

MATH 204 MIDTERM #1

1. Let I be an open interval containing the points a and b with $a < b$. Suppose that the function f is differentiable, and $f'(a) > 0$. Show that a cannot be a maximizer for the function $f : I \rightarrow \mathbb{R}$.

Solution. Suppose a is a maximizer for $f : I \rightarrow \mathbb{R}$, then for all $x \in I$, $f(x) \leq f(a)$. But then, for $x \geq a$,

$$\frac{f(x) - f(a)}{x - a} \leq 0,$$

which implies that

$$f'(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

But we are told that $f'(a) > 0$, so this is a contradiction.

2. Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ have the property that $|f(u) - f(v)| < 2|u - v|^2$ for all u, v in \mathbb{R} . Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is constant.

Hint: Use the definition of derivative to compute f' .

Solution. Write $v = x$, $u = x + h$. Then,

$$|f(x + h) - f(x)| \leq 2h^2,$$

so

$$-2h \leq \frac{f(x + h) - f(x)}{h} \leq 2h$$

Thus, by the squeezing theorem,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = 0.$$

This is true for all $x \in \mathbb{R}^n$. Now, for any a and b with $a < b$ we have, by the mean value theorem there exists c with $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0.$$

Thus $f(b) = f(a)$, and so f is constant.

3. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow \infty$.

Solution. By the mean value theorem,

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f'(c),$$

where $c \in [x, x+1]$. Thus,

$$g(x) = f'(c)$$

for some $c \in [x, x+1]$. As $x \rightarrow \infty$, $c \rightarrow \infty$, so by assumption $f'(c) \rightarrow 0$. Thus, as $x \rightarrow \infty$, $g(x) \rightarrow 0$.

4. Suppose f is defined in a neighborhood of x and suppose $f''(x)$ exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Solution. In this problem, x is a constant. Let $u(h) = f(x+h) + f(x-h) - 2f(x)$. Then,

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h \rightarrow 0} \frac{u(h)}{h^2} = \lim_{h \rightarrow 0} \frac{u'(h)}{2h},$$

where for the last equality we used L'Hospital's rule. But

$$u'(h) = f'(x+h) - f'(x-h).$$

We can rewrite

$$u'(h) = f'(x+h) - f'(x) - (f'(x-h) - f'(x)).$$

Then,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u'(h)}{2h} &= \lim_{h \rightarrow 0} \frac{1}{2} \left[\frac{f'(x+h) - f'(x)}{h} - \frac{f'(x-h) - f'(x)}{h} \right] \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} + \frac{1}{2} \lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x)}{-h}, \end{aligned}$$

but both of the limits in the last line are $f''(x)$ by definition. Then, we get

$$\lim_{h \rightarrow 0} \frac{u'(h)}{2h} = f''(x),$$

which is what we needed to prove.

5. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x & \text{if } x = 1/n, \text{ where } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Compute $(U) \int_0^1 f$ and $(L) \int_0^1 f$. Is f integrable on $[0, 1]$? Justify your answer.

Solution. Suppose $\epsilon > 0$. Let S denote the set of numbers of the form $1/n$, $n \in \mathbb{N}$, which are outside of $[0, \epsilon]$. Note that the set S is finite: in fact the number of elements of S is at most $1/\epsilon$.

We now construct a partition P (depending on ϵ) which has $[0, \epsilon]$ as its first interval, (i.e. $x_0 = 0$ and $x_1 = \epsilon$, and also surrounds each element of S by an interval of size ϵ^2). By definition,

$$U(f, P) = \sum_{j=1}^n M_j(f)(x_j - x_{j-1}), \quad (1)$$

where $M_j(f) = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$. The term $j = 1$ in (1) (corresponding to the interval $[0, \epsilon]$) is

$$M_1(f)(x_1 - x_0) = M_1(f)\epsilon \leq \epsilon^2$$

(since $M_1(f) \leq \epsilon$). The rest of the terms in the sum (1) are of two types: those that contain a point of S and those that don't. If $[x_{j-1}, x_j]$ does not contain a point of S then $M_j(f) = 0$, so these terms contribute 0 to $U(f, P)$. If $[x_{j-1}, x_j]$ contains a point of S , then $M_j(f) \leq 1$ and by construction $x_j - x_{j-1} \leq \epsilon^2$. The number of terms of this type is $|S| \leq \frac{1}{\epsilon}$, so the total contribution of terms of this type is bounded by $\epsilon^2|S| \leq \epsilon$. Then, putting everything together, we have

$$U(f, P) \leq \epsilon^2 + \epsilon.$$

Since ϵ is arbitrary, we have

$$(U) \int_0^1 f = \inf_P U(f, P) = 0.$$

It is easy to see that for any partition P , $L(f, P) = 0$, so

$$(L) \int_0^1 f = \sup_P L(f, P) = 0.$$

Thus the upper and lower integrals are both 0. This implies that f is integrable on $[0, 1]$, and $\int_0^1 f = 0$.

6. Suppose $f \geq 0$, f is continuous on $[a, b]$ and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Suppose there exists $x_0 \in (a, b)$ such that $f(x_0) > 0$. We will derive a contradiction. Since f is continuous at x_0 , there exists $\delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta)$, we have $|f(x) - f(x_0)| < f(x_0)/2$ (I have just used $\epsilon = f(x_0)/2$ in the definition of continuity of f). Let $c = \max(x_0 - \delta, a)$, $d = \min(x_0 + \delta, b)$. Then $[c, d] \subset [a, b]$, and for $x \in (c, d)$,

$$-f(x_0)/2 \leq f(x) - f(x_0) \leq f(x_0)/2,$$

which implies in particular, that for $x \in (c, d)$, $f(x) \geq f(x_0)/2$. (This is a variant of the sign preservation property). Then, by the comparison theorem for integrals,

$$\int_c^d f(x) dx \geq \int_c^d \frac{f(x_0)}{2} dx = \frac{f(x_0)(d - c)}{2} > 0. \quad (2)$$

But then,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^d f(x) dx + \int_d^b f(x) dx$$

The first and the third integrals are non-negative, again by the comparison property of integrals, and the second integral is positive by (2). Thus we proved that $\int_a^b f > 0$, which is a contradiction.