

Torelli buildings and their automorphisms

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Preface

It is well known that each compact orientable surface R gives rise to a simplicial complex $\mathcal{C}(R)$ playing in the theory of Teichmüller modular groups a role similar to the role of Tits buildings in theories of algebraic and arithmetic groups. In this paper we introduce, for each closed orientable surface S , an analogue of Tits buildings more suitable for investigation of the Torelli group $\mathcal{J}(S)$ of S . It is a simplicial complex with some additional structure. We denote this complex with its additional structure by $\mathfrak{B}(S)$ and call it the *Torelli building* of S . The main result of this paper shows that Torelli buildings of surfaces of genus ≥ 5 have only obvious automorphisms, and identifies the group of automorphisms of $\mathfrak{B}(S)$. Namely, we prove that for such surfaces S every automorphism of $\mathfrak{B}(S)$ is induced by a diffeomorphism of S . This theorem is an analogue of the theorem of the second author [I7, I8] about automorphisms of complexes of curves $\mathcal{C}(R)$. The proof is based on this theorem about automorphisms of complexes of curves, and uses methods inspired by its proof [I7, I8].

Our theorem about automorphisms of Torelli buildings is intended for applications to automorphisms and virtual automorphisms of Torelli groups. These applications are also inspired by the results and methods of [I7, I8], and are proved following the scheme developed by the second author [I1, I4, I7] for studying automorphisms of Teichmüller modular groups. Compared to Teichmüller modular groups, Torelli groups are much less accessible and much less is known about them.

The work reported in this paper was largely completed about ten years ago.

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Our main result about Torelli groups is the identification of all isomorphisms between subgroups of finite index in the Torelli group $\mathcal{J}(S)$ of a closed surface S . Namely, every such isomorphism is equal to (the restriction of) to a conjugation by an element of the extended mapping class group $\text{EMod}(S)$ of S . This result immediately allows to find the automorphism group and the outer automorphism group of $\mathcal{J}(S)$, and to find its abstract commensurator. The deduction of our theorems about Torelli groups from the theorem about automorphisms of Torelli building is independent of the methods of the present paper and uses only its main result. It will be published on some other occasion.

The results of this paper, together with their applications to Torelli groups, were announced in [FI].

1. Introduction

The notations introduced in this section will be used throughout the whole paper. For a compact connected surface R we will denote by $g(R)$ its genus. Throughout the paper we will denote by S a closed orientable surface and by $g = g(S)$ its genus.

The main results of this paper are proved under the assumption $g = g(S) \geq 5$. But most of our proofs are valid under weaker assumptions $g \geq 4$ or $g \geq 3$. It is quite reasonable to believe that the assumption $g \geq 5$ can be weakened to $g \geq 4$ throughout the whole paper at the cost of additional technicalities. The authors were not really interested in proving the main result with the weakest possible restrictions on g , and preferred the most natural (at least for the authors) arguments. The latter always worked for $g \geq 5$.

The *Teichmüller modular group** $\text{Mod}(S)$ of S is defined as the group of isotopy classes of orientation-preserving diffeomorphisms $S \rightarrow S$. We will need also the *extended Teichmüller modular group* $\text{EMod}(S)$ of S which is defined as the group of the isotopy classes of *all* diffeomorphisms $S \rightarrow S$. Let us fix an orientation of S . Then the algebraic intersection number provides a nondegenerate, skew-symmetric, bilinear form on $H = H_1(S, \mathbf{Z})$, called the *intersection form*. The natural action of $\text{Mod}(S)$ on H preserves the intersection form. If fix a symplectic basis in H , then we can identify the group of symplectic automorphisms of H with the integral symplectic group $\text{Sp}(2g, \mathbf{Z})$ and the action of $\text{Mod}(S)$ on H leads to a natural surjective homomorphism $\text{Mod}(S) \rightarrow \text{Sp}(2g, \mathbf{Z})$.

The *Torelli group* of S , denoted by $\mathcal{J}(S)$, is defined as the kernel of this homomorphism; that is, $\mathcal{J}(S)$ is the subgroup of $\text{Mod}(S)$ consisting of elements which act trivially on $H_1(S, \mathbf{Z})$. The groups $\mathcal{J}(S)$ plays an important role in algebraic geometry and low-dimensional topology. At the same time $\mathcal{J}(S)$ is an interesting group in its own right.

*Admittedly, another term, uninformative and colorless, is more widely used.

The modern theory of Torelli groups starts with the work of D. Johnson in the 1980-ies (see, for example, surveys [Jo1] and [Ha1]). The first spectacular results of D. Johnson (preceded by some preliminary technical results) is his theorem [Jo2] to the effect that the group $\mathcal{J}(S)$ is finitely generated if the genus $g(S) \geq 3$. Thirty years later it is not known whether or not $\mathcal{J}(S)$ admits a finite presentation for $g(S) \geq 3$. It seems that D. Johnson's work remains to be poorly understood: we understand his proofs, but we do not understand why they work. Unfortunately for mathematicians interested in Torelli groups, but reportedly not for himself, D. Johnson left mathematics in the late 1980-ies. He left the academic world already in the late 1970-ies.

A natural problem is to find all automorphisms of groups $\mathcal{J}(S)$ or, better, all isomorphisms between subgroups of finite index of $\mathcal{J}(S)$. This problem motivated this paper, which devoted to the first part of a solution. As a motivation for the reader, let us outline our main results in this direction.

As usual, for any (discrete) group G we denote by $\text{Aut}(G)$ the group of all automorphisms of G . If H is a normal subgroup of a group G , then G acts on H by conjugations. Namely, if $g \in G$, then the formula $h \mapsto g^{-1}hg$, where $h \in H$, defines an automorphism $c_g: H \rightarrow H$. One immediately checks that the map $g \mapsto c_g$ is a group homomorphism, which we will denote by $\text{co}: G \rightarrow \text{Aut}(G)$.

Since $\mathcal{J}(S)$ is a normal subgroup of $\text{EMod}(S)$, the Teichmüller modular group $\text{EMod}(S)$ acts on $\mathcal{J}(S)$ by conjugation. In particular, the homomorphism $\text{co}: \text{EMod}(S) \rightarrow \text{Aut}(\mathcal{J}(S))$ is defined. The main result about automorphisms is the following theorem.

1.1. Theorem. *If $g = g(S) \geq 5$, then $\text{co}: \text{EMod}(S) \rightarrow \text{Aut}(\mathcal{J}(S))$ is an isomorphism.*

Note that the conclusion of the Theorem 1.1 is false if $g = 2$, since in this case $\mathcal{J}(S)$ is a countably generated free group by a theorem of Mess [Me]. Theorem 1.1 and its proof are inspired by the work of the second author [I1, I4] about automorphisms of $\text{Mod}(S)$, and J. McCarthy's [McC] version of it. The methods of papers [I1, I4], [McC] are not powerful enough to deal with automorphisms of Torelli groups. In our work we followed the methods developed later by the second author [I7, I8]. These methods automatically lead to stronger results and allow to deal not only with automorphisms of $\mathcal{J}(S)$, but also with isomorphisms between subgroups of finite index of $\mathcal{J}(S)$. Namely, they lead to the following theorem.

1.2. Theorem. *If $g = g(S) \geq 5$, then every isomorphism $\Gamma_1 \rightarrow \Gamma_2$ between two subgroups Γ_1, Γ_2 of finite index in $\mathcal{J}(S)$ is induced by the conjugation c_g by an element $g \in \text{EMod}(S)$.*

This theorem allows immediately find the so-called *abstract commensurator* of $\mathcal{J}(S)$. In the present context it is most natural to refer the reader to [I8] for a definition.

1.3. Theorem. *If $g = g(S) \geq 5$, then the abstract commensurator $\text{Comm}(\mathcal{J}(S))$ of $\mathcal{J}(S)$ is isomorphic to $\text{ESp}(2g, \mathbf{Z})$, where $\text{ESp}(2g, \mathbf{Z})$ is the group of automorphisms of \mathbf{Z}^{2g} preserving the standard symplectic form up to a sign.*

Remark. The extended symplectic group $\text{ESp}(2g, \mathbf{Z})$ contains the standard symplectic group $\text{Sp}(2g, \mathbf{Z})$ as a subgroup of index 2.

An easy application of this result is a new proof of the following special case of a theorem of the second author [I5].

1.4. Theorem. *If $g = g(S) \geq 5$, then the Torelli group $\mathcal{J}(S)$ is not isomorphic to any arithmetic group.*

While this theorem is neither new, nor the most general one of this type, our new proof shed an additional light on it (cf. [I8], Theorem 6 and the discussion following it).

A key ingredient in the proof of Theorems 1.1 and 1.2 is an analogue of the theorem about automorphisms of the complexes of curves due to the second author [I7, I8]. Recall that the *complex of curves* $\mathcal{C}(R)$ of an orientable surface R is the simplicial complex having as the set of vertices the set of isotopy classes of all non-trivial (i.e. not bounding a disc in R and not homotopic to a component of the boundary ∂R) circles in R ; a set of vertices forms a simplex if they can be represented by disjoint circles. See the subsections 2.6 and 2.7 for more details about simplicial complexes and complexes of curves, including the definition of simplicial complexes.

An analogue of the complex of curves in the theory of Torelli groups is a simplicial complex having as vertices the isotopy classes of separating circles on R and pairs of the isotopy classes of non-separating circles C, C' on R such that the union $C \cup C'$ is separating. It is convenient to equip this simplicial complex equipped with some additional structure. See the subsection 3.1 for the definition. This simplicial complex together with its additional structure is called the *Torelli building* of R and is denoted by $\mathfrak{TB}(R)$.

Main Theorem. *If $g = g(S) \geq 5$, then every automorphism of the Torelli building $\mathfrak{TB}(S)$ of S is induced by a diffeomorphism of S .*

The Main Theorem is an analogue of the theorem of the second author [I7, I8] about automorphisms of the complexes of curves. The the definition of the Torelli building $\mathfrak{TB}(S)$ will be given in Section 3, and after this Main Theorem will be stated again in as Theorem 3.4. Sections 4–6 are devoted to its proof. Section 2 is devoted to an overview of basic definitions and notations used in this paper.

2. Preliminaries

This section is devoted to basic definitions and facts used throughout the paper.

2.1. Surfaces. By a *surface* we understand a compact orientable 2-manifold with boundary. The boundary is allowed to be empty, and surfaces with empty boundary are called *closed* surfaces. Our main results are concerned only with closed surfaces, but surfaces with non-empty boundary inevitably show up in the proofs.

By a *subsurface* of a surface R we understand a codimension 0 submanifold Q of R such that each component of the boundary ∂Q is either equal to a component of ∂R , or disjoint from ∂R . Obviously, if R is a closed connected surface and Q is a subsurface of R , then either $Q = R$, or the boundary ∂R is non-empty.

2.2. Circles. The most basic tool in the topology of surfaces are submanifolds of surfaces diffeomorphic to the standard circle S^1 in \mathbf{R}^2 . These submanifolds are assumed to be disjoint from the boundary (if the boundary is non-empty). We will call such a submanifold of a surface R a *circle in R* . A circle in R is called *non-peripheral* if does not bound an annulus together with a component of the boundary ∂R . If R is a closed surface, then, obviously, all circles in R are non-peripheral. A circle called *non-trivial* if it is *non-peripheral* and does not bound a disc in R . Alternatively, a *non-peripheral* circle can be defined as circle which is not homotopic to a component of the boundary ∂R . Similarly, a *non-trivial* circle can be defined for closed surfaces as a circle which cannot be deformed into a point, and for surfaces with non-empty boundary as a circle which cannot be deformed into the boundary.

2.3. Separating circles. A circle D in a connected surface R is called *separating*, if the result of cutting R along D is not connected, or, what is the same, $R \setminus D$ is not connected. Then both the result of cutting and $R \setminus D$ consist of two components. One more way to define *separating* circles in R is to define them as *homologically trivial* circles, i.e. as circles in R such that their fundamental class, considered as a 1-dimensional homology class in R , is equal to 0. The last approach has the advantage of working in the expected way also for non-connected surfaces. Separating circles are often called also *bounding circles*.

A separating circle D in a connected surface R divides R into two subsurfaces of R . These subsurfaces are equal to the closures of two components of $R \setminus D$, and D is a boundary component of each of them. If R is closed, then D is the boundary of both these subsurfaces. We will often call these subsurfaces *the parts into which D divides S* .

If h is the genus of either of two subsurface bounded by D , then we will say that D is a *separating circle of genus h* . While in general there are two possible values of h , this expression is unambiguous.

2.4. Bounding pairs of circles. A *bounding pair of circles* on a connected surface R is defined as an unordered pair C, C' of disjoint non-isotopic circles in R such that both circles C, C' are non-separating, but the result of cutting R along $C \cup C'$ is not connected.

Then it consists, obviously, of two components. Alternatively, C, C' is a *bounding pair of circles* if $R \setminus (C \cup C')$ is not connected.

We should stress that a bounding pair of circles is a pair of submanifolds C, C' of R , and not the submanifold $C \cup C'$. Strictly speaking, a bounding pair of circles C, C' is the 2-element set $\{C, C'\}$. We will use the notation $\{C, C'\}$ only when this is essential, and will use the term *a bounding pair of circles* C, C' when this will cause no harm. Of course, the pair $\{C, C'\}$ and the submanifold $C \cup C'$ carry the same information. But pairs have some (admittedly, minor) technical advantage. See, for example, the subsection 3.1 and the proof of Lemma 5.2.1 below.

As in the case of separating circles, one can give also a homological definition. It takes an especially simple form for closed surfaces. Namely, an unordered pair C, C' of disjoint circles on a closed connected surface R is a *bounding pair of circles* if the fundamental class of the submanifold $C \cup C'$ with some orientation, considered as a 1-dimensional homology class in R , is equal to zero. Note that $C \cup C'$ consists of two orientable components and hence admits 4 orientations. If C, C' is a bounding pair of circles on a closed surface R , then for 2 of these 4 orientations the fundamental class is equal to 0, and for 2 other is not. If C, C' is not a bounding pair of circles, then all 4 orientations lead to the non-zero fundamental class.

The reader may skip the following paragraph and go directly to the subsection 2.6.

In order to give a homological definition of bounding pairs of circles in the general case of a possibly non-closed and non-connected surface R it is convenient to fix an orientation of R (recall that all our surfaces are orientable). This orientation induces an orientation of the boundary ∂R , and hence an orientation of every boundary component of R . By the fundamental class of a boundary component we will understand the fundamental class of this component with this induced orientation. Now we can define a *bounding pair of circles* in R as an unordered pair C, C' of disjoint circles in R , such that the fundamental class of $C \cup C'$ with *some orientation*, considered as a 1-dimensional homology class in R , is equal to the sum of the fundamental classes of several (different) boundary components. Because we fixed an orientation of R , if C, C' is a bounding pair of circles, then only 1 of the 4 orientations of $C \cup C'$ satisfies this condition. If we replace our fixed orientation of R by the opposite orientation, then orientations of all boundary components of R will be replaced by the opposite orientations, and then another orientation of $C \cup C'$ will satisfy our condition (obviously, the orientation opposite to the first one). And if C, C' is not a bounding pair of circles, then none of the 4 orientations of $C \cup C'$ will satisfy this condition.

2.5. Isotopy classes of circles. By a standard abuse of language, the term *circle* is often used also for an *isotopy class of circles* in the class of submanifolds of the surface in question. In this paper, the distinction between circles on surfaces and their isotopy classes is usually essential, and we will avoid this abuse of notations to the extent possible.

The isotopy classes of circles are vertices of the complex of curves, and, in the case of separating circles in S , also of $\mathfrak{Ib}(S)$. In view of this, these isotopy classes will be called either *vertices* (of a specified complex), or just the isotopy classes. For a circle C in a surface R we will denote *the isotopy class of C in R* by $\langle C \rangle$. The surface R is usually clear from the context, even if C is also a circle in some other relevant surfaces (for example, some subsurfaces of R).

2.6. Simplicial complexes. By a simplicial complex \mathcal{V} we understand a pair consisting of a set V together with a collection of *finite subsets of V* . As usual, we think of \mathcal{V} as a structure on the set V , namely a structure of a simplicial complex. Elements of V are called *the vertices of \mathcal{V}* , and subsets of V from the given collection are called *the simplices of \mathcal{V}* . These data are required to satisfy only one condition: *a subset of a simplex is also a simplex*. For every subset $W \subset V$, the structure \mathcal{V} of a simplicial complex on V leads to a canonical structure of a simplicial complex \mathcal{W} on W . Namely, a subset of W is declared to be a simplex of \mathcal{W} if and only if it is a simplex of \mathcal{V} . The complex \mathcal{W} is called the *full subcomplex of \mathcal{V} spanned by the set W* .

For two simplicial complexes $\mathcal{V}_1, \mathcal{V}_2$ with the sets of vertices V_1, V_2 respectively, an *isomorphism* $F: \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is defined as a bijective map $F: V_1 \rightarrow V_2$ such that Δ is a simplex of \mathcal{V}_1 if and only if $F(\Delta)$ is a simplex of \mathcal{V}_2 . Alternatively, we may require that the images of simplices under both F and F^{-1} are also simplices. As usual, we do not distinguish between an isomorphism $\mathcal{V}_1 \rightarrow \mathcal{V}_2$ and its underlying bijection $F: V_1 \rightarrow V_2$. The notion of a *simplicial map* $\mathcal{V}_1 \rightarrow \mathcal{V}_2$ which is not necessarily an isomorphism is well known, but is not needed for the purposes of this paper.

The *dimension* of a simplex is defined as its cardinality minus 1. An *edge* is defined as a simplex of dimension 1, i.e. as a pair of vertices which is a simplex. A *triangle* is defined as a simplex of dimension 2.

We will say that two vertices v, w are *connected by an edge* if either $\{v, w\}$ is a simplex of dimension 1, or $v = w$. In the second case $\{v, w\} = \{v\}$ and $\{v, w\}$ is a simplex of dimension zero. We will say that v, w are *connected by a sequences of edges* if there is a sequence

$$v_1 = v, v_2, \dots, v_n = w$$

of vertices such that v_i and v_{i+1} are connected by an edge for every $i = 1, 2, \dots, n - 1$. A simplicial complex is called *connected* if every two vertices are connected by a sequence of edges. Obviously, the property of being connected by a sequence of edges is an equivalence relation. An equivalence class of vertices together with its induced structure of a simplicial complex is called a *component of connectedness* of the original complex.

A simplicial complex is called a *flag complex* if the following condition holds: *a finite set Δ of vertices is a simplex if and only if every subset of Δ with cardinality 2 is a simplex*. A flag complex is determined by its sets of vertices and edges, i.e. it is essentially a graph

without loops and multiple edges. All simplicial complexes considered in the paper are, in fact, flag complexes.

The higher dimensional simplices of a simplicial complex are exactly that part of the structure a simplicial complex which makes simplicial complexes interesting and useful, even for flag complexes. In this paper we will work only with the vertices, edges, and triangles. But eventually we will have to refer to results which are proved by using higher dimensional simplices in an essential way. The most important of them is a theorem of the second author [I7, I8] about automorphisms of complexes $\mathcal{C}(S)$.

2.7. The complex of curves. Let R be a compact surface, possibly with non-empty boundary. The *complex of curves* $\mathcal{C}(R)$ is a simplicial complex in the sense of the subsection 2.6. Its vertices are the isotopy classes $\langle C \rangle$ of non-trivial circles C in R . A collection of such isotopy classes is declared to be a simplex if and only if it is either empty, or the isotopy classes from this collection can be represented by pair-wise disjoint circles. In other words, if C_1, C_2, \dots, C_n are pair-wise disjoint circles, then the set $\{\langle C_1 \rangle, \langle C_2 \rangle, \dots, \langle C_n \rangle\}$ is a simplex of $\mathcal{C}(R)$, and all simplices can be constructed in this way by the definition (the empty simplex corresponds to $n = -1$).

If C is a non-trivial separating (respectively, non-separating) circle on R , we will call the corresponding vertex $\langle C \rangle$ of $\mathcal{C}(R)$ a *separating* (respectively, *non-separating*) vertex.

$\mathcal{C}(R)$ is a *flag complex*. This fact is well known to the experts. For the benefit of the others, let us outline why this is the case.

Let us restrict our attention by compact connected surfaces R of negative Euler characteristic (the few exceptional surfaces with non-negative Euler characteristic can be easily dealt with directly). It is well known that such a surface R admits a hyperbolic structure. The latter can be defined as a riemannian metric on a surface having constant negative curvature (which is usually required to be equal to -1 , but this not important in our context). Moreover, one can assume that the hyperbolic structure (riemannian metric) on R has geodesic boundary. Then every non-trivial circle in R is isotopic to a unique geodesic circle in R , and, moreover, if two circles are isotopic to some disjoint circles, then the geodesic circles isotopic to them are either disjoint or equal. Therefore, if we fix a hyperbolic structure on R , the geodesic circles may serve as canonical representatives of isotopy classes of non-trivial circles, and a set $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of isotopy classes is a simplex of $\mathcal{C}(S)$ if and only if the canonical representatives of $\gamma_1, \gamma_2, \dots, \gamma_n$ are pair-wise disjoint. Clearly, this implies that $\mathcal{C}(R)$ is a flag complex.

Isomorphisms and automorphisms of complexes of curves. Let $F: R \rightarrow Q$ be a diffeomorphism. Then F induces an isomorphism $\mathcal{C}(R) \rightarrow \mathcal{C}(Q)$ by the rule $F_{cc}: \langle C \rangle \mapsto \langle F(C) \rangle$. Clearly, F_{cc} is a map from the set of vertices of $\mathcal{C}(R)$ to the set of vertices $\mathcal{C}(Q)$ which is bijective. Moreover, both F_{cc} and F_{cc}^{-1} take simplices to simplices. We will denote the resulting automorphism by $F_{cc}: \mathcal{C}(R) \rightarrow \mathcal{C}(Q)$. If $R = Q$, then F_{cc} is an automorphism of

$\mathcal{C}(\mathbb{R})$. The following theorem of the second author [I7, I8] tells us that there are no other automorphisms, at least if $g(\mathbb{R}) \geq 2$.

2.8. Theorem. *If $g(\mathbb{R}) \geq 2$, then every automorphism $\mathcal{C}(\mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R})$ is equal to F_{cc} for some diffeomorphism $F: \mathbb{R} \rightarrow \mathbb{R}$.*

This theorem was extended to the cases of $g(\mathbb{R}) = 1$ or 0 by M. Korkmaz [Ko1], [Ko2] and, independently but somewhat later, by F. Luo [Lu]. These extensions are not needed for the present paper.

3. Torelli buildings

The *Torelli building* of S is a simplicial complex with some additional structure. By a standard abuse of language, we will denote by $\mathfrak{Tb}(S)$ both this simplicial complex and the Torelli building proper, i.e. the same complex with this additional structure.

3.1. The simplicial complex structure of $\mathfrak{Tb}(S)$. We start with the definitions the vertices and simplices of $\mathfrak{Tb}(S)$.

Vertices. There are two types of vertices of $\mathfrak{Tb}(S)$.

The vertices of the first type are defined as the isotopy classes of non-trivial separating circles in S . These vertices are the vertices of the complex of curves $\mathcal{C}(S)$ of the form $\langle D \rangle$, where D is a non-trivial separating circle in S . Following the conventions of the subsection 2.7 for the complexes of curves, we will call them *separating vertices*. Occasionally we will call them *SC-vertices*.

The vertices of the second type are defined as the edges of $\mathcal{C}(S)$ of the form $\{\langle C_0 \rangle, \langle C_1 \rangle\}$, where $\{C_0, C_1\}$ is a bounding pairs of circles. These vertices will be called *bounding pairs*, or, occasionally, the *BP-vertices*. Because the components of a bounding pair of circles are disjoint, and hence their isotopy classes are connected by an edge in $\mathcal{C}(S)$, every bounding pair of circles P gives rise to a bounding pair consisting of the isotopy classes of its two elements. Extending our notations for the isotopy classes of circles, we will denote this bounding pair by $\langle P \rangle$. So, if $P = \{C_0, C_1\}$, then $\langle P \rangle = \{\langle C_0 \rangle, \langle C_1 \rangle\}$.

Remark. Our terminology does not allow using the term *bounding pair* as a short form of the term *bounding pair of circles*. This is a minor hassle for the authors, but, hopefully, not for the readers.

Simplices. Given a set V consisting of separating circles and bounding pairs of circles in S , we will denote by $\langle V \rangle$ the set $\{\langle A \rangle \mid A \in V\}$. In other terms, $\langle V \rangle$ is the image of V under the map $\bullet \mapsto \langle \bullet \rangle$.

Suppose that we are given a set W_1 of separating circles in S and a set W_2 of bounding pairs of circles in S . Consider the set V all circles in sight, i.e. the union of W_1 with the set of elements of bounding pairs of circles from W_2 . If all circles in the set V are pair-wise disjoint, then the set $\langle W_1 \cup W_2 \rangle$ is declared to be a *simplex of $\mathfrak{Tb}(S)$* . Of course, sets W_1, W_2 are allowed to be empty. There are no other simplices in $\mathfrak{Tb}(S)$.

$\mathfrak{Tb}(S)$ *is a simplicial complex*. Clearly, a subset of a simplex is a simplex. Therefore, the set of vertices of both types together with the set of simplices forms a simplicial complex. This is our simplicial complex $\mathfrak{Tb}(S)$.

3.2. The additional structure. The additional structure on $\mathfrak{Tb}(S)$ consists of two parts: the marking of each vertex by its type, and a designation of some triangles as *marked triangles*.

Markings of vertices. The first part is a function from the set of vertices to the set $\{\mathcal{SC}, \mathcal{BP}\}$, where \mathcal{SC} and \mathcal{BP} are symbols corresponding to *separating circles* and *bounding pairs* respectively. This function assigns to a vertex the symbol corresponding to its type. Namely, the symbol \mathcal{SC} is assigned to isotopy classes of separating circles, and the symbol \mathcal{BP} to bounding pairs.

Alternatively, one can say simply that \mathcal{SC} -vertices are marked by \mathcal{SC} , and \mathcal{BP} -vertices are marked by \mathcal{BP} .

Marked triangles. Consider an unordered triple C_0, C_1, C_2 of disjoint non-separating circles in S . If circles C_0, C_1, C_2 are pair-wise non-isotopic and pairs $\{C_0, C_1\}, \{C_1, C_2\}, \{C_2, C_0\}$ are bounding pairs of circles in S , then the triple C_0, C_1, C_2 is called a *marked triples of circles*.

For a marked triple of circles C_0, C_1, C_2 the pairs $\{\langle C_0 \rangle, \langle C_1 \rangle\}, \{\langle C_1 \rangle, \langle C_2 \rangle\}, \{\langle C_2 \rangle, \langle C_0 \rangle\}$ are bounding pairs, and hence \mathcal{BP} -vertices of Torelli building $\mathfrak{Tb}(S)$. Since the circles in the set $V = \{C_0, C_1, C_2\}$ are pair-wise disjoint, these three \mathcal{BP} -vertices are vertices of a simplex Δ of $\mathfrak{Tb}(S)$. In addition, these \mathcal{BP} -vertices are all different, since circles C_0, C_1, C_2 are pair-wise non-isotopic. It follows that Δ is a simplex of dimension 2, i.e. a triangle.

Any triangle of $\mathfrak{Tb}(S)$ resulting from this construction is called a *marked triangle*, and there are no other marked triangle. This designation of some triangles in $\mathfrak{Tb}(S)$ as marked triangles is the second part of our additional structure of $\mathfrak{Tb}(S)$.

A description of marked triples of circles. Let C_0, C_1, C_2 be a marked triple of circles. Consider two different elements i, j of the set $\{0, 1, 2\}$. Let k be the third element of this set. Then $C_i \cup C_j$ divides S into two parts, one of which contains C_k , and the other does not. Let us denote by $S_{i,j}$ the part not containing C_k .

The union of subsurfaces $S_{0,1}$, $S_{1,2}$, and $S_{2,0}$ is a closed subsurface of S . Since S is a connected closed surface, this implies that S is equal to this union, i.e.

$$S = S_{0,1} \cup S_{1,2} \cup S_{2,0}.$$

The pair-wise intersections of our subsurfaces are $S_{0,1} \cap S_{1,2} = C_1$, $S_{1,2} \cap S_{2,0} = C_2$, and $S_{2,0} \cap S_{0,1} = C_0$. Each subsurface $S_{0,1}$, $S_{1,2}$, $S_{2,0}$ is a surface with exactly two boundary components. Since the boundary components of each of these subsurfaces are non-isotopic in S by the definition of marked triples of circles, none of these subsurfaces is an annulus. This implies that each of them is a surface of genus ≥ 1 .

This description implies, in particular, that $g = g(S) \geq 4$ if there is a marked triangle in $\mathfrak{Tb}(S)$. Since most of our arguments depend on the existence of marked triangles, we usually have to assume that $g(S) \geq 4$.

3.3. The Torelli buildings. The simplicial complex $\mathfrak{Tb}(S)$ together with its marking of vertices and the set of marked triangles is called the *Torelli building* of S . The Torelli building of S is intended to play in the theory of Torelli groups a role similar to the role of Tits buildings in the theory of arithmetic groups and the role of the complexes of curves in the theory of Teichmüller modular groups. Our main result about $\mathfrak{Tb}(S)$ is a complete description of automorphisms of $\mathfrak{Tb}(S)$; see Theorem 3.4 below.

Recall (see the subsection 2.7) that every diffeomorphism $F: S \rightarrow S$ induces a bijective self-map F_{cc} of set of vertices of $\mathcal{C}(S)$. Recall that every vertex of $\mathfrak{Tb}(S)$ is either a \mathcal{SC} -vertex, or a \mathcal{BP} -vertex, that \mathcal{SC} -vertices are at the same time (some) vertices of $\mathcal{C}(S)$, and that \mathcal{BP} -vertices are some pairs of vertices of $\mathcal{C}(S)$. Since $F_{cc}(\langle C \rangle) = \langle F(C) \rangle$ for every non-trivial circle C in S , the map F_{cc} takes \mathcal{SC} -vertices of $\mathfrak{Tb}(S)$ to \mathcal{SC} -vertices, and takes pairs of vertices of $\mathcal{C}(S)$ which are \mathcal{BP} -vertices of $\mathfrak{Tb}(S)$ into \mathcal{BP} -vertices. Therefore F_{cc} induces a self-map of the set of the vertices of $\mathfrak{Tb}(S)$. Obviously, this self-map is a bijection and, moreover, is an automorphism of the Torelli building $\mathfrak{Tb}(S)$. We will denote this automorphism by $F_{to}: \mathfrak{Tb}(S) \rightarrow \mathfrak{Tb}(S)$. It turns out that all automorphisms of $\mathfrak{Tb}(S)$ have the form F_{to} , at least if $g(S) \geq 5$.

3.4. Main Theorem. *If $g = g(S) \geq 5$, then every automorphism of the Torelli building $\mathfrak{Tb}(S)$ has the form $F_{to}: \mathfrak{Tb}(S) \rightarrow \mathfrak{Tb}(S)$, where $F: S \rightarrow S$ is a diffeomorphism.*

Theorem 3.4 is proved by showing that for $g \geq 5$ every automorphism of the Torelli building $\mathfrak{Tb}(S)$ canonically induces an automorphism of the complex of curves $\mathcal{C}(S)$, and invoking the theorem of the second author to the effect that all automorphisms of $\mathcal{C}(S)$ are induced by diffeomorphisms of S . At the same time Theorem 3.4 is a Torelli analogue of this theorem about automorphisms of $\mathcal{C}(S)$.

A key role in the proof of Theorem 3.4 is played by the connectedness properties of two other simplicial complexes. The first one is the simplicial complex $T_{sep}(R)$ defined

only for connected surfaces R with two boundary components. For such a surface R , the complex $T_{\text{sep}}(R)$ has as vertices the isotopy classes of circles separating two boundary components of R from each other, i.e. the isotopy classes of circles D such that two components of ∂R are contained in different components of $R \setminus D$. The simplices of $T_{\text{sep}}(R)$ are defined in the same way as the simplices of the complex of curves.

3.5. Theorem (Connectedness of $T_{\text{sep}}(R)$). *Let R be a connected surface with 2 boundary components. If $g(R) \geq 3$, then the complex $T_{\text{sep}}(R)$ is connected.*

The second complex defined for all connected surfaces R . Let R be such a surface, and let us define a *genus 1 circle* in R as a non-trivial separating circle in R bounding in R a torus with one hole in R . The *complex of genus 1 curves* is defined as the full subcomplex of $\mathcal{C}(R)$ spanning the set of the isotopy classes of genus 1 circles. It is denoted by $T_1(R)$.

3.6. Theorem (Connectedness of $T_1(R)$). *Let R be a compact connected surface. If $g(R) \geq 3$, then the complex $T_1(R)$ is connected.*

Both connectedness theorems do not hold without the assumption $g(R) \geq 3$; in fact, if $g(R) = 2$, then complexes $T_{\text{sep}}(R)$ and $T_1(R)$ have an infinite number of vertices and no edges, and if $g(R) \leq 1$, these complexes are empty, i.e. have no vertices either.

4. Proof of connectedness theorems

We will prove Theorem 3.6 first (see the subsection 4.2), and then deduce Theorem 3.5 from it (see the subsection 4.3). This deduction will be based on ideas similar to the ideas of the proof of Theorem 3.6.

4.1. Auxiliary complexes and their connectedness. Our proofs of Theorems 3.6, 3.5 are based on connectedness of some auxiliary complexes, discussed in this subsection. Let R be a compact connected surface, possibly with non-empty boundary, in contrast with S .

Complex $\mathcal{C}_0(R)$. The first one is the subcomplex $\mathcal{C}_0(R)$ of the complex of curves $\mathcal{C}(R)$ having as vertices the isotopy classes of non-separating circles and as simplices collections of isotopy classes of several disjoint circles which are jointly nonseparating, i. e. such that the complement of their union in R is connected.

4.1.1. Lemma. *The complex $\mathcal{C}_0(R)$ is connected if the genus of R is ≥ 3 .*

This result was proved independently by J. Harer [Har1, Har2] and the second author [I2]. J. Harer and the second author also proved and used various versions of this result;

see [Har1, Har2], [I3, I5, I9]. The proof proceeds by proving the connectedness of the complex of curves $C(S)$ first and then using some combinatorial arguments.

Complexes $\mathcal{H}(R; y_1, y_2)$ and $\mathcal{H}_0(R; y_1, y_2)$. Suppose that R is a connected surface with at least 2 boundary components and that y_1, y_2 are 2 points belonging to *different* components of the boundary ∂R . Both complexes $\mathcal{H}(R; y_1, y_2)$ and $\mathcal{H}_0(R; y_1, y_2)$ have as vertices the isotopy classes of embedded arcs connecting y_1 with y_2 ; the isotopies are assumed to be point-wise fixed on the boundary ∂R . The simplices of $\mathcal{H}(R; y_1, y_2)$ are collections of isotopy classes of arcs with disjoint interiors. The simplices of $\mathcal{H}_0(R; y_1, y_2)$ are collections of isotopy classes of arcs with disjoint interiors and *jointly non-separating* in R . These complexes were introduced by J. Harer [Har1] and was used by him and by the second author (see, for example, [I5, I9]) in the proofs of homological stability theorems for groups Mod_R .

The complex $\mathcal{H}(R; y_1, y_2)$ is introduced here in order to place the following lemma in its proper context and to discuss the somewhat subtle situation with its proof (see remarks after the lemma); it will not be used in this paper directly.

4.1.2. Lemma. *The complex $\mathcal{H}_0(R; y_1, y_2)$ is connected if the genus of R is ≥ 2 .*

This is a special case of Theorem 1.4 in [Har1]. The proof proceeds by proving first that the complexes $\mathcal{H}(R; y_1, y_2)$ are highly connected; this is the content of Theorem 1.6 in [Har1]. The connectivity properties of $\mathcal{H}(R; y_1, y_2)$ imply (weaker) connectivity properties of $\mathcal{H}_0(R; y_1, y_2)$ by a standard by now argument of Harer (see [Har1], proofs of Theorems 1.1, 1.3 and 1.4).

The proof of Theorem 1.6 in [Har1] turned out to be flawed, as was discovered by N. Wahl [Wa1] in connection with an investigation of homological stability for mapping class groups of non-orientable surfaces. Namely, Harer [Har1] does not treat the case of two points belonging to the same component of boundary. While we do not need this case of Harer's theorem, Harer's proof proceeds by an inductive argument in which both cases are needed to carry on the induction.

A correct proof of Theorem 1.6 from [Har1] in the full generality was provided by N. Wahl [Wa1]; see [Wa1], Theorem 2.3.

Complexes $\mathcal{H}(R, C, x)$ and $\mathcal{H}_0(R, C, x)$. These complexes are, in fact, versions of complexes $\mathcal{H}(R; y_1, y_2)$ and $\mathcal{H}_0(R; y_1, y_2)$. Suppose that R is a connected surface and C is a non-separating circle on R . Suppose that a point $x \in C$ is fixed. Both complexes have as vertices the isotopy classes of circles D on R such that $C \cup D = \{x\}$ and D intersects C (at the point x) transversely; the isotopies are assumed to be pointwise fixed on C . The simplices of $\mathcal{H}(R, C, x)$ are collections of isotopy classes of circles D_1, D_2, \dots, D_n such that the complements $D_1 \setminus x, D_2 \setminus x, \dots, D_n \setminus x$ are disjoint. The simplices of $\mathcal{H}_0(R, C, x)$ are collections of isotopy classes of arcs such that the complements $D_1 \setminus x, D_2 \setminus x, \dots, D_n \setminus x$ are disjoint, and the complement of $C \cup D_1 \cup D_2 \cup \dots \cup D_n$ in R is connected.

4.1.3. Lemma. *The complex $\mathcal{H}_0(R, C, x)$ is connected if the genus of R is ≥ 3 .*

Proof. Let us cut our surface R along the circle C . Let R_C be the resulting surface, and let x_1, x_2 be two points on the boundary ∂R_C resulting from the point x . Clearly, the points x_1, x_2 belong to two different components of ∂R_C which consists of two components. Moreover, the complex $H_0(R, C, x)$ is obviously isomorphic to $H_0(R_C; x_1, x_2)$ and the genus of R_C is at least 2 if the genus of R is at least 3. Hence the lemma follows from Lemma 4.1.2. ■

4.2. Proof of Theorem 3.6. Recall that R is a compact connected orientable surface. As is well known and obvious, for every genus 1 circle C there are two circles D, E transversely intersecting at a single point such that C is isotopic to the boundary of a regular neighborhood of the union $D \cup E$, or, what is the same, such that $D \cup E$ is contained in the torus with one hole bounded by C . The circle C is determined up to isotopy by circles D and E ; slightly abusing the notations, we will write $C = C(D, E)$. Note that for a given circle C such a pair D, E is far from unique even up to isotopy.

Let C, C' be two genus 1 circles which we would like to connect by a sequence of edges of the complex of genus 1 circles. Let us choose pairs of circles D, E and D', E' such that $C = C(D, E)$ and $C' = C(D', E')$. Notice that circles D, D' are non-separating. By Lemma 4.1.1 the complex $\mathcal{C}_0(R)$ is connected, and hence there is a sequence of non-separating circles

$$D_1 = D, D_2, \dots, D_n = D'$$

such that D_i is connected with D_{i+1} by an edge of $\mathcal{C}_0(R)$ for each i . In other words, for each i , the circles D_i, D_{i+1} are isotopic to two jointly non-separating disjoint circles. In fact, after replacing the circles D_i by isotopic circles, if necessary (for example, one can fix a hyperbolic metric with geodesic boundary on R and take geodesic circles isotopic to the circles D_i), we may assume that the circles D_i, D_{i+1} are themselves disjoint and jointly non-separating, i.e. the complement of $D_i \cup D_{i+1}$ is connected.

Since the complements $R \setminus (D_i \cup D_{i+1})$ are connected, we can choose for each $i = 1, \dots, n-1$ a circle E_i transversely intersecting D_i at a single point and disjoint from D_{i+1} . Let $E_n = E'$, and let us assume that $E_1 = E$. Note that the circles E_i and E_{i+1} may be not disjoint, even up to isotopy. We will deal with this difficulty by introducing some additional circles F_{i+1} .

Note that the union $D_i \cup E_i$ is disjoint from D_{i+1} , and hence a regular neighborhood N_i of $D_i \cup E_i$ is disjoint from D_{i+1} . Since the complement of $D_i \cup D_{i+1}$ is connected, the result R_{cut} of cutting R along the union $D_i \cup D_{i+1}$ is also connected. After the cutting, E_i turns into an arc A_i connecting two different components of the boundary ∂R_{cut} . Since R_{cut} is connected, the complement of this arc in R_{cut} is also connected. After the cutting, the regular neighborhood N_i turns into a regular neighborhood of the union of A_i with these two boundary components. It follows that the complement of N_i in R_{cut} is

connected. Moreover, the genus of $R_{\text{cut}} \setminus N_i$ is equal to the genus of R_{cut} . Since the genus of R is ≥ 3 , the genus of R_{cut} is ≥ 1 , and hence the genus of $R \setminus (N_i \cup D_{i+1})$ is also ≥ 1 .

The last statement implies that there is a circle F_{i+1} transversely intersecting D_{i+1} at a single point and disjoint from N_i . In particular, F_{i+1} is disjoint from E_i . We may assume that, in addition, F_{i+1} intersects D_{i+1} at the same point as E_{i+1} .

Clearly, $C = C(D, E) = C(D_1, E_1)$, $C' = C(D', E') = C(D_n, E_n)$, and for every $i = 1, \dots, n-1$ the circles $C(D_i, E_i)$ and $C(D_{i+1}, F_{i+1})$ are disjoint up to isotopy. If for every $i = 1, \dots, n-1$ we will be able to connect the isotopy classes $\langle C(D_{i+1}, F_{i+1}) \rangle$ and $\langle C(D_{i+1}, E_{i+1}) \rangle$ by a chain of edges in the complex of genus 1 circles, we will be able to solve our problem: the chain formed by union of such chains and edges connecting $\langle C(D_i, E_i) \rangle$ with $\langle C(D_{i+1}, F_{i+1}) \rangle$ will connect the isotopy classes $\langle C \rangle$ and $\langle C' \rangle$.

So, let us fix i , $1 \leq i \leq n-1$. Curves F_{i+1} and E_{i+1} define two vertices of the complex $H_0(R, D_{i+1}, \chi)$, where χ is the point of intersection of E_{i+1} and F_{i+1} with D_{i+1} . By Lemma 4.1.3 the complex $H_0(R, D_{i+1}, \chi)$ is connected. Hence we can find a sequence of circles

$$G_1 = F_{i+1}, G_2, \dots, G_m = E_{i+1}$$

such that the circles G_j, G_{j+1} are isotopic by an isotopy fixed on D_{i+1} to two disjoint circles G'_j, G'_{j+1} such that the complement of union $D_{i+1} \cup G'_j \cup G'_{j+1}$ in R does not separate R for $1 \leq j \leq m-1$. As usual, we may assume that curves G_j, G_{j+1} are themselves disjoint and the complement of union $D_{i+1} \cup G_j \cup G_{j+1}$ in R does not separate R .

Consider now the regular neighborhood of the union $D_{i+1} \cup G_j \cup G_{j+1}$. Since the regular neighborhood of $D_{i+1} \cup G_j$ is a torus with one hole, and the regular neighborhood of $D_{i+1} \cup G_j \cup G_{j+1}$ results from it by attaching a band to the boundary, the latter regular neighborhood is a torus with two holes. This torus with two holes has a connected complement in R because $D_{i+1} \cup G_j \cup G_{j+1}$ does not separate R . Since the genus of R is ≥ 3 , this complement contains a genus 1 circle H_j . Now we can connect $\langle C(D_{i+1}, G_j) \rangle$ and $\langle C(D_{i+1}, G_{j+1}) \rangle$ by a chain consisting of two edges: one connecting $\langle C(D_{i+1}, G_j) \rangle$ with $\langle H_j \rangle$ and the other connecting $\langle H_j \rangle$ with $\langle C(D_{i+1}, G_{j+1}) \rangle$. By taking all these two-edge chains together we will get a chain connecting $\langle C(D_{i+1}, F_{i+1}) \rangle$ with $\langle C(D_{i+1}, E_{i+1}) \rangle$. As was explained in the previous paragraph, this implies that we can connect $\langle C \rangle$ with $\langle C' \rangle$. This completes the proof of Theorem 3.6. ■

4.3. Proof of Theorem 3.5. Recall that in this theorem R is a compact connected surface with exactly 2 boundary components. As before, we will call a circle C on R a *genus 1 circle* if C bounds in R a torus with one hole. Let B_1 and B_2 be the components of ∂R . Let us call a circle C in R *special* if C bounds together with either B_1 or B_2 a torus with two holes. Since every circle separating B_1 from B_2 is disjoint from some special circle, it sufficient to prove the connectedness of full subcomplex $T_{\text{spec}}(R)$ of $T_{\text{sep}}(R)$ having the

set of the isotopy classes special circles as the set of vertices. We will call the complex $T_{\text{spec}}(\mathbb{R})$ the *complex of special circles* in \mathbb{R} .

For a genus 1 circle D on \mathbb{R} , we will denote by τ_D be the torus with one hole bounded by D in \mathbb{R} . Suppose that J is an embedded arc in \mathbb{R} connecting τ_D with boundary $\partial\mathbb{R}$ and intersecting τ_D and $\partial\mathbb{R}$ only at its endpoints. Let $B_J = B_1$ or B_2 be the component of $\partial\mathbb{R}$ connected with τ_D by J . Then the boundary of a regular neighborhood of the union $\tau_D \cup J \cup B_J$ is clearly a special circle. By a slight abuse of notation, we will denote this circle by $C(D, J)$. It is clear that every special circle has the form $C(D, J)$ for some D, J . The remaining part of the proof is parallel to the proof of Theorem 3.6, with the arcs J playing a role similar to the circles E_i in that proof.

Suppose that C, C' are two special circles which we would like to connect by a chain of edges in $T_{\text{spec}}(\mathbb{R})$. Since every special circle bounding together with B_2 a torus with two holes is disjoint from some special curve bounding a torus with two holes together with B_1 , we may assume that both curves C and C' bound a torus with two holes together with B_1 .

Let us begin by choosing some genus 1 curves D, D' and arcs J, J' such that $C = C(D, J)$ and $C' = C(D', J')$. By Theorem 3.6 the complex $T_1(\mathbb{R})$ of genus 1 circles is connected. Hence there is a sequence of genus 1 circles

$$(4.1) \quad D_1 = D, D_2, \dots, D_n = D'$$

such that D_i is connected with D_{i+1} by an edge of $T_1(\mathbb{R})$ for every $i = 1, 2, \dots, n-1$. In other words, for every $i = 1, 2, \dots, n-1$ the circles D_i, D_{i+1} are isotopic to two disjoint circles. As usual, we can assume that circles D_i, D_{i+1} are themselves disjoint. Let us choose an arc J_i connecting τ_{D_i} with B_1 for every $i = 1, 2, \dots, n$ (in particular, $B_{J_i} = B_1$ for every $i = 1, 2, \dots, n$). We may assume that $J_1 = J$ and $J_n = J'$.

Let us fix for a moment some $i = 1, 2, \dots, n-1$. Since the circles D_i and D_{i+1} are disjoint, either subsurfaces τ_{D_i} and $\tau_{D_{i+1}}$ are disjoint, or one of them is contained in the other (because each of them is a torus with one hole). In the second case their boundaries D_i, D_{i+1} are isotopic and the corresponding vertices of $T_1(\mathbb{R})$ are equal. In this case we can shorten our chain (4.1) by removing either D_i or D_{i+1} . Therefore, we can assume that subsurfaces τ_{D_i} and $\tau_{D_{i+1}}$ are disjoint for every $i = 1, 2, \dots, n-1$.

The rest of the proof would be easier if we would be able to assume that for every $i = 1, 2, \dots, n-1$ the arcs J_i and J_{i+1} are disjoint at least up to an isotopy (of course, only the isotopies fixed on B_1 and on $\tau_{D_i}, \tau_{D_{i+1}}$ respectively are allowed). But this is not possible in general. In order to deal with this difficulty, let us choose for every $i = 1, 2, \dots, n-1$ an arc A_{i+1} connecting $\tau_{D_{i+1}}$ with B_1 and disjoint from $\tau_{D_i} \cup J_i \cup B_1$ (such an arc exists because now we are assuming that τ_{D_i} and $\tau_{D_{i+1}}$ are disjoint).

CLAIM. *For every $i = 1, 2, \dots, n-1$ the vertices $\langle C(D_i, J_i) \rangle$ and $\langle C(D_{i+1}, A_{i+1}) \rangle$ can be connected in $T_{\text{spec}}(\mathbb{R})$ by a chain consisting of no more than 2 edges.*

Proof of the claim. As we noted above, every regular neighborhood of $\tau_{D_i} \cup J_i \cup B_1$ is a torus with one hole. We can choose a regular neighborhood N_i of $\tau_{D_i} \cup J_i \cup B_1$ such that it is disjoint from $\tau_{D_{i+1}}$ and that its intersection with the arc A_{i+1} is a segment contained in A_{i+1} and having one endpoint in B_1 and the other endpoint in $C(D_i, J_i)$. Then we can construct a regular neighborhood L_i of

$$\tau_{D_i} \cup J_i \cup B_1 \cup \tau_{D_{i+1}} \cup A_{i+1}$$

by adding to N_i a regular neighborhood $\nu_{D_{i+1}}$ of $\tau_{D_{i+1}}$ (which is a subsurface of R containing the subsurface $\tau_{D_{i+1}}$ and isotopic to it) and a band connecting N_i with $\nu_{D_{i+1}}$. Since both surfaces N_i and $\nu_{D_{i+1}}$ are surfaces of genus 1, the genus of L_i is 2.

Since the genus of R is assumed to be ≥ 3 , there is a special curve H_i bounding a torus with two holes together with the component B_2 of ∂R and disjoint from L_i . In particular, H_i is disjoint from the union $\tau_{D_i} \cup J_i \cup \tau_{D_{i+1}} \cup A_{i+1} \subset L_i$. It follows that $\langle C(D_i, J_i) \rangle$ can be connected with $\langle C(D_{i+1}, A_{i+1}) \rangle$ by a chain consisting of two edges: one connecting $\langle C(D_i, J_i) \rangle$ with $\langle H_i \rangle$ and the other connecting $\langle H_i \rangle$ with $\langle C(D_{i+1}, A_{i+1}) \rangle$. This proves the claim. \square

It remains to connect $\langle C(D_{i+1}, A_{i+1}) \rangle$ with $\langle C(D_{i+1}, J_{i+1}) \rangle$ by a chain in $T_{\text{spec}}(R)$ for all $1 \leq i \leq n-1$. By using, if necessary, an isotopy of R preserving $\tau_{D_{i+1}}$ set-wise (and not necessarily fixed on the boundary), we can assume that J_{i+1} and A_{i+1} have the same endpoints, which we denote by x_1, x_2 . Let $R_{i+1} = (R \setminus \tau_{D_{i+1}}) \cup D_{i+1}$. Consider the complex $H_0(R_{i+1}; x_1, x_2)$. Since the genus of R is assumed to be ≥ 3 , the genus of R_{i+1} is ≥ 2 , and hence the complex $H_0(R_{i+1}; x_1, x_2)$ is connected by Lemma 4.1.2. Therefore there is a sequence of arcs

$$K_1 = J_{i+1}, K_2, \dots, K_m = A_{i+1}$$

such that K_j is connected with K_{j+1} by an edge of $H_0(R_{i+1}; x_1, x_2)$ for every $j = 1, 2, \dots, m-1$. In other words, for every $j = 1, 2, \dots, m-1$ the arcs K_j, K_{j+1} are isotopic in R_{i+1} to two disjoint arcs by isotopies point-wise fixed on ∂R_{i+1} . As usual, we can assume that the interiors of K_j, K_{j+1} are disjoint for every $j = 1, 2, \dots, m-1$. Then, in addition, $K_j \cup K_{j+1}$ does not separate R_{i+1} because $\langle K_j \rangle$ is connected with $\langle K_{j+1} \rangle$ by an edge of $H_0(R_{i+1}; x_1, x_2)$.

Since the genus R_{i+1} is ≥ 2 and $K_j \cup K_{j+1}$ does not separate R_{i+1} , for every $j = 1, 2, \dots, m-1$ the surface $R_{i+1} \setminus (K_j \cup K_{j+1})$ has genus ≥ 1 . Therefore $R_{i+1} \setminus (K_j \cup K_{j+1})$ contains a circle G_j bounding together with B_2 a torus with two holes. Clearly, G_j is disjoint from the union $K_j \cup K_{j+1}$. Considered as a circle on R , this circle G_j is disjoint from the union $\tau_{D_{i+1}} \cup K_j \cup K_{j+1}$. It follows that $\langle C(D_{i+1}, K_j) \rangle$ can be connected with $\langle C(D_{i+1}, K_{j+1}) \rangle$ in $T_{\text{spec}}(R)$ by a chain consisting of two edges: the edge connecting $\langle C(D_{i+1}, K_j) \rangle$ with $\langle G_j \rangle$ and the edge connecting $\langle G_j \rangle$ with $\langle C(D_{i+1}, K_{j+1}) \rangle$. By joining all these two-edge chains together, we get a chain connecting $\langle C(D_{i+1}, A_{i+1}) \rangle$ with $\langle C(D_{i+1}, J_{i+1}) \rangle$. As we noticed above, this completes the proof of Theorem 3.5. \blacksquare

5. Encodings

This section is devoted to a key tool need for the proof Theorem 3.4, namely, to encodings of isotopy classes of non-separating circles in S in terms of the Torelli building $\mathfrak{Ab}(S)$.

5.1. Encodings of non-separating circles. Let γ be a vertex of $\mathfrak{Ab}(S)$ represented by a bounding pair of non-separating circles C_0, C_1 . We would like to find a description of the isotopy class $\langle C_0 \rangle$ of the circle C_0 alone in terms of the Torelli building $\mathfrak{Ab}(S)$. The idea is to use a separating circle D on S which is up to isotopy disjoint from C_1 and which cannot be made by an isotopy disjoint from C_0 . In terms of the complex of curves $C(S)$, the isotopy class $\gamma_0 = \langle C_0 \rangle$ is a vertex of $C(S)$, and our condition means that the isotopy class $\delta = \langle D \rangle$ (which is also a vertex of $C(S)$) is connected by an edge of $C(S)$ with $\langle C_1 \rangle$ and is not connected by an edge of $C(S)$ with $\langle C_0 \rangle$.

Obviously, the vertex $\langle C_0 \rangle$ of $C(S)$ can be recovered from the *ordered* pair (γ, δ) , where $\delta = \langle D \rangle$. The ordered pair (γ, δ) can be considered as an *encoding* of the vertex γ_0 . Clearly, there are many encodings of γ_0 , and we need to be able to tell when two pairs encode the same vertex. In addition, we need to be able to tell when a pair is an encoding of some non-separating circle.

5.2. Encoding by admissible pairs. For technical reasons we will use only some pairs (γ, δ) as encodings, namely, only the pairs which we will call *admissible* pairs. They are defined as follows.

Admissible pairs. Let γ be a vertex of $\mathfrak{Ab}(S)$ corresponding to a bounding pair of circles and let δ be a vertex of $\mathfrak{Ab}(S)$ corresponding to a separating circle. We call the pair (γ, δ) an *admissible pair* if there are two vertices γ', β of $\mathfrak{Ab}(S)$ corresponding to bounding pair of circles (i.e. marked by the symbol \mathcal{BP} in the Torelli building $\mathfrak{Ab}(S)$) such that the following two conditions hold.

- (i) $\{\gamma, \gamma', \beta\}$ is a marked triangle of $\mathfrak{Ab}(S)$.
- (ii) δ is connected by an edge of $\mathfrak{Ab}(S)$ with β .
- (iii) δ is *not* connected by an edge with either γ or γ' .

If the the conditions (i), (ii), and (iii) hold, we say that the (ordered) triple (γ, γ', β) *justifies the admissibility* of the pair (γ, δ) . Note that these conditions are stated entirely in terms of the Torrelli geometry structure of $\mathfrak{Ab}(S)$.

Since the definition of an admissible pair requires the existence of a marked triangle, admissible pairs exist only if $g(S) \geq 4$ (see the end of the subsection 3.2).

5.2.1. Lemma. Let C_0, C_1, D , and γ, δ be three circles on S and two vertices of $\mathfrak{Ab}(S)$ with the properties described in the section 5.1. Suppose that the pair (γ, δ) is admissible, and let (γ, γ', β) be any triple justifying the admissibility of (γ, δ) . Then the isotopy class $\gamma_0 = \langle C_0 \rangle$ of C_0 is the unique element of the set $\gamma \cap \gamma' = \gamma \setminus \beta = \gamma' \setminus \beta$, and (γ, δ) is an encoding of γ_0 .

Proof. Let (γ, γ', β) be some triple justifying the admissibility of (γ, δ) . Then $\{\gamma, \gamma', \beta\}$ is a marked triangle and is determined by three circles as in the definition of the marked triangles in section 3.1. Since γ is the vertex corresponding to the bounding pair of circles C_0, C_1 , we can assume that two of these three circles are C_0, C_1 . Let C_2 be the third circle.

Then β is the vertex of $\mathfrak{Ab}(S)$ corresponding to the bounding pair of circles C_i, C_j , where i, j is a pair of two distinct elements of the set $\{0, 1, 2\}$. Since δ is not connected by an edge of $\mathfrak{C}(S)$ with $\gamma_0 = \langle C_0 \rangle$, the circle D is not isotopic to a circle disjoint from C_0 . On the other hand δ is connected by an edge of $\mathfrak{Ab}(S)$ with β and hence D is isotopic to a circle disjoint from both C_i and C_j . By comparing the last two statements we conclude that neither i , nor j could be equal to 0. It follows that $\{i, j\} = \{1, 2\}$, $\{C_i, C_j\} = \{C_1, C_2\}$, and β is the vertex of $\mathfrak{Ab}(S)$ corresponding to the bounding pair of circles C_1, C_2 .

In other words, β is equal to the bounding pair $\{\langle C_1 \rangle, \langle C_2 \rangle\}$. Since γ corresponds to the bounding pair of circles C_0, C_1 , it is equal to the bounding pair $\{\langle C_0 \rangle, \langle C_1 \rangle\}$. Since $\{\gamma, \gamma', \beta\}$ is a marked triangle, γ' is equal to the bounding pair $\{\langle C_0 \rangle, \langle C_1 \rangle\}$. Taken together, these descriptions of γ, γ', β imply that $\gamma \cap \gamma' = \gamma \setminus \beta = \gamma' \setminus \beta$ and the isotopy class $\langle C_0 \rangle$ is the unique element of this set.

It remains to prove that (γ, δ) is an encoding of γ_0 . Indeed, since $\{\gamma, \gamma', \beta\}$ is a marked triangle and δ is not connected by an edge of $\mathfrak{Ab}(S)$ with β , the circle D is isotopic to a circle disjoint from $C_1 \cup C_2$. In particular, D is isotopic to a circle disjoint from C_1 . On the other hand, δ is not connected by an edge with γ . It follows that D is not isotopic to a circle disjoint from C_0 . Hence (γ, δ) is indeed an encoding of γ_0 . This completes the proof of the lemma. ■

5.2.2. Corollary. If (γ, δ) is an admissible pair, then (γ, δ) is an encoding of the isotopy class of a non-separating circle. ■

5.3. Moves of admissible pairs. We will prove that two admissible pairs encode the same isotopy class of non-separating circles if and only if these two pairs can be connected by a sequence of moves of two types. We start with the definition of these moves.

Moves of type I. If a triple (γ, γ', β) justifies the admissibility of the pair (γ, δ) , then there is a *move of type I* from (γ, δ) to (γ', δ) (and there are no other moves of type I). Alternatively, we say such pairs (γ, δ) and (γ', δ) are *connected by a move of type I*.

Moves of type II. If a triple (γ, γ', β) simultaneously justifies the admissibility of the pairs (γ, δ) and (γ, δ') , then there is a *move of type II* replacing (γ, δ) by (γ, δ') (and there

are no other moves of type II). Alternatively, we say such pairs (γ, δ) and (γ, δ') are connected by a move of type II.

5.3.1. Lemma. *The isotopy class of non-separating circles encoded by an admissible pair is not changed by moves of both types.*

Proof. We may assume that we are in the situation of Lemma 5.2.1. Then by this lemma γ_0 is the unique element of $\gamma \cap \gamma'$. This immediately implies our claim for moves of type I. In addition, this implies that if a triple justifying the admissibility of (γ, δ) is known, we can find γ_0 by using only this triple without a recourse to δ . This implies our claim for moves of type II. ■

5.4. Theorem (Changing δ). *Suppose that the genus of S is ≥ 5 . Let (γ, δ') and (γ, δ'') be two admissible pairs with the same first vertex and encoding the same isotopy class of non-separating curve. Then there is a sequence of moves of the type II starting at (γ, δ') and ending at (γ, δ'') .*

Proof. Let $(\gamma, \gamma', \beta')$ and $(\gamma, \gamma'', \beta'')$ be some triples justifying the admissibility of (γ, δ') and (γ, δ'') respectively. Suppose that C_0, C_1 is a bounding pair of circles such that γ is equal to the corresponding bounding pair. We can assume that (γ, δ') and (γ, δ'') encode the isotopy class $\gamma_0 = \langle C_0 \rangle$ of C_0 . Since the triangles $\{\gamma, \gamma', \beta'\}$ and $\{\gamma, \gamma'', \beta''\}$ are marked, they correspond to triples of circles C_0, C_1, C'_2 and C_0, C_1, C''_2 respectively for some non-separating circles C'_2 and C''_2 .

By Lemma 5.2.1, γ_0 is the unique element of each of the intersections $\gamma \cap \gamma'$ and $\gamma \cap \gamma''$. In view of the definition of marked triangles, this implies that γ', γ'' are the vertices of $\mathfrak{Ab}(S)$ corresponding to the bounding pairs of circles $\{C_0, C'_2\}, \{C_0, C''_2\}$ respectively. Similarly, β', β'' are the vertices of $\mathfrak{Ab}(S)$ corresponding to the bounding pairs of circles $\{C_1, C'_2\}, \{C_1, C''_2\}$ respectively.

Since C_0, C_1 is a bounding pair of circles, the set difference $S \setminus (C_0 \cup C_1)$ consists two components. We will denote the closures of these components by S' and S'' . Clearly, S' and S'' are subsurfaces of S with the common boundary $C_0 \cup C_1$. The rest of the proof splits into two cases.

Case 1. *The circles C'_2 and C''_2 are contained in different components of $S \setminus (C_0 \cup C_1)$. We may assume that C'_2 is contained S' and C''_2 is contained in S'' .*

In this case C_0 and C'_2 bound a subsurface of S' of genus at least 1 with boundary equal to $C_0 \cup C'_2$. Clearly, this subsurface is disjoint from C_1 . Its genus is ≥ 1 , because otherwise it is an annulus and C_0 and C_2 are isotopic, which is not allowed by the definition of marked triangles. Similarly, C_0 and C''_2 bound a subsurface of S'' disjoint from C_1 . Its genus is also ≥ 1 and its boundary is equal to $C_0 \cup C''_2$. These two subsurfaces are situated on different sides of C_0 because C'_2 and C''_2 are situated on different sides

of $C_0 \cup C_1$ (i.e. are contained in different components of $S \setminus (C_0 \cup C_1)$). The union Q of these two subsurfaces is a subsurface of S disjoint from C_1 . Its boundary is equal to $C'_2 \cup C''_2$ and its genus is ≥ 2 . Clearly, C_0 is a nontrivial separating circle in Q . Moreover, C_0 separates the components C'_2 and C''_2 of ∂Q . As is well known, this implies that there is a nontrivial separating circle D_1 in Q which is not isotopic in Q to a circle disjoint from C_0 . One can obtain such a circle D_1 by applying to C_0 a Dehn twist along any circle not isotopic to a circle disjoint from C_0 . The point of this argument is that it is easier to imagine or draw a non-separating circle with this property than a separating one.

CLAIM. *Let $\delta_1 = \langle D_1 \rangle$. Then both triangles $\{\gamma, \gamma', \beta'\}$ and $\{\gamma, \gamma'', \beta''\}$ justify the admissibility of the pair (γ, δ_1) .*

Proof of the claim. The condition (i) from the definition of admissibility in the section 5.2 holds because both triangles $\{\gamma, \gamma', \beta'\}$ and $\{\gamma, \gamma'', \beta''\}$ are marked.

Let us verify the condition (ii) from the definition of admissibility. Since D_1 is a circle in Q , it is disjoint from the boundary $\partial Q = C'_2 \cup C''_2$ and, in particular, is disjoint from C'_2 and C''_2 . In addition, since D_1 is contained in Q and Q is disjoint from C_1 , the circle D_1 is also disjoint from C_1 . Since β', β'' are the vertices of $\mathfrak{Tb}(S)$ corresponding to the bounding pairs $\{C_1, C'_2\}, \{C_1, C''_2\}$ respectively, these disjointness properties of D_1 imply that δ_1 is connected by edges of $\mathfrak{Tb}(S)$ with both β', β'' . This implies the condition (ii).

Finally, let us verify the condition (iii). By the choice of D_1 it is not isotopic in Q to a circle disjoint from C_0 . As is well known, this implies that D_1 is not isotopic to such a circle in S either. It follows that δ_1 is not connected by an edge of $\mathfrak{Tb}(S)$ with γ' or γ'' . This implies the condition (iii). This completes the proof of the claim. \square

The above claim immediately implies that there is a move of the type II replacing (γ, δ') by (γ, δ_1) and another move of the type II replacing (γ, δ_1) by (γ, δ'') . Therefore, (γ, δ') is connected to (γ, δ'') by two moves. This proves the lemma in Case 1. \blacksquare

Case 2. *The circles C'_2 and C''_2 are contained in the same component of $S \setminus (C_0 \cup C_1)$. We may assume that this component is S' . There are two subcases to consider depending on the genus of S'' (the part *not* containing C'_2 and C''_2).*

Subcase 2-A. *The genus of S'' is ≥ 2 .* Since the genus of S'' is assumed to be ≥ 2 , there is a non-trivial circle E in S'' separating two boundary components C_0 and C_1 of S'' . Such a circle is not isotopic to C_1 in S'' (because it is non-trivial in S'') and hence is not isotopic to C_1 in S . In addition, it is not isotopic to C'_2 in S because C'_2 is contained in S' . Finally, while the circle E is separating in S'' , it is non-separating in S , because two sides of E can be connected (by paths) in S'' to C_0 and C_1 , and C_0 and C_1 can be connected in S' .

Let U_0, U_1 be the components of $S'' \setminus E$ containing, respectively, C_0 and C_1 . Let $S_0 = U_0 \cup C_0$ and $S_1 = U_1 \cup C_1$. Clearly, S_0 and S_1 are subsurfaces of S having the unions $C_0 \cup E$ and

$C_1 \cup E$ as, respectively, their boundaries. In addition, $C_0 \cup E$ separates S into subsurfaces S_1 and $S' \cup S_0$, and $C_1 \cup E$ separates S into subsurfaces S_0 and $S' \cup S_1$.

By the assumption, the union $C_0 \cup C_1$ is separating S (into S' and S''). Therefore, all three pairs $C_0 \cup C_1$, $C_0 \cup E$, and $C_1 \cup E$, are bounding pairs of circles, and the corresponding vertices γ , γ' and β of $\mathfrak{Ab}(S)$ form a marked triangle.

Let $Q = S' \cup S_0$. By exactly the same reasons as in Case 1, there is a non-trivial separating circle D_1 in Q which is not isotopic in Q to a circle disjoint from C_0 . Such a circle D_1 is not isotopic to a circle disjoint from C_0 also in S . Since D_1 is a circle in Q , it is disjoint from C_1 and E . Let $\delta_1 = \langle D_1 \rangle$. Obviously, the pair (γ, δ_1) encodes the same isotopy class $\gamma_0 = \langle C_0 \rangle$ as (γ, δ) and (γ, δ') . Since the circle D_1 is disjoint from C_1 and E , the vertex δ_1 is connected with β (recall that β is the bounding pair corresponding to the C_1, E) by an edge in $\mathfrak{Ab}(S)$. It is clear now that the triple (γ, γ', β) justifies the admissibility of (γ, δ_1)

Since E is contained in S'' and both circles C'_2, C''_2 are contained in S' , the already proved Case 1 of the lemma implies that of the union $C_0 \cup C_1$, we can use the result of the previous paragraph and conclude that one can connect (γ, δ) with (γ, δ_1) by two moves and also connect (γ, δ_1) with (γ, δ') by two moves. It follows that one can connect (γ, δ) with (γ, δ') by four moves. All these moves are moves of the second type. This proves the lemma in Subcase 2-A. ■

Subcase 2-B. The genus of S'' is 1. Since S'' has exactly two boundary components, in this case S'' is a torus with two holes. Since the surface S is the union of its subsurfaces S' and S'' intersecting only along $\partial S' = \partial S'' = C_0 \cup C_1$, we may consider S as the result of gluing of S'' to S' along boundary circles C_0, C_1 . As is well known, when we start with a connected surface with boundary and glue to it another connected surface along 2 boundary components, the genus increases by $1 + h$, where h is the genus of the glued surface. In our case this means that the genus of S is equal to the genus of S' plus 2. Since the genus of S is assumed to be ≥ 5 , this implies that the genus of S' is ≥ 3 .

Since $\{C_0, C'_2\}$ is a bounding pair of circles, $C_0 \cup C'_2$ divides S into two subsurfaces. Let us denote them by P, Q . Since C_1 is disjoint from both circles C_0, C'_2 , one of the subsurfaces P, Q contains C_1 and hence contains a neighborhood of C_1 in S . In particular, it contains some points of the interior of S'' . We may assume that this is the subsurface Q . Then Q contains the whole subsurface S'' (because ∂Q is disjoint from the interior of S''). This implies that P is contained in S' . In turn, this implies that $\{C_0, C'_2\}$ is a bounding pair of circles in S' and hence C'_2 separates the boundary components C_0, C_1 of S' in S' . By the same argument, C''_2 also separates the boundary components C_0, C_1 of S' in S' .

Since the genus of S' is ≥ 3 , we can apply Theorem 3.5 to S' in the role of R . By this theorem, there is a sequence

$$E_1 = C'_2, E_2, \dots, E_n = C''_2$$

of non-trivial circles in S' separating the boundary components of S' and such that E_i and E_{i+1} are isotopic in S' to disjoint circles for all $i = 1, 2, \dots, n-1$. As usual, we may assume that the circles E_i and E_{i+1} are actually disjoint for all $i = 1, 2, \dots, n-1$.

For each $i = 1, 2, \dots, n$ let us denote by R_i the subsurface of S' bounded by C_0 and E_i . Since E_i is nontrivial in S' , the genus of R_i is ≥ 1 . This implies that the genus of $S'' \cup R_i$ is ≥ 2 . Obviously, C_0 is a separating circle in $S'' \cup R_i$ (it separates $S'' \cup R_i$ into two parts S'' and R_i). Since the genus of $S'' \cup R_i$ is ≥ 2 , this implies that there is a non-separating circle D_i in $S'' \cup R_i$ which is not isotopic to a circle disjoint from C_0 . Let $\delta_i = \langle D_i \rangle$.

For each $i = 1, 2, \dots, n$ the pairs $\{C_0, C_1\}$, $\{C_0, E_i\}$, and $\{C_1, E_i\}$ are bounding pairs of circles (recall that E_i separates C_0 from C_1 in S'). Recall that γ is the vertex of $\mathfrak{Ab}(S)$ corresponding to the bounding pair of circles $\{C_0, C_1\}$. Let γ_i, β_i be the vertices of $\mathfrak{Ab}(S)$ corresponding to the bounding pair of circles $\{C_0, E_i\}$, $\{C_1, E_i\}$ respectively. Obviously, $\{\gamma, \gamma_i, \beta_i\}$ is a marked triangle in $\mathfrak{Ab}(S)$.

CLAIM. For each $i = 1, 2, \dots, n$ the pair (γ, δ_i) is admissible. In fact, the triple $(\gamma, \gamma_i, \beta_i)$ justifies the admissibility of (γ, δ_i) .

Proof of the claim. Since $\{\gamma, \gamma_i, \beta_i\}$ is a marked triangle for each $i = 1, 2, \dots, n$, the condition (i) from the section 5.2 holds.

For each $i = 1, 2, \dots, n$, the circle D_i is a circle in $S'' \cup R_i$ and hence is disjoint from $\partial(S'' \cup R_i) = E_i \cup C_1$. Therefore, δ_i is connected with β_i by an edge of $\mathfrak{Ab}(S)$ and the condition (ii) from the section 5.2 holds.

By the choice of the circle D_i , it is not isotopic to a circle disjoint from C_0 in $S'' \cup R_i$, and hence also in S . This implies that δ_i is not connected by an edge of $\mathfrak{Ab}(S)$ with either γ or γ_i . Therefore, the condition (iii) holds. This completes the proof of the claim. \square

A SEQUENCE OF MOVES. Finally, we can construct a sequence of moves of type II starting at (γ, δ') and ending at (γ, δ'') .

Since $E_1 = C'_2$ and $E_n = C''_2$, we have $\gamma_1 = \gamma'$, $\gamma_n = \gamma''$, $\beta_1 = \beta'$, and $\beta_n = \beta''$. Hence $(\gamma, \gamma_1, \beta_1) = (\gamma, \gamma', \beta')$ and $(\gamma, \gamma_n, \beta_n) = (\gamma, \gamma'', \beta'')$. It follows that $(\gamma, \gamma_1, \beta_1)$ justifies the admissibility of (γ, δ') in addition to (γ, δ_1) . Hence (γ, δ') is connected with (γ, δ_1) by a move of type II. Similarly, $(\gamma, \gamma_n, \beta_n)$ justifies the admissibility of (γ, δ'') in addition to (γ, δ_n) and hence (γ, δ_n) is connected with (γ, δ'') by a move of type II.

For each $i = 1, 2, \dots, n-1$ there are two possible cases: the circle E_{i+1} may be contained in R_i or not. In the first case D_{i+1} is a circle not only in $S'' \cup R_{i+1}$, but also in $S'' \cup R_i$. In addition, D_{i+1} is not isotopic to a circle disjoint from C_0 . Therefore, one can use D_{i+1} in the role of D_i , and the above claim implies that the triple $(\gamma, \gamma_i, \beta_i)$ justifies the admissibility of the pair (γ, δ_{i+1}) in addition to the pair (γ, δ_i) . In particular, in this case the pairs (γ, δ_i) and (γ, δ_{i+1}) are connected by a move of type II. Similarly, in the second

case one can use D_i in the role of D_{i+1} the triple $(\gamma, \gamma_{i+1}, \beta_{i+1})$ justifies the admissibility of the pair (γ, δ_i) in addition to the pair (γ, δ_{i+1}) . In particular, in this case the pairs $(\gamma, \delta_{i=1})$ and (γ, δ_i) are connected by a move of type II.

By combing the results of the last two paragraphs, we conclude that there is a sequence of moves of type II starting at (γ, δ') and ending at (γ, δ'') . This completes the proof in the case 2-B, and hence of the lemma ■

Remark. The assumption that the genus of S is ≥ 5 is used at only one place in the above proof. Namely, it is used only in the first paragraph of the proof in the Subcase 2-B in order to ensure that the genus of S' is ≥ 3 . It is fairly reasonable to expect that the proof can be extended at least to the case of surfaces of genus 4 at the cost of introducing some additional types of moves.

5.5. Theorem (Connecting encoding pairs by moves). *Suppose that the genus of S is ≥ 5 . Suppose that a non-separating circle can be encoded by two (different) admissible pairs (γ, δ) and (γ', δ') . Then there is a sequence of moves of types I and II, starting at (γ, δ) and ending at (γ', δ') .*

Proof. Let C_0 be a non-separating circle such that $\gamma_0 = \langle C_0 \rangle$ is the isotopy class encoded by the both pairs. Then for some non-separating circles C_1, C'_1 the vertices γ, γ' of $\mathfrak{Ab}(S)$ correspond to bounding pairs of circles $\{C_0, C_1\}$ and $\{C_0, C'_1\}$ respectively.

Let R be the result of cutting S along C_0 . Clearly, R is surface with two boundary components, and its genus is equal to the genus of S minus 1. In particular, the genus of R is ≥ 4 . By Theorem 3.5 there is a sequence

$$E_1 = C_1, E_2, \dots, E_n = C'_1$$

of nontrivial circles in R separating its boundary components and such that E_i and E_{i+1} are isotopic in R to disjoint circles for all $i = 1, 2, \dots, n-1$. As usual, we may assume that the circles E_i and E_{i+1} are actually disjoint for all $i = 1, 2, \dots, n-1$. Let γ_i be the vertex of $\mathfrak{Ab}(S)$ corresponding to the bounding pair of circles C_0, E_i . In particular, $\gamma_1 = \gamma$ and $\gamma_n = \gamma'$.

CLAIM. *There are non-trivial circles D_1, D_2, \dots, D_n , such that for each $i = 1, 2, \dots, n-1$:*

- (i) *the pair (γ_i, δ_i) , where $\delta_i = \langle D_i \rangle$, encodes γ_0 ;*
- (ii) *(γ_i, δ_i) is connected with (γ_{i+1}, δ_i) by a move of type I;*
- (iii) *(γ_{i+1}, δ_i) is connected with $(\gamma_{i+1}, \delta_{i+1})$ by a sequence of moves of type II;*
- (iv) *(γ, δ) is connected with $(\gamma, \delta_1) = (\gamma_1, \delta_1)$ by a sequence of moves of type II;*
- (v) *$(\gamma_n, \delta_n) = (\gamma', \delta_n)$ is connected with (γ', δ') by a sequence of moves of type II.*

Proof of the claim. Let $i = 1, 2, \dots, \text{ or } n - 1$.

Since both E_i and E_{i+1} separate the boundary components of R , the union $E_i \cup E_{i+1}$ divides R into 3 parts. One of these parts is a subsurface of R with boundary $E_i \cup E_{i+1}$. We will denote this subsurface by P_i . Two other parts are the components of the closure of $R \setminus P_i$ in R . Clearly, these components are subsurfaces of R having two boundary components each. One of these subsurfaces has E_i as a boundary component, the other has E_{i+1} . We will denote these subsurfaces by V_i and W_i (it does not matter which one is denoted by V_i and which one by W_i). Since E_i and E_{i+1} are non-trivial circles in R , neither of subsurfaces V_i, W_i is an annulus. Since each of surfaces V_i, W_i has two boundary components, this implies that the genus of surfaces V_i, W_i is ≥ 1 .

Clearly, $P_i, V_i,$ and W_i are also subsurfaces of S . While V_i and W_i are disjoint as subsurfaces of R , their intersection as subsurfaces of S is equal to C_0 . Let Q_i be the union of V_i and W_i considered as subsurfaces of S . It is a subsurface of S with the boundary ∂Q_i equal to the boundary $\partial P_i = E_i \cup E_{i+1}$. Since the genus of surfaces V_i, W_i is ≥ 1 , the genus of Q_i is ≥ 2 . Obviously, $C_0 \subset Q_i$ and hence C_0 is a circle in Q_i . Clearly, the set difference $Q_i \setminus C_0$ consists of two components. Namely, the components of $Q_i \setminus C_0$ are $V_i \setminus C_0$ and $W_i \setminus C_0$. Since, as we saw, neither $V_i,$ nor W_i is an annulus, the circle C_0 is a non-trivial separating circle in Q_i (the circle C_0 does not bound a disc in S and, hence, in Q_i). Since the genus of Q_i is ≥ 2 , this implies that Q_i contains a circle not isotopic to a circle disjoint from C_0 . Let us choose such a circle as the promised circle D_i . As in the statement of the claim, we will denote by δ_i the isotopy class $\langle D_i \rangle$.

Since the circle D_i is contained in Q_i , it is disjoint from C_1 . In addition, D_i is not isotopic to a circle disjoint from C_0 . This implies that (γ_i, δ_i) encodes γ_0 , and hence implies (i).

Let β_i be the vertex of $\mathfrak{Ab}(S)$ corresponding to the bounding pair of circles E_i, E_{i+1} . Clearly, $\{\gamma_i, \gamma_{i+1}, \beta_i\}$ is a marked triangle. Moreover, the triple $(\gamma_i, \gamma_{i+1}, \beta_i)$ justifies the admissibility of both pairs (γ_i, δ_i) and (γ_{i+1}, δ_i) . Therefore, (γ_i, δ_i) is connected with (γ_{i+1}, δ_i) by a move of type I. This proves (ii).

Finally, Lemma 5.4 implies (iii), (iv), and (v) because in these statements only the second element of the encoding pair is changed. This completes the proof of the claim. \square

By combining the moves of type I from the statement (ii) of the claim with the sequences of moves of type II from the statements (iii), (iv), and (v) of the claim, we get a sequence of moves connecting (γ, δ) with (γ', δ') . This completes the proof of the lemma. \blacksquare

5.6. Corollary. *Suppose that the genus of S is ≥ 5 . Two admissible pairs encode the same isotopy class of non-separating curve if and only if they can be connected by a sequence of moves of types I and II.*

Proof. The corollary immediately follows from Lemmas 5.3.1 and 5.5. \blacksquare

6. Encodings, edges of $\mathcal{C}(S)$, and automorphisms

From now on, for the isotopy classes of non-separating circles we will consider only encodings by admissible pairs, and will call them simply *encodings*.

Let D be a separating circle in S , and let $\delta = \langle D \rangle$. The isotopy class δ is a vertex of $\mathcal{C}(S)$ and at the same time is a vertex of $\mathfrak{Tb}(S)$. It is convenient to define an *encoding of δ* considered as a vertex of $\mathcal{C}(S)$ as the same isotopy class δ , but considered as a vertex of $\mathfrak{Tb}(S)$. In contrast with non-separating vertices, separating circles have only one encoding. With this definition, we have the notion of an encoding for the isotopy classes of all non-trivial circles in S , or, what is the same, for all vertices of $\mathcal{C}(S)$.

The goal of this section is to find properties of encodings of vertices of $\mathcal{C}(S)$ which are equivalent to the property of being connected in $\mathcal{C}(S)$, and which can be stated entirely in terms of the Torelli building $\mathfrak{Tb}(S)$. As an application, we will prove Theorem 3.4 at the end of this section. See the subsection 6.4.

6.1. Recognizing circles of genus 1. Let δ be a vertex of $\mathcal{C}(S)$, i.e. $\delta = \langle D \rangle$ for a non-trivial circle D in S . For the problem of deciding if δ is connected with another vertex γ of $\mathcal{C}(S)$ by an edge of $\mathcal{C}(S)$ in terms of encodings of δ, γ , the case of genus 1 circles D presents substantial additional difficulties (in comparison with other circles in S). The main reason for this is the fact that a torus with one hole contains neither non-trivial separating circles, nor bounding pairs of circles.

We will need to deal with such vertices $\delta = \langle D \rangle$ separately. In order to do this, we, first of all, need to be able to recognize such vertices in terms of properties their encodings in the Torelli building $\mathfrak{Tb}(S)$. In more details, we need to find properties of encodings of a vertex $\delta = \langle D \rangle$ of $\mathcal{C}(S)$ which are equivalent to D being a genus 1 circle, and which are stated entirely in terms of the Torelli building $\mathfrak{Tb}(S)$. This is the main goal of this subsection, achieved in Lemma 6.1.4 below.

The first step in achieving this goal is trivial: encodings of separating and non-separating vertices are of different nature. Therefore, D can be assumed to be a separating circle from the very beginning. Still, we will use this assumption only when it is needed.

The graph $\mathcal{L}_{\text{sep}}(\delta)$. Our main tool will be the graph $\mathcal{L}_{\text{sep}}(\delta)$ which we will define now. Let $L_{\text{sep}}(\delta)$ be the set of separating vertices of $\mathfrak{Tb}(S)$ different from δ and connected by an edge of $\mathfrak{Tb}(S)$ with δ . Let us turn $L_{\text{sep}}(\delta)$ into a graph $\mathcal{L}(\delta)$ by connecting two vertices

in $L_{\text{sep}}(\delta)$ by an edge if and only if they are *not* connected by an edge in $\mathfrak{Tb}(S)$. The graph $\mathcal{L}_{\text{sep}}(\delta)$ is called the *separating link* of δ (in $\mathfrak{Tb}(S)$).

In order to deal with graphs $\mathcal{L}_{\text{sep}}(\delta)$, we need auxiliary graphs $G_{\text{sep}}(R)$ defined for a compact connected surface R with exactly one boundary component. Let $V_{\text{sep}}(R)$ be the set of isotopy classes of separating circles on R . Clearly, $V_{\text{sep}}(R)$ is a subset of the set of vertices of $\mathcal{C}(R)$. Let us define $G_{\text{sep}}(R)$ as the graph having $V_{\text{sep}}(R)$ as its set of vertices and having two different vertices connected by an edge if and only if these vertices are *not connected by an edge* of $\mathcal{C}(R)$.

6.1.1. Observation. *Let Q be a compact connected surface with 1 boundary component, and let D be a separating circle in Q . Then one of the subsurfaces into which D divides Q has $D \cup \partial Q$ as its boundary, and the other is disjoint from ∂Q and its boundary is D .*

Proof. Since ∂Q is connected, one of these subsurfaces contains ∂Q . Clearly, the boundary of this subsurface is $D \cup \partial Q$. It follows that the other subsurface is disjoint from ∂Q and has D as its boundary. ■

6.1.2. Lemma. *Suppose that the surface R is a subsurface of another surface Q with ≤ 1 boundary component. Let D be a circle contained in $R \setminus \partial R$. Then the circle D is a separating circle in R if and only if it is a separating circle in Q .*

Proof. Suppose that D is a separating circle in R . By Observation 6.1.1, D divides R into two subsurfaces, and one of these subsurfaces has $D \cup \partial R$ as its boundary, while the other is disjoint from ∂R and has D as its boundary. The second subsurface is also a subsurface of Q . It follows that D is a separating circle in Q . This proves the “only if” direction of the lemma.

By the same Observation 6.1.1, if D is a separating circle in Q , then D divides Q into two subsurfaces such that one of them has $D \cup \partial Q$ as its boundary, while the other one has D as its boundary. Clearly, if P is a subsurface of Q disjoint from ∂Q with $\partial P = D \subset R$, then P is contained in R . It follows that if D is a separating circle in Q , then D is the boundary of a subsurface of R and hence D is a separating circle in R . This proves the “if” direction of the lemma, and hence completes the proof. ■

6.1.3. Lemma. *Let R be a compact connected surface with 1 boundary component. If $g(R) \geq 2$, then $V_{\text{sep}}(R) \neq \emptyset$ and the graph $G_{\text{sep}}(R)$ is connected.*

Proof. It is much more easy to connect two vertices in $G_{\text{sep}}(R)$ than to connect them in $\mathfrak{Tb}(R)$. The experts may skip the rest of the proof.

The non-emptiness statement is trivial. Note that the assumption $g(R) \geq 2$ is exactly what is needed for it to be true (for surfaces with one boundary component).

In order to prove the connectedness statement of the lemma, it is sufficient to prove that the isotopy classes $\delta = \langle D \rangle$, $\delta' = \langle D' \rangle$ of two arbitrary non-trivial separating circles D , D' in R are connected by a sequence of edges of $G_{\text{sep}}(R)$.

Case 1. *The circles D , D' are not isotopic to disjoint circles.* Then their isotopy classes $\delta = \langle D \rangle$, $\delta' = \langle D' \rangle$ are not connected by an edge of $\mathcal{C}(R)$, and hence are connected by an edge of $G_{\text{sep}}(R)$. This completes the proof in Case 1. ■

Case 2. *The circles D , D' are isotopic to disjoint circles.* In this case we may assume that the circles D , D' are themselves disjoint. Since D is a separating circle in R , it divides R into two subsurfaces such that one of them has $D \cup \partial R$ as its boundary, while the other has D as its boundary and is disjoint from ∂R . Let Q_1 be the first subsurface, and let Q be the second, so, in particular, $\partial Q = D$. Since D' is disjoint from $\partial Q = D$, either $D' \subset Q$ or D' is disjoint from Q .

Subcase 2-A. $D' \subset Q$. Since D' is a separating circle in R , Lemma 6.1.2 implies that D' is also a separating circle in Q . This implies that D' divides Q into two subsurfaces having D as the common component of their boundaries. One of them has ∂Q as a boundary component, the other is disjoint from ∂Q . Let Q_3 be the first subsurface, and let Q_2 be the second. In particular, $\partial Q_3 = D'$, and $\partial Q_2 = D' \cup \partial Q = D' \cup D$.

It follows that R is the union of 3 subsurfaces Q_3 , Q_2 , Q_1 with boundaries $\partial Q_3 = D'$, $\partial Q_2 = D' \cup D$, and $\partial Q_1 = D \cup \partial R$. Moreover, $Q_3 \cap Q_2 = D'$, $Q_2 \cup Q_1 = D$, and $Q_3 \cap Q_1 = \emptyset$. It follows that the configuration of surfaces R , Q_3 , Q_2 , Q_1 and circles D , D' up to diffeomorphisms depends only on the genera $g(Q_3)$, $g(Q_2)$, and $g(Q_1)$. Therefore, we may assume that we are dealing with a standard model configuration. Then it is very easy to exhibit a separating circle A in R such that A is not isotopic to a circle disjoint from either D or D' . We leave this task to the reader. If A is such a circle, then $\alpha = \langle A \rangle$ is not connected by edges of $\mathcal{C}(R)$ with $\delta = \langle D \rangle$ and $\delta' = \langle D' \rangle$. By the definition, this means that α is connected with both δ and δ' by edges of $G_{\text{sep}}(R)$. Hence, δ is connected with δ' by a sequence consisting of these two edges. This completes the proof in Subcase 2-A. ■

Remark. Since circles D , D' are non-trivial, in Subcase 2-A either D is isotopic to D' , or each of the subsurfaces Q_3 , Q_2 , Q_1 has genus ≥ 1 . Therefore, in Subcase 2-A either $\delta = \delta'$ (and then there is nothing to prove), or $g(R) \geq 3$.

Subcase 2-B. D' is disjoint from Q . Since D' is a separating circle in R , D' divides R into two subsurfaces such that one of them has $D' \cup \partial R$ as its boundary, while the other has D' as its boundary and is disjoint from ∂R . Let Q'_1 be the first subsurface, and let Q' be the second, so, in particular, $\partial Q' = D'$. Since D is disjoint from $\partial Q' = D'$, either $D \subset Q'$ or D is disjoint from Q' . If $D \subset Q'$, then the arguments of the Subcase 2-A apply, with the roles of D and D' interchanged. Hence we may assume that D is disjoint from Q' .

CLAIM. *If D' is disjoint from Q , and D is disjoint from Q' , then Q is disjoint from Q' .*

Proof of the claim. Consider the union $P = Q \cup Q'$. Since the boundaries $D = \partial Q$ and $D' = \partial Q'$ are disjoint, P is subsurface of R with the boundary $D \cup D'$. If P is connected, then either $P \subset Q$, or $P \subset Q_1$. Since $D' \subset \partial P \subset P$, the inclusion $P \subset Q$ contradicts to the assumption that D' is disjoint from Q . If $P \subset Q_1$, then $Q \subset P \subset Q_1$, contradicting to the fact that $Q \cap Q_1 = D$. It follows that P is not connected. Since Q and Q' are connected, this is possible only if Q and Q' are disjoint. This proves our claim. \square

If Q is disjoint from Q' , the the closure of $R \setminus (Q \cup Q')$ is a subsurface of R with the boundary $D \cup D' \cup \partial R$. Let us denote this subsurface by Q_0 . Then R is the union of 3 subsurfaces Q , Q' , and Q_0 with boundaries $\partial Q = D$, $\partial Q' = D'$, and $\partial Q_0 = D \cup D' \cup \partial R$. Moreover, $Q \cap Q' = \emptyset$, $Q \cup Q_0 = D$, and $Q' \cap Q_0 = D'$. It follows that the configuration of surfaces R , Q , Q' , Q_0 and circles D , D' up to diffeomorphisms depends only on the genera $g(Q)$, $g(Q')$, and $g(Q_0)$. Therefore, we may assume that we are dealing with a standard model configuration. Then it is very easy to exhibit a separating circle A in R such that A is not isotopic to a circle disjoint from either D or D' . We leave this task to the reader. Given such a circle A , we can finish the proof in Subcase 2-B in the same manner as we did in the Subcase 2-A. This completes the proof in Subcase 2-B and, hence, the proof of the lemma. \blacksquare

6.1.4. Lemma. *Suppose that $g = g(S) \geq 3$. Let D be a separating circle in S , and let $\delta = \langle D \rangle$. Then the graph $\mathcal{L}_{\text{sep}}(\delta)$ is connected if and only if D bounds in S a torus with one hole.*

Proof. Let S_1 and S_2 be the two parts into which D separates S . Both S_1 and S_2 are subsurfaces of S with boundary D . Every vertex of $\mathcal{L}_{\text{sep}}(\delta)$ can be represented by a separating circle A disjoint from D . Obviously, every such a circle A is contained either in S_0 or in S_1 . Since $\langle A \rangle \neq \delta$, the circle A does not bound an annulus in S together with D and hence if $A \subset S_i$, then A is a non-trivial circle in S_i (where $i = 0$ or 1).

It follows that the set of vertices of $\mathcal{L}_{\text{sep}}(\delta)$ is equal to the union $L_1 \cup L_2$ of sets L_1, L_2 , where for each $i = 1, 2$ the set L_i consists of the vertices which can be represented by separating non-trivial circles in $S_i \setminus D$. Clearly, such a representative of any vertex in L_1 is disjoint from such a representative of any vertex in L_2 . Therefore, every element of L_1 is connected by an edge of $\mathcal{C}(S)$ with every element of L_2 . By the definition of $\mathcal{L}_{\text{sep}}(\delta)$, this means that vertices from L_1 are not connected with vertices of L_2 by edges of $\mathcal{L}_{\text{sep}}(\delta)$. Therefore, if both sets L_1, L_2 are non-empty, then the graph $\mathcal{L}_{\text{sep}}(\delta)$ has at least 2 connected components.

Together with Lemma 6.1.3 this implies that if $G(S)$ is connected, then either $g(S_1) \leq 1$ or $g(S_2) \leq 1$. We may assume that $g(S_1) \leq 1$. Since S_1 is a connected surface with exactly 1 boundary component and is not a disc (because $\partial S_1 = D$ is a non-trivial circle in S), the classification of surfaces implies that in this case S_1 is a torus with one hole. This proves the “only if” direction of the lemma.

In order to prove the “if” direction of the lemma, we may assume that S_1 is a torus with one hole. Then $L_1 = \emptyset$, the set of vertices of the graph $\mathcal{L}_{\text{sep}}(\delta)$ is equal to $L_2 = V_{\text{sep}}(S_2)$, and hence $\mathcal{L}_{\text{sep}}(\delta) = G_{\text{sep}}(S_2)$. Since S_1 is a torus with one hole, $g(S_2) = g(S) - 1 = g - 1 \geq 2$. Therefore, Lemma 6.1.3 applies to S_2 and implies that $\mathcal{L}_{\text{sep}}(\delta) = G_{\text{sep}}(S_2)$ is connected. This completes the proof of the “if” direction of the lemma, and hence completes the proof. ■

6.2. Circles of genus 1 and non-separating circles. In this subsection we assume that $g(S) \geq 3$. The main results of this subsection, namely Lemmas 6.2.4 and 6.2.5, require stronger assumption $g(S) \geq 4$.

Let D_0 be a genus 1 circle in S , and let $\delta_0 = \langle D_0 \rangle$. Since $g(S) \geq 3$, only one of the two parts into which D_0 divides S can be a torus with one hole. Let S_0 be this part, and let S_1 be the other. Our next goal is to find for $i = 0, 1$ properties of encodings of isotopy classes $\gamma = \langle C \rangle$ of non-separating circles C in S which are equivalent to C being isotopic to a circle contained in S_i , and which can be stated entirely in terms of the Torelli building $\mathfrak{B}(S)$ and its vertex δ . Such conditions are provided by Lemmas 6.2.4 and 6.2.5 below.

6.2.1. Lemma. *Suppose that D is a non-trivial separating circle in S . If C_0, C_1 is a bounding pair of circles disjoint from D , then both circles C_0, C_1 are contained in the same component of $S \setminus D$.*

Proof. Let S_0 and S_1 be the closures of two components of $S \setminus D$. Clearly, both S_0 and S_1 are subsurfaces of S having D as their common boundary. Being connected, each of the circles C_0, C_1 is entirely contained either in S_0 or in S_1 .

Suppose that $C_0 \subset S_0$. If C_0 separates S_0 , then C_0 bounds in S_0 a subsurface not containing D , and then C_0 bounds the same subsurface in the whole surface S . This contradicts to the fact that C_0 is non-separating in S , being a circle from a bounding pair of circles in S . It follows that C_0 is a non-separating circle in S_0 . By the same argument C_1 is a non-separating circle either in S_0 or in S_1 , depending on which of these subsurfaces contains C_1 .

Recall that $C_0 \subset S_0$ by our assumption. If $C_1 \subset S_1$, then the union $C_0 \cup C_1$ does not separate S because circles C_0, C_1 are non-separating in S_0, S_1 respectively, and subsurfaces S_0, S_1 are connected by their common boundary. This contradicts to the fact that C_0, C_1 is a bounding pair of circles. It follows that $C_1 \subset S_0$.

Therefore, if $C_0 \subset S_0$, then also $C_1 \subset S_0$. Similarly, if $C_0 \subset S_1$, then $C_1 \subset S_1$. The lemma follows. ■

Remark. The last lemma is valid without any restrictions on $g = g(S)$, but it is non-vacuous only if $g \geq 2$.

6.2.2. Lemma. *Let C_0 be a non-separating circle in S . The circle C_0 is isotopic to a circle contained in S_0 if and only if for each separating circle D in S_1 , the circle C_0 is isotopic to a circle disjoint from D .*

Proof. The “only if” direction of the lemma is trivial. In order to prove the “if” direction of the lemma, we will use Thurston’s theory of surfaces. In fact, we will prove a much stronger result.

Recall that a collection of circles C_1, C_2, \dots, C_n contained in S_1 is said to be *filling the surface S_1* if for every circle C in S the following two conditions are equivalent.

- (i) For each for $i = 1, 2, \dots, n$ the circle S is isotopic a circle disjoint from S_1 .
- (ii) The circle C is isotopic to a circle disjoint from S_1 .

Note that, obviously, (ii) always implies (i). Thurston’s theory provides a powerful tool for constructing filling collections of circles. Actually, every connected surface can be filled by just two circles.

Our assumption $g(S) \geq 3$ implies that $g(S_1) \geq 2$, and hence S_1 contains non-trivial separating circles (see, for example, Lemma 6.1.3). Let E be such a circle. Let $f \in \text{Mod}(S_1)$ be a pseudo-Anosov element, and let $F: S_1 \rightarrow S_1$ be any diffeomorphism representing f . Then for any sufficiently large natural numbers n, m the circle $F^n(E), F^{-m}(E)$ fill S_1 (the fact that D is separating is of no importance here). It follows that any circle in S isotopic to a circle disjoint from $F^n(E)$, and isotopic to a, perhaps, another circle disjoint from $F^{-m}(E)$ is isotopic to a circle disjoint from S_1 and hence contained in S_0 . This completes the proof of the “if” direction of the lemma, and hence proves the lemma. ■

6.2.3. Corollary *Let C_0 be a non-separating circle in S , and let $\gamma_0 = \langle C_0 \rangle$. The circle C_0 in S is isotopic to a circle contained in S_0 if and only if for every separating circle D in S_1 the vertex $\gamma_0 = \langle C_0 \rangle$ of $\mathcal{C}(S)$ is connected by an edge of $\mathcal{C}(S)$ (or, what is the same, of $\mathfrak{Ab}(S)$) with $\delta = \langle D \rangle$. ■*

Remark. Unfortunately, the authors are not aware of any reference for the used above corollary of Thurston’s theory not requiring learning the whole theory. For those who know the whole theory (with proofs), this corollary is obvious. The idea behind it is that the images of $\langle F^n(E) \rangle$ and $\langle F^{-m}(E) \rangle$ in the projective space of measured foliations approach, respectively, the stable and unstable foliations of f when $n, m \rightarrow \infty$.

6.2.4. Lemma. *Suppose that $g(S) \geq 4$. Let C_0 be a non-separating circle in S , and let $\gamma_0 = \langle C_0 \rangle$. The circle C_0 is isotopic to a circle contained in S_0 if and only if the following condition holds.*

- (S_0) For every non-trivial separating circle D in S_1 , there exists a admissible encoding (γ, ε) of $\gamma_0 = \langle C_0 \rangle$ such that γ is connected with $\delta = \langle D \rangle$ by an edge of $\mathfrak{Ab}(S)$.

Proof. Suppose that C_0 is contained in S_0 . Let D be a non-trivial separating circle in S_1 . Since D is separating, D divides S_1 into two parts. One of these parts contains $D_0 = \partial S_1$.

Let us denote this part by R . Then R is a subsurface of S_1 with boundary $\partial R = D_0 \cup D$. Since D is non-trivial, D is not isotopic in S to $D_0 = \partial S_1$. Therefore, R is not an annulus and hence $g(R) \geq 1$. This implies that the genus of the subsurface $R_0 = S_0 \cup R$ is ≥ 2 . Obviously, C_0 is a non-separating circle in R_0 . Since $g(R_0) \geq 2$, there is a bounding pair of circles in R_0 having C_0 as one of its components. Let C_1 be the other circle from this bounding pair of circles, and let $\gamma = \{\langle C_0 \rangle, \langle C_1 \rangle\}$. Since $g(S) \geq 3$, there is a separating circle E in S such that the pair (γ, ε) , where $\varepsilon = \langle E \rangle$, is an encoding of γ_0 . Since $g(S) \geq 4$, we can choose such a circle E that this encoding is admissible. Since $C_0 \cup C_1$ is contained in R_0 and hence disjoint from $D = \partial R_0$, the vertices γ and δ of $\mathfrak{Ab}(S)$ are connected by an edge of $\mathfrak{Ab}(S)$. This proves the “only if” part of the lemma.

In order to prove the “if” part, suppose that γ is connected with δ by an edge of $\mathfrak{Ab}(S)$ for an encoding (γ, ε) of γ_0 . For every such an encoding the vertex γ has the form $\gamma = \{\langle C_0 \rangle, \langle C_1 \rangle\}$ for some non-separating circle C_1 , and since γ is connected by an edge with δ , both circles C_0, C_1 are isotopic to circles disjoint from D . In particular, C_0 is isotopic to a circle disjoint from D , and hence γ_0 is connected by an edge of $\mathcal{C}(S)$ with δ .

Therefore, (S_0) implies that for every separating circle δ in S_1 the vertex γ_0 of $\mathcal{C}(S)$ is connected with δ by an edge of $\mathcal{C}(S)$. Now Corollary 6.2.3 implies that C_0 is isotopic to a circle in S_0 . This proves the “if” part of the lemma, and hence completes the proof. ■

6.2.5. Lemma. *Suppose that $g(S) \geq 4$. Let C_0 be a non-separating circle in S , and let $\gamma_0 = \langle C_0 \rangle$. The circle C_0 is isotopic to a circle contained in S_1 if and only if the following condition holds.*

(S_1) *There exists an admissible encoding (γ, ε) of γ_0 such that the vertices δ_0 and γ of $\mathfrak{Ab}(S)$ are connected by an edge of $\mathfrak{Ab}(S)$.*

Proof. Note that $g(S_1) = g(S) - 1 \geq 3$.

Suppose that C_0 is isotopic to a circle in S_1 . Then we can assume that the circle C_0 itself is contained in S_1 . Since the genus $g(S_1) \geq 3$, there exist a non-separating circle C_1 and a separating circle E in S_1 such that (γ, ε) is an encoding of γ_0 , where $\gamma = \{\langle C_0 \rangle, \langle C_1 \rangle\}$ and $\varepsilon = \langle E \rangle$. Since $g(S) \geq 4$, the circles C_1 and E can be chosen in a such a way that this encoding is admissible. Since C_0, C_1 are circles in S_1 , they are disjoint from $D_0 = \partial S_1$, and hence γ is connected by an edge of $\mathfrak{Ab}(S)$ with δ_0 . This proves the “only if” direction of the lemma.

In order to prove the “if” part, suppose that γ is connected with δ_0 by an edge of $\mathfrak{Ab}(S)$ for an encoding (γ, ε) of γ_0 . For every such an encoding the vertex γ has the form $\gamma = \{\langle C_0 \rangle, \langle C_1 \rangle\}$ for some non-separating circle C_1 , and since γ is connected by an edge with δ_0 , both circles C_0, C_1 are isotopic to circles disjoint from D_0 . We may assume that the circles C_0, C_1 themselves are disjoint from D_0 . Then, by Lemma 6.2.1, either both circles C_0, C_1 are contained in S_0 , or both of them are contained in S_1 . Since the subsurface S_0 is a torus with one hole, it does not contain any bounding pairs of circles.

Since C_0, C_1 is a bounding pair of circles, this implies that C_0, C_1 are contained in S_1 . In particular, C_0 is a circle in S_1 . This proves the “if” direction of the lemma, and hence completes the proof. ■

Remark. The assumption $g(S) \geq 4$ in Lemmas 6.2.4 and 6.2.5 is needed only to ensure the existence of admissible encodings. If $g(S) \leq 3$, then there are no marked triangles in $\mathfrak{Tb}(S)$ and hence there are no admissible encodings.

6.3. Encodings and edges of $\mathcal{C}(S)$. The goal of this subsection is to find properties of encodings of two arbitrary vertices of $\mathcal{C}(S)$ which are equivalent to the property of being connected by an edge of $\mathcal{C}(S)$ and can be stated entirely in terms of the Torelli building $\mathfrak{Tb}(S)$. There are several cases to consider.

If both vertices are the isotopy classes of separating circles, then their encodings are these circles themselves considered as vertices, and by the definition of $\mathfrak{Tb}(S)$ they are connected by an edge in $\mathfrak{Tb}(S)$ if and only if they are connected by an edge in $\mathcal{C}(S)$.

6.3.1. Lemma. *Suppose that $g(S) \geq 4$. Let C_0, D_0 be, respectively, a non-separating and a separating circle in S , and let $\gamma_0 = \langle C_0 \rangle, \delta_0 = \langle D_0 \rangle$. Suppose that D_0 is a genus 1 circle in S .*

Then γ_0 is connected with δ_0 by an edge of $\mathcal{C}(S)$ if and only if either the condition (S_0) of Lemma 6.2.4, or the condition (S_1) of Lemma 6.2.5 holds.

Proof. This is an easy corollary of Lemmas 6.2.4 and 6.2.5. Since D_0 is a genus 1 circle, we are in the situation of the subsection 6.2. Let S_0 and S_1 be the subsurfaces of S defined at the beginning of the subsection 6.2. In particular, S_0 is a torus with one hole.

If γ_0 is connected with δ_0 by an edge of $\mathcal{C}(S)$, then C_0 is isotopic to a circle disjoint from D_0 , and hence to a circle contained in S_0 or S_1 . If C_0 is isotopic to a circle contained in S_0 , Lemma 6.2.4 applies and by this lemma the condition (S_0) holds. If C_0 is isotopic to a circle contained in S_1 , Lemma 6.2.5 applies and by this lemma the condition (S_1) holds. This proves the “only if” direction of the lemma.

If the condition (S_0) from Lemma 6.2.4 holds, then Lemma 6.2.4 implies that C_0 is isotopic to a circle contained in S_0 , and hence γ_0 is connected with δ_0 by an edge of $\mathcal{C}(S)$. The argument for (S_1) is similar. This proves the “if” direction of the lemma, and hence completes the proof. ■

6.3.2. Lemma. *Suppose that $g(S) \geq 4$. Let C_0, D_0 be, respectively, a non-separating and a separating circle in S , and let $\gamma_0 = \langle C_0 \rangle, \delta_0 = \langle D_0 \rangle$. Suppose that D_0 is not a genus 1 circle.*

Then γ_0 is connected with δ_0 by an edge of $\mathcal{C}(S)$ if and only if there exists an admissible encoding (γ, ε) of γ_0 such that γ is connected with δ_0 by an edge of $\mathfrak{Tb}(S)$.

Proof. Let S_0, S_1 be two subsurfaces bounded by D_0 in S . Since D_0 is not a genus 1 circle, both S_0 and S_1 have genus ≥ 2 .

If γ_0 and δ_0 are connected by an edge of $\mathcal{C}(S)$, then C_0 is isotopic to a circle disjoint from D_0 and we may assume that C_0 itself is disjoint from D_0 . Then C_0 is contained in one of the subsurfaces S_0, S_1 . We may assume that C_0 is contained in S_0 . Since $g(S_0) \geq 2$, there is a non-separating circle C_1 in S_0 such that C_0, C_1 is a bounding pair of circles in S_0 . Let $\gamma = \{\langle C_0 \rangle, \langle C_1 \rangle\}$. Since the circles C_0, C_1 are disjoint from D_0 , γ is connected with δ_0 by an edge of $\mathfrak{Ab}(S)$. In addition, since $g(S) \geq 4$, there exists a vertex ε such that (γ, ε) is an admissible encoding of γ_0 (it does not matter if ε is the isotopy class of a circle contained in S_0 or not). This proves the “only if” part of the lemma.

Let us prove the “if” part. If γ and δ_0 are connected by an edge, then $\gamma = \{\langle E_0 \rangle, \langle E_1 \rangle\}$ for some bounding pair of circles E_0, E_1 disjoint from D_0 . Since (γ, δ) encodes γ_0 , the circle C_0 is isotopic to one of these circles E_0, E_1 . In particular, C_0 is isotopic to a circle disjoint from D_0 , i.e. γ_0 and δ_0 are connected by an edge of $\mathcal{C}(S)$. This proves the “if” part of the lemma and hence completes the proof. ■

6.3.3. Lemma. *Suppose that $g = g(S) \geq 5$. Let γ'_0, γ''_0 be two isotopy classes of non-separating circles in S . The following two conditions are equivalent.*

- (C) γ'_0 and γ''_0 are connected by an edge in $\mathcal{C}(S)$.
- (Ab) There exist encodings (γ', δ') of γ'_0 and (γ'', δ'') of γ''_0 such that γ' and γ'' are connected by an edge in $\mathfrak{Ab}(S)$.

Proof. Let $\gamma'_0 = \langle C'_0 \rangle$ and $\gamma''_0 = \langle C''_0 \rangle$ for some non-separating circles C'_0 and C''_0 in S .

(C) *implies* (Ab). Suppose first that the condition (C) holds. Then we may assume that the circles C'_0 and C''_0 are disjoint and are not isotopic (the case of isotopic circles being trivial). There are two cases to consider.

Case 1. $C'_0 \cup C''_0$ *separates* S . Then C'_0, C''_0 is a bounding pair of circles. Let $\gamma = \{\langle C'_0 \rangle, \langle C''_0 \rangle\}$ be the corresponding vertex of $\mathfrak{Ab}(S)$. Then for some separating vertices δ', δ'' of $\mathfrak{Ab}(S)$ the pairs (γ, δ') and (γ, δ'') are encoding γ'_0 and γ''_0 respectively. This proves that (C) implies (Ab) in Case 1. □

Case 2. $C'_0 \cup C''_0$ *does not separate* S . Suppose that the genus g of S is fixed. Then up to a diffeomorphism there is only one (ordered) triple (S, C'_0, C''_0) such that the union $C'_0 \cup C''_0$ is not separating S (and such a triple exists if and only if $g \geq 2$). Therefore, we can deal with this case by considering a model example.

Since $g \geq 5$ by the assumption, $g = g' + g'' + 2$ for some $g' \geq 1$ and $g'' \geq 2$. Let R' be a surface of genus $g' = g(R') \geq 1$ having exactly two boundary components C'_0, C'_1 , and let R'' be a surface of genus $g'' = g(R'') \geq 2$ having exactly two boundary components $C''_0,$

C_1'' . Let R_0 be a surface of genus $g(R_0) = 0$ having exactly 4 boundary components. Let us glue R' and R'' to R_0 by glueing $\partial R'$ to arbitrary 2 of the 4 boundary components of R_0 , and glueing $\partial R''$ to the 2 other boundary components of R_0 . The resulting surface R is a closed surface of genus $g' + g'' + 2$. We may take (R, C'_0, C''_0) as our model of (S, C'_0, C''_0) .

Clearly, $\{C'_0, C'_1\}$ and $\{C''_0, C''_1\}$ are bounding pairs of circles. Let γ' and γ'' be the corresponding vertices of $\mathfrak{Tb}(S)$. Since the circles C'_0, C'_1, C''_0, C''_1 are pair-wise disjoint by the construction, the vertices γ' and γ'' are connected by an edge of $\mathfrak{Tb}(S)$. On the other hand, for some vertices δ', δ'' of $\mathfrak{Tb}(S)$ corresponding to separating circles the pairs (γ, δ') and (γ, δ'') are encoding γ'_0 and γ''_0 respectively. This proves that (C) implies (Tb) in Case 2. \square

(Tb) *implies* (C). Suppose now that the condition (Tb) holds.

Let (γ', δ') and (γ'', δ'') be encodings of γ'_0 and γ''_0 respectively such that γ' and γ'' are connected by an edge in $\mathfrak{Tb}(S)$. Then $\gamma' = \{\langle C'_0 \rangle, \langle C'_1 \rangle\}$ and $\gamma'' = \{\langle C''_0 \rangle, \langle C''_1 \rangle\}$ for some bounding pairs of circles $\{C'_0, C'_1\}$ and $\{C''_0, C''_1\}$. Since γ' is connected by an edge of $\mathfrak{Tb}(S)$ with γ'' , we may assume that circles C'_0, C'_1, C''_0, C''_1 are pair-wise disjoint. Then, in particular, the circles C'_0 and C''_0 are disjoint, and hence γ'_0 and γ''_0 are connected by an edge in $\mathcal{C}(S)$. This proves that (Tb) implies (C), and hence completes the proof of the lemma. \blacksquare

6.4. Proof of Theorem 3.4. Let S be a closed surface of genus ≥ 5 . Let $A: \mathfrak{Tb}(S) \rightarrow \mathfrak{Tb}(S)$ be an automorphism of the Torelli geometry $\mathfrak{Tb}(S)$.

Recall that the set of vertices of $\mathcal{C}(S)$ which are the isotopy classes of separating circles is equal to the set of \mathcal{SC} -vertices of $\mathfrak{Tb}(S)$, i.e. the set of vertices of $\mathfrak{Tb}(S)$ marked by the symbol \mathcal{SC} . Since A is an automorphism of the Torelli building $\mathfrak{Tb}(S)$, A preserves markings of vertices. Therefore, A induces a bijective self-map of the set of vertices of $\mathcal{C}(S)$ which are the isotopy classes of separating circles.

By Corollaries 5.2.2 and 5.6, encodings by admissible pairs establish a canonical 1-to-1 correspondence between the isotopy classes of non-separating circles and the equivalence classes of admissible pairs with respect to the relation of being connected by a sequence of moves (obviously, this is an equivalence relation). Since the admissible pairs and the moves of both types are defined entirely in terms of the Torelli building, A maps admissible pairs to admissible pairs, and if two admissible pairs are related by a move, their images under A are also related by a move of the same type. Therefore, A induces bijective self-maps of the set of admissible pairs respecting the relation of being connected by a sequence of moves. It follows that A induces also a bijective self-map of the set of the equivalence classes of admissible pairs with respect to the above relation. When combined with encodings, the last self-map induced by A induces, in turn, a bijective self-map of the set of the isotopy classes of non-separating circles. In other words,

A induces a bijective self-map of the set of vertices of $\mathcal{C}(S)$ which are the isotopy classes of non-separating circles.

By combining the results of the last paragraphs, we see that A induces a bijective self-map of the set of the isotopy classes of non-trivial circles in S , i.e. of the set of the vertices of $\mathcal{C}(S)$. We will denote this induced self-map by $A_{\mathcal{C}}$. By the construction, the induced map $A_{\mathcal{C}}$ agrees with A and encodings of non-separating vertices the sense of the following lemma.

6.4.1. Lemma *If (γ, δ) is an admissible encoding of a non-separating vertex γ_0 of $\mathcal{C}(S)$, then $A_{\mathcal{C}}(\gamma_0)$ is also a non-separating vertex of $\mathcal{C}(S)$, and $(A(\gamma), A(\delta))$ is an admissible encoding of $A_{\mathcal{C}}(\gamma_0)$. ■.*

Now we have to prove that $A_{\mathcal{C}}$ is not only a bijective self-map of the set of vertices of $\mathcal{C}(S)$, but, moreover, is an automorphism of the complex of curves $\mathcal{C}(S)$. This is done in the following lemma.

6.4.2. Lemma. *$A_{\mathcal{C}}$ preserves the structure of the simplicial complex $\mathcal{C}(S)$. In other words, $A_{\mathcal{C}}$ maps every set of vertices which is a simplex of $\mathcal{C}(S)$ to a set which is also a simplex of $\mathcal{C}(S)$.*

Proof. Since $\mathcal{C}(S)$ is a flag complex (see the subsection 2.7), it is sufficient to prove that if two vertices of $\mathcal{C}(S)$ are connected by an edge, then their images under the induced map $A_{\mathcal{C}}$ are also connected by an edge. There are several cases to consider.

Case 1. Two separating vertices. By the definitions of $\mathcal{C}(S)$ and $\mathfrak{Sb}(S)$, two separating vertices, i.e. two isotopy classes of non-trivial separating circles, are connected by an edge in $\mathcal{C}(S)$ if and only if they are connected by an edge in $\mathfrak{Sb}(S)$. It follows that $A_{\mathcal{C}}$ takes edges connecting two such vertices into similar edges. □

Case 2. Two non-separating vertices. Let γ'_0, γ''_0 be two non-separating vertices of $\mathcal{C}(S)$. By Lemma 6.3.3 the vertices γ'_0 and γ''_0 are connected by an edge of $\mathcal{C}(S)$ if and only if the condition (\mathfrak{Sb}) of this lemma holds. Lemma 6.4.1 implies that if this condition holds for γ'_0, γ''_0 , then it holds also for $A_{\mathcal{C}}(\gamma'_0), A_{\mathcal{C}}(\gamma''_0)$ in the roles of γ'_0, γ''_0 respectively. By using Lemma 6.3.3 again, we conclude that if γ'_0, γ''_0 are connected by an edge, then $A_{\mathcal{C}}(\gamma'_0), A_{\mathcal{C}}(\gamma''_0)$ are also connected by an edge. □

Case 3. A non-separating vertex and a separating vertex. Let C_0, D_0 be, respectively, a non-separating and a separating circle in S , and let $\gamma_0 = \langle C_0 \rangle, \delta_0 = \langle D_0 \rangle$. There are two subcases depending on if D_0 is a genus 1 circle or not.

First of all, by Lemma 6.1.4 D_0 is a genus 1 circle if and only if the graph $\mathfrak{L}_{\text{sep}}(\delta_0)$ is connected. In view of the definition of $\mathfrak{L}_{\text{sep}}(\delta_0)$ and Lemma 6.4.1, if $\mathfrak{L}_{\text{sep}}(\delta_0)$ is connected, then $\mathfrak{L}_{\text{sep}}(A_{\mathcal{C}}(\delta_0))$ is also connected. It follows that $A_{\mathcal{C}}$ preserves the set of separating vertices of $\mathcal{C}(S)$ which are the isotopy classes of genus 1 circles.

Subcase 3-A. D_0 is a genus 1 circle. As we just saw, then $A_{\mathcal{C}}(\delta)$ is also an isotopy class of genus 1 circle. By Lemma 6.3.1, γ_0 is connected with δ_0 by an edge of $\mathcal{C}(S)$ if and only if either the condition (\mathcal{S}_0) of Lemma 6.2.4, or the condition (\mathcal{S}_1) of Lemma 6.2.5 holds. By arguments completely similar to the arguments in Cases 1 and 2, we conclude that if γ_0, δ_0 are connected by an edge, then $A_{\mathcal{C}}(\gamma_0), A_{\mathcal{C}}(\delta_0)$ are also connected by an edge. \square

Subcase 3-B. D_0 is not a genus 1 circle. This subcase is similar to the previous one. The only difference is that the reference to Lemma 6.3.1 should be replaced by a reference to Lemma 6.3.2. This completes the proof of the lemma. \blacksquare

End of the proof of Theorem 3.4. By the last lemma $A_{\mathcal{C}}$ is an automorphism of $\mathcal{C}(S)$. Therefore, Theorem 2.8 implies that $A_{\mathcal{C}} = F_{\mathcal{C}\mathcal{C}}$ for some diffeomorphism $F: S \rightarrow S$. Recall that for every non-trivial circle C , the map $F_{\mathcal{C}\mathcal{C}}$ maps the vertex $\langle C \rangle$ of $\mathcal{C}(S)$ to the vertex $\langle F(C) \rangle$, and that $F_{\mathcal{C}\mathcal{C}}$ is a bijective self-map of the set of all vertices of $\mathcal{C}(S)$. See the subsection 2.7. The bijection $F_{\mathcal{C}\mathcal{C}}$, in turn, induces an automorphism $F_{\mathcal{T}\mathcal{b}}: \mathfrak{A}\mathcal{b}(S) \rightarrow \mathfrak{A}\mathcal{b}(S)$. See the subsection 3.3 for the details. In order to complete the proof of the theorem, it is sufficient to prove that $A = F_{\mathcal{T}\mathcal{b}}$.

Let $\gamma = \{\gamma_0, \gamma_1\}$ be an arbitrary $\mathcal{B}\mathcal{P}$ -vertex of $\mathfrak{A}\mathcal{b}(S)$. Then there exists an admissible pair (γ, δ_0) encoding the vertex γ_0 of $\mathcal{C}(S)$. By Lemma 6.4.1, this implies $(A(\gamma), A(\delta_0))$ is an encoding of $A_{\mathcal{C}}(\gamma_0)$. But $A_{\mathcal{C}} = F_{\mathcal{C}\mathcal{C}}$ and hence $A_{\mathcal{C}}(\gamma_0) = F_{\mathcal{C}\mathcal{C}}(\gamma_0)$. It follows that $(A(\gamma), A(\delta_0))$ is an encoding of $F_{\mathcal{C}\mathcal{C}}(\gamma_0)$. In particular, $F_{\mathcal{C}\mathcal{C}}(\gamma_0)$ is one of the elements of the bounding pair $A(\gamma)$ (recall that bounding pairs are, in particular, 2-element sets of vertices of $\mathcal{C}(S)$). By a similar argument, $F_{\mathcal{C}\mathcal{C}}(\gamma_1)$ is also one of the elements of the bounding pair $A(\gamma)$. But the isotopy classes $F_{\mathcal{C}\mathcal{C}}(\gamma_0), F_{\mathcal{C}\mathcal{C}}(\gamma_1)$ are different, because the isotopy classes γ_0, γ_1 are different, being elements of a bounding pair (namely, of the bounding pair γ). Therefore, the pair $A(\gamma)$ is equal to the pair $\{F_{\mathcal{C}\mathcal{C}}(\gamma_0), F_{\mathcal{C}\mathcal{C}}(\gamma_1)\}$. The latter pair is nothing else as $F_{\mathcal{T}\mathcal{b}}(\gamma)$. It follows that $A(\gamma) = F_{\mathcal{T}\mathcal{b}}(\gamma)$.

C_0 is not isotopic to C_1 , and hence $F(C_0)$ is not isotopic to $F(C_1)$. It follows that the pair $A(\gamma)$ should be equal to the pair $\{F_{\mathcal{C}\mathcal{C}}(\gamma_0), F_{\mathcal{C}\mathcal{C}}(\gamma_1)\}$. The latter pair is nothing else as $F_{\mathcal{T}\mathcal{b}}(\gamma)$. It follows that $A(\gamma) = F_{\mathcal{T}\mathcal{b}}(\gamma)$.

Since γ was an arbitrary $\mathcal{B}\mathcal{P}$ -vertex of $\mathfrak{A}\mathcal{b}(S)$, this means A agrees with $F_{\mathcal{T}\mathcal{b}}$ on the set of $\mathcal{B}\mathcal{P}$ -vertices. Since, as we noted above, A agrees with $F_{\mathcal{T}\mathcal{b}}$ also on the set of $\mathcal{S}\mathcal{C}$ -vertices, this implies that $A = F_{\mathcal{T}\mathcal{b}}$, and hence completes the proof of the theorem. \blacksquare

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