by playing the coloring c we are building and Player 2 plays according to Φ, and a corresponding k_0 necessarily included in Player 2's final move in that run, with $c(k_0) = 0$. Then start defining $c(s) = 1$ at stage s and find a further finite approximation F_1 extending F_0 and a corresponding k_1 with $c(k_1) = 1$. At that point, by letting $c(t) = 2$ for all sufficiently large t, we ensure that there is no infinite thin set for c containing both k_0 and k_1.

In general, we can think of this construction as producing a sequence $F_0 \subset F_1 \subset \cdots$ of approximations to runs of a game $G(Q \to P)$. (In the above cases, we can stop after one or two such approximations, but for the general case we think of the construction as finding infinitely many approximations.) Each F_i determines some part of Player 2's last move. We can easily adapt the above construction so that these parts are longer and longer initial segments, so that their union is a set. We think of this set as Ψ^X, where X is the instance of P that we construct, and Ψ has the property that Ψ^Y is total and infinite for all instances Y of P. The key property of P that makes the construction work is captured by the following definition. (If solutions to instances of P are not themselves subsets of $\mathbb{N}$, we think of them as coded by such sets, as mentioned in Remark 1.4. We then say that such a solution is infinite if the set coding it is infinite, which for the problems considered in this paper is always the case.)

Definition 4.12. A problem P *has diagonalization opportunities* if all of its solutions are infinite and, given any Turing functional Ψ such that Ψ^Y is total and infinite for all instances Y of P, there are a τ and an instance $X \succ \tau$ of P such that no solution to X extends Ψ^τ.

We now have the following general result.

Theorem 4.13. *Let Q be undiagonalizable and P have diagonalization opportunities. Then $P \nleqslant_{\mathrm{gW}} Q$.*

Proof. In this proof, we regard all instances and solutions as subsets of $\mathbb{N}$. Write $X^{[i]}$ for $\{n : \langle i, n\rangle \in X\}$. Assume for a contradiction that $P \leqslant_{\mathrm{gW}} Q$, so that, by Proposition 4.2, Player 2 has a computable winning strategy Φ for $G(Q \to P)$. Say that R is a *possible run* if there is a legal run of this game where Player 1 plays $R^{[0]}, \ldots, R^{[k]}$, Player 2 plays according to its winning strategy, and the game ends after exactly $k+1$ many moves. From a possible run R, we can determine the corresponding k, and hence compute $\widehat{\Phi}^{\bigoplus_{i\leqslant k} R^{[i]}}$, which is a solution to $R^{[0]}$. Thus there is a Turing functional Γ such that if R is a possible run then Γ^R is a solution to $R^{[0]}$.

For $s = (\sigma_0, \ldots, \sigma_n)$ and $s' = (\sigma'_0, \ldots, \sigma'_{n'})$, write $s \preccurlyeq s'$ to mean that $n' \geqslant n$ and $\sigma_i \preccurlyeq \sigma'_i$ for all $i \leqslant n$, and write Γ^s for $\Gamma^{\bigoplus_{i\leqslant n}\sigma_i}$.

Let Θ be a Turing functional such that if Y is an instance of Q then Θ^Y is the set of all σ that can be extended to a solution to Y. For a set X, let S_X be the set of all sequences $s = (\sigma_0, \ldots, \sigma_n)$ for which there is a $k \leqslant n$ such that

1. $\sigma_0 \prec X$,
2. $\Phi^{s\restriction i+1}(0)\downarrow = 0$ for all $i < k$,
3. $\Phi^{s\restriction k+1}(0)\downarrow = 1$, and
4. $\sigma_{i+1} \in \Theta^{\widehat{\Phi}^{s\restriction i+1}}$ for all $i < k$.

We say that a run R extends s if $R^{[i]} \succ \sigma_i$ for all $i \leqslant n$. Note that if X is an instance of P then S_X is uniformly X-c.e., every element of S_X can be extended to a possible run R with $R^{[0]} = X$, and for every possible run R with $R^{[0]} = X$, every sufficiently long initial segment of R is in S_X.

Let Ψ be the functional such that Ψ^X is defined as follows. Search for an $s_0 \in S_X$ and an n_0 such that $\Gamma^{s_0}(m)\downarrow$ for all $m \leqslant n_0$ and $\Gamma^{s_0}(n_0) = 1$. If such an s_0 and n_0 are found then let $\Psi^X(m) = \Gamma^{s_0}(m)$ for all $m \leqslant n_0$ and search for an $s_1 \succcurlyeq s_0$ in S_X and an $n_1 > n_0$ such that $\Gamma^{s_1}(m)\downarrow$ for all $m \leqslant n_1$ and $\Gamma^{s_1}(n_1) = 1$. If such an s_1 and n_1 are found then let $\Psi^X(m) = \Gamma^{s_1}(m)$ for all $m \in (n_0, n_1]$. Continue to define the s_i and n_i in this way. If X is an instance of P then each s_i can be extended to a possible run R such that $R^{[0]} = X$, and Γ^R must be total and infinite, so s_{i+1} and n_{i+1} will be found. Thus in this case Ψ^X is total and infinite.

Let τ be as in Definition 4.12 and let $X \succ \tau$ be an instance of P such that no solution to X extends Ψ^τ. Letting s_i be as above, there is an i such that $\Psi^\tau \preccurlyeq \Gamma^{s_i}$. There is a possible run R extending s_i such that $R^{[0]} = X$. Then $\Gamma^R \succ \Psi^\tau$, so Γ^R is not a solution to $R^{[0]}$, which is a contradiction. $\square$

Remark 4.14. Brattka, Hendtlass, and Kreuzer [4] defined the notion of ω-indiscriminative multi-valued functions on represented spaces using the all or co-unique choice operation $\mathrm{ACC}_{\mathbb{N}}$. The latter can be restated as a problem $P_{\mathrm{ACC}_{\mathbb{N}}}$ with infinite solutions, for example by letting an instance be either $\emptyset$ or a singleton $\{\langle n, k\rangle\}$, and letting a solution to an instance X be any set Y of the form $\{\langle m, k\rangle : k \in \mathbb{N}\}$ such that $X \cap Y = \emptyset$. Then a problem Q is ω-indiscriminative if and only if $P_{\mathrm{ACC}_{\mathbb{N}}} \leqslant_{\mathrm{W}} Q$. Brattka, Hendtlass, and Kreuzer [4, Proposition 4.3] showed that if Q is densely realized (i.e., if for every instance X of Q, every σ can be extended to a solution to X) then Q is ω-indiscriminative. It is easy to see that $P_{\mathrm{ACC}_{\mathbb{N}}}$ has diagonalization opportunities, so Theorem 4.13 extends that result in the context of Π^1_2 principles.

The class of densely realized problems mentioned above includes all problems where the class of solutions to any given instance is a collection of subsets of $\mathbb{N}$ closed under finite difference, for example n-RAND, the statement that for every X, there is a set that is n-random relative to X, and indeed RAND, the statement that for every X there is a set that is arithmetically random (i.e., n-random for every n) relative to X. (See Downey and Hirschfeldt [13] for more on algorithmic randomness.)

A different kind of example is given by the Atomic Model Theorem AMT, which states that every complete atomic theory has an atomic model. Given such a theory T, we can uniformly determine whether σ is an initial segment of a model of T, in which case σ is also an initial segment of an atomic model of T. Thus AMT is undiagonalizable. The same is true of the related principles OPT and AST considered in Hirschfeldt, Shore, and Slaman [29].

As we have seen, an example of a problem with diagonalization opportunities is RT^1_2. Thus, although RT^1_2 is provable in RCA_0, it is not Weihrauch reducible in the generalized sense to any undiagonalizable problem, such as RAND, COH, or AMT. (Of course, the same holds of RT^n_k for any $n \geqslant 1$ and $k \geqslant 2$.)

Corollary 4.15. *If Q is undiagonalizable, then* $\mathrm{RT}^1_2 \nleqslant_{\mathrm{gW}} Q$.

Several other examples of problems with diagonalization opportunities can be found among the consequences of RT^2_2, for example SRT^2_2 and the principles ADS, CAC, SADS, and SCAC studied in Hirschfeldt and Shore [28]. The following are a few more examples.

Recall the principle WWKL, defined at the end of Section 1. Let Ψ be as in Definition 4.12 and let τ be an initial segment of the full binary tree such that $\Psi^\tau(0)\downarrow$. Let T be a binary tree extending τ such that A is a path on T if and only if $A(0) \neq \Psi^\tau(0)$. Then T is an instance of WWKL such that no solution to T extends Ψ^τ, so WWKL has diagonalization opportunities.

Let Θ_e be the eth Turing functional in a fixed effective listing of such functionals. The Diagonally Nonrecursive Principle DNR states that for every X, there is a function f such that $f(e) \neq \Theta^X_e(e)$ for all e. Let Ψ be as in Definition 4.12, and let e be such that $\Theta_e = \Psi$. Let τ be any string such that $\Psi^\tau(e)\downarrow$ and let $X \succ \tau$. Then $\Psi^\tau(e) = \Psi^X(e) = \Theta^X_e(e)$, so no solution to X extends Ψ^τ, and hence DNR has diagonalization opportunities.

Giusto and Simpson [23, Lemma 6.18] showed that WWKL implies DNR over RCA_0. Their proof in fact shows that DNR $\leqslant_{\mathrm{sW}}$ WWKL. Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1] showed that WWKL $\nleqslant_\omega$ DNR. Yu and Simpson [61, Section 2] showed that WKL $\nleqslant_\omega$ WWKL.

As shown by Kučera [37, Lemma 3], for every Π^0_1 class C of positive measure and every 1-random set R, there is a final segment of R contained in C. Relativizing this result, we see that WWKL $\leqslant_{\mathrm{sc}}$ 1-RAND. Kučera

[37] (see also [13, Theorem 8.8.1]) also showed that every 1-random set computes a function f as in the definition of DNR, and his proof shows that in fact DNR $\leqslant_{\mathrm{sc}}$ 1-RAND. On the other hand, Brattka, Hendtlass, and Kreuzer [4, Proposition 7.3] showed that DNR $\nleqslant_{\mathrm{W}}$ 1-RAND. Indeed, the above remarks show that the following holds.

Corollary 4.16. WWKL $\nleqslant_{\mathrm{gW}}$ RAND *and* DNR $\nleqslant_{\mathrm{gW}}$ RAND.

A function $f : [\mathbb{N}]^n \to \mathbb{N}$ is *k-bounded* if $|f^{-1}(i)| \leqslant k$ for all i. A set R is a *rainbow* for f if R is infinite and f is injective on $[R]^n$. The Rainbow Ramsey Theorem for n and k, RRT^n_k, states that every k-bounded function on $[\mathbb{N}]^n$ has a rainbow. Csima and Mileti [11, Theorem 3.1] showed that $\mathrm{RRT}^2_k \leqslant_{\mathrm{c}}$ 2-RAND. However, RRT^2_2 has diagonalization opportunities: Let Ψ be as in Definition 4.12. Let τ be an initial segment of an injective function on $[\mathbb{N}]^2$ such that Ψ^τ contains at least two elements x and y. Let $f \succ \tau$ be a 2-bounded function on $[\mathbb{N}]^2$ such that $f(x, z) = f(y, z)$ for all sufficiently large z. Then no infinite rainbow for f can extend Ψ^τ. Thus we have the following result.

Corollary 4.17. $\mathrm{RRT}^2_2 \nleqslant_{\mathrm{gW}}$ RAND.

One may argue that in the above examples, the real issue is not effectivity but continuity. That is, we can define the notion of a continuous strategy by replacing the Turing functional in Definition 4.3 by any continuous map $2^\omega \to 2^\omega$, and use that to define a notion of "continuously reducible in the generalized sense". If we then redefine the notion of diagonalizable instance by replacing "any Turing functional Ψ" by "any continuous Ψ", we can redo all the work in this subsection for this weaker notion of reducibility in essentially the same way. However, a continuous map $2^\omega \to 2^\omega$ is just a Turing functional relative to some oracle, so working in the cone above this oracle, there is no difference between the two notions of reducibility. Thus it appears that, for natural problems like the ones discussed above, there is no real difference between these two approaches.

4.4. **Counting instances.** Another thing we can do with our reduction games is calibrate the exact number of instances of Q needed to obtain P (uniformly or not). Let us write $P \leqslant^n_\omega Q$ if Player 2 has a winning strategy in $G(Q \to P)$ that guarantees victory in $n+1$ or fewer moves, and $P \leqslant^n_{\mathrm{gW}} Q$ if Player 2 has a computable winning strategy in $G(Q \to P)$ that guarantees victory in $n+1$ or fewer moves. Notice that $P \leqslant^0_\omega Q$ if and only if P is computably true (i.e., every instance X of P has an X-computable solution), while $P \leqslant^0_{\mathrm{gW}} Q$ if and only if P is uniformly computably true (i.e., there is a procedure for obtaining a solution to an instance X of P from X). RT^1_2 is an example of a principle that is computably true but not uniformly so. Similarly, $P \leqslant^1_\omega Q$ if and only if $P \leqslant_{\mathrm{c}} Q$, and $P \leqslant^1_{\mathrm{gW}} Q$ if and only if $P \leqslant_{\mathrm{W}} Q$.

For instance, the proof of Proposition 4.8 shows that $\mathrm{SRT}^2_2 \leqslant^2_{\mathrm{gW}} \mathrm{D}^2_2$. We also have the following more detailed analysis of the equivalence, under both $\leqslant_\omega$ and $\leqslant_{\mathrm{gW}}$, of KL, RT^n_k for $n \geqslant 3$ and $k \geqslant 2$, $\mathrm{RT}^n_{<\infty}$ for $n \geqslant 3$, and RT.

Proposition 4.18. *Let $n \geqslant 1$. Then $\mathrm{RT}^n_{<\infty} \leqslant^{n-1}_\omega$ KL but (for $n > 1$) $\mathrm{RT}^n_2 \nleqslant^{n-2}_\omega$ KL. Thus RT $\nleqslant^k_\omega$ KL and RT $\nleqslant^k_\omega \mathrm{RT}^n_{<\infty}$ for all k and all $n \geqslant 1$.*

Proof. For $n = 1$ we are just stating that $\mathrm{RT}^1_{<\infty}$ is computably true, so let $n \geqslant 2$. As mentioned before Proposition 4.5, PRE $\leqslant_{\mathrm{W}}$ KL, where PRE is the principle stating that for every $n, k \geqslant 2$, every k-coloring of $[\mathbb{N}]^n$ has an infinite prehomogeneous set. With $n-1$ applications of PRE, we can reduce a coloring c of $[\mathbb{N}]^n$ to a coloring of $\mathbb{N}$. This coloring has a computable infinite homogeneous set, which is also homogeneous for c. Thus $\mathrm{RT}^n_{<\infty} \leqslant^{n-1}_\omega$ KL.

For the second statement, we use the fact that every instance Y of KL has a solution Z such that $Y \oplus Z$ is low over Y'. (This fact follows from the relativized form of the low basis theorem, since every Y-computable finitely bounded tree is Y'-computably bounded.) From this fact, it follows easily by induction that in the game $G(\mathrm{KL} \to \mathrm{RT}^n_2)$, if Player 1 plays a computable instance of RT^n_2 on its first move, then it is always possible for Player 1 to ensure that Player 2's kth move is low over $\emptyset^{(k-1)}$. By Jockusch [30, Theorem 5.1], there is a computable instance of RT^n_2 with no $\emptyset^{(n-1)}$-computable solution, so if Player 1 begins with such an instance and plays as above, Player 2 cannot win until move n. Thus $\mathrm{RT}^n_2 \nleqslant^{n-2}_\omega$ KL. ☐

It follows from Proposition 4.18 that $\mathrm{RT}^n_2 \nleqslant^{n-2}_{\mathrm{gW}}$ KL. In the positive direction, we have the following fact.

Proposition 4.19. *Let $n \geqslant 1$. Then $\mathrm{RT}^n_{<\infty} \leqslant^n_{\mathrm{gW}}$ KL.*

Proof. As in the previous proof, with $n-1$ applications of PRE, we can reduce a coloring c of $[\mathbb{N}]^n$ to a coloring of $\mathbb{N}$, in a uniform way, and by Theorem 2.10(7), $\mathrm{RT}^1_{<\infty} \leqslant_{\mathrm{W}}$ KL. ☐

It is easy to see that $\mathrm{RT}^1_2 \nleqslant^0_{\mathrm{gW}}$ KL, and by Theorem 2.10(2), $\mathrm{RT}^2_2 \nleqslant^1_{\mathrm{gW}}$ KL. However, we do not know whether $\mathrm{RT}^n_k \leqslant^{n-1}_{\mathrm{gW}}$ KL for $n \geqslant 3$, where $k \geqslant 2$ may be $<\infty$.

Theorem 4.20. *Let $n \geqslant 3$, let $k \geqslant 2$, and let $j \geqslant 1$, and suppose that*

$$m \in (n + (j-1)(n-2), n + j(n-2)].$$

Then $\mathrm{RT}^m_k \leqslant^{j+1}_{\mathrm{gW}} \mathrm{RT}^n_k$ but $\mathrm{RT}^m_2 \nleqslant^j_\omega \mathrm{RT}^n_k$.

Proof. We begin by showing that $\mathrm{RT}^m_k \leqslant^{j+1}_\omega \mathrm{RT}^n_k$, and then discuss uniformity. Suppose that in the game $G(\mathrm{RT}^n_k \to \mathrm{RT}^m_k)$, Player 1 begins by playing an instance c of RT^m_k. By the relativized form of Lemma 5.9 in

Jockusch [30], for each A there is an A-computable instance Y of RT^n_k such that for any solution Z, we have $A^{(n-2)} \leqslant_{\mathrm{T}} A \oplus Z$. Thus Player 2 can force Player 1 to play a set Z_2 on its second move such that $c^{(n-2)} \leqslant_{\mathrm{T}} c \oplus Z_2$, then force it to play a set Z_3 on its third move such that $c^{(2(n-2))} \leqslant_{\mathrm{T}} c \oplus Z_2 \oplus Z_3$, and so on. So Player 2 can gain access to $c^{((j-1)(n-2))}$ before its jth move. By the relativized form of Corollary 2.2, with one more move, Player 2 can gain access to a set of PA degree over $c^{(j(n-2))}$. By the proof of Lemma 3.2, Player 2 then has access to an infinite set P such that the c-color of a tuple in $[P]^m$ depends only on its least $m - j(n-2) \leqslant n$ many colors. With one more move, Player 2 can force Player 1 to play a solution to the induced coloring of $[P]^n$, which is also a solution to c. Player 2 then plays this set as its $(j+2)$nd move and wins.

We now need to show that the above strategy for Player 2 is computable. We claim that the proof of the relativized form of Corollary 2.2 can be carried out uniformly, or, more precisely, that there are Turing functionals Φ and Ψ such that for each X, the set Φ^X codes a computable 2-coloring of $[\mathbb{N}]^n$ such that if H is an infinite homogeneous set for this coloring, then $\Psi^{X \oplus H}$ is a completion of the partial function $e \mapsto \Phi_e^{X^{(n-2)}}(e)$. Assume the claim for now. Then, since $X^{(n-2)}$ can be computed uniformly from such a completion, Player 2's first j many moves under the above strategy can be performed computably. Player 2's jth move is a completion of $e \mapsto \Phi_e^{c^{(j(n-2))}}(e)$. From such a completion, we can uniformly compute completions of $e \mapsto \Phi_e^{c^{(i)}}(e)$ for each $i \leqslant j(n-2)$, which allows us to carry out the construction in the proof of Lemma 3.2 uniformly to obtain a set P as above. (The key point here is that Lemma 5.4 of [30] is proved by building a finitely branching tree all of whose paths are infinite prehomogeneous sets for a given coloring d. This tree is built uniformly from d, and a path on such a tree can be obtained uniformly from a completion of $e \mapsto \Phi_e^{d'}(e)$.) Now Player 2's final 2 moves can also be made computably.

It remains to prove the claim. We repeat the argument in the proof of Corollary 2.2, but now paying attention to uniformity. As in that proof, we will ensure that the coloring coded by Φ^X is unbalanced. In fact, we will ensure that every infinite homogeneous set for that coloring is homogeneous to 1; we say that the coloring is *unbalanced toward* 1. For $n = 3$, the existence of the functionals Φ and Ψ follows from the proof of Lemma 3.1. Now let $n \geqslant 3$ and assume by induction that we have Turing functionals $\widehat{\Phi}$ and $\widehat{\Psi}$ such that for each Y, the set $\widehat{\Phi}^Y$ codes a 2-coloring c^Y of $[\mathbb{N}]^n$ that is unbalanced toward 1 and such that if H is an infinite homogeneous set for c^Y, then $\widehat{\Psi}^{Y \oplus H}$ is a completion of the partial function $e \mapsto \Phi_e^{Y^{(n-2)}}(e)$. In particular, for each X, if H is an infinite homogeneous set for $c^{X'}$, then $\widehat{\Psi}^{X' \oplus H}$ is a completion of $e \mapsto \Phi_e^{X^{(n-1)}}(e)$. The argument in the proof of Lemma 5.2 of [30] shows that we can uniformly obtain an X-computable 2-coloring d^X of $[\mathbb{N}]^{n+1}$ that is unbalanced toward 1 and such that every infinite set homogeneous for d^X is homogeneous for $c^{X'}$. Since $n \geqslant 3$, we can also transform c^X into a 2-coloring e^X of $[\mathbb{N}]^{n+1}$ that is unbalanced toward 1 and such that we can uniformly compute X' from $X \oplus H$ for any infinite homogeneous set H for e^X. (We just let the e^X-color of an $(n+1)$-tuple be the c^X-color of its n many least elements.) By the proof of Lemma 5.10 of [30], from d^X and e^X we can uniformly obtain a 2-coloring f^X of $[\mathbb{N}]^{n+1}$ that is unbalanced toward 1 and such that the infinite homogeneous sets for f^X are precisely the infinite sets homogeneous for both d^X and e^X. Let Φ^X code f^X. Given an infinite homogeneous set H for f^X, the fact that H is homogeneous for d^X, and hence for $c^{X'}$, implies that $\widehat{\Psi}^{X' \oplus H}$ is a completion of $e \mapsto \Phi_e^{X^{(n-1)}}(e)$. But since H is homogeneous for e^X, we can uniformly compute X' from $X \oplus H$. Thus we can define a Turing functional Ψ such that, for any such H, the set $\Psi^{X \oplus H}$ is a completion of $e \mapsto \Phi_e^{X^{(n-1)}}(e)$.

To show that $\mathrm{RT}^m_2 \nleqslant^j_\omega \mathrm{RT}^n_k$, we use the relativized form of Theorem 12.1 in Cholak, Jockusch, and Slaman [7], which states that every instance Y of RT^n_k has a solution Z such that $(Y \oplus Z)'' \leqslant_{\mathrm{T}} Y^{(n)}$. From this result, it follows easily by induction that in the game $G(\mathrm{RT}^n_k \to \mathrm{RT}^m_2)$, if Player 1 plays a computable instance of RT^m_2 on its first move, then it is always possible for Player 1 to ensure that Player 2's ith move has $\emptyset^{((n+(i-2)(n-2))}$-computable double jump. As mentioned above, there is a computable instance of RT^m_2 such that every solution Z computes $\emptyset^{(m-2)}$, and hence has $Z'' \geqslant_{\mathrm{T}} \emptyset^{(m)} >_{\mathrm{T}} \emptyset^{(n+(j-1)(n-2))}$. If Player 1 begins with such an instance and plays as above, Player 2 cannot win until move $j+2$. □

We turn now to comparing versions of RT with fixed size of tuples but different numbers of colors. We have seen that $\mathrm{RT}^n_3 \leqslant^2_{\mathrm{gW}} \mathrm{RT}^n_2$ but $\mathrm{RT}^n_3 \nleqslant^1_{\mathrm{gW}} \mathrm{RT}^n_2$. In the $n = 1$ case at least, this fact has the following generalization.

Theorem 4.21. *Let $j \geqslant 2$. If $k \in (j^m, j^{m+1}]$ then $\mathrm{RT}^1_k \leqslant^{m+1}_{\mathrm{gW}} \mathrm{RT}^1_j$ but $\mathrm{RT}^1_k \nleqslant^m_{\mathrm{gW}} \mathrm{RT}^1_j$. Similarly, $\mathrm{RT}^1_{<\infty} \nleqslant^m_{\mathrm{gW}} \mathrm{RT}^1_j$ for all m.*

Proof. For the positive direction, we use the usual reduction from $\mathrm{RT}^1_{j^{m+1}}$ to RT^1_j, which we can think of inductively. The base case is $m = 0$, for which a single application of RT^1_j suffices. Given a j^{m+1}-coloring c, we partition its colors into j many sets $G_0, \dots, G_{j-1}$, each of size j^m, and let d be the j-coloring defined by letting $d(x) = i$ if $c(x) \in G_i$. Solving this instance of RT^1_j gives us an infinite homogeneous set H for d, and $c \upharpoonright H$ is a j^m-coloring (which we can encode as a coloring of $\mathbb{N}$), so by induction we can find an infinite homogeneous set for $c \upharpoonright H$ (and hence for c) with m applications of RT^1_j. This procedure is clearly uniform (both in the instances of $\mathrm{RT}^1_{j^{m+1}}$ and in m), so we conclude that $\mathrm{RT}^1_{j^{m+1}} \leqslant^{m+1}_{\mathrm{gW}} \mathrm{RT}^1_j$, and hence $\mathrm{RT}^1_k \leqslant^{m+1}_{\mathrm{gW}} \mathrm{RT}^1_j$ for all $k \leqslant j^{m+1}$.

For the negative direction, let Φ be a computable strategy for Player 2 in $G(\mathrm{RT}^1_j \to \mathrm{RT}^1_k)$, where $k > j^m$. Recall that if X is the join of Player 1's first m moves in a run of this game, then Φ^X is of the form $V \oplus \widehat{\Phi}^X$, where $V = \{0\}$ if Player 2 declares victory on the mth move, in which case $\widehat{\Phi}^X$ is a solution to Player 1's first move, and $V = \emptyset$ otherwise, in which case $\widehat{\Phi}^X$ codes a j-coloring of $\mathbb{N}$. We write d^X for this coloring.

Build a k-coloring c of $\mathbb{N}$ as follows. We have auxiliary sets $A_{i_0,\dots,i_l}$, where $l < m$ and $i_p < j$ for each $p \leqslant l$. Begin defining c arbitrarily. For uniformity of notation, let $A_\emptyset = c$, and start computing $\Phi^{A_\emptyset}$. If we find that $\Phi^{A_\emptyset}(0) = 1$ (which indicates that Player 2 declares victory on the first move) then wait until we find an x such that $\widehat{\Phi}^{A_\emptyset}(x) = 1$ and ensure that $c(s) \neq c(x)$ for all sufficiently large s.

Otherwise, once we find that $\Phi^{A_\emptyset}(0) = 0$, start putting all x such that $d^{A_\emptyset}(x) = i$ into A_i. Once we have started building a set $A_{i_0,\dots,i_l}$, if we find that $\Phi^{\bigoplus_{j \leqslant l+1} A_{i_0,\dots,i_{j-1}}}(0) = 1$ and $\widehat{\Phi}^{\bigoplus_{j \leqslant l+1} A_{i_0,\dots,i_{j-1}}}(x) = 1$ for some x, ensure that $c(s) \neq c(x)$ for all sufficiently large s.

On the other hand, if we find that $\Phi^{\bigoplus_{j \leqslant l+1} A_{i_0,\dots,i_{j-1}}}(0) = 0$, and $l < x-1$, then start putting all x such that $d^{\bigoplus_{j \leqslant l+1} A_{i_0,\dots,i_{j-1}}}(x) = i$ into $A_{i_0,\dots,i_l,i}$.

The total number of colors that we can ever want to avoid in this construction is j^m, so we never run out of colors. Now consider the following run of our game, where Player 2 plays according to Φ. Player 1 begins by playing $A_\emptyset = c$. By construction, Player 2 cannot declare victory on the first move, so the A_i are built. There is an $i_0 < j$ such that A_{i_0} is an infinite homogeneous set for $d^{A_\emptyset}$, and Player 1 plays A_{i_0}. Again by construction, Player 2 cannot declare victory on this move, so the $A_{i_0,i}$ are built. Again, there is an $i_1 < j$ such that A_{i_0,i_1} is an infinite homogeneous set for $d^{A_\emptyset \oplus A_{i_0}}$, and Player 1 plays A_{i_0,i_1}. Continuing in this way, we have a run of our game in which Player 2, playing according to Φ, cannot declare victory before move $m+2$. Since Φ is arbitrary, $\mathrm{RT}^1_k \nleqslant^m_{\mathrm{gW}} \mathrm{RT}^1_j$, which of course implies that $\mathrm{RT}^1_{<\infty} \nleqslant^m_{\mathrm{gW}} \mathrm{RT}^1_j$. $\square$

The positive part of the above proof works for higher exponents, so we have the following result.

Proposition 4.22. *Let $j, n \geqslant 2$. If $k \in (j^m, j^{m+1}]$ then $\mathrm{RT}^n_k \leqslant^{m+1}_{\mathrm{gW}} \mathrm{RT}^n_j$.*

The exponent-lifting technique of Section 3.1 does not seem immediately applicable to establishing corresponding lower bounds on the m such that $\mathrm{RT}^n_k \leqslant^m_{\mathrm{gW}} \mathrm{RT}^n_j$ for $n \geqslant 2$ and $k > j \geqslant 2$, since some of the sets $A_{i_0,\dots,i_j}$ in the proof of Theorem 4.21 might be infinite, and hence, if we try to perform a version of the construction in that proof inside a set P as in the proof of Theorem 3.3, we might get infinitely many convergent computations $\widehat{\Phi}^{\bigoplus_{j \leqslant l+1}(A_{i_0,\dots,i_{j-1}} \cap P[s])}(x) = 1$ for which $\widehat{\Phi}^{\bigoplus_{j \leqslant l+1}(A_{i_0,\dots,i_{j-1}} \cap P)}(x) \neq 1$.

Patey [45] gave an exact characterization of the least m such that $(\mathrm{S})\mathrm{RT}^n_k \leqslant^m_\omega \mathrm{RT}^n_j$ for $n \geqslant 2$ and $k > j \geqslant 2$, answering a question stated in an earlier version of this paper. For $n \geqslant 3$, this m is 2, independent of j and k. The $n = 2$ case is more complicated: Let $\pi(x, y)$ be the least $a \geqslant 1$ such that $x = ay - b$ for some $b < y$. For $j \geqslant 2$, let $m_{1,j} = 0$, and for $k > 1$, let $m_{k,j} = m_{\pi(k,j),j} + 1$. Then $m_{k,j}$ is the least m such that $(\mathrm{S})\mathrm{RT}^2_k \leqslant^m_\omega \mathrm{RT}^2_j$.

Remark 4.23. Another way to define $\leqslant^n_{\mathrm{gW}}$ is to use the operation $\bullet$ discussed in Section 5.2 of Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12]. For problems P and Q and a fixed Turing functional Θ, the problem $Q \bullet P$ is defined as follows. A set A is an instance of $Q \bullet P$ if A is an instance of P and, for every solution B to A as an instance of P, we have that $\Theta^{A \oplus B}$ is an instance of Q. (In practice, Θ will normally be chosen so that every instance A of P has the latter property.) A solution to an instance A of $Q \bullet P$ is a pair (B, C) such that B is a solution to A as an instance of P, and C is a solution to the instance $\Theta^{A \oplus B}$ of Q. (See [12] for a discussion of the relationship of this notion with function composition in the context of computable analysis, and the compositional product in the Weihrauch lattice.)

For notational simplicity, let us consider the $n = 2$ case. It is easy to check that if $P \leqslant_{\mathrm{W}} Q \bullet Q$ then $P \leqslant^2_{\mathrm{gW}} Q$: Suppose that Φ and Ψ witness that $P \leqslant_{\mathrm{W}} Q \bullet Q$. In the game $G(Q \to P)$, once Player 1 plays an instance X of P, Player 2 can respond with Φ^X. Player 1 then plays a solution Z to Φ^X, thought of as an instance of Q. Player 2 can then play $\Theta^{\Phi^X \oplus Z}$, which is an instance of Q. Player 1 then plays a solution W to this instance. Now (Z, W) is a solution to Φ^X as an instance of $Q \bullet Q$, so $\Psi^{(Z,W)}$ is a solution to P. Player 2 can thus play $\Psi^{(Z,W)}$ on its third move and declare victory.

In the other direction, we need a slight technical adjustment. Suppose that $P \leqslant^2_{\mathrm{gW}} Q$ and let Γ be Player 2's computable winning strategy for $G(Q \to P)$. Let $\widehat{Q}$ be the problem whose instances are pairs (X, Y) where X is any set and Y is an instance of Q, such that a solution to such an instance is a pair (X, Z) where Z is a solution to Y. Let Φ be a Turing functional that, on oracle X, returns (X, Γ^X). Let Θ be a Turing functional that, on oracle $(X, Y) \oplus (X, Z)$, returns $\Gamma^{X \oplus Z}$. Let Ψ be a Turing functional that, on oracle $X \oplus (Z, W)$ returns $\Gamma^{X \oplus Z \oplus W}$. Define $\bullet$ using Θ. Let X be an instance of P. Then Φ^X is an instance of $Q \bullet \widehat{Q}$. A solution to this instance has the form (Z, W), where Z is a solution to (X, Φ^X) as an instance of $\widehat{Q}$, and W is a solution to the instance $\Gamma^{X \oplus Z}$ of Q. Then $\Psi^{X \oplus (Z,W)}$ is a solution to X, as it is equal to $\Gamma^{X \oplus Z \oplus W}$, which is Player 2's third, and therefore winning, move in a run of $G(Q \to P)$ played according to Γ. Thus Φ and Ψ witness that $P \leqslant_{\mathrm{W}} Q \bullet \widehat{Q}$.

A similar analysis can be made for larger values of n. Of course, the real power of the $\bullet$ operation comes when considering multiple principles. For instance, saying that $\mathrm{RT}^2_2 \leqslant_{\mathrm{W}} \mathrm{SRT}^2_2 \bullet \mathrm{COH}$ gives us more information than saying that $\mathrm{RT}^2_2 \leqslant^2_{\mathrm{gW}} \mathrm{SRT}^2_2 \wedge \mathrm{COH}$. On the other hand, we cannot use the $\bullet$ operation to define $\leqslant_{\mathrm{gW}}$ in general, as there are principles P and Q such that $P \leqslant_{\mathrm{gW}} Q$ but $P \nleqslant^n_{\mathrm{gW}} Q$ for all n.

4.5. **Non-ω-models.** We need not limit our game-based approach to ω-models. Let Σ^0_1-PA be first-order Peano arithmetic with the induction scheme restricted to Σ^0_1 formulas. For a first order model M of Σ^0_1-PA and $X_0, \dots, X_n \subseteq M$, let $M[X_0, \dots, X_n]$ be the second order model whose first order part is M and whose second order part consists of all subsets of M that are Δ^0_1-definable over M with $X_0, \dots, X_n$ as additional parameters. Let P be of the form $\forall X\, [\Theta(X) \rightarrow \exists Y\, \Psi(X, Y)]$, with Θ and Ψ arithmetic. An *instance of P over M* is an $X \subseteq M$ such that $\Theta(X)$ holds in $M[X]$, and a *solution to X (over M)* is a $Y \subseteq M$ such that $\Psi(X, Y)$ holds in $M[X, Y]$.

Definition 4.24. For problems P and Q, the *generalized reduction game* $\widehat{G}(Q \rightarrow P)$ is a two-player game that proceeds as follows. If at any point one of the players does not have a legal move, then the game ends with a victory for the other player.

On the first move, Player 1 plays a countable first order model M of Σ^0_1-PA and an instance X_0 of P over M with $M[X_0] \vDash \mathrm{RCA}_0$, and Player 2 either plays a solution to X_0 in $M[X_0]$ and declares victory, in which case the game ends, or responds with an instance Y_1 of Q in $M[X_0]$.

For $n > 1$, on the nth move (if the game has not yet ended), Player 1 plays a solution X_{n-1} to the instance Y_{n-1} of Q with $M[X_0, \dots, X_{n-1}] \vDash \mathrm{RCA}_0$. Then Player 2 either plays a solution to X_0 in $M[X_0, \dots, X_{n-1}]$ and declares victory, in which case again the game ends, or plays an instance Y_n of Q in $M[X_0, \dots, X_{n-1}]$.

Player 2 wins this play of the game if it ever declares victory, or if Player 1 has no legal move at some point in the game. Otherwise, Player 1 wins.

It is easy to adapt the proof of Proposition 4.2 to show that if $\mathrm{RCA}_0 + Q \vdash P$ then Player 2 has a winning strategy in $\widehat{G}(Q \rightarrow P)$, while otherwise Player 1 has a winning strategy in $\widehat{G}(Q \rightarrow P)$. We can use our generalized games to count applications as above, by writing $\mathrm{RCA}_0 + Q \vdash^k P$ to mean that Player 2 has a winning strategy in $\widehat{G}(Q \rightarrow P)$ that guarantees a win in at most $k + 1$ many moves.

This notion seems particularly interesting in connection with principles equivalent to ACA_0, such as RT^n_k for $n \geqslant 3$ and $k \geqslant 2$. For example, our discussion above transfers to non-ω-models to show that $\mathrm{RCA}_0 + \mathrm{RT}^3_2 \vdash^2 \mathrm{RT}^4_2$ but $\mathrm{RCA}_0 + \mathrm{RT}^3_2 \nvdash^1 \mathrm{RT}^4_2$. It is well known that adding Σ^0_1-comprehension to RCA_0 yields ACA_0, but our notion allows us to count how many applications of Σ^0_1-comprehension are needed in a particular proof in ACA_0. For example, Jockusch [30, Theorems 5.1 and 5.5] showed that for $n \geqslant 2$, every instance X of RT^n_k has a solution computable in $X^{(n)}$, but there is a computable instance with no solution computable in $\emptyset^{(n-1)}$. The proof of the first of these facts carries through in RCA_0 to show that RT^n_k can be proved in RCA_0 together with a single application of the principle $\forall X\, \exists Y\, [Y = X^{(n)}]$, from which it follows easily that $\mathrm{RCA}_0 + \Sigma^0_1\text{-CA} \vdash^n \mathrm{RT}^n_k$. The second fact above implies that $\mathrm{RT}^n_k \not\leqslant^{n-1}_\omega \Sigma^0_1\text{-CA}$, and hence $\mathrm{RCA}_0 + \Sigma^0_1\text{-CA} \nvdash^{n-1} \mathrm{RT}^n_k$.

A simple example showing that $P \leqslant^n_\omega Q$ (or even $P \leqslant^n_{\mathrm{gW}} Q$) and $\mathrm{RCA}_0 + Q \vdash P$ do not together imply $\mathrm{RCA}_0 + Q \vdash^n P$ is given by the principles $\Pi^0_1\mathrm{G}$ and $\Pi^0_1\mathrm{GA}$ from Definition 1.6. It is easy to see that $\Pi^0_1\mathrm{GA}$ is uniformly computably true, so in particular $\Pi^0_1\mathrm{GA} \leqslant^0_{\mathrm{gW}} \Pi^0_1\mathrm{G}$. Of course, we also have $\mathrm{RCA}_0 + \Pi^0_1\mathrm{G} \vdash^1 \Pi^0_1\mathrm{GA}$. However, as shown in [27, Theorem 3.3], $\mathrm{RCA}_0 \nvdash \Pi^0_1\mathrm{GA}$. Let $\mathcal{M}$ be a countable model of $\mathrm{RCA}_0 + \neg\Pi^0_1\mathrm{GA}$, let M be the first order part of $\mathcal{M}$, and let $X \in \mathcal{M}$ be an instance of $\Pi^0_1\mathrm{GA}$ over M with no solution in $\mathcal{M}$. In the game $\widehat{G}(\Pi^0_1\mathrm{G} \to \Pi^0_1\mathrm{GA})$, Player 1 can begin by playing M and X, and Player 2 cannot then win on its first move. Thus $\mathrm{RCA}_0 + \Pi^0_1\mathrm{G} \nvdash^0 \Pi^0_1\mathrm{GA}$.

It would of course be interesting to have natural examples for larger n of problems P and Q such that $\mathrm{RCA}_0 + Q \vdash P$ and $P \leqslant^n_\omega Q$, but $\mathrm{RCA}_0 + Q \nvdash^n P$.

It is also straightforward to define the notion of a computable strategy for Player 2 in a generalized reduction game (using Δ^0_1-definable functionals), and hence a notion of gu-reducibility that is not restricted to ω-models, which we may denote by $\mathrm{RCA}_0 + Q \vdash_{\mathrm{W}} P$, as well as the corresponding instance-counting version $\mathrm{RCA} + Q \vdash^n_{\mathrm{W}} P$. In this context, $\mathrm{RCA}_0 + Q \vdash^1 P$ and $\mathrm{RCA}_0 + Q \vdash^1_{\mathrm{W}} P$ are the analogs to $P \leqslant_{\mathrm{c}} Q$ and $P \leqslant_{\mathrm{W}} Q$, respectively. We leave the further study of these notions to future work.

5. Relationships between versions of RT and KL: a summary

We summarize the relationships between some versions of Ramsey's Theorem and König's Lemma under $\leqslant_\omega$, $\leqslant_{\mathrm{c}}$, $\leqslant_{\mathrm{sc}}$, $\leqslant_{\mathrm{W}}$, $\leqslant_{\mathrm{sW}}$, and $\leqslant_{\mathrm{gW}}$ in Figures 5.1–5.6. (We do not include the Thin Set Theorem, as we have not considered it beyond Theorem 3.6.) No other implications than the ones shown (or implied by transitivity) hold. Dotted arrows represent hierarchies of principles. For example, the dotted arrow in Figure 5.2 represents the fact that $\mathrm{RT}^4_2 <_{\mathrm{c}} \mathrm{RT}^5_2 <_{\mathrm{c}} \mathrm{RT}^6_2 <_{\mathrm{c}} \cdots <_{\mathrm{c}} \mathrm{RT}$. We list several open questions, old and new, at the end of this section.

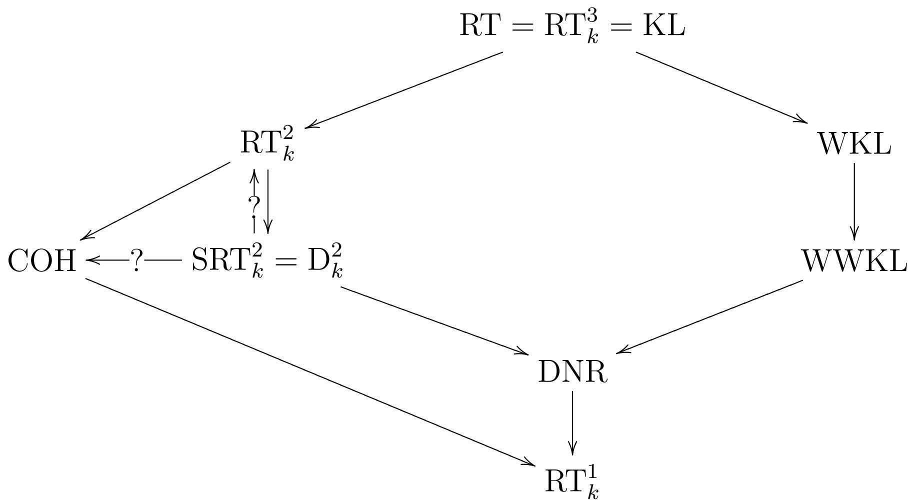

FIGURE 5.1. Versions of RT and KL under $\leqslant_\omega$ $(k \geqslant 2)$

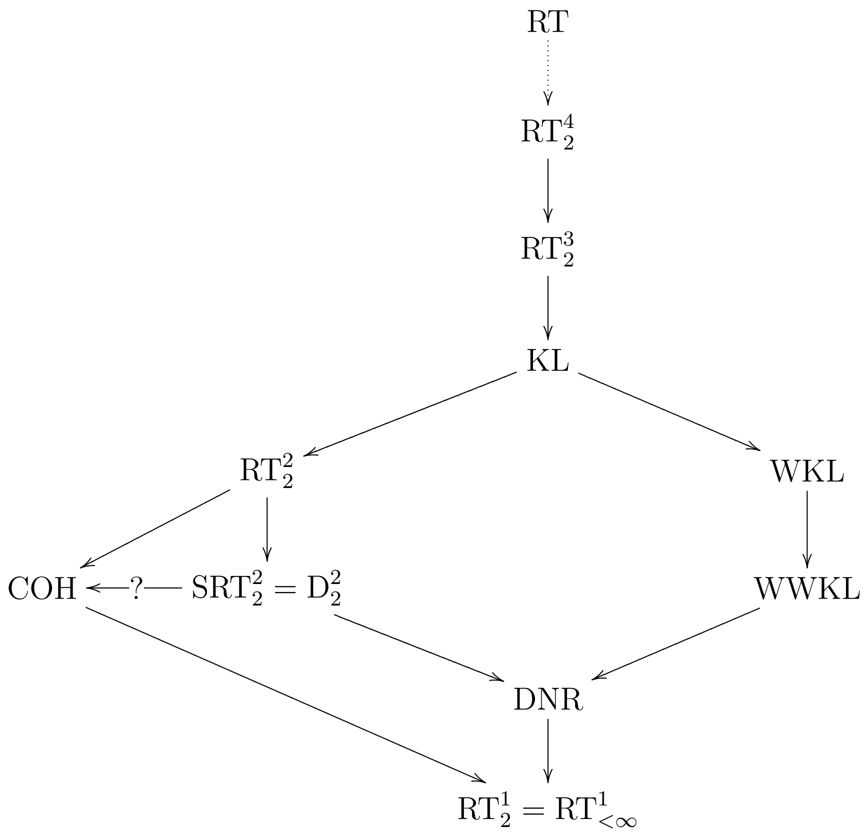

FIGURE 5.2. Versions of RT and KL under $\leqslant_{\mathrm{c}}$

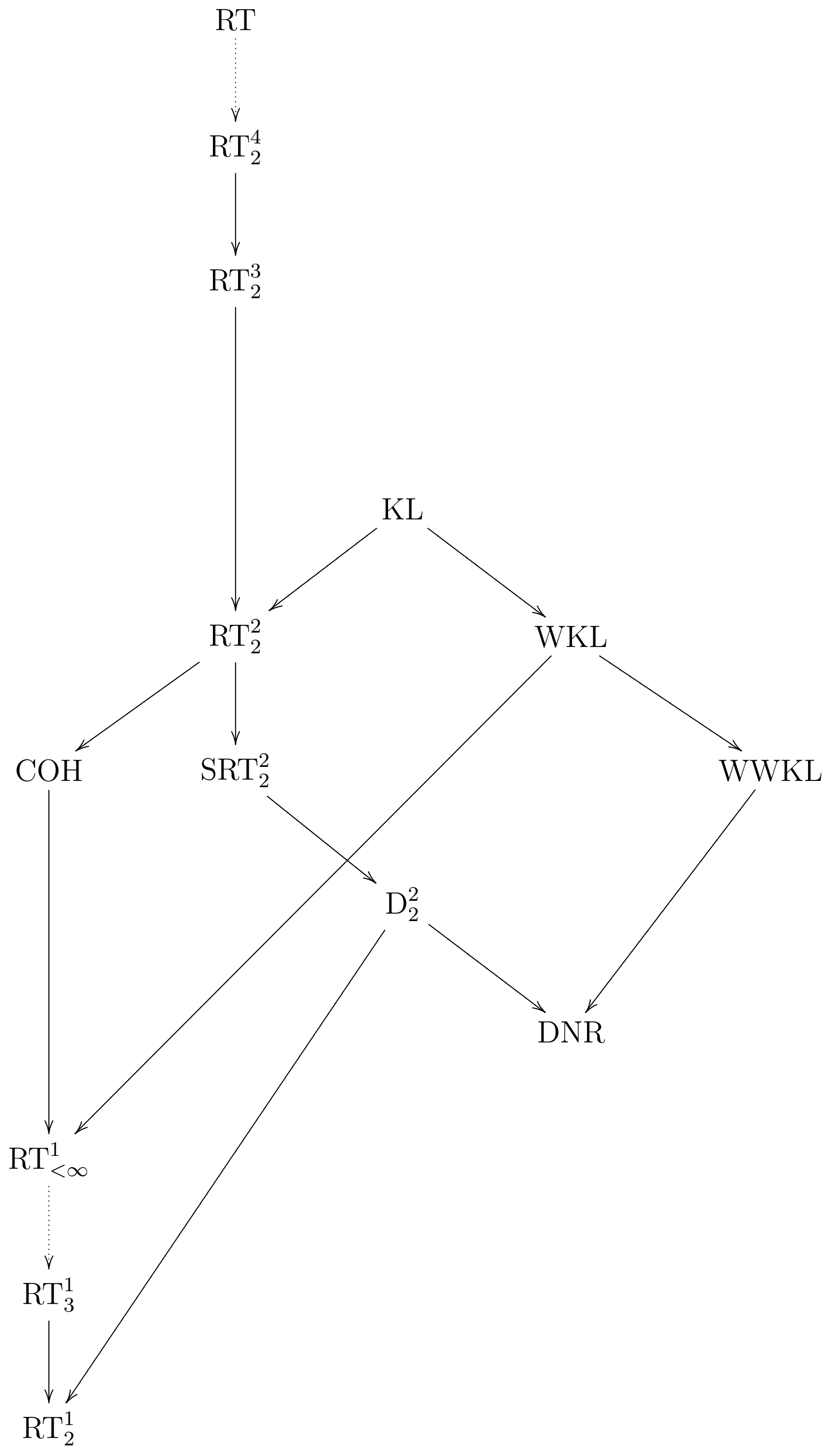

FIGURE 5.3. Versions of RT and KL under $\leqslant_{\mathrm{sc}}$

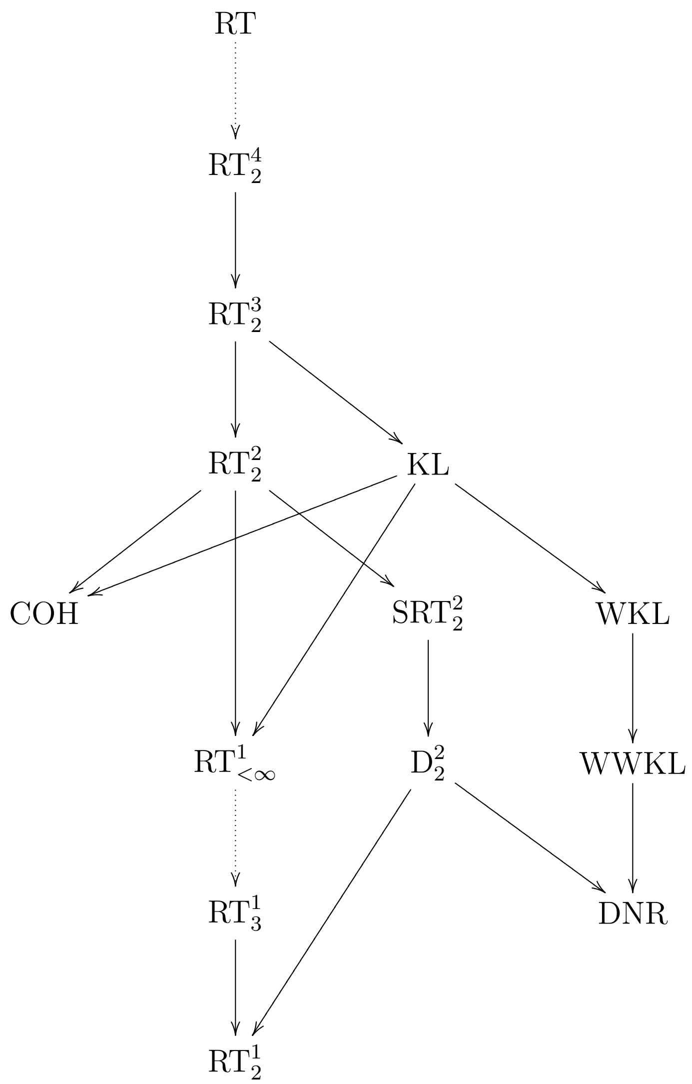

FIGURE 5.4. Versions of RT and KL under $\leqslant_{\mathrm{W}}$

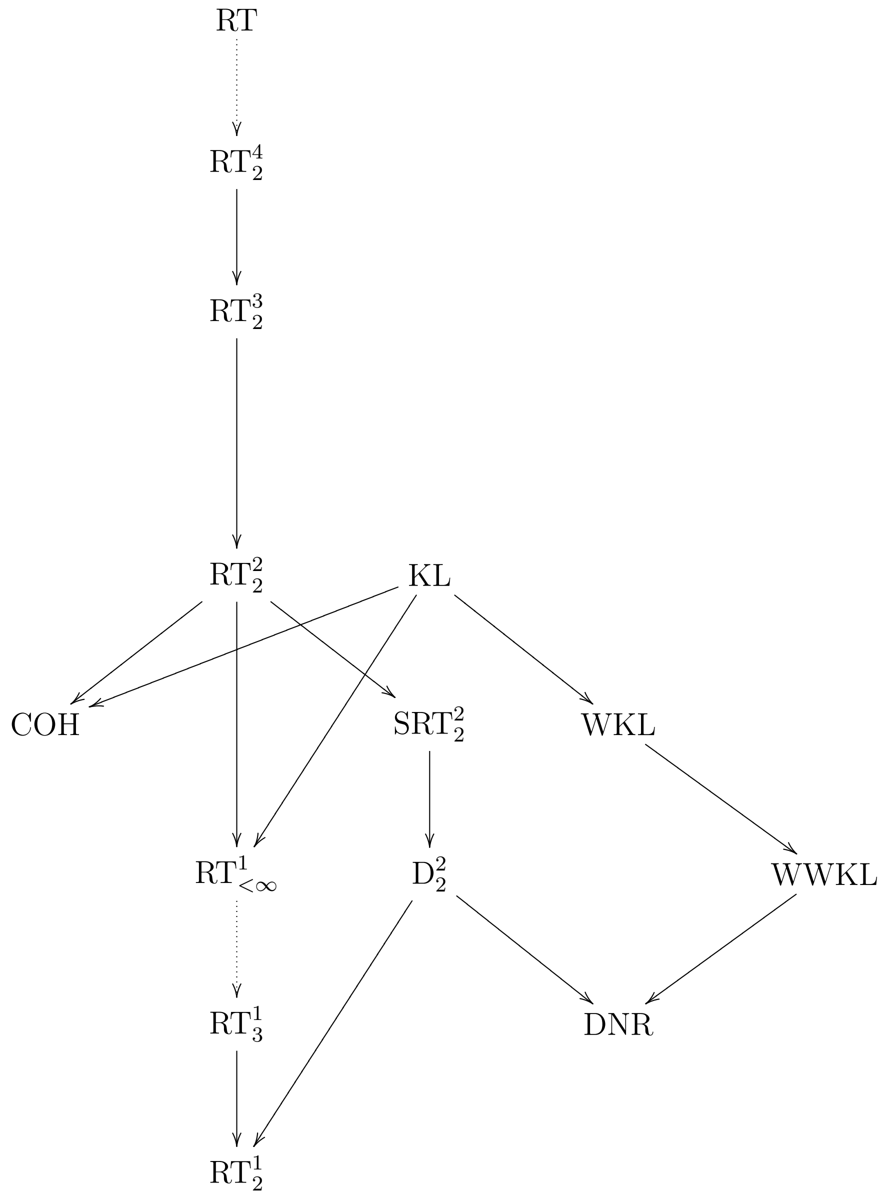

FIGURE 5.5. Versions of RT and KL under $\leqslant_{\mathrm{sW}}$

$$\begin{tikzcd}
 & & \mathrm{RT} = \mathrm{RT}^3_k = \mathrm{KL} \arrow[dl] \arrow[dr] & \\
 & \mathrm{RT}^2_k \arrow[dl] \arrow[d, shift left] & & \mathrm{WKL} \arrow[d] \\
\mathrm{COH} & \mathrm{SRT}^2_k = \mathrm{D}^2_k \arrow[l, "?"'] \arrow[u, shift left, "?"] \arrow[dr] \arrow[ddr] & & \mathrm{WWKL} \arrow[dl] \\
 & & \mathrm{DNR} & \\
 & & \mathrm{RT}^1_k &
\end{tikzcd}$$

FIGURE 5.6. Versions of RT and KL under $\leqslant_{\mathrm{gW}}$ ($k \geqslant 2$)

Figure 5.1 extends Figure 2.1. Since the number of colors does not matter in this case, the subscript k here stands for any number greater than 1 or for $<\infty$. For justifications of the implications and nonimplications in this figure, see [25].

In Figure 5.2, we leave out RT^n_k, SRT^2_k, and D^2_k for $n \geqslant 2$ and $k > 2$. As noted above, Patey [46] has shown that $\mathrm{RT}^n_k \nleqslant_{\mathrm{c}} \mathrm{RT}^n_j$ for all $n \geqslant 2$ and $k > j \geqslant 2$. His argument also shows that $\mathrm{SRT}^2_k \nleqslant_{\mathrm{c}} \mathrm{RT}^2_j$ for all $k > j \geqslant 2$. In this figure, all implications are obvious or follow from those in Figure 5.4, except for that between KL and RT^2_2, which is established in Corollary 2.4; and that between D^2_2 and SRT^2_2, which follows from the same proof that shows that D^2_2 implies SRT^2_2 over RCA_0 (see Cholak, Jockusch, and Slaman [7, Lemma 7.10]). All nonimplications in this figure follow either from those in Figure 5.1 or from computability-theoretic results of Jockusch [30].

In Figure 5.3 too, we leave out RT^n_k, SRT^2_k, and D^2_k for $n \geqslant 2$ and $k > 2$; Patey's results also apply here. As mentioned following Theorem 3.7, Dzhafarov, Patey, Solomon, and Westrick [19] showed that if if $2 \leqslant j < k$ then $\mathrm{RT}^1_k \nleqslant_{\mathrm{sc}} \mathrm{SRT}^2_j$. They also showed that $\mathrm{COH} \nleqslant_{\mathrm{sc}} \mathrm{SRT}^2_{<\infty}$. In this figure, all implications are obvious or follow from those in Figure 5.5, except for those between

1. KL and RT^2_2, which is established in Corollary 2.6,
2. WKL and $\mathrm{RT}^1_{<\infty}$, which follows from Proposition 2.7, and

3. COH and $\mathrm{RT}^1_{<\infty}$, which is established in Theorem 2.10(9).

All nonimplications in this figure follow from those in Figure 5.2, except for those between

1. RT (and hence RT^4_2 and RT^3_2) and WWKL, which is due to Monin and Patey [42] (strengthening the nonimplication between RT and (W)KL established in Corollary 2.9),
2. SRT^2_2 and COH, which is due to Dzhafarov [16] (strengthening the nonimplication between D^2_2 and COH that he proved in [15]),
3. SRT^2_2 and RT^1_3 (and hence $\mathrm{RT}^1_{<\infty}$), which is due to Dzhafarov, Patey, Solomon, and Westrick [19] (strengthening the nonimplication between D^2_2 and RT^1_3 proved by Dzhafarov [15]),
4. D^2_2 and SRT^2_2, which is due to Dzhafarov [16],
5. WWKL (and hence DNR) and RT^1_2 (and hence RT^1_3 and $\mathrm{RT}^1_{<\infty}$), which is established in Theorem 2.10(10), and
6. RT^1_j and RT^1_k for $k > j \geqslant 2$, which is Theorem 3.7, again due to Dzhafarov [15].

In Figure 5.4, we again leave out RT^n_k, SRT^2_k, and D^2_k for $n \geqslant 2$ and $k > 2$; see Theorems 3.3 and 3.4. In this figure, all implications are obvious or follow from those in Figure 5.5, except for the one between RT^3_2 and KL, which is established in Corollary 2.3. All nonimplications in this figure follow from those in Figure 5.2, except for those between

1. KL and D^2_2 (and therefore $\mathrm{(S)RT}^2_2$), which is established in Theorem 2.10(2),
2. SRT^2_2 (and therefore D^2_2) and COH, which is due to Dzhafarov [16],
3. SRT^2_2 (and therefore D^2_2) and RT^1_3 (and therefore $\mathrm{RT}^1_{<\infty}$), which is established in Theorem 2.10(4),
4. D^2_2 and SRT^2_2, which is due to Dzhafarov [16],
5. COH and RT^1_2 (and therefore RT^1_3 and $\mathrm{RT}^1_{<\infty}$), which is established in Theorem 2.10(5),
6. WKL (and therefore WWKL and DNR) and RT^1_2 (and therefore RT^1_3 and $\mathrm{RT}^1_{<\infty}$), which is established in Theorem 2.10(1), and
7. RT^1_j and RT^1_k for $k > j \geqslant 2$, which is the easy case of Theorem 3.3.

In Figure 5.5 we once more leave out RT^n_k, SRT^2_k, and D^2_k for $n \geqslant 2$ and $k > 2$; again see Theorems 3.3 and 3.4. In this figure, all implications are obvious except for those between

1. RT^2_2 and $\mathrm{RT}^1_{<\infty}$, which is established in Theorem 2.10(8),
2. RT^2_2 and COH, which follows from the proof of Theorems 12.4 and 12.5 in Cholak, Jockusch, and Slaman [7], as noted also by Brattka and Rakotoniaina [5],
3. KL and $\mathrm{RT}^1_{<\infty}$, which is established in Theorem 2.10(7),
4. KL and COH, which is established in Theorem 2.10(6),
5. D^2_2 and RT^1_2, which is established in Theorem 2.10(3),

6. D^2_2 and DNR, which follows from the proof of Theorem 2.3 in Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [26] (see also [5, Corollary 5.25]), and
7. WWKL and DNR, which follows from the proof of Lemma 6.18 in Giusto and Simpson [23].

All nonimplications in this figure follow from those in Figures 5.3 and 5.4.

In light of Propositions 4.5 and 4.7, in Figure 5.6, k stands for any number greater than 1 or for $<\infty$. All implications in this figure follow from those in Figure 5.4 except for the equivalences on the top and bottom lines, which are established in Propositions 4.5, 4.6, and 4.7; and the implication between D^2_k and SRT^2_k, which is established in Proposition 4.8. All nonimplications in this figure follow from the ones in Figure 5.1, except for those between

1. WKL (and therefore WWKL and DNR) and RT^1_2, which is established in Corollary 4.10, and
2. COH and RT^1_2, which is established at the beginning of Section 4.3.

Figure 5.2 shows that most of the implications in Figure 5.1 are in fact $\leqslant^1_\omega$-reductions (or $\leqslant^0_\omega$-reductions in the case of reductions to RT^1_k). The exceptions are

1. some of the reductions witnessing the equivalences represented by the top line of Figure 5.1 (see Proposition 4.18 and Theorem 4.20) and
2. the ones between principles with the same exponent (greater than 1) and different numbers of colors (by Patey [46]).

Figure 5.4 shows that most of the implications in Figure 5.6 are in fact $\leqslant^1_{\mathrm{gW}}$-reductions. The exceptions are

1. some of the reductions witnessing the equivalences represented by the top line of Figure 5.6 (again see Proposition 4.18 and Theorem 4.20),
2. those between principles with the same exponent and different numbers of colors (see Theorems 3.3 and 3.4), and
3. the one between D^2_k and SRT^2_k (see the comment on the work of Dzhafarov [16] in Definition 1.3, and the paragraph preceding Proposition 4.18).

The following questions remain open.

Question 5.1. Is $\mathsf{RT}^2_2 \leqslant_\omega \mathsf{SRT}^2_2$? (Equivalently, is $\mathsf{COH} \leqslant_\omega \mathsf{SRT}^2_2$?)

Question 5.2. Is $\mathsf{RT}^2_2 \leqslant_{\mathrm{gW}} \mathsf{SRT}^2_2$? (As in the previous question, this question is equivalent to asking whether $\mathsf{COH} \leqslant_{\mathrm{gW}} \mathsf{SRT}^2_2$, since we can obtain RT^2_2 uniformly from $\mathsf{COH} + \mathsf{SRT}^2_2$, by the proof of Lemma 7.11 of [7].)

Question 5.3. Is $\mathsf{COH} \leqslant_{\mathrm{c}} \mathsf{SRT}^2_2$?

Question 5.4. Let $n \geqslant 3$ and $k \geqslant 2$, where k may also be $<\infty$. Is $\mathrm{RT}^n_k \leqslant^{n-1}_{\mathrm{gW}} \mathrm{KL}$?

Question 5.5. Let $n \geqslant 2$ and $k > j \geqslant 2$, where k may also be $<\infty$. For which m do we have $\mathrm{RT}^n_k \leqslant^m_{\mathrm{gW}} \mathrm{RT}^n_j$? (See the mention of the work of Patey [45] following Proposition 4.22 for the $\leqslant^m_\omega$ case.)

There are also many principles related to Ramsey's Theorem not shown in the diagrams in this section. In some cases, their positions in these diagrams follow from known results (or from the proofs of these results), but there are likely several interesting questions involved in filling out these diagrams. For more information on some of these principles, see [14, 25].

Another area for further work is the development of useful extensions of the reducibilities discussed here to principles with more complicated syntactic forms than the Π^1_2 principles we have considered.

Appendix A. On a proof by Cholak, Jockusch, and Slaman

The following result is stated as Theorem 12.2 in Cholak, Jockusch, and Slaman [7].

Theorem A.1. *Let $n, k \geqslant 2$ and let $C_0, C_1, \dots$ be sets such that $C_i \nleqslant_{\mathrm{T}} \emptyset^{(n-2)}$ for all i. Each computable k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set H such that $H' \ngeqslant_{\mathrm{T}} \emptyset^{(n)}$ and $C_i \nleqslant_{\mathrm{T}} H$ for all i.*

The proof given in that paper is by induction on n. The base case $n = 2$ of that proof is correct, but the inductive case has a flaw. The argument for $n+1$ fixes a computable coloring c of $[\mathbb{N}]^{n+1}$ and $C_0, C_1, \dots$ such that $C_i \nleqslant_{\mathrm{T}} \emptyset^{(n-1)}$ for all i. It then chooses an infinite prehomogeneous set A for this coloring with $A' \leqslant_{\mathrm{T}} \emptyset''$, and claims that $C_i \nleqslant_{\mathrm{T}} A^{(n-2)}$ for each i. This claim is correct if $n \geqslant 3$, but for $n = 2$ it becomes a claim that $C_i \nleqslant_{\mathrm{T}} A$ for all i. However, the only condition on C_i in this case is that $C_i \nleqslant_{\mathrm{T}} \emptyset'$. There are computable colorings of $[\mathbb{N}]^3$ with no infinite $\emptyset'$-computable prehomogeneous sets, as may be seen easily from the existence of a computable 2-coloring of pairs with no infinite $\emptyset'$-computable homogeneous sets ([30, Theorem 3.1]). Taking $C_i = A$ for such a coloring shows that the claim does not hold in general.

To correct this argument, it is enough to prove the theorem for $n = 3$, as then the inductive argument can proceed as before. We give a proof based on that of the $n = 2$ case, beginning with an auxiliary lemma, which is an extension of Theorem 12.1 in [7], and is proved by a similar argument. We write $\deg(X)$ for the (Turing) degree of a set X.

Lemma A.2. *Let $k, n \geqslant 2$, let e be a computable k-coloring of $[\mathbb{N}]^n$, and let $\mathbf{p}$ be a degree such that $\mathbf{p} \gg \mathbf{0^{(n-1)}}$. Then there is an infinite homogeneous set H for e such that $\deg(H)' \leqslant \mathbf{p}$.*

Proof. The lemma is proved in relativized form by induction on n. The base step $n = 2$ follows from Corollary 12.6 in [7]. (A corrected proof of this result appears in [8].) For the inductive step, assume the lemma holds in relativized form for n. For notational simplicity, we prove the unrelativized form for $n + 1$. Suppose that e is a computable k-coloring of $[\mathbb{N}]^{n+1}$ and that $\mathbf{p} \gg \mathbf{0^{(n)}}$. By Jockusch [30, Lemma 5.4] there is an infinite prehomogeneous set A for e such that $A' \leqslant_{\mathrm{T}} 0''$. Note that $\mathbf{p} \gg \mathbf{0^{(n)}} \geqslant \deg(A^{(n-1)})$. Let d be the coloring of $[A]^n$ induced by e, so that d is A-computable. By the inductive hypothesis (relative to A), there is an infinite homogeneous set H for d such that $\deg(H)' \leqslant \mathbf{p}$. Clearly H is also homogeneous for e, so the inductive step is complete. □

Lemma A.3. *Let $k \geqslant 2$ and let $C_0, C_1, \dots$ be sets such that $C_i \nleqslant_{\mathrm{T}} \emptyset'$ for all i. Each computable k-coloring of $[\mathbb{N}]^3$ has an infinite homogeneous set H such that $H' \ngeqslant_{\mathrm{T}} \emptyset'''$ and $C_i \nleqslant_{\mathrm{T}} H$ for all i.*

Proof. Fix a k-coloring c of $[\mathbb{N}]^3$. Let $\mathbf{c_i} = \deg(C_i)$. Let $\mathbf{d_0} = \mathbf{0''}$. If $\mathbf{d_i} \vee \mathbf{c_i} \ngeqslant \mathbf{0'''}$ then let $\mathbf{e_i} = \mathbf{c_i}$; otherwise let $\mathbf{e_i} = \mathbf{0''}$. Let $\mathbf{d_{i+1}} = \mathbf{d_i} \vee \mathbf{e_i}$. By Kleene and Post [36]/Lacombe [39]/Spector [57] (see Odifreddi [44, Theorem V.4.3]), the ideal generated by the $\mathbf{d_i}$ has an exact pair $\mathbf{f}, \mathbf{g}$ such that the $\mathbf{e_i}$ are uniformly computable in both $\mathbf{f}$ and $\mathbf{g}$. (More precisely, there are $E_i \in \mathbf{e_i}$ such that the E_i are uniformly computable from any element of $\mathbf{f}$, and $\widehat{E}_i \in \mathbf{e_i}$ such that the $\widehat{E}_i$ are uniformly computable from any element of $\mathbf{g}$.) Since $\mathbf{0'''}$ is not in this ideal, at least one of $\mathbf{f}$ and $\mathbf{g}$ is not above $\mathbf{0'''}$, say $\mathbf{g} \ngeqslant \mathbf{0'''}$. Note that, since $\mathbf{d_0} = \mathbf{0''}$, we have $\mathbf{g} \geqslant \mathbf{0''}$. Also, $\mathbf{e_i} \nleqslant \mathbf{0'}$ for all i. By Posner and Robinson [47, Theorem 3] relativized to $\mathbf{0'}$, there is a $\mathbf{d} \geqslant \mathbf{0'}$ such that $\mathbf{g} = \mathbf{d'} = \mathbf{d} \vee \mathbf{e_i}$ for all i. By Friedberg [20], there is an $\mathbf{a}$ such that $\mathbf{a'} = \mathbf{d}$. Then $\mathbf{a''} = \mathbf{g}$, so

$$\mathbf{a''} \ngeqslant \mathbf{0'''}.$$

If $\mathbf{e_i} = \mathbf{c_i}$ then $(\mathbf{a'} \vee \mathbf{c_i})' = (\mathbf{d} \vee \mathbf{e_i})' = \mathbf{g'} \geqslant \mathbf{0'''}$. If $\mathbf{e_i} \neq \mathbf{c_i}$ then $(\mathbf{a'} \vee \mathbf{c_i})' \geqslant \mathbf{a''} \vee \mathbf{c_i} = \mathbf{g} \vee \mathbf{c_i} \geqslant \mathbf{d_i} \vee \mathbf{c_i} \geqslant \mathbf{0'''}$. Thus

$$(\mathbf{a'} \vee \mathbf{c_i})' \geqslant \mathbf{0'''}$$

for all i. These two displayed properties of $\mathbf{a}$ are all that we will use below.

Let $\mathbf{p}$ be a degree that is PA over $\mathbf{a''}$ and hyperimmune-free over $\mathbf{a''}$. By Jockusch [30, Lemma 5.9] relative to $\mathbf{a}$, there is an $\mathbf{a}$-computable 2-coloring d of $[\mathbb{N}]^3$ such that for any infinite homogeneous set H for d, we have $\deg(H) \vee \mathbf{a} \geqslant \mathbf{a'}$. We can combine the colorings c and d into an $\mathbf{a}$-computable $2k$-coloring e of $[\mathbb{N}]^3$ such that any homogeneous set for e is also homogeneous for both c and d. (Just let $e(s) = 2c(s) + d(s)$.) By the $n = 3$ case of Lemma A.2 (relativized to $\mathbf{a}$), there is an infinite homogeneous set H for e such that $(\deg(H) \vee \mathbf{a})' \leqslant \mathbf{p}$. We claim that H satisfies the conclusion of Lemma A.3. Let $\mathbf{b} = \deg(H) \vee \mathbf{a}$. Then $\mathbf{b} \geqslant \mathbf{a'}$ and $\mathbf{b'}$ is hyperimmune-free over $\mathbf{a''}$.

If $\mathbf{b}' \geqslant \mathbf{0}'''$ then, since we also have $\mathbf{b}' \geqslant \mathbf{a}''$, it follows that $\mathbf{b}' \geqslant \mathbf{a}'' \vee \mathbf{0}'''$ and hence $\mathbf{a}'' \vee \mathbf{0}'''$ is hyperimmune-free over $\mathbf{a}''$. But $\mathbf{a}'' \vee \mathbf{0}'''$ is c.e. relative to $\mathbf{a}''$, so $\mathbf{a}'' \geqslant \mathbf{0}'''$, which is not the case. Thus $\mathbf{b}' \not\geqslant \mathbf{0}'''$, and hence $H' \not\geqslant_{\mathrm{T}} \emptyset'''$.

Now suppose that $\mathbf{c_i} \leqslant \mathbf{b}$. Then $\mathbf{b} \geqslant \mathbf{b} \vee \mathbf{c_i} \geqslant \mathbf{a}' \vee \mathbf{c_i}$, so $\mathbf{b}' \geqslant (\mathbf{a}' \vee \mathbf{c_i})' \geqslant \mathbf{0}'''$, which is not the case. Thus $\mathbf{c_i} \not\leqslant \mathbf{b}$ for all i, and hence $C_i \not\leqslant_{\mathrm{T}} H$ for all i. This completes the proof of Lemma A.3 and hence also the proof of Theorem A.1. □

References

[1] K. Ambos-Spies, B. Kjos-Hanssen, S. Lempp, and T. A. Slaman, Comparing DNR and WWKL, J. Symbolic Logic 69 (2004) 1089–1104.

[2] V. Brattka and G. Gherardi, Effective choice and boundedness principles in computable analysis, Bull. Symbolic Logic 17 (2011) 73–117.

[3] V. Brattka and G. Gherardi, Weihrauch degrees, omniscience principles and weak computability, J. Symbolic Logic 76 (2011) 143–176.

[4] V. Brattka, M. Hendtlass, and A. P. Kreuzer, On the uniform computational content of computability theory, to appear, `arXiv:1501.00433`.

[5] V. Brattka and T. Rakotoniaina, On the uniform computational content of Ramsey's Theorem, to appear, `arXiv:1508.00471`.

[6] P. A. Cholak, M. Giusto, J. L. Hirst, and C. G. Jockusch, Jr., Free sets and reverse mathematics, in S. G. Simpson, ed., Reverse Mathematics 2001, Lecture Notes in Logic 21, Association for Symbolic Logic, La Jolla, CA and A K Peters, Ltd., Wellesley, MA, 2005, 104–119.

[7] P. A. Cholak, C. G. Jockusch, Jr., and T. A. Slaman, On the strength of Ramsey's Theorem for pairs, J. Symbolic Logic 66 (2001) 1–55

[8] P. A. Cholak, C. G. Jockusch, Jr., and T. A. Slaman, Corrigendum to [7], J. Symbolic Logic 74 (2009) 1438–1439.

[9] C. T. Chong, S. Lempp, and Y. Yang, On the role of collection principles for Σ^0_2 formulas in second-order reverse mathematics, Proc. Amer. Math. Soc. 138 (2010) 1093–1100.

[10] C. T. Chong, T. A. Slaman, and Y. Yang, The metamathematics of stable Ramsey's Theorem for pairs, J. Amer. Math. Soc. 27 (2014) 863–892.

[11] B. F. Csima and J. R. Mileti, The strength of the rainbow Ramsey theorem, J. Symbolic Logic 74 (2009) 1310–1324.

[12] F. G. Dorais, D. D. Dzhafarov, J. L. Hirst, J. R. Mileti, and P. Shafer, On uniform relationships between combinatorial problems, Trans. Amer. Math. Soc. 368 (2016) 1321–1359.

[13] R. G. Downey and D. R. Hirschfeldt, Algorithmic Randomness and Complexity, Theory and Applications of Computability, Springer, New York, 2010.

[14] D. D. Dzhafarov (maintainer), The RM Zoo (website), `rmzoo.uconn.edu`.

[15] D. D. Dzhafarov, Cohesive avoidance and strong reductions, Proc. Amer. Math. Soc. 143 (2015) 869–876.

[16] D. D. Dzhafarov, Strong reductions between combinatorial principles, to appear in J. Symbolic Logic, `arXiv:1504.01405`.

[17] D. D. Dzhafarov and G. Igusa, Notions of robust information coding, to appear in Computability, `arXiv:1406.2982`.

[18] D. D. Dzhafarov and C. G. Jockusch, Jr., Ramsey's theorem and cone avoidance, J. Symbolic Logic 74 (2009) 557–578.

[19] D. D. Dzhafarov, L. Patey, R. Solomon, and L. B. Westrick, Ramsey's Theorem for singletons and strong computable reducibility, to appear in Proc. Amer. Math. Soc., `arXiv:1602.04481`.

[20] R. M. Friedberg, A criterion for completeness of degrees of unsolvability, J. Symbolic Logic 22 (1957) 159–160.

[21] H. M. Friedman, two postings to the Foundations of Mathematics mailing list, `http://www.cs.nyu.edu/pipermail/fom/1999-July/003257.html` and `http://www.cs.nyu.edu/pipermail/fom/1999-July/003263.html`.

[22] G. Gherardi and A. Marcone, How incomputable is the separable Hahn-Banach Theorem?, Notre Dame J. Formal Logic 50 (2009) 393–425.

[23] M. Giusto and S. G. Simpson, Located sets and reverse mathematics, J. Symbolic Logic 65 (2000) 1451–1480.

[24] P. Hájek, Interpretability and fragments of arithmetic, in P. Clote and K. Krajíček, eds., Arithmetic, Proof Theory, and Computational Complexity (Prague, 1991), Oxford Logic Guides 23, Oxford Univ. Press, New York, 1993, 185–196.

[25] D. R. Hirschfeldt, Slicing the Truth: On the Computable and Reverse Mathematics of Combinatorial Principles, Lecture Note Series, Institute for Mathematical Sciences, National University of Singapore, vol. 28, World Scientific, Singapore, 2014.

[26] D. R. Hirschfeldt, C. G. Jockusch, Jr., B. Kjos-Hanssen, S. Lempp, and T. A. Slaman, The strength of some combinatorial principles related to Ramsey's Theorem for pairs, in C. Chong, Q. Feng, T. A. Slaman, W. H. Woodin, and Y. Yang, eds., Computational Prospects of Infinity, Part II: Presented Talks, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 15, World Scientific, Singapore, 2008, 143–146.

[27] D. R. Hirschfeldt, K. Lange, and R. A. Shore, Induction, bounding, weak combinatorial principles, and the Homogeneous Model Theorem, to appear in Mem. Amer. Math. Soc., `math.uchicago.edu/~drh/Papers/hmt.html`.

[28] D. R. Hirschfeldt and R. A. Shore, Combinatorial principles weaker than Ramsey's Theorem for pairs, J. Symbolic Logic 72 (2007) 171–206.

[29] D. R. Hirschfeldt, R. A. Shore, and T. A. Slaman, The Atomic Model Theorem and type omitting, Trans. Amer. Math. Soc. 361 (2009) 5805–5837.

[30] C. G. Jockusch, Jr., Ramsey's Theorem and recursion theory, J. Symbolic Logic 37 (1972) 268–280.

[31] C. G. Jockusch, Jr., A. A. Lewis, and J. B. Remmel, Π^0_1-classes and Rado's selection principle, J. Symbolic Logic 56 (1991) 684–693.

[32] C. G. Jockusch, Jr. and T. G. McLaughlin, Countable retracing functions and Π^0_2 predicates, Pacific J. Math. 30 (1969) 67–93.

[33] C. G. Jockusch, Jr. and P. E. Schupp, Generic computability, Turing degrees, and asymptotic density, J. London Math. Soc. 85 (2012) 472–490.

[34] C. G. Jockusch, Jr. and R. I. Soare, Π^0_1 classes and degrees of theories, Trans. Amer. Math. Soc. 173 (1972) 33–56.

[35] C. G. Jockusch, Jr. and R. I. Soare, Degrees of members of Π^0_1 classes, Pacific J. Math. 40 (1972) 605–616.

[36] S. C. Kleene and E. L. Post, The uppersemilattice of degrees of recursive unsolvability, Ann. Math. 59 (1954) 379–407.

[37] A. Kučera, Measure, Π^0_1 classes, and complete extensions of PA, in H.-D. Ebbinghaus, G. H. Müller, and G. E. Sacks, eds., Recursion Theory Week, Oberwolfach, 1984, Lect. Notes in Math. 1141, Springer, Berlin, 1985, 245–259.

[38] S. Kurtz, Notions of weak genericity, J. Symbolic Logic 48 (1983) 764–770.

[39] D. Lacombe, Sur le semi-réseau constitué par lies degrés d'indecidabilité récursive, Compt. Rend. Ac. Sci. 239 (1954) 1108–1109 (in French).
[40] J. Liu, RT^2_2 does not imply WKL_0, J. Symbolic Logic 77 (2012) 609–620.
[41] J. R. Mileti, Partition Theorems and Computability Theory, PhD Dissertation, University of Illinois at Urbana-Champaign, 2004.
[42] B. Monin and L. Patey, Π^0_1 encodability and omniscient reductions, to appear, `arXiv:1603.01086`.
[43] A. Montalbán, Open questions in reverse mathematics, Bull. Symbolic Logic 17 (2011) 431–454.
[44] P. Odifreddi, Classical Recursion Theory. The Theory of Functions and Sets of Natural Numbers, Studies in Logic and the Foundations of Mathematics 125, North-Holland Publishing Co., Amsterdam, 1989.
[45] L. Patey, The Reverse Mathematics of Ramsey-Type Theorems, PhD Dissertation, Université Paris Diderot (Paris VII), 2016.
[46] L. Patey, The weakness of being cohesive, thin or free in reverse mathematics, to appear in Israel J. Math., `arXiv:1502.03709`.
[47] D. B. Posner and R. W. Robinson, Degrees joining to $\mathbf{0}'$, J. Symbolic Logic 46 (1981) 714–722.
[48] T. Rakotoniaina, On the Computational Strength of Ramsey's Theorem, PhD Dissertation, University of Cape Town, 2015.
[49] G. E. Sacks, Higher Recursion Theory, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990.
[50] D. S. Scott, Algebras of sets binumerable in complete extensions of arithmetic, in Proceedings of the Symposium on Pure and Applied Mathematics, vol. V, American Mathematical Society, Providence, RI, 1962, 117–121.
[51] D. Seetapun and T. A. Slaman, On the strength of Ramsey's Theorem, Notre Dame J. Formal Logic 36 (1995) 570–582.
[52] R. A. Shore, Reverse mathematics : The playground of logic, Bull. Symbolic Logic 16 (2010) 378–402.
[53] S. G. Simpson, Subsystems of Second Order Arithmetic, First edition, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.
[54] S. G. Simpson, Subsystems of Second Order Arithmetic, Second edition, Perspectives in Logic, Cambridge University Press, Cambridge and Association for Symbolic Logic, Poughkeepsie, NY, 2009.
[55] R. I. Soare, Recursively Enumerable Sets and Degrees, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.
[56] R. M. Solovay, Hyperarithmetically encodable sets, Trans. Amer. Math. Soc. 239 (1978) 99–122.
[57] C. Spector, On degrees of recursive unsolvability, Ann. of Math. (2) 64 (1956) 581–592.
[58] W. Wang, Some logically weak Ramseyan theorems, Adv. Math. 261 (2014) 1–25.
[59] K. Weihrauch, The degrees of discontinuity of some translators between representations of the real numbers, Technical Report TR-92-050, International Computer Science Institute, Berkeley, 1992.
[60] K. Weihrauch, The TTE-interpretation of three hierarchies of omniscience principles, Informatik Berichte FernUniversität Hagen 130, Hagen, 1992.
[61] X. Yu and S. G. Simpson, Measure theory and weak König's lemma, Arch. Math. Logic 30 (1990) 171–180.

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