THE SMOOTH MORDELL-WEIL GROUP AND MAPPING CLASS GROUPS OF ELLIPTIC SURFACES

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Abstract. This is a paper in smooth 4-manifold topology, inspired by the Mordell-Weil Theorem in number theory. More precisely, we prove a smooth version of the Mordell-Weil Theorem and apply it to the 'unipotent radical' case of a Thurston-type classification of mapping classes of simply-connected 4-manifolds M_d that admit the structure of an elliptic complex surface of arithmetic genus $d \geq 1$. Applications include Nielsen realization theorems for M_d.

By combining this with known results, we obtain the following remarkable consequence: if the singular fibers of such an elliptic fibration are of the simplest (i.e. nodal) type, then the fibered structure is unique up topological isotopy. In particular, any diffeomorphism of $M_d, d \geq 3$ is topologically isotopic to a diffeomorphism taking fibers to fibers.

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1. Introduction

This is a paper in smooth 4-manifold topology, inspired by the Mordell-Weil Theorem in number theory. More precisely, we prove a smooth version of the Mordell-Weil Theorem and apply it to the 'unipotent radical' case of a Thurston-type classification of mapping classes of simply-connected elliptic surfaces. We also give methods to compute a number of related subgroups of mapping class groups of these surfaces. Applications include Nielsen realization theorems for simply-connected elliptic surfaces. In order to state our main results, we first briefly review the corresponding chapter in the theory of complex surfaces.

1.1. **Elliptically fibered surfaces.**

In this paper a (holomorphic) *genus one fibration* of a smooth, compact complex surface M is a holomorphic map $\pi : M \to \mathbb{P}^1$ whose general fiber is of genus one. We will always assume that (i) $\pi_1(M) = 0$, (ii) π is *relatively minimal*, i.e. no fiber contains a rational curve of self-intersection -1, and (iii) that the fiber class of π in $H_2(M)$ is primitive (=indivisible). This implies among other things that π has at least one singular fiber but has no multiple fibers, and that all singular fibers are of Kodaira type.

Such a fibration has only finitely many critical values, and each such value has a well-defined multiplicity > 0 given by the Euler characteristic of the fiber; the resulting divisor on $\mathbb{P}^1$ is the *discriminant* of π. This multiplicity is 1 precisely when the fiber over it is a rational nodal curve; topologically this is a 2-sphere with two distinct points identified. We say that the genus one fibration is *generic* if all singular fibers are of that type, in other words, if the discriminant divisor is reduced. As the name suggests, every genus one fibration can, by an arbitrary small deformation, be deformed into a generic one. An *elliptic fibration* is a genus one fibration equipped with a section of π.

The most important topological invariant of a genus one fibration $\pi : M \to \mathbb{P}^1$ is the Euler characteristic of M; this is always a positive integer divisible by 12 and the quotient d is called *of arithmetic genus* of M. This number determines the place of M in Kodaira's classification of complex surfaces: for $d = 1$ the surface is rational (of Kodaira dimension $-\infty$). Indeed, such a surface is obtained by blowing up the fixed-point set of a general pencil of cubic curves in $\mathbb{P}^2$ and comes with many sections (every exceptional divisor is one; see Example 2.4 for more details). For $d = 2$ it is a K3 surface (of Kodaira dimension 0) and for $d > 2$ its Kodaira dimension is 1.

Moishezon proved that the simply-connected, genus one fibrations of a fixed arithmetic genus d form a single connected family. This makes d a complete smooth invariant, so that we can think of such genus one fibrations as structures having the same underlying closed oriented 4-manifold M_d. An elliptic fibration $\pi_d : M_d \to \mathbb{P}^1$ of arithmetic genus d can be obtained by pulling back π_d along a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d whose discriminant is disjoint with that of π_d. From this one can see that the intersection pairing on $H_2(M_d)$ makes it unimodular of signature $(2d - 1, 10d - 1)$.

1.2. **The holomorphic Mordell-Weil group.**

As explained in more detail in §2.2 below, each smooth fiber of a holomorphic genus one fibration $\pi : M \to \mathbb{P}^1$ inherits an affine structure, and the notion of 'translation' of a fiber makes sense. The *holomorphic Mordell-Weil group* is the group

$$\mathrm{MW}_{\mathrm{hol}}(\pi) := \{f \in \mathrm{Aut}(M) : f \text{ acts on each smooth fiber by translation}\},$$

where $\mathrm{Aut}(M)$ denotes the group of biholomorphic automorphisms of M. The 'passage to the jacobian' does not change $\mathrm{MW}_{\mathrm{hol}}(\pi)$ but turns it into one of an elliptic fibration (rather than a genus one fibration) and in that case it is a well-studied, classical object (see e.g. the book [SS]): it is the group of rational points of an elliptic curve over the field of rational functions on $\mathbb{P}^1$. The Mordell-Weil Theorem for function fields (proved by Lang-Néron) implies that $\mathrm{MW}_{\mathrm{hol}}(\pi)$ is finitely generated ([1]).

In its interpretation as an automorphism group, $\mathrm{MW}_{\mathrm{hol}}(\pi)$ acts faithfully on $H_2(M) := H_2(M;\mathbb{Z})$; in its interpretation as a group of sections it is naturally a subquotient of $H_2(M)$, and so comes equipped with a *height pairing* induced by the intersection form on M. The fiber class $e \in H_2(M)$ is isotropic for the intersection pairing and primitive by assumption. Hence $e^\perp/\mathbb{Z}e$ is in a natural manner a lattice. The group $\mathrm{MW}_{\mathrm{hol}}(\pi_d)$ modulo its torsion can be identified with the part of this lattice that is of Hodge type $(1,1)$; this is a negative-definite sublattice of rank at most $10d-2$; see Thm. 6.20 in [SS]. In general the group $\mathrm{MW}_{\mathrm{hol}}(\pi)$ depends on π and on the complex structure on M.

1.3. The smooth Mordell-Weil group.

For a genus one fibration $\pi : M \to \mathbb{P}^1$, the group

$$\mathrm{Trans}(\pi) := \{f \in \mathrm{Diff}(M) : f \text{ acts on each smooth fiber of } \pi \text{ by translation}\}.$$

is abelian and infinite-dimensional. We define the *smooth Mordell-Weil group* $\mathrm{MW}(\pi)$ of π to be its maximal discrete quotient

$$\mathrm{MW}(\pi) := \pi_0(\mathrm{Trans}(\pi)).$$

Here the implicit assumption is that π is holomorphic, but the isomorphism type of $\mathrm{Trans}(\pi)$ and $\mathrm{MW}(\pi)$ only depends on the fiberwise diffeomorphism type of $\pi : M \to \mathbb{P}^1$. As in the holomorphic case, the set of connected components of the space of smooth sections of π is a torsor for $\mathrm{MW}(\pi)$, so that the elements of $\mathrm{MW}(\pi)$ are represented by fiberwise translations by the difference of two smooth sections. See Figure 1.

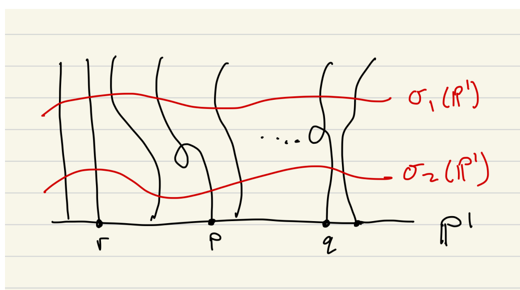

FIGURE 1. A schematic of an elliptic fibration over $\mathbb{P}^1$. The generic fiber, as over $r \in \mathbb{P}^1$ in the figure, is a smooth elliptic curve; the fibers over $p, q \in \mathbb{P}^1$ in the figure represent rational nodal curves. While such fibers are generic, a typical fiber of an elliptic fibration can be much more complicated. Two smooth sections σ_1, σ_2 are indicated. They determine a fiber-preserving diffeomorphism by translating each fiber $\pi^{-1}(b)$ by $z \mapsto z + \sigma_2(b)$, where addition is given in the unique group structure on $\pi^{-1}(b)$ with $\sigma_1(b)$ as identity.

Our first result is a smooth analogue of the Lang-Neron theorem. (We emphasize that all the theorems stated here are subject to our blanket assumptions (i)–(iii).)

Theorem 1.1 (Smooth Mordell-Weil Theorem). *Let $\pi : M \to \mathbb{P}^1$ be any genus one fibration of a compact, complex surface M. Then $\mathrm{MW}(\pi)$ is finitely-generated.*

[1] Recall that we are assuming that M is simply connected; otherwise finite generation does not hold.

A more general and refined version of this result is given as Theorem 3.3 below. We will now focus on the generic case: in what follows $\pi_d : M_d \to \mathbb{P}^1$ will always stand for a generic (holomorphic) genus one fibration of arithmetic genus d. We write Λ_d for the lattice $H_2(M_d)$ and $e \in \Lambda_d$ for the fiber class. This class is primitive and with $e \cdot e = 0$. The following theorem then gives a complete description of the smooth Mordell-Weil group.

Theorem 1.2 (**Computation of MW(π_d)**). *Let $\pi_d : M_d \to \mathbb{P}^1$ be a generic genus one fibration of arithmetic genus $d \geq 1$. Then $\mathrm{MW}(\pi_d)$ acts faithfully on Λ_d. Further, $\mathrm{MW}(\pi_d)$ comes with a height pairing (making it a lattice) that extends the one defined above, and there is an isomorphism of lattices*

$$\mathrm{MW}(\pi_d) \cong e^{\perp}/\mathbb{Z}e.$$

The right hand side is an even unimodular lattice of signature $(2d-2, 10d-2)$ and hence isomorphic to $d\mathbf{E}_8(-1) \perp (2d-2)\mathbf{U}$, where $\mathbf{E}_8(-1)$ is the negative-definite $\mathbf{E}_8$ lattice and $\mathbf{U}$ is the rank 2 hyperbolic lattice (both are even unimodular).

Theorem 1.2 is proved on page 20.

Remark 1.3 (Holomorphic vs. smooth). Theorem 1.2 implies that

$$\mathrm{rank}(\mathrm{MW}_{\mathrm{hol}}(\pi_d)) \leq 10d-2 < 12d-4 = \mathrm{rank}(\mathrm{MW}(\pi_d)) \text{ for } d \geq 2$$

highlighting the difference between the holomorphic and smooth categories. In particular, the natural map $\mathrm{MW}_{\mathrm{hol}}(\pi_d) \to \mathrm{MW}(\pi_d)$ induced by the inclusion $\mathrm{MW}_{\mathrm{hol}}(\pi_d) \to \mathrm{Trans}(\pi_d)$ cannot be onto when $d \geq 2$ (it is an isomorphism for $d = 1$).

1.4. $\mathrm{MW}(\pi_d)$ **as a mapping class group.** Our interest in the smooth Mordell-Weil group $\mathrm{MW}(\pi_d)$ arises from its close relationship with the *(smooth) mapping class group* of M_d, defined by

$$\mathrm{Mod}(M_d) := \pi_0(\mathrm{Diff}(M_d)).$$

Since every $f \in \mathrm{Diff}(M_d)$ preserves orientation and hence the intersection pairing on M_d, the map $\rho_d([f]) := f_*$ defines a representation $\rho_d : \mathrm{Mod}(M_d) \to \mathrm{O}(\Lambda_d)$. The action of $\mathrm{MW}(\pi_d)$ on Λ_d factors as a composition

$$\mathrm{MW}(\pi_d) \to \mathrm{Mod}(M_d) \to \mathrm{O}(\Lambda_d)$$

which is injective by Theorem 1.2, so that we can regard $\mathrm{MW}(\pi_d)$ as a subgroup $\mathrm{Mod}(M_d)$. It is not known whether the kernel of $\mathrm{Mod}(M_d) \to \mathrm{O}(\Lambda_d)$ is trivial or not. Theorem 1.2 implies that $\mathrm{MW}(\pi_d)$ has trivial intersection with this kernel. In fact a stronger result holds.

Corollary 1.4. *For each $d \geq 1$ the map $\mathrm{MW}(\pi_d) \to \mathrm{O}(\Lambda_d)$ is injective, so that the following are equivalent for a fiberwise translation $F \in \mathrm{Trans}(\pi_d) \subset \mathrm{Diff}(M_d)$:*

(1) F acts as the identity in Λ_d,
(2) F is topologically isotopic to the identity,
(3) F is smoothly isotopic to the identity,
(4) F is smoothly fiberwise isotopic to the identity,
(5) F is smoothly isotopic to the identity via fiberwise translations in M.

The equivalence of (1) and (2) in Corollary 1.4 is given in [GGHKP]; and the implications (5)⇒(4)⇒(3)⇒(2)⇒ (1) are trivial. Our contribution is Theorem 1.2, which gives (1)⇒(5).

The *Nielsen Realization Problem* asks whether a given homomorphism $i : G \to \mathrm{Mod}(M_d)$ lifts to a homomorphism $\tilde{i} : G \to \mathrm{Diff}(M_d)$; in this case we say that *G is realized by* this lift.

Theorem 1.5 (**Nielsen Realization, I**). *For a generic genus one fibration* $\pi : M_d \to \mathbb{P}^1$ *of arithmetic genus* $d \geq 1$*, the Nielsen Realization Problem is solvable for the injection* $\mathrm{MW}(\pi_d) \to \mathrm{Mod}(M_d)$*; in fact, we can lift it to a homomorphism* $\mathrm{MW}(\pi_d) \to \mathrm{Trans}(\pi_d)$.

We prove Theorem 1.5 on Page 14.

Remarks 1.6. We make a few comments on Theorem 1.5.

(1) The Nielsen Realization Problem has a negative answer for arbitrary subgroups of $\mathrm{Mod}(M_d)$; indeed there are finite subgroups $G \subset \mathrm{Mod}(M_2)$ that cannot be realized by a group of diffeomorphisms of M_2; see [FL1].

(2) Although each element of $\mathrm{Trans}(\pi_d)$ leaves invariant some complex structure, when $d \geq 2$ the whole group $\mathrm{MW}(\pi_d)$ cannot leave invariant any complex structure; see Remark 1.3 above.

(3) We can choose the lift in Theorem 1.5 such that the action of $\mathrm{MW}(\pi_d)$ on M_d via this lift is free on the smooth part of each fiber with the topological closure of each orbit equalling that fiber. So then this lift determines π.

(4) When $\pi : M \to \mathbb{P}^1$ is not generic, that is, when non-nodal singular fibers are allowed, then $\mathrm{MW}_{\mathrm{hol}}(\pi)$ (and hence $\mathrm{MW}(\pi)$) can contain torsion; see, e.g. [SS]. It would be interesting to compute $\mathrm{MW}(\pi)$ in these cases.

1.5. **Thurston-type classification and Eichler transformations.** The group $\mathrm{MW}(\pi_d)$ is a major case in a Thurston-type classification of elements of $\mathrm{Mod}(M_d)$, by which we mean finding for each class in $\mathrm{Mod}(M_d)$ an explicit representative that preserves a geometric structure and/or is optimal in some way.

The real vector space $\mathbb{R} \otimes \Lambda_d = H_2(M_d; \mathbb{R})$ admits a *spinor orientation*, which is an orientation on the tautological bundle of the Grassmannian of its (maximal) positive-definite $(2d-1)$-planes in $\mathbb{R} \otimes \Lambda_d$. Denote by $\Gamma_d, d \geq 2$ the index 2 subgroup of $\mathrm{O}(\Lambda_d)$ consisting of those automorphisms preserving this spinor orientation and let $\Gamma_1 := \mathrm{O}(H_1)$. Friedman-Morgan [FM97] proved that the image of $\rho_d : \mathrm{Mod}(M_d) \to \mathrm{O}(\Lambda_d)$ lies in Γ_d.

For $d = 2$ (the case of a K3 surface), the authors considered in [FL1] the case when f_* is of finite order. Certain examples with f_* infinite order semisimple were explored in depth by McMullen [Mc] from a dynamical perspective. In this paper we consider the case when $f_* \in \mathrm{O}(H_m)$ is unipotent. We show in Proposition 2.1 below that such an element must fix some primitive isotropic vector $e \in \Lambda_d$. Remarkably, Friedman-Morgan proved [FM97] that when $d > 2$ every element of Γ_d has this property; see below. We will show in Corollary 2.2 that any unipotent element of Γ_d can be represented by some $f \in \mathrm{Diff}(M_d)$ leaving invariant some elliptic fibration.

Denote by $\Gamma_{d,e}$ the stabilizer in $\mathrm{O}(\Lambda_d)$ of e. Since e isotropic it is contained in $e^\perp$. Set

$$\Lambda_d(e) := e^\perp / \mathbb{Z}e$$

The $\Gamma_{d,e}$ action on $\Lambda_d(e)$ induces a representation $\Gamma_{d,e} \to \mathrm{O}(\Lambda_d(e))$ with image the index 2 subgroup $\Gamma_d(e)$ fixing the spinor orientation. This gives a (non-canonically split) short exact sequence

$$0 \to \Lambda_d(e) \to \Gamma_{d,e} \to \Gamma_d(e) \to 1.$$

The elements of $\Lambda_d(e)$ are represented as *Eichler transformations*: every $\tilde{c} \in e^\perp$ defines an *Eichler transformation* $E(e, \tilde{c}) \in \Gamma_{d,e}$ via

$$E(e, \tilde{c})(x) := x + (x \cdot e)\tilde{c} - (x \cdot \tilde{c})e - \tfrac{1}{2}(\tilde{c} \cdot \tilde{c})(x \cdot e)e.$$

This transformation fixes e and only depends on the image c of $\tilde{c}$ in $\Lambda_d(e)$, or better yet, on the 2-vector $e \wedge \tilde{c} \in \wedge^2 \Lambda_d$, so that we can write $E(e \wedge c)$ instead. In fact, any element of Γ_d that fixes e and acts trivially on $H(e)$ is of the form $E(e \wedge c)$ for a unique $c \in \Lambda_d(e)$.

The group $\Lambda_d(e) < \Gamma_d$ is generated by Eichler transformations $E(e \wedge c)$ with $c^2 = -2$. The following theorem gives a "best representative" for these elements that is different than the one given in 1.5. Recall that to an embedded 2-sphere C with self-intersection -2 there is associated *order* 2 Dehn twist $T(C) \in \mathrm{Mod}(M_d)$.

Theorem 1.7 (Eichler representatives). *Let $\pi_d : M_d \to \mathbb{P}^1$ be a generic genus one fibration of arithmetic genus $d \geq 1$ and with fiber class $e \in \Lambda_d$. Given any $c \in H(e)$ with $c \cdot c = -2$, there exists:*

(1) an open disk $U_c \subset \mathbb{P}^1$ with two embedded 2-spheres $C, C' \subset \pi^{-1}U_c$ representing resp. c and $e - c$ such that π_d has precisely two singular fibers over $p, q \in U_c$, and these define the same vanishing cycle in a smooth fiber over U_c. See Figure 2.

(2) a diffeomorphism $f_c : M_d \to M_d$ such that :

(a) f_c acts by fiberwise translations, has support in $\pi^{-1}U_c$ and induces in Λ_d the Eichler transformation $E(e \wedge c)$ associated to c,

(b) the Dehn twists associated to C resp. C' can be represented by fiber-preserving diffeomorphisms $\tilde{\tau}(C)$ resp. $\tilde{\tau}(C')$ that lift a diffeomorphism of $\mathbb{P}^1$ supported on U_c that interchanges p and q and represents a 'simple braid', such that $\tilde{\tau}(C')\tilde{\tau}(C)^{-1}$ is isotopic to f_c by a fiber-preserving isotopy. See Figure 3.

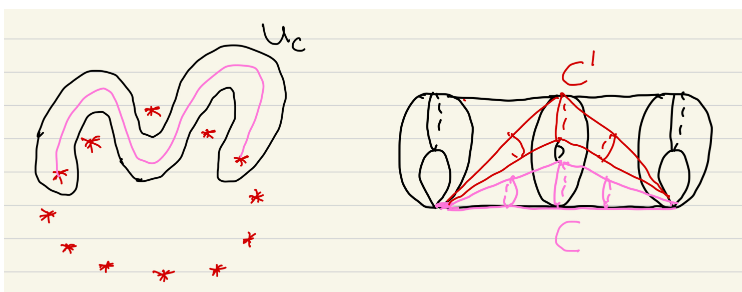

FIGURE 2. On the left: each $c \in H(e)$ with $c^2 = -2$ determines a pink path γ_c between the projections to $\mathbb{P}^1$ of two nodal fibers; the disk U_c is a tubular neighborhood of γ_c. On the right: a picure of $\pi^{-1}(\gamma_c)$, together with 2-spheres C, C'; these exist since the two singular fibers in $\pi^{-1}(\gamma_c)$ have a common vanishing cycle. Each of C, C' is made up of two "thimbles", in the terminology of Lefschetz.

We prove Theorem 1.7 at the end of §5. It gives another way to realize the group $\Lambda_d(e)$ by a group of fiber-preserving diffeomorphisms.

Corollary 1.8 (Nielsen Realization by Eichler transformations). *For each $d \geq 1$, the lattice $\Lambda_d(e)$ admits a basis $\mathscr{C}$ consisting of (-2)-vectors. For any such basis, the associated set $\{f_c\}_{c\in\mathscr{C}}$ (in the notation of Theorem 1.7) of fiberwise-translations determines an injective homomorphism $\Lambda_d(e) \to \mathrm{Trans}(\pi_d)$ whose composite with the projection $\mathrm{Trans}(\pi_d) \to \mathrm{MW}(\pi_d)$ is an isomorphism of lattices. This isomorphism is the inverse of the "Eichler representation" of $\mathrm{MW}(\pi_d)$ on Λ_d.*

The normal forms given in Theorem 1.5 and Theorem 1.7 are of two contrasting flavors: in the first, a free abelian group of fiberwise-translations of M_d representing $\Lambda_d(e)$ acts freely on M_d; in the second, elements of $\Lambda_d(e)$ are represented by diffeomorphisms with small support (in particular that are the identity on large open sets).

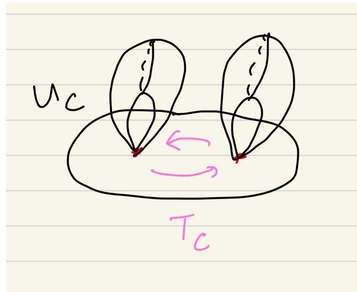

FIGURE 3. In the situation described in Figure 2, the Dehn twist $\tilde{\tau}(C)$ about C (resp. C') is a lift of an element of the spherical braid group; it "braids" two singular fibers of π with the same vanishing cycle. The product $\tilde{\tau}(C')\tilde{\tau}(C)^{-1}$ is isotopic to the fiber-preserving diffeomorphism f_c representing the Eichler transformation $E(e \wedge c) \in \mathrm{O}(H_M)$.

1.6. **Fiber-preserving diffeomorphisms.** As an application of the results above, we are able to give a geometric realization for the full stabilizer $\Gamma_{d,e}$. To explain this, let $e \in \Lambda_d$ denote the class of a fiber of the elliptic fibration $\pi_d : M_d \to \mathbb{P}^1$, and let $\varepsilon \in H^2(M_d)$ denote its Poincaré dual. The first Chern class of the canonical bundle of M_d (endowed with a complex structure that makes π_d holomorphic) is $(d-2)\varepsilon$. When $d \neq 2$ this class is nonzero and more is true: the curves E on M_d with the property that $(d-2)E$ is a canonical divisor are precisely the fibers of π. Friedman-Morgan [FM97] have shown that this class is for $d > 2$ a differentiable invariant up to sign: every self-diffeomorphism of M_d preserves that class up to sign. Subsequently, it was proved that this is the only restriction, so that $\rho_d : \mathrm{Mod}(M_d) \to \Gamma_d$ is surjective for for $d \leq 2$ (an older result of Borcea, [Bo]); and for $d \geq 3$ is equal to the Γ_d-stabilizer $\Gamma_{d,e}$ of e in Λ_d (a more recent result of Lönne, [L21]). We prove in Corollary 5.2 below Lönne's result can be lifted to the mapping class group of π_d in the following sense:

Theorem 1.9 (**Geometric realization**). *Each element of $\Gamma_{d,e}$ is realized by a diffeomorphism of M_d that takes elliptic fibers (of π_d) to elliptic fibers.*

This has the following remarkable consequence.

Corollary 1.10. *If $d \geq 3$ then any two generic genus one fibrations $\pi, \pi' : M_d \to \mathbb{P}^1$ are* topologically isotopic*: there exists a one-parameter family $\{h_t \in \mathrm{Homeo}(M_d)\}_{0 \leq t \leq 1}$ with h_0 the identity and h_1 taking π'-fibers to π-fibers. In particular, every diffeomorphism of M_d is topologically isotopic to a diffeomorphism that takes elliptic fibers to elliptic fibers.*

Proof. Moishezon [M] proved that there exists $h \in \mathrm{Diff}(M_d)$ taking π'-fibers to π-fibers. By Theorem 1.9 there exists $f \in \mathrm{Diff}(\pi)$ so that $(f \circ h)_* = \mathrm{I} \in \mathrm{O}(\Lambda_d)$. By a recent result of Gabai-Gay-Hartman-Krushkal-Powell, building on earlier work of Kreck, Perron and Quinn (see [GGHKP] and the references contained therein), $f \circ h$ is topologically isotopic to the identity. □

1.7. **Outline.** In order to prove Theorem 1.2 we introduce in §3.4 'relative Mordell-Weil groups', which should also be a useful tool for future computations. In §3.6 we find formulas for (relative) Mordell-Weil groups under fiberwise connect-sum. A core component of this is the computation, in §4, of the relative Mordell-Weil group of a certain compact 4-manifold with nonempty boundary, equipped with a genus one fibration with two singular fibers of the 'equinodal type'. The gluing formulas then allow us to compute the differentiable Mordell-Weil group for any elliptic fibration (with all nodal fibers) of any 4-manifold. Note that none of the above concepts exists in the classical case of complex-holomorphic Mordell-Weil groups, as biholomorphic maps cannot fix a nonempty open subset of M. Other tools used in this paper include spectral sequences, the theory of quadratic forms and the theory of reflection groups.

1.8. **Acknowledgements.** We would like to thank R. Hain, C. McMullen and M. Powell for helpful comments on an earlier version of this paper.

2. Background material

In this section we state and prove some background results that will be used throughout the paper.

2.1. **Realizing unipotent transformations.** We begin by proving the claim made in Subsection 1.5 (but we will use this only for $d \leq 2$).

Proposition 2.1. *Any unipotent $U \in \mathrm{O}(\Lambda_d)$ fixes some primitive isotropic vector $e \in \Lambda_d$.*

Proof. Note that the fixed-point lattice Λ_d^U of U in Λ_d cannot be definite, for then its orthogonal complement would contain another fixed vector. It then suffices to show that Λ_d^U has rank ≥ 5, for Meyer's theorem then implies that Λ_d^U represents zero.

We put $H_{d,\mathbb{Q}} := \mathbb{Q} \otimes H_M$. By the Jacobson-Mozorov theorem there exists a homomorphism of $\mathbb{Q}$-algebraic groups which on $\mathbb{Q}$-points is given by a $\rho : \mathrm{SL}_2(\mathbb{Q}) \to \mathrm{O}(H_{d,\mathbb{Q}})$ with $\rho(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) = U$. We decompose $H_{d,\mathbb{Q}}$ into irreducible $\mathrm{SL}_2(\mathbb{Q})$-representations: if V_k stands for the irreducible $\mathrm{SL}_2(\mathbb{Q})$-representation of degree $k+1$ and I_k for the space of $\mathrm{SL}_2(\mathbb{Q})$-homomorphisms $V_k \to H_{d,\mathbb{Q}}$, then the natural map

$$\oplus_{k\geq 0} V_k \otimes I_k \to H_{d,\mathbb{Q}}$$

is an isomorphism of $\mathrm{SL}_2(\mathbb{Q})$-representations and defines the isotypical decomposition $H_{d,\mathbb{Q}} = \oplus_{k\geq 0} H_{d,\mathbb{Q}}^{(k)}$. That decomposition will be orthogonal with respect to the given form on $H_{d,\mathbb{Q}}$ and hence induces in each $H_{d,\mathbb{Q}}^{(k)}$ a nondegenerate $\mathrm{SL}_2(\mathbb{Q})$-invariant quadratic form.

For k even, V_k comes with a nondegenerate $\mathrm{SL}_2(\mathbb{Q})$-invariant quadratic form of signature $(k/2, 1 + k/2)$. This determines on I_k a nondegenerate quadratic form so that the isomorphism $V_k \otimes I_k \cong H_{d,\mathbb{Q}}^{(k)}$ is one of quadratic spaces. From this we see that $H_{d,\mathbb{Q}}^{(k)}$ has a positive definite subspace of dimension $\geq \frac{1}{2} k \dim I_k$.

For k is odd, V_k comes with a nondegenerate alternating form. This determines on I_k a nondegenerate alternating form such that $V_k \otimes I_k \cong H_{d,\mathbb{Q}}^{(k)}$ is an isomorphism of quadratic spaces. In particular, the Witt index of $H_{d,\mathbb{Q}}^{(k)}$ is zero, and so $H_{d,\mathbb{Q}}^{(k)}$ will have a positive definite subspace of dimension $\geq \frac{1}{2}(k+1) \dim I_k$.

Since $H_{d,\mathbb{Q}}$ is of dimension $12d - 2$ and has signature $(2d - 1, 10d - 1)$, we must have $\sum_k (k+1) \dim I_k = 12d - 2$ and $\sum_k \frac{1}{2} k \dim I_k \leq 2d - 1$. It follows that

$$\dim I_0 \geq \textstyle\sum_k \big((k+1) - 2k\big) \dim I_k \geq (12d - 2) - 4(2d - 1) = 4d + 2 \geq 6,$$

which implies that $\mathrm{rank}(\Lambda_d^U) \geq 6$. □

Corollary 2.2. *Every unipotent element of $\mathrm{O}(H_2)^+$ is realized by a diffeomorphism of M_2 which preserves the π_2-fibers.*

Proof. Proposition 2.1 implies that such unipotent $\varphi \in O(H_2)^+$ preserves a primitive isotropic vector. Since $O(H_2)^+$ acts transitively on such vectors and is the image of $\mathrm{Mod}(M_2)$, we can without loss of generality assume that ϕ preserves e. It follows from Theorem 1.9 of [FL2] that φ can be realized by some fiber preserving diffeomorphism. □

Remark 2.3. The restriction in Corollary 2.2 to $d = 2$ is for a reason: when $d > 2$, the fiber class e is by the theorem of Friedman-Morgan mentioned earlier (up to sign) intrinsic to M_d and hence preserved up to sign by every diffeomorphism. On the other hand for $d = 1$ it is not true as stated, for then $\Gamma_1 = O(H_1)$ and the lattice H_1 is isomorphic to both $\mathbf{I} \perp \mathbf{I}(-1) \perp \mathbf{E}_8(-1)$ and $\mathbf{I} \perp 9\mathbf{I}(-1)$ (odd unimodular lattices of the same indefinite signature are isomorphic to each other). Both have $\mathbf{I} \perp \mathbf{I}(-1)$ as their first two summands. The difference of its two basis elements define via these isomorphisms primitive isotropic vectors e, e' of H_1 with $H_1(e) \cong \mathbf{E}_8(-1)$ and $H_1(e') \cong 8\mathbf{I}(-1)$. These represent the class of an elliptic fibration resp. of a conic bundle and every primitive isotropic vector of H_1 is $O(H_1)$-equivalent to e or e'. The orthogonal transformation which via $H_1 \cong \mathbf{I} \perp 9\mathbf{I}(-1)$ has the diagonal form $(1, 1, -1, \dots, -1)$ does not fix a primitive isotropic vector representing an elliptic fibration. We make this concrete in Example 2.4 below.

Example 2.4 (*Rational elliptic surface*). Let F and F' be smooth cubic curves in $\mathbb{P}^2$ that intersect in nine distinct points $p_0, \dots, p_8$. Then the cubics in $\mathbb{P}^2$ passing through $p_0, \dots, p_8$ make up a pencil that is generated by F and F'. Assume that each member of the pencil is irreducible and is separated on the blowup $Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $\{p_i\}$, so that Y is endowed with the structure of an elliptic fibration $\pi : Y \to \mathbb{P}^1$. Each exceptional divisor E_i over p_i appears as a section of π.

For each $1 \le i \le 8$, there is a unique element of $\mathrm{MW}_{\mathrm{hol}}(\pi)$ that takes E_0 to E_i; we therefore simply denote that element by $E_i - E_0$. These differences can also be viewed as cycles on Y. In that sense they are independent in $H_2(Y)$. In fact, $H_2(Y)$ is generated by the classes $\{e_i := [E_i]\}_{i=0}^8$ together with the class ℓ of a line in $\mathbb{P}^2$ not passing through $p_0, \dots, p_8$ (viewed as a class on Y). This implies that the $\{E_i - E_0\}_{i=1}^8$ are also independent as elements of $\mathrm{MW}_{\mathrm{hol}}(\pi)$. But the latter is strictly larger (one contains the other as a sublattice of index 9): for example, the line through p_1 and p_2 does not contain any other p_i and hence its strict transform L_{12} in Y is a section. Note that its class is $\ell - e_1 - e_2$. So $L_{12} - E_3$ lies in $\mathrm{MW}_{\mathrm{hol}}(\pi)$ but its class $\ell - e_1 - e_2 - e_3$ is not an integral linear combination of the $\{e_i - e_0\}_{i=1}^8$. It is known (see also the discussion below) that

$$\mathrm{MW}_{\mathrm{hol}}(\pi) \cong [F]^\perp / \mathbb{Z}[F]$$

where we note that $[F] = 3\ell - \sum_{i=0}^8 e_i$ is isotropic, so that indeed $[F] \in [F]^\perp$. Since taking the quotient by $\mathbb{Z}[F]$ allows us to eliminate e_0, a basis of $[F]^\perp/\mathbb{Z}[F]$ is given by the images of $\ell - e_1 - e_2 - e_3, e_1 - e_2, \dots, e_7 - e_8$. This is the standard basis of the even, unimodular, rank 8, negative-definite lattice $\mathbf{E_8}(-1)$. So $\{L_{12} - E_3, E_1 - E_2, \dots, E_7 - E_8\}$ is a basis for $\mathrm{MW}_{\mathrm{hol}}(\pi)$.

A conic fibration as mentioned in Remark 2.3 is represented by the strict transforms of the lines in $\mathbb{P}^2$ passing through p_1. Its fiber class is $\ell - e_1$.

2.2. **The translation subgroup.** Recall that an *affine structure* on manifold M is specified by an atlas whose coordinate changes are affine-linear and which is maximal for that property; it is equivalent to giving a flat, torsion-free connection on the tangent bundle TM.

A closed genus one surface admits such a affine structure. When endowed with it, the resulting affine surface E becomes a principal homogeneous space for the identity component its automorphism group. We denote that group (which is isomorphic to the 2-torus T^2) by $\mathrm{Trans}(E)$ and refer to it as the *translation group* of E. A complex structure on E determines such an affine structure, because there is then a unique flat metric compatible with the complex structure that gives that fiber unit volume. Then $\mathrm{Trans}(E)$ is the identity component of the automorphism group of the Riemann surface E and hence is identified with the Jacobian of E.

The situation is similar for a once or twice punctured 2-sphere: an affine structure makes such a surface a torsor of the identity component of its automorphism group, which is then isomorphic to the additive group resp. the multiplicative group of $\mathbb{C}$. There are no other orientable connected surfaces with that property.

2.3. **Various mapping class groups of genus one fibrations.** In order to understand more subtle properties of fiber-preserving diffeomorphisms of M, we introduce some related subgroups of $\mathrm{Diff}(M)$. Let $\pi : M \to \mathbb{P}^1$ be a genus one fibration. Define

$$\begin{aligned}
\mathrm{Diff}(\pi) :=&\{F \in \mathrm{Diff}(M) :\ F \text{ takes fibers of } \pi \text{ to fibers of } \pi\}\\
\mathrm{Diff}(M/\mathbb{P}^1) :=&\{F \in \mathrm{Diff}(\pi) : F \text{ preserves each fiber of } \pi\}\\
\mathrm{Diff}^0(M/\mathbb{P}^1) :=&\{F \in \mathrm{Diff}(M/\mathbb{P}^1) : \text{the restriction of } F \text{ to each smooth fiber is}\\
&\text{isotopically trivial}\}
\end{aligned}$$

For a singular (Kodaira) fiber of π, the affine structure extends to the part where the fiber is smooth (so *a fortiori* of multiplicity one). The fiberwise translations then extend to this fiber by a certain group multiplication. So

$$\mathrm{Trans}(\pi) \subset \mathrm{Diff}^0(M/\mathbb{P}^1) \subset \mathrm{Diff}(M/\mathbb{P}^1) \subset \mathrm{Diff}(\pi) \subset \mathrm{Diff}(M) \tag{2.1}$$

Applying π_0 to each group in (2.1) gives a string of homomorphisms

$$\mathrm{MW}(\pi) \to \mathrm{Mod}^0(M/\mathbb{P}^1) \to \mathrm{Mod}(M/\mathbb{P}^1) \to \mathrm{Mod}(\pi) \to \mathrm{Mod}(M). \tag{2.2}$$

Note that the image of $\mathrm{Mod}^0(M/\mathbb{P}^1)$ in $\mathrm{O}(H_M)$ lies in $H(e)$ for $e \in H_M$ the fiber class of the fibration π.

Remark 2.5. If $\pi : M \to \mathbb{P}^1$ is an elliptic fibration, then $\mathrm{Mod}^0(M/\mathbb{P}^1)$ of $\mathrm{Mod}(M/\mathbb{P}^1)$ has index two. In that case the nontrival coset is represented by the involution that acts in each fiber as 'minus the identity' relative to the given section. If all fibers are integral, then such an involution acts in $H(e)$ as minus the identity.

Let $\pi : X \to B$ be a (locally trivial) fiber bundle in the smooth category whose fibers are closed genus one surfaces. Assume that both X and B are oriented; this will then orient every fiber. In case each fiber comes endowed with an affine structure (depending smoothly on the base point), then the structure group of π is the semi-direct product $T^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. This determines a subgroup $\mathrm{Diff}^+_{\mathrm{aff}}(\pi) \subset \mathrm{Diff}^+(\pi)$ as the subgroup of preserving this affine structure. The subgroup that in addition preserves each fiber,

$$\mathrm{Diff}^+_{\mathrm{aff}}(X/B) := \mathrm{Diff}^+(X/B) \cap \mathrm{Diff}^+_{\mathrm{aff}}(\pi),$$

contains the group $\mathrm{Trans}(X/B)$ of fiberwise translations as a normal, abelian subgroup. We have denoted the connected component group of $\mathrm{Trans}(X/B)$ by $\mathrm{MW}(\pi)$; for any of the other diffeomorphism groups, we denote its connected component group by replacing Diff by Mod.

We will need the following "families version" of a theorem of Earle-Eells. Let $\mathrm{Diff}^0(X/B)$ denote the subgroup of $\mathrm{Diff}(X/B)$ of diffeomorphisms that induce in each fiber X_b the identity on $H_1(X_b)$. (Warning: $\mathrm{Diff}^0(X/B)$ need not be the identity component of $\mathrm{Diff}(X/B)$).

Proposition 2.6. *The inclusion* $\mathrm{Trans}(\pi) \subset \mathrm{Diff}^0(X/B)$ *induces a surjection on arcwise connected components.*

Proof. This is known to be so when B is a point; in fact, a theorem due to Earle-Eells ([EE], last Corollary of §11) asserts that there exists a strong deformation retraction

$$r : [0,1] \times \mathrm{Diff}^0(T^2) \to \mathrm{Diff}^0(T^2)$$

of $\mathrm{Diff}^0(T^2)$ onto the translation group $\mathrm{Trans}(T^2)$ (so that it is a copy of T^2), i.e., r_t is the identity on $\mathrm{Trans}(T^2)$ for all $t \in [0,1]$, r_0 is the identity and r_1 has image $\mathrm{Trans}(T^2)$.

Choose a smooth triangulation Σ of B whose vertex set contains the discriminant. Let $f \in \mathrm{Diff}^0(X/B)$. The first step is to find a path in $\mathrm{Diff}^0(X/B)$ that connects f with an f_0 that is the identity over a regular neighborhood of the 0-skeleton of Σ. For the smooth fibers we can invoke the theorem of Earle-Eells, but the singular fibers need special care: first choose a path in $\mathrm{Diff}^0(X/B)$ that connects f with an f_0' that is the identity near the singular points of the fibers. Then f_0' induces in the smooth part of each singular fiber (which is a copy of $\mathbb{C}$ or $\mathbb{C}^\times$) a compactly supported diffeomorphism. It is well-known that the compactly supported diffeomorphisms of such a surface have the identity element as a strong deformation retract. The construction of f_0 is then straightforward.

The rest of the argument follows a standard pattern: if $f_{k-1} \in \mathrm{Diff}^0(X/B)$ is a fiberwise translation on a regular neighborhood of the $(k-1)$-skeleton for some $k \in \{1, 2\}$, then we prove that there exists a path in $\mathrm{Diff}^0(X/B)$ from f_{k-1} to some f_k that is a fiberwise translation over a regular neighborhood of the k-skeleton. This will of course imply the proposition. If σ and a k-simplex of Σ and we choose a trivialization $X_{|\sigma|} \to T^2$ of $\pi_{|\sigma|} : X_{|\sigma|} \to |\sigma|$ as an affine family, then f_{k-1} defines a map $|\sigma| \to \mathrm{Diff}^0(T^2)$ which takes values in $\mathrm{Trans}(T^2)$ on a neighborhood of the boundary. The Earle-Eells theorem then provides the desired path over σ from f_{k-1} to a map taking its values in $\mathrm{Trans}(X/B)$. We omit the details. □

2.4. **Translations near a Kodaira fiber.** We now recall what happens near a singular. Assume that B is an open complex disk, $o \in B$ and that the genus one fibration $\pi : X \to B$ has X_o as its unique singular fiber (of Kodaira type). Denote by $\mathrm{Trans}(X_o)$ the group of automorphisms of X_o that extend to a B-automorphism of X acting as fiberwise translations in the smooth fibers. Then $\mathrm{Trans}(X_o)$ acts transitively on the part of X_o where it is smooth (we use here smooth in the sense of algebraic geometry and so this implies reduced). This smooth part need not be connected and hence $\mathrm{Trans}(X_o)$ may have torsion. The Kodaira theory shows that this torsion group is naturally isomorphic to a discriminant group: the class $e \in H_2(X)$ generates the radical of the intersection pairing on $H_2(X)$, so that $\overline{H}_2(X) := H_2(X)/\mathbb{Z}e$ is nondegenrate and then the torsion group in question is naturally identified with the finite abelian group

$$\mathrm{Discr}\, H_2(X) := \overline{H}_2(X)^\vee / \overline{H}_2(X).$$

This group is trivial if and only when X_o is integral (a nodal or a cuspidal curve, but of course also when it is smooth) or is of Kodaira type II^* (as $\hat{E}_8$-fiber). The identitity component of $\mathrm{Trans}(X_o)$ is of multiplicative type in case X_o is of type I_r (the affine Coxeter group is then of affine type $\hat{A}_{r-1}$) and is of additive type otherwise.

We thus obtain for any genus one fibration $\pi : X \to B$ an sheaf of abelian topological groups $\mathscr{T}_{\mathrm{hol}}(\pi)$ on B. We define its C^∞-counterpart as $\mathscr{T}(\pi) := \mathcal{E}_B \otimes_{\mathcal{O}_B} \mathscr{T}_{\mathrm{hol}}(\pi)$, where $\mathcal{E}_B$

is the sheaf of smooth $\mathbb{R}$-valued functions on B. So here the translation may depend in a differentiable manner on the base point. We will write $\mathrm{Trans}(\pi)$ for the group of global sections of $\mathscr{T}(\pi)$.

3. The differentiable Mordell-Weil group

In this section we introduce a smooth version of the holomorphic Mordell-Weil group. This group already made an implicit appearance in the preceding section, but here we give the formal definition and develop this notion in a more systematic manner.

3.1. **Smooth Mordell-Weil groups.** In the rest of this section B will always stand for an *oriented* smooth surface of finite type with compact (possibly empty) boundary ∂B and interior $\mathring{B}$. If it has a complex structure, we will explicitly say so.

The preceding leads to the following notions. A *genus one fibration* over B is a proper differentiable map $\pi : X \to B$, where X is an oriented 4-manifold with boundary $\partial X = \pi^{-1}\partial B$ of which each smooth fiber comes with an affine structure structure smoothly depending on the base point that makes it a torsor for a torus. We demand that the discriminant D_π of π (i.e., the set of critical values) is contained in $\mathring{B}$ and that each of its points has a neighborhood over which π is isomorphic to a Kodaira degeneration by an orientation preserving diffeomorphism that is fiberwise affine. In that case, the sheaf $\mathscr{T}(\pi)$ and its group $\mathrm{Trans}(\pi)$ of global sections is still defined.

Definition 3.1 (**Relative Mordell-Weil group).** *With notation as above, the (smooth)* Mordell-Weil group $\mathrm{MW}(\pi)$ *is the group of connected components of the group* $\mathrm{Trans}(\pi)$ *of diffeomorphisms of B that are fiberwise translations (as this is clearly an abelian group, we write this group additively).*

More generally, if $K \subset B$ is a closed subset, then $\mathrm{MW}(\pi, \pi_K)$ *stands for the group of connected components of the group of diffeomorphisms of M that are fiberwise translations and are the identity over a neighborhood of K; in case $K = \partial B$, we often write $\partial\pi$ for π_K.*

If there are no singular fibers (so that we have an affine torus bundle), then the above definition makes of sense for any base manifold. With this mind, it is not hard to see that there is an exact sequence

$$\mathrm{MW}(\pi, \partial\pi) \to \mathrm{MW}(\pi) \to \mathrm{MW}(\partial\pi).$$

Although our main interest will be when the singular fibers are all nodal, we also sometimes have to deal with other Kodaira fibers. We note that a section σ of π will meet every fiber transversally and will do so in a smooth point of that fiber (so that the component on which lies must have multiplicity one in the fiber class): a local equation for the fiber over $p \in S$, pulled back along σ, will have a simple zero at s.

3.2. **A cohomological characterization of** $\mathrm{MW}(\pi)$**.** Let $\pi : X \to B$ be as above. If $X_b = \pi^{-1}b$ is smooth then the translation group $\mathrm{Trans}(X_b)$ is naturally isomorphic to the torus $H_1(X_b;\mathbb{R})/H_1(X_b;\mathbb{Z})$, which we identify via Poincaré duality with $H^1(X_b;\mathbb{R})/H^1(X_b;\mathbb{Z})$. This almost remains true if X_b is a nodal curve with singular point p, for then

$$\begin{aligned}\mathrm{Trans}(X_b) &\cong H_1(X_b \smallsetminus \{p\};\mathbb{C})/H_1(X_b \smallsetminus \{p\}) \cong \\ &\cong H^1(X_b,\{p\};\mathbb{C})/H^1(X_b,\{p\}) \cong H^1(X_b;\mathbb{C})/H^1(X_b) \cong \mathbb{C}^\times\end{aligned}$$

where the second isomorphism is given by Alexander duality. This group contains the circle group $H^1(X_b;\mathbb{R})/H^1(X_b)$ as a maximal compact subgroup, and the inclusion is a homotopy equivalence.

In case X_b is a cuspidal curve, $\mathrm{Trans}(X_b) \cong \mathbb{C}$ and $H^1(X_b;\mathbb{C})/H^1(X_b)$ is trivial, so if we think of $H^1(X_b;\mathbb{C})/H^1(X_b)$ as the identity element of $\mathrm{Trans}(X_b)$, then this still a homotopy equivalence. This shows that if $\mathscr{T}^c(\pi)$ denotes the subsheaf of $\mathscr{T}(\pi)$ for which the translations belong to a maximal compact subgroup, then the quotient $\mathscr{T}(\pi)/\mathscr{T}^c(\pi)$ has its support on the discriminant D_π with fibres copies of $\mathbb{R}$ or $\mathbb{C}$. This implies that $\mathscr{T}^c(\pi) \subset \mathscr{T}^c(\pi)$ induces a map $H^q(B, \mathscr{T}^c(\pi)) \to H^q(B, \mathscr{T}(\pi))$ that is an isomorphism for $q > 0$ and induces for $q = 0$ an isomorphism on their connected component groups (note that these spaces of sections are topological groups).

Remark 3.1. In what follows we adhere to topological conventions when dealing with sheaf cohomology: if (B, C) is a topological pair with C closed in B and $\mathscr{F}$ is an abelian sheaf on the open subset $B \smallsetminus C$, then $H^q(B, C; \mathscr{F})$ stands for $H^q(B; j_!\mathscr{F})$, where $j : B \smallsetminus C \subset B$ and $j_!\mathscr{F}$ is the extension of $\mathscr{F}$ to B by zero. This means that if $i : C \subset B$ and $\mathscr{G}$ is an abelian sheaf on B, then there is a long exact sequence

$$\cdots \to H^q(B, C; j^*\mathscr{G}) \to H^q(B, \mathscr{G}) \to H^q(C, i^*\mathscr{G}) \to H^{q+1}(B, C; j^*\mathscr{G}) \to \cdots$$

We here often abuse notation a bit by suppressing j^* and i^* in the coefficient sheaves.

Proposition 3.2 (**Cohomological characterization of MW groups).** *Assume that π has only integral fibers (all singular fibers are all nodal or cuspidal). Then there are natural isomorphisms*

$$\mathrm{MW}(\pi) \cong H^1(B, R^1\pi_*\mathbb{Z}) \;\; \textit{and} \;\; \mathrm{MW}(\pi, \partial\pi) \cong H^1(B, D_\pi; R^1\pi_*\mathbb{Z}).$$

Moreover, for $q > 0$, there are natural isomorphisms

$$H^q(B, \mathscr{T}(\pi)) \cong H^{q+1}(B, R^1\pi_*\mathbb{Z}), \quad H^q(B, \partial B; \mathscr{T}(\pi)) \cong H^{q+1}(B, D_\pi; R^1\pi_*\mathbb{Z}).$$

Proof. There is (essentially by definition) a short exact sequence

$$0 \to R^1\pi_*\mathbb{Z} \to \mathscr{E}_B \otimes_{\mathbb{Z}} R^1\pi_*\mathbb{Z} \to \mathscr{T}^c(\pi) \to 0. \tag{3.1}$$

Since $\mathscr{E}_B$ is a soft sheaf, so is $\mathscr{E}_B \otimes_{\mathbb{Z}} R^1\pi_*\mathbb{Z}$ and hence $H^q(B, \mathscr{E}_B \otimes_{\mathbb{Z}} R^1\pi_*\mathbb{Z}) = 0$ for $q > 0$. It follows that the long exact cohomology sequence associated to (3.1) gives the exact sequence of topological groups

$$0 \to H^0(B, R^1\pi_*\mathbb{Z}) \to H^0(B, \mathscr{E}_B \otimes_{\mathbb{Z}} R^1\pi_*\mathbb{Z}) \to H^0(B, \mathscr{T}^c(\pi)) \to H^1(B, R^1\pi_*\mathbb{Z}) \to 0$$

and that for $q > 0$:

$$H^q(B, \mathscr{T}^c(\pi)) \cong H^q(B, \mathscr{T}(\pi)) \cong H^{q+1}(B, R^1\pi_*\mathbb{Z}).$$

Since $H^0(\pi, \mathscr{E}_B \otimes_{\mathbb{Z}} R^1\pi_*\mathbb{Z})$ is a vector space (hence connected) and $H^1(B, R^1\pi_*\mathbb{Z})$ is discrete, it follows that $\mathrm{MW}(\pi) = \pi_0 H^0(B, \mathscr{T}(\pi)) \cong \pi_0 H^0(B, \mathscr{T}^c(\pi))$ gets identified with $H^1(B, R^1\pi_*\mathbb{Z})$. The assertions regarding $\mathrm{MW}(\pi, \partial\pi)$ and $H^q(B, \partial B; \mathrm{Trans}(\pi))$ are obtained similarly. □

Proposition 3.2 shows that in that situation (all fibers integral) the groups $\mathrm{MW}(\pi)$ and $\mathrm{MW}(\pi, \partial\pi)$ are finitely generated if the discriminant D_π is finite. According to the following theorem, this is true in general.

Theorem 3.3. *Assume the discriminant $D := D_\pi$ finite. Then there are short exact sequences*

$$H^1(B, D; R^1\pi_*\mathbb{Z}) \to \mathrm{MW}(\pi) \to \oplus_{b\in D} \operatorname{Discr} H_2(X_b) \to 0,$$

$$H^1(B, \partial B \cup D; R^1\pi_*\mathbb{Z}) \to \mathrm{MW}(\pi, \partial\pi) \to \oplus_{b\in D} \operatorname{Discr} H_2(X_b) \to 0.$$

In particular, $\mathrm{MW}(\pi)$ *and* $\mathrm{MW}(\pi, \partial\pi)$ *are finitely generated.*

Theorem 3.3 in particular implies Theorem 1.1.

Proof of Theorem 3.3. We only do this for $\mathrm{MW}(\pi)$; the proof for $\mathrm{MW}(\pi, \partial\pi)$ is similar. Let $D \subset B$ stand for the discriminant of π. Then there is a natural map $\mathrm{MW}(\pi, \pi_D) \to \mathrm{MW}(\pi)$. For every fiber X_b (in particular, for a singular one), the connected component group of the stalk $\mathscr{T}(\pi)_b$ is a torsion group which acts simply transitively on the set of irreducible components of X_b with multiplicity one. The evident map $\mathrm{MW}(\pi) \to \oplus_{b\in D}\pi_0\mathscr{T}(\pi)_b$ is onto. Its kernel consist of the connected component group of the sections that are zero at D. It is easy to see that there is an exact sequence

$$\mathrm{MW}(\pi, \pi_D) \to \mathrm{MW}(\pi) \to \oplus_{b\in D}\pi_0\mathscr{T}(\pi)_b \to 0.$$

This shows that $\mathrm{MW}(\pi)$ is finitely generated, as it is thus sandwiched between two finitely generated groups. □

3.3. **Sections of π and the action of $\mathrm{MW}(\pi)$ on $H^2(X)$.** If π admits a section σ then the space $\Gamma(\pi)$ of sections of π is a torsor under the group of fiberwise translations. In this case the set $\pi_0(\Gamma(\pi))$ of connected components of $\Gamma(\pi)$ is an $\mathrm{MW}(\pi)$-torsor. But sections of π need not exist. If $\sigma \in H^2(X)$ is the class of a section and $e \in H_2(X)$ is the image of the fundamental class of a general fiber, then $\langle\sigma|e\rangle = 1$ and so a necessary condition is that the linear form $\alpha \in H^2(X) \mapsto \langle\alpha|e\rangle \in \mathbb{Z}$ takes the value 1. This is equivalent to e being *primitive* in the sense that it spans a copy of $\mathbb{Z}$ as a direct summand of $H_2(X)$.

The following theorems gives (among other things) a sufficient condition for the existence of a section.

Proposition 3.4 (**Existence of sections and action of $\mathrm{MW}(\pi)$ I**)**.** *Let $\pi : X \to B$ be a genus one fibration over a compact, connected, oriented surface B with nonempty boundary ∂B. Assume that each fiber of π is integral. Denote by $e \in H_2(X)$ the homology class of a general fiber and by $H^2(X)^e$ its annihilator in $H^2(X)$ (i.e., the cohomology classes that vanish on e).*

Then π admits a section. For every $\tau \in \mathrm{MW}(\pi)$ there exists a unique $\gamma_\tau \in H^2(X)$ such that for all $\xi \in H^2(X)$:

$$\tau(\xi) = \xi + \langle\xi|e\rangle\gamma_\tau$$

and the resulting map $\tau \in \mathrm{MW}(\pi) \mapsto \gamma_\tau \in H^2(X)^e$ is an isomorphism of abelian groups.

Proposition 3.5 (**Existence of sections and action of $\mathrm{MW}(\pi)$ II**)**.** *Let $\pi : M \to \mathbb{P}^1$ be a genus one fibration of arithmetic genus $d \geq 1$ with integral fibers (so $M \cong M_d$). Then π admits a section and for every $\tau \in \mathrm{MW}(\pi)$ there exists a unique $c_\tau \in e^\perp/\mathbb{Z}e$ (where $e \in H_2(M)$ is the fiber class) such that τ acts on $H_2(M)$ as the Eichler transformation $E(e \wedge c_\tau)$ given by*

$$E(e \wedge c_\tau)(x) = x + (x \cdot e)\hat{c}_\tau - (x \cdot \hat{c}_\tau)e - \tfrac{1}{2}(c_\tau \cdot c_\tau)(x \cdot e)e$$

for all $x \in H_2(M)$, where $\hat{c}_\tau \in e^\perp$ is a lift of c_τ. The resulting map

$$\tau \in \mathrm{MW}(\pi) \to c_\tau \in e^\perp/\mathbb{Z}e$$

is an isomorphism of abelian groups. In particular, $\mathrm{MW}(\pi)$ has naturally the structure of a lattice (which is even unimodular and of signature $(2d-2, 10d-2)$).

Remark 3.6. Since $e^\perp/\mathbb{Z}e \cong \Lambda_d(e)$ is an even lattice , $\frac{1}{2}(c_\tau \cdot c_\tau) \in \mathbb{Z}$ and so $E(e \wedge c_\tau)$ preserves $H_2(M)$.

Before we get into the proofs, we note that Proposition 3.5 implies Theorem 1.2:

Proof of Theorem 1.2. Choose a basis of the lattice $\mathrm{MW}(\pi_d)$ and represent each basis element by a translation. Since $\mathrm{Trans}(\pi_d)$ is abelian, this determines a group homomorphism $\mathrm{MW}(\pi_d) \to \mathrm{Trans}(\pi_d)$ that is a section of $\mathrm{Trans}(\pi_d) \to \mathrm{MW}(\pi_d)$. □

Since the proofs of these propositions have arguments in common, we prove them almost simultaneously. We therefore let the setting of Proposition 3.5 B stand for $\mathbb{P}^1$ and X for M.

Proof of Propositions 3.4 and 3.5. We prove the theorems in a number of steps. Our assumption that the fibers of π are integral implies that $R^2\pi_*\mathbb{Z}$ is a trivial local system of rank one on B so that a natural isomorphism $H^0(B, R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$ is given by integration over (=natural pairing with) the fiber class, i.e., the map $\xi \in H^2(X) \mapsto \langle \xi | e \rangle \in \mathbb{Z}$.

Step 1: The Mordell-Weil group $\mathrm{MW}(\pi)$ *acts trivially on the grading of* $H^2(X)$ *with respect to the Leray filtration defined by* π.

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(B, R^q\pi_*\mathbb{Z}) \Rightarrow H^{p+q}(X).$$

Any diffeomorphism of X that is a fiberwise translation acts trivially on the cohomology of the fibers acts on this spectral sequence via its action on $\pi_*\mathbb{Z}$ and its higher direct images. That action is clearly trivial.

Step 2: If $H^2(B, R^1\pi_*\mathbb{Z})$ *vanishes then* π *admits a section.*

Since π has no multiple fibers, we can find an open cover $\mathscr{U}$ of B such that for each $U \in \mathscr{U}$, π_U admits a section s_U. Then $s_{U'} - s_U$ defines a section of $\mathrm{Trans}(\pi)$ over $U \cap U'$. Together they make up a Čech 1-cycle relative the covering $\mathscr{U}$ and hence define an element of $\check{H}^1(\mathscr{U}, \mathrm{Trans}(\pi))$. If this a coboundary, then there exist $\{t_U \in H^0(U, \mathrm{Trans}(\pi))\}_{U \in \mathscr{U}}$ such that $s_{U'} - s_U = t_{U'} - t_U$, which means that the collection $\{s_U + t_U\}_{U \in \mathscr{U}}$ defines a global section. Since we are allowed to pass to a refinement of $\mathscr{U}$, it suffices to show that $\check{H}^1(B, \mathrm{Trans}(\pi))$ is trivial. For paracompact spaces Čech cohomology is ordinary cohomology, and so this is by Proposition 3.2 equal to $H^2(B, R^1\pi_*\mathbb{Z})$. Hence Step 2 follows.

Step 3: Proof of Proposition 3.4.

Then B contains an embedded graph $G \subset B$ as a deformation retract which contains all the critical values. This implies that the restriction map $E_2^{p,q} = H^p(B, R^q\pi_*\mathbb{Z}) \to H^p(G, R^q\pi_*\mathbb{Z})$ is an isomorphism so that $E_2^{p,q}$ is trivial unless $p \in \{0, 1\}$ and $q \in \{0, 1, 2\}$. In particular, π admits a section by Step 2. It also follows that the Leray sequence degenerates on this page, so that there is an exact sequence

$$0 \to H^1(B, R^1\pi_*\mathbb{Z}) \to H^2(X) \to H^0(B, R^2\pi_*\mathbb{Z}) \to 0.$$

The map $H^2(X) \to H^0(B, R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$ is given by integration over the fiber class e. It is surjective, because it takes the value 1 on a section (we here regard the class of a section as an element of $H_2(X, \partial X) \cong H^2(X)$). Proposition 3.2 identifies $\mathrm{MW}(\pi)$ with its kernel of this map. Any $\tau \in \mathrm{MW}(\pi)$ induces a transformation of $H^2(X)$ which preserves the above short exact sequence and acts trivially on both $H^1(B, R^1\pi_*\mathbb{Z})$ and $H^0(B, R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$. Hence it is of the form $\xi \in H^2(X) \mapsto \xi + \langle \xi \,|\, e \rangle \gamma_\tau$ for some $\gamma_\tau \in H^1(B, R^1\pi_*\mathbb{Z})$. So if ξ is the class of a section, then its image under τ is $\xi + \gamma_\tau$. The definition shows that $\tau \mapsto \gamma_\tau$ is in fact the isomorphism $\mathrm{MW}(\pi) \cong H^1(B, R^1\pi_*\mathbb{Z})$ found above.

Step 4: Proof of Proposition 3.5

We first show that the differential $\mathbb{Z} \cong H^0(\mathbb{P}^1, R^2\pi_*\mathbb{Z}) \to H^2(\mathbb{P}^1, R^1\pi_*\mathbb{Z})$ in the Leray spectral sequence is the zero map. Indeed, we already observed that integration over e defines an isomorphism $H^0(\mathbb{P}^1, R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$. Since the Leray spectral sequence degenerates on its third page, the kernel of this differential is a quotient of $H^2(M)$ and it is via this quotient that the integration map $\xi \in H^2(M) \mapsto \langle\xi|e\rangle$ is defined. Since e is primitive, the latter is onto and hence this kernel is all of $H^0(\mathbb{P}^1, R^2\pi_*\mathbb{Z})$. So the differential $H^0(\mathbb{P}^1, R^2\pi_*\mathbb{Z}) \to H^2(\mathbb{P}^1, R^1\pi_*\mathbb{Z})$ must be the zero map.

This implies that $H^2(\mathbb{P}^1, R^1\pi_*\mathbb{Z})$ embeds in $H^3(M)$. But $H^3(M) \cong H_1(M) = 0$ and hence $H^2(\mathbb{P}^1, R^1\pi_*\mathbb{Z}) = 0$. So π admits a section by Step 2. This in turn implies that the natural map $H^2(\mathbb{P}^1) \to H^2(M)$ is injective. Hence the Leray filtration of $H^2(M)$ is given by

$$H^2(M) = L_2H^2 \supset L_1H^2 \supset L_0H^2 = H^2(\mathbb{P}^1) \supset L_{-1}H^2 = 0$$

with $L_1H^2/L_0H^2 \cong H^1(\mathbb{P}^1, R^1\pi_*\mathbb{Z})$ and $L_2H^2/L_1H^2 \cong H^0(\mathbb{P}^1, R^2\pi_*\mathbb{Z})$. The preceding shows that L_1H^2 is the annihilator of e and that $L_0H^2 = H^2(\mathbb{P}^1)$ is spanned by the Poincaré dual ε of e. If $\sigma \in H_2(M)$ is the class of a section, then its Poincaré dual and ε span a unimodular lattice of rank 2 of signature $(1, 1)$ (even or odd according the parity of $\sigma\cdot\sigma = -d$). Since the intersection pairing on $H_2(M)$ is unimodular, its orthogonal complement is also unimodular. This orthogonal complement maps isomorphically onto $\varepsilon^\perp/\mathbb{Z}\varepsilon$ and hence the latter is unimodular.

Any orthogonal transformation of the unimodular lattice $H_2(M)$ whose induced transformation on $H^2(M)$ acts trivially on the successive quotients of the above filtration (defined by the isotropic vector ε) is an Eichler transformation as stated in the theorem with c canonically defined in $\Lambda_d(e)$. In particular, we find for every $\tau \in \mathrm{MW}(\pi)$ an element $c_\tau \in \Lambda_d(e)$. For σ as above, $\tau(\sigma) \equiv \sigma + c_\tau \pmod{\mathbb{Z}\varepsilon}$. Again the definition shows that $\tau \mapsto c_\tau$ is in fact the isomorphism $\mathrm{MW}(\pi) \cong H^1(\mathbb{P}^1, R^1\pi_*\mathbb{Z}) \cong e^\perp/\mathbb{Z}e$. □

Remark 3.7. In the Kähler setting, the lattice $\varepsilon^\perp/\mathbb{Z}\varepsilon$ has a natural Hodge structure of weight 2, and a theorem of Shioda [SS] then implies that the torsion-free quotient of $\mathrm{MW}_{\mathrm{hol}}(\pi)$ can be identified with the $(1, 1)$-part of this lattice. In fact, he shows that in general (where we admit reducible fibers), it is the quotient of $\varepsilon^\perp$ by the span of the cohomology classes supported by the singular fibers.

Corollary 3.8 (**Smooth Mordell-Weil group of a rational elliptic fibration**). *If* $\pi : M \to \mathbb{P}^1$ *is a rational elliptic fibration with all fibers integral, then its holomorphic Mordell-Weil group* $\mathrm{MW}_{\mathrm{hol}}(M)$ *maps isomorphically to its smooth counterpart* $\mathrm{MW}(\pi)$*. In particular, the latter is a lattice is isometric to* $\mathbf{E_8}(-1)$*.*

As observed in Remark 1.3, this is no longer true when $d \geq 2$.

Proof. As Example 2.4 shows, the holomorphic Mordell-Weil group is identified with $e^\perp/\mathbb{Z}e$. The corollary then follows from Theorem 3.5. □

With Proposition 3.5 in hand, we can now prove the following.

Proof of Theorem 1.5. By Proposition 3.5 there is an isomorphism $\mathrm{MW}(\pi_d) \cong \Lambda_d(e)$. Now choose a basis of $\Lambda_d(e)$ and lift that basis to a group of fiberwise translations in M_d. Then the group generated by these is abelian and can be considered as Nielsen realization of $\mathrm{MW}(\pi_d)$ (and of its image in $\mathrm{O}(\Lambda_d)$). □

3.4. **Relative Mordell-Weil groups.** We continue with the setting and notation of Proposition 3.4: $\pi : X \to B$ is a genus one fibration over a compact, connected surface with nonempty boundary with integral fibers and no singular fibers over ∂B. We denote e its fiber class (that we consider as an element of $H_2(\partial X)$ but identify with its image in $H_2(X)$) and by $H^2(X)^e$ its annihilator in $H^2(X)$ as before. It is clear that the natural map $H_2(X) \to H_2(X, \partial X) \cong H^2(X)$ has its image in $H^2(X)^e$ and factors through $H_2(X)/\mathbb{Z}e$.

Proposition 3.9. *Assume we are in the setting of Proposition 3.4. Then the sequence*

$$H_2(X)/\mathbb{Z}e \to H^2(X)^e \to H^2(\partial X)^e,$$

where the first map is a factor of the map $H_2(X) \to H^2(X)$ defined by the intersection pairing, is exact and is naturally isomorphic to

$$\mathrm{MW}(\pi, \partial\pi) \to \mathrm{MW}(\pi) \to \mathrm{MW}(\partial\pi). \tag{3.2}$$

In particular, $\mathrm{MW}(\pi, \partial\pi)$ *is free abelian.*

If in addition $H_1(X) = 0$ and ∂B is connected (so diffeomorphic to a circle), then $\mathrm{MW}(\pi) \to \mathrm{MW}(\partial\pi)$ *is onto and the above sequence is part of the (self-dual) exact sequence*

$$0 \to H^1(X_b)^{\partial B} \to H_2(X)/\mathbb{Z}e \to H^2(X)^e \to H_1(X_b)_{\partial B} \to 0,$$

where $b \in \partial B$ and $H^1(X_b)^{\partial B}$ resp. $H_1(X_b)_{\partial B}$ stands for the invariant cohomology resp. coinvariant homology relative to the monodromy over ∂B.

Proof. From Proposition 3.2 we learn that the sequence (3.2) can be identified with a sequence

$$H^1(B, \partial B; R^1\pi_*\mathbb{Z}) \to H^1(B; R^1\pi_*\mathbb{Z}) \to H^1(\partial B; R^1\pi_*\mathbb{Z})$$

Each of these terms appears in Leray spectral sequences. For example,

$$E_2^{p,q} = H^p(B, \partial B; R^q\pi_*\mathbb{Z}) \Rightarrow H^{p+q}(X, \partial X) \cong H_{4-p-q}(X),$$

where the last isomorphism is Alexander duality. Since all fibers are integral, this sequence degenerates, for the term $E_2^{p,q}$ vanishes unless $p \in \{1, 2\}$ and $q \in \{0, 1, 2\}$ (a reducible fiber would give nonzero elements of $H^0_c(\mathring{B}, R^2\pi_*\mathbb{Z})$). The isomorphism $H^2(B, \partial B; \pi_*\mathbb{Z}) = H^2(B, \partial B) \cong H_0(B) \cong \mathbb{Z}$, then yields a short exact sequence

$$0 \to \mathbb{Z} \to H_2(X) \to \mathrm{MW}(\pi, \partial\pi) \to 0.$$

with $1 \in \mathbb{Z}$ mapping to the fiber class $e \in H_2(X)$. It follows that $\mathrm{MW}(\pi, \partial\pi) \cong H_2(X)/\mathbb{Z}e$. A similar argument identifies $H^1(B; R^1\pi_*\mathbb{Z})$ with $H^2(X)^e$ and $H^1(\partial B; R^1\pi_*\mathbb{Z})$ with $H^2(\partial X)^e$.

Now assume $H_1(X) = 0$ and ∂B is connected. Then $H_3(X, \partial X) = 0 \cong H^1(X) = 0$. The exact homology sequence for the pair $(X, \partial X)$ plus Alexander/Poincaré duality yields the exact sequence

$$0 \to H_2(\partial X) \to H_2(X) \to H^2(X) \to H^2(\partial X) \to 0.$$

This sequence is self-dual and induces the exact sequence

$$0 \to H^2(\partial X)/\mathbb{Z}e \to H_2(X)/\mathbb{Z}e \to H^2(X)^e \to H^2(\partial X)^e \to 0.$$

The Wang sequence for a fiber bundle over a circle identifies $H^2(\partial X)/\mathbb{Z}e$ with $H^1(X_b)^{\partial B}$ and $H^2(\partial X)^e$ with $H_1(X_b)_{\partial B}$. □

Remark 3.10. If we are in the special case of Proposition 3.9 (where $H_1(X) = 0$ and ∂B connected), then that proposition shows that $\mathrm{MW}(\partial\pi) \cong H_1(X_b)_{\partial B}$ and that there is an extension of groups

$$1 \to H^1(X_b)^{\partial B} \to \mathrm{MW}(\pi, \partial\pi) \to \mathrm{Im}(H_2(X) \to H^2(X)) \to 1.$$

We shall make the isomorphism geometrically explicit in Lemma 3.11. For now we explain how the embedding of $H^1(X_b)^{\partial B}$ in $\mathrm{MW}(\pi, \partial\pi)$ is geometrically defined.

Let U be a collar neighborhood of ∂B that does not contain any critical value and identify it with $[0,1)\times\partial B$. Choose a retraction $r : X_U \to X_{\partial B}$ such that $(\pi_U, r) : X_U \to U \times X_{\partial B}$ is a diffeomorphism and let $\varphi : [0,1] \to \mathbb{R}$ be a smooth function that is 0 near 0 and 1 near 1. We think of $a \in H_1(X_b)^{\partial B}$ as a locally constant family of classes $\{a_s \in H_1(X_s)\}_{s\in\partial B}$ with $a = a_b$. Then a fiberwise translation for π with support in U is defined by letting it over U be given (in terms of the above identifications) by

$$(t, x) \in [0,1) \times X_s \mapsto (t, x + \varphi(t)a_s) \in [0,1) \times X_s, \text{ where } s \in \partial B.$$

This defines an element of $\mathrm{MW}(\pi, \partial\pi)$ and up to a sign (which we did not bother to determine) this equals the image of a.

3.5. **Torus fibrations over a circle.** We found in Proposition 3.9 that if ∂B is connected (and hence a copy of the circle) and $b \in \partial B$, then $\mathrm{MW}(\partial\pi)$ is the group of co-invariants of the monodromy action on $H_1(X_b)$. As promised, we shall make this a bit more explicit (and geometric). Let us state and prove this in the appropriate generality, as there is no extra cost in doing so.

Lemma 3.11. *Let $f : Y \to T$ be a locally trivial fiber bundle over the unit circle $T \subset \mathbb{C}$ whose fibers are torsors of affine n-tori (so that its structural group is $\mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n$). Let $A \in \mathrm{GL}(H_1(Y_1)) \cong \mathrm{GL}(n,\mathbb{Z})$ be the monodromy of this bundle. Then its Mordell-Weil group $\mathrm{MW}(f)$ is naturally isomorphic to the group $H_1(Y_1)_A$ of monodromy co-invariants and this group acts simply transitively on $\pi_0(\Gamma(f))$.*

Proof. Since the fiber Y_1 is connected, f will certainly admit a section. Let σ_0 be one such. To give a fiberwise translation is then give a section and vice versa and so this identifies $\mathrm{MW}(f)$ with the connected component group of the space of sections of f.

The section σ_0 identifies Y with the bundle whose fiber over $s \in T$ is $H_1(Y_s;\mathbb{R})/H_1(Y_s;\mathbb{Z})$ (we may regard this as the Jacobian bundle) and we can reconstruct f from A by taking the product $[0,2\pi]\times H_1(Y_1;\mathbb{R})$ and dividing out by $H_1(Y_1;\mathbb{Z})$ acting on $H_1(Y_1;\mathbb{R})$ by translations and identifying $(0,x) \in \mathbb{R} \times H_1(Y_1;\mathbb{R})$ with $(2\pi, Ax)$ so that the orbit space then projects onto $\mathbb{R}/2\pi\mathbb{Z} \cong T$. Any $v \in H_1(Y_1;\mathbb{Z})$ determines a section σ_v of f by assigning to $e^{\sqrt{-1}t}$ the image of $(t, (2\pi)^{-1}tv) \in \mathbb{R}\times H_1(Y_1;\mathbb{R})$ in Y. We leave it to the reader to check that $v \in H_1(Y_1;\mathbb{Z}) \mapsto [\sigma_v] \in \mathrm{MW}(f)$ is onto and that $[\sigma_v] = 0$ if and only if $v = A(v') - v'$ for some $v' \in H_1(Y_1;\mathbb{Z})$. □

3.6. **Mordell-Weil groups of fiberwise connected sums.** In this section we describe a connected sum construction for genus one fibered 4-manifolds and determine its effect on Mordell-Weil groups. We shall illustrate this by the glueing of two rational elliptic fibrations producing an elliptic K3 manifold.

Assume that we are in the setting of Proposition 3.5, where we have a genus one fibration $\pi : X \to B$ with B a copy of $\mathbb{P}^1$ and X simply connected (a copy of some M_d). Let $b \in B$ be a regular value of π. The *real oriented blowup* of $b \in B$ produces a surface with boundary $\hat{B}_b$ diffeomorphic with a closed disk and a smooth map $f : \hat{B}_b \to \mathbb{P}^1$ that is a diffeomorphism over $B \smallsetminus \{b\}$ and where $\partial\hat{B}_b := f^{-1}b$ is the circle of rays in the tangent plane T_bB (to be thought of as a boundary of $B \smallsetminus \{b\}$). It is clear that the pull-back $\hat{\pi} : \hat{X} := f^*X \to \hat{B}_b$ of π is a genus one fibration whose restriction to $f^{-1}b$ is the projection $\partial\hat{B}_b \times X_b \to \partial\hat{B}$.

Suppose we are given another such pair $(\pi' : X' \to B', b' \in B')$. Then the choice of an affine isomorphism $u : X_b \cong X'_{b'}$ and an orientation *reversing* isomorphism $v : T_bB \cong T_{b'}B'$

allows us to glue the two fibrations $\hat{X} \to \hat{B}$ and $\hat{X}' \to \hat{B}'$ to produce a genus one fibration that we might denote by

$$(\pi, b)\#_{(u,v)}(\pi', b') : (X, b)\#_{(u,v)}(X', b') \to (B, b)\#_v(B', b'),$$

were we to insist on a notation that expresses all the dependencies. We will however simply write $\pi'' : X'' \to B''$. Since the possible choices for b, b' and v vary in a connected family, the diffeomorphism type of π'' only depends on the isotopy class of u. This connected sum is of the same type as its terms: B'' is a copy of $\mathbb{P}^1$ and X'' is simply connected. The discriminant of π'' appears here as the disjoint union of discriminants of π and π'. It follows that the arithmetic genus of X'' is the sum of the arithmetic genera of X and X'. This construction, to which we shall refer as a *fiber connected sum*, was essentially introduced by Moishezon [M]. The base B'' contains copies $\hat{B}$ and $\hat{B}'$ over which this bundle reproduces resp. $\hat{\pi}$ and $\hat{\pi}'$ and these copies intersect in a copy S of the circle that we shall identify with $\partial\hat{B}$. The following theorem tells us how $\mathrm{MW}(\pi'')$ is related to $\mathrm{MW}(\pi)$ and $\mathrm{MW}(\pi')$.

Theorem 3.12. *The restriction of π'' to S determines a primitive isotropic sublattice $I \subset \mathrm{MW}(\pi'')$ of rank 2. We then have natural isomorphisms:*

$$I \cong H^1(X_b), \quad I^\perp/I \cong \mathrm{MW}(\pi) \oplus \mathrm{MW}(\pi'), \quad \mathrm{MW}(\pi'')/I^\perp \cong I^\vee \cong H_1(X_b).$$

with the middle isomorphism being an isomorphism of lattices.

Proof. The cohomology exact sequence of the sheaf $R^1\pi_*\mathbb{Z}$ for the pair $(B, \{b\})$ gives the short exact sequence

$$0 \to H^1(X_b) \xrightarrow{\delta} H^1(B, \{b\}; R^1\pi_*\mathbb{Z}) \to H^1(B, R^1\pi_*\mathbb{Z}) \to 0$$

and also shows that $H^0(B, \{b\}; R^1\pi_*\mathbb{Z}) \cong H^0(B, R^1\pi_*\mathbb{Z}) = 0$. Next we consider a piece of cohomology exact sequence of the sheaf $R^1\pi''_*\mathbb{Z}$ for the pair (B'', S):

$$\text{(3.3)} \quad H^0(B'', S; R^1\pi''_*\mathbb{Z}) \to H^0(S; R^1\pi''_{S*}\mathbb{Z}) \xrightarrow{\delta''} H^1(B'', S; R^1\pi''_*\mathbb{Z}) \to \\ \to H^1(B''; R^1\pi''_*\mathbb{Z}) \to H^1(S; R^1\pi''_*\mathbb{Z}) \to \cdots$$

The bundle π''_S is trivial, so $H^0(S; R^1\pi''_{S*}\mathbb{Z}) \cong H^1(X_b)$ and $H^1(S; R^1\pi''_{S*}\mathbb{Z}) \cong H^1(S)\otimes H^1(X_b)$. The group $H^q(B'', S; R^1\pi''_*\mathbb{Z})$ decomposes as $H^q(B, \{b\}; R^q\pi_*\mathbb{Z}) \oplus H^q(B', \{b'\}'; R^1\pi'_*\mathbb{Z})$. So for $q = 0$ this group is trivial and for $q = 1$, δ'' becomes $(\delta, -\delta')$, where δ is the coboundary in the short exact sequence for the pair $(B, \{b\})$ above and δ' is the coboundary of a similar exact sequence for the pair $(B', \{b'\})$. If we substitute this in the sequence (3.3) and if we apply Proposition 3.2 to identify $H^1(B, R^1\pi_*\mathbb{Z})$, $H^1(B', R^1\pi_*\mathbb{Z})$ and $H^1(B'', R^1\pi_*\mathbb{Z})$ with $\mathrm{MW}(\pi)$, $\mathrm{MW}(\pi')$ and $\mathrm{MW}(\pi'')$, then find the exact sequence

$$0 \to H^1(X_b) \xrightarrow{(\delta, -\delta')} \mathrm{MW}(\pi) \oplus \mathrm{MW}(\pi') \to \mathrm{MW}(\pi'') \to H^1(S) \otimes H^1(X_b) \to \cdots$$

It has the property that $(\delta, 0)$ maps $H^1(X_b)$ isomorphically onto a sublattice of $\mathrm{MW}(\pi'')$. This sublattice is isotropic for geometric reasons, for if we identify $\mathrm{MW}(\pi'')$ with a subquotient of $H^2(X'')$, then the image of $H^1(X_b)$ is represented by the image of the composite $H^1(X_b) \cong H_1(X_b) \cong H_1(X_b) \otimes H_1(S) \xrightarrow{\times} H_2(X''_S) \to H_2(X'') \cong H^2(X'')$ (which also lies in the perp of the fibre class). The theorem follows. $\square$

An immediate corollary is the following.

Corollary 3.13. $\mathrm{MW}(\pi'')$ *is as a lattice isometric to* $\mathrm{MW}(\pi) \oplus \mathbf{U}^{\oplus 2} \oplus \mathrm{MW}(\pi')$.

Applying the gluing formula of Corollary 3.13, we are now able to compute the Mordell-Weil group of the generic elliptic fibration π_d.

Proof of Theorem 1.2. The generic genus one fibration π_d is (up to isomorphism) obtained by starting out with π_1 and iterating the above connected sum construction. Since $\mathrm{MW}(\pi_1) \cong \mathbf{E}_8(-1)$, such a connected sum decomposition gives via Corollary 3.13 a corresponding decomposition of smooth Mordell-Weil lattices :

$$\mathrm{MW}(\pi_d) \cong \Lambda_d(e) \cong \mathbf{E}_8(-1) \perp \underbrace{2\mathbf{U} \perp \mathbf{E}_8(-1) \perp \cdots \perp 2\mathbf{U} \perp \mathbf{E}_8(-1)}_{d-1},$$

so with d copies of $\mathbf{E}_8(-1)$ and $(d-1)$ copies of $2\mathbf{U}$; here the first isomorphism comes from Theorem 3.5. □

4. Fibrations over a disk with two nodal fibers

We here study the most basic situation, namely a genus one fibration over a closed disk B which has only two (interior) singular fibers, both of which are nodal. The special case where the two vanishing cycles on the fibers have intersection number 1, and so the monodromy around ∂B has order 6, was studied by Gompf in [G].

We begin with looking at the general case, setting up notation as we proceed, and then focus on the case when the two nodal fibers define the same vanishing cycle. Its topology, and in particular the group of mapping classes it gives rise to, turns out to be remarkably rich and subtle in structure.

So we are given a genus one fibration $\pi : X \to B$ over a closed disk with only two singular fibers, both of which are nodal and lie over interior points of B. Without loss of generality we shall assume that B is a closed complex disk of radius $r > 1$ centered at 0 and that the singular fibers lie over 1 and -1.

4.1. **First properties.** It is not difficult to see that $X_{[-1,1]} := \pi^{-1}[-1,1]$ is a deformation retract of X. So the homotopy type of π is that of its restriction $X_{[-1,1]} \to [-1,1]$ over $[-1,1]$. We shall describe a simple geometric model for this restriction.

As before, we let T stand for the unit circle in $\mathbb{C}^\times$, regarded an an oriented abelian Lie group. An orientation-preserving affine diffeomorphism h_0 of T^2 onto the smooth fiber X_0 determines circle subgroups $T_\pm \subset T^2$ such that if E is the quotient space of $[-1,1]\times T^2$ obtained by collapsing $\{-1\} \times T_{-1}$ and $\{1\} \times T_1$, then h_0 extends to a homeomorphism $h : E \cong X_{[-1,1]}$ over $[-1,1]$ that is a differentiable isomorphism of affine bundles over $(-1,1)$.

It is then clear that for each sign $\pm$, the space $X_\pm := \pi^{-1}[0,\pm 1]$ has the fiber $X_{\pm 1}$ as a deformation retract and that the natural map $H_1(X_0) \to H_1(X_\pm) \cong H_1(X_{\pm 1}) \cong \mathbb{Z}$ is a surjection. Its kernel $V_\pm \subset H_1(X_0)$, which via h corresponds to the image of $H_1(T_\pm) \to H_1(T^2)$, is the *vanishing homology* of this degeneration. See Figure 4.

The inclusion $X_0 \subset X_\pm$ is up to homotopy obtained by the 2-cellular extension of T^2 obtained by 'filling in' the circle subgroup $T_\pm$ with a 2-disk. If we fill in both disks we recover the inclusion $X_0 \subset X_{[-1,1]} \sim X$ up to homotopy. It is then clear that the map $H_1(X_0) \to H_1(X_{[-1,1]}) \cong H_1(X)$ is onto with kernel $V_- + V_+$ and that the map $H_2(X_0) \to H_2(X)$ injective with cokernel naturally identified with $V_+ \cap V_-$. The following two propositions explain why our interest is mainly in the special case when $V_- = V_+$.

Proposition 4.1. *The Mordell-Weil group* $\mathrm{MW}(\pi)$ *is naturally isomorphic to the cyclic group* $H_1(X) \cong H_1(X_0)/(V_+ + V_-)$. *So that group is infinite if and only if* $V_- = V_+$, *and is trivial if and only if* $V_- + V_+ = H_1(X_0)$.

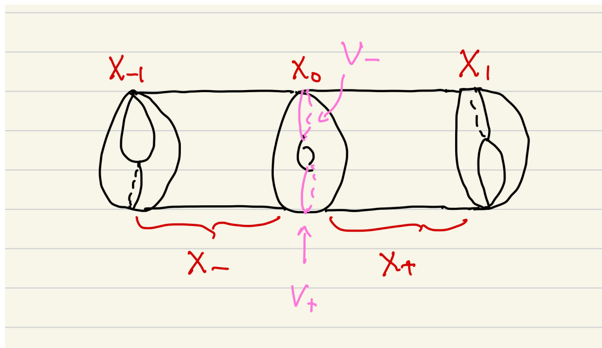

FIGURE 4. The manifold $X_{[-1,1]}$. In this case the vanishing homology subspaces $V_\pm \subset H_1(X_0)$ are generated by homologous classes.

Proof. The inclusion $[-1,1] \subset B$ induces an isomorphism $\mathrm{MW}(\pi) \cong H^1(B; R^1\pi_*\mathbb{Z}) \cong H^1([-1,1]; R^1\pi_*\mathbb{Z})$. The restriction of $R^1\pi_*\mathbb{Z}$ to $[-1,1]$ is simple: over $(-1,1)$ it is constant equal to $H^1(X_0)$ and its stalk in ± 1 is $H^1(X_{\pm 1})$. So $H^1([-1,1]; R^1\pi_*\mathbb{Z})$ is the cokernel of the natural map $H^1(X_1) \oplus H^1(X_{-1}) \to H^1(X_0)$. Alexander duality turns this into $V_+ \oplus V_- \to H_1(X_0)$ defined by the inclusions $V_\pm \subset H_1(X_0)$. □

In case V_- and V_+ are distinct (so that $\mathrm{MW}(\pi) \cong H_1(X)$ is finite), we can say a bit more:

Proposition 4.2. *When V_- and V_+ are distinct, then* $\mathrm{MW}(\pi, \partial\pi)$ *is trivial. The natural map* $\mathrm{MW}(\pi) \to \mathrm{MW}(\partial\pi)$ *is (therefore) injective with cokernel naturally identified with $H_1(X)$.*

Furthermore, if q is the order of $H_1(X) \cong H_1(X_0)/(V_- + V_+)$, then the trace of the monodromy along ∂B on H_1 of a fiber equals $2 - q^2$. In particular, this monodromy is hyperbolic unless $q = 1$ (i.e., when $V_- + V_+ = H_1(X_0)$), in which case it has order 6.

Proof. The Mayer-Vietoris sequence of the pair (X_-, X_+) gives the exact sequence

$$0 \to H_2(X_0) \to H_2(X) \to H_1(X_0) \to H_1(X_-) \oplus H_1(X_+) \to H_1(X).$$

The map $H_1(X_0) \to H_1(X_\pm)$ is given by ± 1 times the natural map $H_1(X_0) \to H_1(X_0)/V_\pm$. Since $V_- \neq V_+$, it follows that $H_1(X_0) \to H_1(X_-) \oplus H_1(X_+)$ is injective. We deduce that $H_2(X)$ is generated by e so that $\mathrm{MW}(\pi, \partial\pi)$, which by Proposition 3.9 can be identified with $H_2(X)/\mathbb{Z}e$, is trivial. This proposition also identifies $\mathrm{MW}(\pi) \to \mathrm{MW}(\partial\pi)$ with $H^2(X)^e \to H^2(\partial X)^e$. The latter map fits in the exact sequence (derived from that of the pair $(X, \partial X)$)

$$H^2(X)^e \to H^2(\partial X)^e \to H^3(X, \partial X) \to H^3(X).$$

Since X has the homotopy type of a 2-dimensional CW complex, $H^3(X) = 0$ and $H^3(X, \partial X) \cong H_1(X)$ by Alexander duality. This proves the first assertion.

For the second statement, we choose a basis e_1, e_2 of $H_1(X_0)$ such that V_+ is spanned by $\delta_+ = e_1$ and V_- by $\delta_- = pe_1 + qe_2$ (where p and q must be relatively prime). Then the monodromy is the composite of two Picard-Lefschetz transformations, namely $x \mapsto x + (x\cdot\delta_+)\delta_+$ followed by $x \mapsto x + (x\cdot\delta_-)\delta_-$. A straightforward computation shows that the trace of this composite is $2 - q^2$, as asserted. So for $|q| = 1$, the trace is 1 and any element of $\mathrm{SL}_2(\mathbb{Z})$ with that property has order 6, whereas for $|q| \geq 2$, the trace is ≤ -3 and any element of $\mathrm{SL}_2(\mathbb{Z})$ with that property is hyperbolic. □

Remark 4.3. The homeomorphism $h : E \cong X_{[-1,1]}$ is *not* unique up to relative isotopy. This is because the group of isotopy classes of self-homemorphisms of $X_{[-1,1]}$ relative to the projection onto $[-1,1]$ can be nontrivial. Indeed that group is naturally isomorphic with $H_1(X)$. To be precise, note that if $f:[-1,1]\to T^2$ is a smooth loop that is constant equal to the identity element on a neighborhood of $\{-1,1\}$, then we can postcompose h with the the map induced by

$$(s,u)\in[-1,1]\times T^2\mapsto(s,f(s)u)\in[-1,1]\times T^2$$

and this may represent a different isotopy class. Such an f defines an element of $H_1(T^2)$ and it is not hard to check that the change in relative isotopy class only depends on that element. All the relative isotopy classes of such h are thus obtained. In fact, $H_1(X_0)\cong H_1(T^2)$ permutes the choices transitively and since $H_1(T_\pm)$ gets killed over ±1, that action will be through the cokernel of $H_1(X_0)/(V_-+V_+)\cong H_1(X)$. On may verify that the latter action is simply transitive.

From now on we assume that the two nodal fibers of π have the same vanishing homology, so that $V_-=V_+$.

We assume that if $\delta,\delta'\in H_1(T^2)$ denote the generators defined by the factors, then $V_-=V_+=\mathbb{Z}\delta$. So then $H_1(X)\cong V_+$ and $H_2(X_0)$ embeds in $H_2(X)$ with cokernel identified with V_+ (hence is free abelian of rank two). We can exhibit a generator of that cokernel explicitly by representing the two vanishing cycles in T^2 by the same circle $T\times\{1\}$: this defines a 2-sphere, namely the image of $[-1,1]\times(T\times\{1\})$ under the projection $[-1,1]\times T^2\to E$. This is indeed a topological oriented 2-sphere C in $X_{[-1,1]}$. The fundamental classes of C and of the central fiber X_0 make up a free basis of $H_2(X)$. We shall see in the next subsection that we can arrange C is smooth with $C\cdot C=-2$.

The monodromy around both 1 and -1 is given by

$$\tau_\delta: a\in H_1(T^2)\mapsto a+(a\cdot\delta)\delta\in H_1(T^2).$$

Hence the monodromy of π over ∂B is τ_δ^2 (connect $0\in B$ with the point of ∂B via an interval on the negative imaginary axis to identify a fiber over ∂B with T^2). By Lemma 3.11 this identifies $\mathrm{MW}(\partial\pi)$ with the cokernel of $\tau_\delta^2-1: a\mapsto 2(a\cdot\delta)\delta$, i.e., with $(\mathbb{Z}/2)\delta\oplus\mathbb{Z}\delta'$.

The situation is quite different form that of Proposition 4.2:

Proposition 4.4. *The exact sequence* $\mathrm{MW}(\pi,\partial\pi)\to\mathrm{MW}(\pi)\to\mathrm{MW}(\partial\pi)$ *is naturally isomorphic to the sequence*

$$H_2(C)\xrightarrow{2\times}H_1(X_0)/V_+\to H_1(X_0)/2V_+,$$

where the first map sends a generator of $H_2(C)$ to twice a generator of $H_1(X_0)/V_+$, so that the image of the second map is the order two subgroup $V_+/2V_+$.

Proof. We already established in Proposition 4.1 and Proposition 4.2 isomorphisms $\mathrm{MW}(\pi)\cong H_1(X_0)/(V_++V_-)=H_1(X_0)/V_+$ and $\mathrm{MW}(\partial\pi)\cong H_1(X_0)/2V_+$. For the rest of the argument, we recall that the MW-exact sequence in question appears as part of a long exact sequence

$$\cdots\to H^0(B,\partial B;R^1\pi_*\mathbb{Z})\to H^1(B,\partial B;R^1\pi_*\mathbb{Z})\to H^1(B;R^1\pi_*\mathbb{Z})\to H^1(\partial B;R^1\pi_*\mathbb{Z})\to$$
$$\to H^2(B,\partial B;R^1\pi_*\mathbb{Z})\to H^2(B;R^1\pi_*\mathbb{Z})\to\cdots$$

Since $H^0(B,\partial B;R^1\pi_*\mathbb{Z})$ consists of the sections of $R^1\pi_*\mathbb{Z}$ that vanish on ∂B, this group is trivial. The group $H^2(B;R^1\pi_*\mathbb{Z})$ is also trivial, because it restricts isomorphically to $H^2([-1,1];R^1\pi_*\mathbb{Z})$ and that group is zero too, so that there is an exact sequence

$$\begin{array}{ccccccccc} 0 \to H^1(B,\partial B;R^1\pi_*\mathbb{Z}) & \longrightarrow & H^1(B;R^1\pi_*\mathbb{Z}) & \longrightarrow & H^1(\partial B;R^1\pi_*\mathbb{Z}) & \longrightarrow & H^2(B,\partial B;R^1\pi_*\mathbb{Z}) \to 0 \\ \| & & \| & & \| & & \\ \mathrm{MW}(\pi,\partial\pi) & & \mathrm{MW}(\pi) & & \mathrm{MW}(\partial\pi) & & \end{array}$$

We are going to determine the as yet unknown $H^1(B, \partial B; R^1\pi_*\mathbb{Z})$ and $H^2(B, \partial B; R^1\pi_*\mathbb{Z})$. For this we use the Leray spectral sequence

$$E_2^{p,q} = H^p(B, \partial B; R^q\pi_*\mathbb{Z}) \Rightarrow H^{p+q}(X, \partial X) \cong H_{4-p-q}(X),$$

where we keep in mind that $H_1(X) \cong H_1(X_0)/V_+$, that $H_2(X_0)$ embeds in $H_2(X)$ with cokernel $H_2(C)$ and $H^k(X) = 0$ for $k > 2$.

Note that $E^{p,q}$ vanishes unless $0 \leq p, q \leq 2$. We already observed that $E_2^{0,1} = 0$, so that $E_2^{2,0} = H^2(B, \partial B; R^0\pi_*\mathbb{Z}) = H^2(B, \partial B; \mathbb{Z}) \cong \mathbb{Z}$ must embed in $H^2(X)$. Also, $E_2^{0,2} = H^0(B, \partial B; R^2\pi_*\mathbb{Z}) \cong H^0(B, \partial B; \mathbb{Z}) = 0$ and hence the cokernel of $H^2(B, \partial B; \mathbb{Z}) \to H^2(X, \partial X; \mathbb{Z})$ can be identified with $E^{1,1} = H^1(B, \partial B; R^1\pi_*\mathbb{Z})$. Duality identifies $\pi^* : H^2(B, \partial B; \mathbb{Z}) \to H^2(X, \partial X; \mathbb{Z})$ with the Gysin map $\pi^! : H_0(B) \to H_2(X)$, which assigns to $1 \in H_0(B)$ the fiber class. In other words, the cokernel in question is identified with that of $H_2(X_0) \to H_2(X)$, i.e., with $H_2(C)$.

This also shows that $E_2^{2,1} = H^2(B, \partial B; R^1\pi_*\mathbb{Z})$ is isomorphic to $H^3(X, \partial X) \cong H_1(X) \cong V_+$. So the exact sequence above becomes

$$0 \to H_2(C) \to H_1(X_0)/V_+ \to H_1(X_0)/2V_+ \to V_+ \to 0$$

and hence the maps in it must have the asserted properties. □

4.2. **A Weierstraß model and a circle action.** A Weierstraß model for an equinodal pair is given by the family of plane cubics defined by

$$y^2 + x^3 + x^2 + \varepsilon(s^2 - 1),$$

where we let $\varepsilon > 0$ be small, and where we must add the section at infinity in order to make this a (degenerating) family of elliptic curves.

This sphere can be made explicit in terms of our Weierstraß model: for $s \in (-1, 1)$, the cubic equation $x^3 + x^2 + \varepsilon(s^2 - 1) = 0$ has three distinct real roots: one close to -1 and two close to zero and the real locus of the cubic curve $y^2 + x^3 + x^2 = \varepsilon(1 - s^2)$ in $\mathbb{R}^2$ has two connected components, one of which is compact (so a differentiable circle) which meets the x-axis in the two roots close to zero and the other is a properly embedded copy of $\mathbb{R}$ that is compactified by the point at infinity. If we let $s \to \pm 1$, then the compact component shrinks to a point, so that if we let s run over the interval $[-1, 1]$, then these compact components trace out a 2-sphere. This 2-sphere, which is easily checked to be smooth, is our C. This C in turn, appears as a vanishing 2-cycle of the following degeneration of cubic surfaces

$$t = (y^2 + x^3 + x^2) + \varepsilon s^2, \quad |t| \leq \varepsilon,$$

which for $t = 0$ acquires a quadratic singularity at $(x, y, s) = (0, 0, 0)$. A (Picard-Lefschetz) vanishing cycle in dimension two has self-intersection -2, and so it follows that $C \cdot C = -2$.

Remark 4.5. One can check that the homology classes $c \in H_2(X)$ of C and $e \in H_2(X)$ of a smooth fiber make up a free basis of $H_2(X)$. Their intersection numbers are $c \cdot c = -2$, $e \cdot e = 0$ and $c \cdot e = 0$. In particular, e spans the radical of the intersection pairing on $H_2(X)$.

Remark 4.6. It is worth noting that the the group $\mathrm{Trans}(\pi)$ of fiberwise translations of π contains as a subgroup a copy T_π of the circle group T. It fixes the nodes in the singular fibers, but is otherwise free and we can take the 2-sphere C to be such that each fiber of the projection $\pi|C : C \to [-1, 1]$ is a T_π-orbit. This action commutes with the geometric monodromy and its orbit space defines a T-bundle over B (which must be trivial since B is contractible). A trivialization $X \to T$ of this bundle induces an isomorphism $H_1(X) \to H_1(T) \cong \mathbb{Z}$.

4.3. **The braid action.** Since the embedded smooth sphere C has self-intersection number -2, it determines a 2-dimensional Dehn twist $T(C)$ as an element of the mapping class group of the pair $(X, \partial X)$. It is well-known that $T(C)$ only depends on the isotopy class of C and has square isotopic to the identity. We will represent $T(C)$ as the monodromy of the function $t = (y^2 + x^3 + x^2) + \varepsilon s^2$ above and this will in fact produce a relative mapping class $\tau(C)$ in $\mathrm{Mod}(\pi, \partial\pi)$ that lifts $T(C) \in \mathrm{Mod}(X, \partial X)$.

Recall that the connected component group of the group diffeomorphisms of B that are the identity near ∂B and preserve $\{-1, 1\}$ is infinite cyclic with a distinguished (positive) generator. We can represent this generator by the simple braid, i.e., as the monodromy τ of the bundle pair over the circle $(T \times B, \mathscr{D}) \to T$, where $\mathscr{D}$ is the set of $(u, s) \in T \times B$ satisfying $s^2 = u$. Its square is the Dehn twist along a ∂B and generates the mapping class group of the pair $(B, \partial B \cup \{-1, 1\})$. We will lift this to a map $T \times X \to T \times B$ over T (making it a family of genus one fibrations of X parametrized by T) in such a manner that $\mathscr{D}$ is its set of critical values and the fiber over $1 \in T$ is the given $\pi : X \to B$. Although it ought to be possible to express this entirely in differential-topological terms, it is much easier to exploit the complex structure and rely on a bit of singularity theory. To this end, we return to the 2-parameter family of cubic curves that we used to define the vanishing sphere C, except that we rescale the t-parameter:

$$y^2 + x^3 + x^2 = \varepsilon t - \varepsilon s^2.$$

We here let s run over B and let t run over an open complex disk Δ of radius > 1 (as before, we assume $\varepsilon > 0$ small). Write $\mathcal{X}$ for the preimage of $B \times \Delta$, including its section at infinity and regard $\mathcal{X} \to \Delta$ as a family of elliptic fibrations $\{\pi_t : \mathcal{X}_t \to B\}_{t \in \Delta}$, which for $t = 1$ returns $\pi : X \to B$. The singular fibers are defined by $t = s^2$, so if t runs over the circle of radius 1 in Δ the critical values of π_t traverse a basic braid $\mathscr{D}$.

Use (x, y, s) as global coordinates for $\mathcal{X}$ minus its section at infinity, so that t is there given by $t = \varepsilon^{-1}(y^2 + x^3 + x^2) + s^2$. All fibers of this map are smooth surfaces except the central fiber $\mathcal{X}_0$, which has an ordinary double point. As is well-known, the monodromy of this family, regarded as a diffeomorphism of $\mathcal{X}_1 = X$ onto itself, is a 2-dimensional Dehn twist defined by the vanishing 2-sphere C. Since the action T_π on X extends to one on $\mathcal{X}$, we find:

Corollary 4.7. *Let τ be a diffeomorphism of B that is the identity near ∂B, exchanges* 1 *and* -1 *and represents the simple braid. Then C determines a lift of τ to an automorphism of π that commutes with the action of T_π. Its image $\tau(C)$ in* $\mathrm{Mod}(\pi, \partial\pi)$ *is of infinite order, but its image in* $\mathrm{Mod}(X, \partial X)$ *is the* 2*-dimensional Dehn twist $T(C)$ and hence of order two.*

4.4. **Enumerating the classes of sections.** According to Proposition 4.1, $\mathrm{MW}(\pi)$ is naturally isomorphic with the infinite cyclic group $H_1(X_0)/\mathbb{Z}\delta \cong \mathbb{Z}\delta'$

Since the set $\pi_0(\Gamma(\pi))$ is a $\mathrm{MW}(\pi)$-torsor, we can enumerate its elements by the integers. We make this concrete as follows. The image of a section of $\pi : X \to B$ in $\pi_0(\Gamma(\pi))$ is determined by its restriction to $[-1, 1]$. Since a section avoids the singular points, it will via h define a section of the trivial bundle $[-1,1]\times T^2 \to T^2$ with the property that its value in ± 1 avoids $T_\pm = T\times\{\pm 1\}$. So this is given by a differentiable map $f=(f',f''):[-1,1]\to T\times T$ with $f''(\pm 1)\neq 1$. All what matters for the class of this section is the connected component of f in the space of such maps. Since we do not impose a boundary condition on f', it is in fact only the connected component of f'' that counts. As $T\smallsetminus\{1\}$ contains -1 as a deformation retract, we may just as well agree that $f''(\pm 1)=-1$. So such sections are enumerated by the homotopy classes of loops in T with base point -1. These homotopy classes are naturally indexed by the integers $n\in\mathbb{Z}$. Indeed, each such homotopy class contains a unique representative of the form

$$f''_n : s\in[-1,1]\mapsto -e^{n(s+1)\pi\sqrt{-1}}\in T.$$

We denote by σ_n a section of π that extends the section over $[-1,1]$ defined by one for which the second component equals f''_n. In order that σ_n be smooth some precaution is necessary: we should replace the multiplication factor $e^{n(s+1)\pi\sqrt{-1}}$ by $e^{n\varphi(s)\pi\sqrt{-1}}$, where $\varphi:[-1,1]\to[0,2]$ is constant 0 near -1 and constant 2 near 1. Observe that σ_0 does not meet C, because via the homeomorphism $X_{[-1,1]}\cong E$, the second T-coordinate is constant -1 on σ_0 and constant $+1$ on C. So if we write $[\sigma_n]$ for the image of σ_n in $H_2(X,\partial X)\cong H^2(X)$, then $\langle\sigma_0|c\rangle=0$.

We identified $\mathrm{MW}(\partial\pi)$ with $(\mathbb{Z}/2)\delta\oplus\mathbb{Z}\delta'$ and so $\mathrm{MW}(\partial\pi)$ has a unique element of order two. Proposition 4.4 implies:

Proposition 4.8. *The image of $\sigma_n-\sigma_0$ in $\mathrm{MW}(\partial\pi)$ is torsion and only depends on the parity of n: it is zero or of order two depending on whether n is even or odd.*

The preceding suggests that we consider the diffeomorphism of $[-1,1]\times T^2$ over $[-1,1]$ onto itself defined by

$$F:(s;u_1,u_2)\mapsto(s;u_1,u_2e^{\varphi(s)\pi\sqrt{-1}}),$$

where $\varphi:[-1,1]\to\mathbb{R}$ is as before: constant 0 near -1 and constant 2 near 1. As this is the identity over neighborhood of $\{-1,1\}$, it induces a homeomorphism of E onto itself that extends as a smooth fiberwise translation of X. We denote that diffeomorphism by Φ.

The identity $f''_n(s)e^{n\varphi(s)\pi\sqrt{-1}}=f''_{n+1}(s)$ implies that Φ takes the class σ_n to that of σ_{n+1}. Since the classes of $\sigma_n|\partial B$ and $\sigma_{n+1}|\partial B$ are different, $\Phi|\partial B$ will be nontrivial (of order two), but $\Phi^2|\partial B$ is in the identity component. We conclude:

Proposition 4.9. *The Mordell-Weil group $\mathrm{MW}(\pi)$ is infinite cyclic with a generator represented by Φ; this generator takes the class of σ_n to the class of σ_{n+1}. The section σ_0 is disjoint from C. The relative Mordell-Weil group $\mathrm{MW}(\pi,\partial\pi)$ is generated by the class of Φ^2.*

We put $C_n:=\Phi^n(C)$. Since $\sigma_n=\Phi^n\sigma_0$, it follows that C_n and σ_n are disjoint. The following corollary shows among other things that σ_n is the only section with that property.

Corollary 4.10. *The class of $C_n=\Phi^n(C)$ is equal to $c+ne$. In particular, $\sigma_m\cdot C_n=\sigma_{m-n}\cdot C=m-n$. A basis of $H_2(X,\partial X)\cong H^2(X)$ is given by the classes of the sections σ_0 and σ_1.*

Proof. By construction, Φ induces in E the map defined by F. Recall that the oriented topological 2-sphere S is the image of $[-1,1]\times T^1\times\{1\}$ in E. There is a projection of E onto the quotient $\check{T}^2$ of T^2 that is obtained by collapsing $T\times\{1\}$ to a point. Under this projection, S is mapped to a point, but the central fiber T^2 maps with degree one on $\check{T}^2$.

Since $F_*(S)$ also maps to $\check{T}^2$ with degree one, it follows that $\Phi_*(c) = c + e$. If we combine this with the fact that Φ_* fixes e, we find that $[\Phi^n(C)] = (\Phi_*)^n(c) = c + ne$.

We check the second assertion by evaluating σ_0 and σ_1 on the basis (e, c) of $H_2(X)$ and prove that the resulting 2×2-matrix has absolute determinant 1: indeed, $\sigma_0 \cdot e = 1$, $\sigma_0 \cdot c = 0$ and $\sigma_1 \cdot e = 1$, $\sigma_1 \cdot c = \Phi(\sigma_0) \cdot c = \sigma_0 \cdot \Phi_*^{-1}(c) = \sigma_0 \cdot (c - e) = -1$. □

Note the classes $[C_n] = c + ne$ yield up to sign all elements in $H_2(X)$ of self-intersection number -2. So the Mordell-Weil group $\mathrm{MW}(\pi)$ acts simply transitively on this collection of isotopy classes of embedded 2-spheres in X (when we regard them as unoriented submanifolds) as well as on the set $\pi_0\Gamma(\pi)$ of isotopy classes of sections. The group $\mathrm{MW}(\pi, \partial\pi)$ has therefore two orbits in these sets.

Remark 4.11 (Relation to degenerations). This decomposition into two orbits can be understood in terms a degeneration of π. For this purpose it is convenient to represent the vanishing cycles $T_\pm$ in T^2 not by the same circle $T \times \{1\}$ as we did above, but by two isotopic ones, namely by taking $T_\pm = T \times \{\pm 1\}$. If we then let the critical values $\{-1, 1\}$ of π then both move to 0 (thereby shrinking the interval that connects them to a singleton), we obtain an elliptic fibration $\pi' : X' \to B$ whose only singular fiber X'_0 is topologically obtained from T^2 by contracting both $T \times \{1\}$ and $T \times \{-1\}$. (This limiting process can in fact be carried out in the holomorphic category; the central singular fiber is then of Kodaira type I_2: it has two irreducible components, each being a Riemann sphere with self-intersection number -2 and meeting the other transversally in two points.)

The set of sections $\Gamma(\pi')$ that meet a given connected component of the smooth part of X'_0 make up a connected set and so $\pi_0\Gamma(\pi')$ has only two elements. Indeed, $\mathrm{MW}(\pi')$ is of order two with the nontrivial element exchanging these two. The natural map $\mathrm{MW}(\pi') \to \mathrm{MW}(\partial\pi') = \mathrm{MW}(\partial\pi)$ is injective with image the torsion subgroup of $\mathrm{MW}(\partial\pi)$ and $\mathrm{MW}(\pi', \partial\pi')$ is trivial.

If we make the above choices for $T_\pm$, then these 2-spheres also live in our E as topological spheres $S_\pm$, namely as the images of the two maps

$$(s, t) \in [-1, 1] \times T^1 \mapsto (\pm s, \pm e^{\pi\sqrt{-1}s}, t) \in [-1, 1] \times T^2.$$

When suitably oriented, the image of S_+ resp. S_- in X under the embedding $E \hookrightarrow X$ is isotopic to C resp. $-C_{-1}$ (whose class is $e - c$). The two $\mathrm{MW}(\pi, \partial\pi)$-orbits in $\pi_0\Gamma(\pi)$ degenerate into the two elements of $\pi_0\Gamma(\pi')$.

In Corollary 4.7 we noted that the smoothly embedded 2-sphere C (with self-intersection -2) defines $\tau(C) \in \mathrm{Mod}(\pi, \partial\pi)$. It is clear that

$$\tau(C_n) = \tau(\Phi^n C) = \Phi^n \tau(C) \Phi^{-n}.$$

Since each $\tau(C_n)$ lifts the same diffeomorphism of B, any two such 'differ' by an element of $\mathrm{MW}(\pi, \partial\pi)$. In particular, $\tau(C_1)\tau(C)^{-1}$ defines an element of $\mathrm{MW}(\pi, \partial\pi)$ and hence represents an even power of Φ.

In what follows we make use of a "variation homomorphism", defined as follows. Let (X, Y) be a topological pair with $Y \subset X$ closed and $h : X \to X$ a homeomorphism that is the identity on Y. If z is a simplicial k-chain on X such that ∂z has its support on Y, then $h_* z - z$ is a k-cycle on X. Thus $h_* - 1$ induces what is called the *variation map*

$$\mathrm{var}(h) : H_\bullet(X, Y) \to H_\bullet(X).$$

It only depends on the relative homotopy class of h. One justification for this notion is that if $j : (X,Y) \subset (X',Y')$ is an embedding which induces an isomorphism on relative homology, then if $h' : X' \to X'$ is the extension of h to the identity, then $\mathrm{var}(h')$ is expressible in terms of $\mathrm{var}(h)$ as the composite

$$\mathrm{var}(h') : H_\bullet(X',Y') \xleftarrow{j_*\cong} H_\bullet(X,Y) \xrightarrow{\mathrm{var}(h)} H_\bullet(X) \xrightarrow{j_*} H_\bullet(X').$$

We further note that var satisfies the cocycle condition: if $h_1, h_2 : (X,Y) \to (X,Y)$ are as above, then $\mathrm{var}(h_1h_2) = \mathrm{var}(h_1)h_{2*} + \mathrm{var}(h_2)$.

Proposition 4.12. *The image of $\tau(C_1)\tau(C)^{-1}$ in $\mathrm{MW}(\pi,\partial\pi)$ is that of Φ^2 (which we recall defines the generator of $\mathrm{MW}(\pi,\partial\pi)$) and its associated variation homomorphism equals*

$$x \in H_2(X,\partial X) \mapsto \langle x,e\rangle c - \langle x,c\rangle e + \langle x,e\rangle e \in H_2(X),$$

hence comes from the Eichler transformation $E(e\wedge c)$. The class of $\sigma_{2m} - \sigma_0$ in $H_2(X)$ (considered as the image of $\mathrm{var}(\Phi^{2m})[\sigma_0]$) is $mc + m^2e$.

Proof. We first determine the variation homomorphism of $T(C_1)^{-1}T(C)$. For $x \in H_2(X,\partial X)$:

$$\begin{aligned}\mathrm{var}(T(C_1)T(C)^{-1})(x) &= \mathrm{var}(T(C_1)T(C)_*(x) + \mathrm{var}(T(C))(x)\\ &= \langle x + \langle x,c\rangle c, c+e\rangle(c+e) + \langle x,c\rangle c = \langle x,e\rangle c - \langle x,c\rangle e + \langle x,e\rangle e.\end{aligned}$$

This is indeed the variation of the Eichler transformation defined by $e\wedge c$. This shows in particular that $\tau(C_1)\tau(C)^{-1}$ takes c to $c+2e$. Since Φ^n_* takes c to $c+ne$, it follows that $\tau(C_1)\tau(C)^{-1} = \Phi^2$.

Recalling that Φ^{2m} acts as the Eichler transformation defined by $me\wedge c$, the last assertion then follows from

$$\mathrm{var}(\Phi^{2m})[\sigma_0] = \langle\sigma_0, me\rangle c - \langle\sigma_0, c\rangle me + \langle\sigma_0, me\rangle me = mc + m^2e. \qquad \square$$

Remark 4.11 above suggests the following alternate way of looking at $\mathrm{Mod}(\pi,\partial\pi)$ and its map to $\mathrm{Mod}(X,\partial X)$.

Corollary 4.13. *A presentation of $\mathrm{Mod}(\pi,\partial\pi)$ has as generators $\tau(C)$ and $\tau(C_1)$ and the relation $\tau(C)^2 = \tau(C_1)^2$, so that $\mathrm{Mod}(\pi,\partial\pi)$ is a central extension*

$$1 \to \mathbb{Z} \to \mathrm{Mod}(\pi,\partial\pi) \to (\mathbb{Z}/2) * (\mathbb{Z}/2) \to 1,$$

the center being generated by $\tau(C)^2$ (here $(\mathbb{Z}/2)*(\mathbb{Z}/2)$ is the infinite dihedral group).

The relative Mordell-Weil group $\mathrm{MW}(\pi,\partial\pi)$ is the infinite cyclic subgroup of $\mathrm{Mod}(\pi,\partial\pi)$ generated by the basic translation $F(C) := \tau(C)^{-1}\tau(C_1)$. In particular,

$$\mathrm{Mod}(\pi,\partial\pi) \cong \mathrm{MW}(\pi,\partial\pi) \rtimes \tau(C)^{\mathbb{Z}}$$

with $\tau(C)$ acting by inversion:

$$\tau(C)F(C)\tau(C)^{-1} = F(C)^{-1}.$$

The image of $\tau(C)$ in the relative mapping class group $\mathrm{Mod}(X,\partial X)$ is the Dehn twist $T(C)$ and hence of order 2. Indeed, $\mathrm{Mod}(X,\partial X)$ is the quotient group of $\mathrm{Mod}(\pi,\partial\pi)$ defined by this relation and hence is the infinite dihedral group $\mathrm{MW}(\pi,\partial\pi)\rtimes\mathbb{Z}/2$). $\square$

The groups $\mathrm{Mod}(\pi)$, $\mathrm{Mod}(\pi,\partial\pi)$, $\mathrm{Mod}(X,\partial X)$ that appear here are of course naturally associated to π, but C is not privileged over any of the spheres $\pm C_n$ with $n\in\mathbb{Z}$. On the other hand, as Proposition 4.12 shows, with the choice of C comes the choice of a component of $\Gamma(\pi)$ (containing a section disjoint with C): there is a natural bijection between the collection of (unoriented) spheres $\{C_n\}_{n\in\mathbb{Z}}$ and $\pi_0(\Gamma(\pi))$.

5. Small support realizations of elements of the Mordell-Weil group

Our goal in this section is to prove Theorem 1.7, that each element of $H(e)$ has a representative in $\mathrm{Diff}(M)$ with small support, and of a very special form.

5.1. **Equinodal arcs.** We shall see that the 'equinodal' fibration studied in §4 appears in $\pi_d : M_d \to \mathbb{P}^1$ in many ways. In order make this a bit more formal, let us first introduce a relevant notion.

Definition 5.1 (**Equinodal arc).** *A* equinodal arc *in $\mathbb{P}^1$ (with respect to the genus one fibration π_d) is a smoothly (embedded) arc $\gamma \subset \mathbb{P}^1$ whose relative interior does not meet any critical value and over whose end points there are nodal fibers that define the same vanishing homology in a fiber over an interior point of γ.*

For such an equinodal arc γ there exists a closed regular neighborhood B_γ of γ such that $\pi_d|B_\gamma$ is diffeomorphic to the genus one fibration investigated in Section 4. Corollary 4.13 shows that then the part $\mathrm{MW}_\gamma(\pi_d)$ of the Mordell-Weil group $\mathrm{MW}(\pi_d)$ with support in B_γ is infinite cyclic with a generator F_γ. It also tells us that there exists a smooth oriented 2-sphere C_γ 'suspended' over γ with $C_\gamma \cdot C_\gamma = -2$ such that the associated Dehn twist $T(C_\gamma) \in \mathrm{Mod}(M_g)$ has a natural lift $\tau(C_\gamma) \in \mathrm{Mod}(\pi_d)$ with the latter is represented by a diffeomorphism of M_d over a diffeomorphism of $\mathbb{P}^1$ that has its support in B_γ and is inside B_γ the basic (positive) braid that exchanges the endpoints of γ. They satisfy $\tau(C_\gamma)F_\gamma\tau(C_\gamma)^{-1} = F_\gamma^{-1}$. While $\tau(C_\gamma)^2$ is nontrivial (for it acts on $\mathbb{P}^1$ as a Dehn twist along ∂B_γ), its image $T(C_\gamma)^2$ in $\mathrm{Mod}(M_d)$ is trivial (even in the group of mapping classes of M_d that have their support over B_γ).

If $c_\gamma = [C_\gamma] \in \Lambda_d$, then $c_\gamma \cdot e = 0$ and the above mapping classes act on Λ_d as an orthogonal reflection resp. by an Eichler transformation:

$$\tau(C_\gamma)_* : x \mapsto x + (c_\gamma \cdot x)c_\gamma$$
$$F_{\gamma *} = E(e \wedge c_\gamma) : x \mapsto x + (x \cdot e)c_\gamma - (x \cdot c_\gamma)e + (x \cdot e)e,$$

Let us refer to the image of c_γ in $\Lambda_d(e)$, resp. $E(e \wedge c_\gamma$ in $\Gamma_{d,e}$ as resp. an *equinodal* (-2)-*vector*, an *equinodal Eichler transformation* and to the reflection $\tau(C_\gamma)_*$ as acting in $\Lambda_d(e)$ an *equinodal reflection*.

Theorem 5.1. *These equinodal objects are maximal in the following sense:*

(i) all the (-2)-vectors in $\Lambda_d(e)$ are equinodal and generate $\Lambda_d(e)$, or equivalently, the equinodal Eichler transformations generate the unipotent radical $\Lambda_d(e)$ of $\Gamma_{d,e}$,

(ii) the equinodal reflections in $\mathrm{O}(\Lambda_d(e))^+$ make up a single conjugacy class and generate $\mathrm{O}(\Lambda_d(e))^+$.

Proof. This theorem is known for $d = 1$. In that case $\mathrm{MW}(\pi_1)$ is its holomorphic counterpart and isomorphic to $\mathbf{E}_8(-1)$ and $\mathrm{O}(H_1(e))^+$ is isomorphic to the Weyl group of the root system of (-2)-vectors in $\mathbf{E}_8(-1)$. It is also known for $d = 2$: we established this in [FL2] with the help of the Torelli theorem for K3 surfaces.

We now proceed with induction and assume $d > 2$. For any set $\Delta \subset \Lambda_d(e)$ of (-2)-vectors we denote by $\Gamma(\Delta)$ the subgroup of $\mathrm{O}(\Lambda_d(e))^+$ generated by the reflections in Δ. Write $\Delta_d \subset \Lambda_d(e)$ for the set of equinodal (-2)-vectors.

We make use of the fiber connected sum decomposition of M_d obtained via Theorem 3.12. This gives a decomposition of $\Lambda_d(e)$

$$\Lambda_d(e) \cong \mathbf{E}_8(-1) \perp 2\mathbf{U} \perp \mathbf{E}_8(-1) \perp \cdots \perp 2\mathbf{U} \perp \mathbf{E}_8(-1),$$

where we have d copies of $\mathbf{E}_8(-1)$ and $d-1$ copies of $2\mathbf{U}$. This ordering of the summands should be understood as follows: if $d>1$, then the copy $H_{d-1}(e)_+$ resp. $H_{d-1}(e)_-$ of $H_{d-1}(e)$ that supplements the last three summands $2\mathbf{U} \perp \mathbf{E}_8(-1)$ resp. the first three summands $\mathbf{E}_8(-1) \perp 2\mathbf{U}$ comes from the preimages over disks $D_\pm \subset \mathbb{P}^1$ over whose boundary we have a trivial torus bundle. If we contract $\partial D_\pm$ in $D_\pm$ and use a trivialization to lift that to a contraction of $\pi_d^{-1}\partial D_\pm$ in $\pi_d^{-1}D_\pm$ to a torus, we get a fibration diffeomorphic with π_{d-1}.

The induction hypothesis then implies that $\Delta_\pm := \Delta_d \cap H_{d-1}(e)_\pm$ consists of all the (-2)-vectors in $H_{d-1}(e)_\pm$ and generates $H_{d-1}(e)_\pm$ and that $\Gamma(\Delta_d) \supset \mathrm{O}(H_{d-1}(e)_+)^+ \cup \mathrm{O}(H_{d-1}(e)_-)^+$. Note that $H_{d-1}(e)_+ \cap H_{d-1}(e)_-$ is of type $H_{d-2}(e)$ and hence is generated by (-2)-vectors. This implies that Δ_d generates all of $\Lambda_d(e)$. It also shows that $\Delta_+ \cup \Delta_-$ lies in a $\Gamma(\Delta_+ \cup \Delta_-)$-orbit. As $H_{d-1}(e)_+$ contains $\mathbf{E}_8(-1) \perp 2\mathbf{U}$ and hence a copy of $\mathbf{A}_2(-1) \perp 2\mathbf{U}$, it follows that the pair $(\Lambda_d(e), \Delta_+ \cup \Delta_-)$ is a *complete vanishing lattice* in the sense of Ebeling [E]. His main theorem 2.3 (*op. cit.*) states that then $\Gamma(\Delta_+ \cup \Delta_-) = \mathrm{O}(\Lambda_d)^+$. So *a fortiori*, $\Gamma(\Delta_d) = \mathrm{O}(\Lambda_d)^+$. It is well-known that the (-2)-vectors in a lattice of type Λ_d make up a single $\mathrm{O}(\Lambda_d)^+$-orbit; this also follows from Ebelings' s proposition (2.5). □

Corollary 5.2. *For each $d \geq 1$, the representation of* $\mathrm{Mod}(\pi_d)$ *on* Λ_d *has image* $\Gamma_{d,e}$.

Proof. Proposition 3.5 asserts that the subgroup $\mathrm{MW}(\pi_d)$ maps (in fact, isomorphically) onto the unipotent radical ($\cong \Lambda_d(e)$ of $\Gamma_{d,e}$ and Theorem 5.1 tells us $\mathrm{Mod}(\pi_d)$ maps onto $\mathrm{O}(\Lambda_d(e))^+$. Hence $\mathrm{MW}(\pi_d)$ maps onto $\Gamma_{d,e}$. □

Remark 5.3. The representation of $\mathrm{Mod}(\pi_d)$ on Λ_d is not faithful: in Corollary 4.13 we found a (-2)-sphere C defining a relative mapping class $\tau(C) \in \mathrm{Mod}(\pi_d)$ of infinite order whose image in $\mathrm{Mod}(M_d)$ is the 2-dimensional Dehn twist $T(C)$, which has order 2 (and which induces an orthogonal reflection in Λ_d). This also shows that the forgetful homomorphism $\mathrm{Mod}(\pi_d) \to \mathrm{Mod}(M_d)$ is not injective.

Proof of Corollary 1.8. We first show that the lattice $\Lambda_d(e)$ admits a basis $\mathscr{C}$ of (-2)-vectors. Since $\Lambda_d(e)$ is the isomorphic to the orthogonal direct sum of $\mathbf{E}_8(-1)$ and $d-1$ copies of $\mathbf{E}_8(-1) \perp 2(-1)\mathbf{U}$, it suffices to prove this for these two lattices. This is clear for $\mathbf{E}_8(-1)$, for any root basis $\alpha_1, \dots, \alpha_8$ will do. If e, f resp. e', f' are standard (isotropic) bases for the first resp. second $\mathbf{U}$-summand, then $\{\alpha_1, \dots, \alpha_8, \alpha+e, \alpha+f, \alpha+e', \alpha+f'\}$ is basis of $\mathbf{E}_8(-1) \perp 2(-1)\mathbf{U}$ consisting of (-2)-vectors.

By Theorem 5.1 every $c \in \mathscr{C}$ is an equinodal (-2)-vector, so associate to some equinodal arc γ_c. By definition there exists a translation $f_c\, \mathrm{Trans}(\pi_d)$ with support contained in a regular neighborhood of γ such that F_γ induces the Eichler transformation $E(c \wedge e)$. Then the homomorphism $\Lambda_d(e) \to \mathrm{Trans}(\pi_d)$ of abelian groups that takes c to f_c gives the desired Nielsen realization. □

Proof of Theorem 1.7. Given c as in the hypothesis of the theorem, Theorem 5.1 produces an equinodal arc $\gamma_c \subset \mathbb{P}^1$ and its tubular neighborhood U_c, giving part (1). Part (2a) of the theorem is part of the statement of Proposition 4.12. Part (2b) is Corollary 4.7. □

Remark 5.4. For $h = (h_M, h_{\mathbb{P}^1}) \in \mathrm{Diff}(\pi_d) \subset \mathrm{Diff}(M_d) \times \mathrm{Diff}(\mathbb{P}^1)$, the mapping torus of $h_{\mathbb{P}^1}$ determines a spherical braid. Its $\mathrm{Diff}^+(\mathbb{P}^1)$-conjugacy class only depends on the image of h in $\mathrm{Mod}(\pi_d)$ and can be understood as an element of the orbifold fundamental group of $\mathfrak{S}_{12d}\backslash\mathcal{M}_{12d}$ (the moduli space of $12d$-element subsets of $\mathbb{P}^1$ given up to projective equivalence). We thus have defined a homomorphism

$$\mathscr{B} : \mathrm{Mod}(\pi_d) \to \pi_1^{\mathrm{orb}}(\mathfrak{S}_{12d}\backslash\mathcal{M}_{12d}, [D]) \tag{5.1}$$

where D is the discriminant of π_d. The right-hand side of (5.1) is a quotient of the mapping class group of the pair $(\mathbb{P}^1, D)$ by $\pi_1(\mathrm{SO}_3) \cong \{\pm 1\}$). The kernel of $\mathscr{B}$ is $\mathrm{Mod}(M_d/\mathbb{P}^1)$, the connected component group of $\mathrm{Diff}(M_d/\mathbb{P}^1)$, which contains $\mathrm{MW}(\pi_d)$ as a subgroup of index two (see Remark 2.5).

The image of $\mathscr{B}$ certainly contains the simple spherical braid around a regular neighborhood boundary of an equinodal arc. It also contains the third power of the spherical braids defined by the regular neighborhood boundary of what we might call an an *anti-equinodal arc* γ: such an arc connects two points of the discriminant that define on a fiber over its interior vanishing cycles that span its first homology of that fiber as in Proposition 4.2 (if we let γ shrink to a singleton, then its preimage becomes a Kodaira fiber of cuspidal type and hence the monodromy over a regular neighbourhood boundary has order 6). In either case, the squares of these elements become trivial in $\mathrm{Mod}(M_d)$.

Question 5.5. Is the image of $\mathscr{B}$ generated by the simple spherical braids associated to the equinodal arcs and the third power of the simple spherical braids associated to the anti-equinodal arcs? Is the subgroup generated by their squares equal to the kernel of the natural map $\mathrm{Mod}(\pi_d) \to \mathrm{Mod}(M_d)$?

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