Geometry of the Wiman-Edge pencil and the Wiman curve

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Abstract

The *Wiman-Edge pencil* is the universal family $C_t, t \in \mathcal{B}$ of projective, genus 6, complex-algebraic curves admitting a faithful action of the icosahedral group $\mathfrak{A}_5$. The curve C_0, discovered by Wiman in 1895 [11] and called the *Wiman curve*, is the unique smooth, genus 6 curve admitting a faithful action of the symmetric group $\mathfrak{S}_5$. In this paper we give an explicit uniformization of $\mathcal{B}$ as a non-congruence quotient $\Gamma\backslash\mathfrak{H}$ of the hyperbolic plane $\mathfrak{H}$, where $\Gamma < \mathrm{PSL}_2(\mathbb{Z})$ is a subgroup of index 18. We also give modular interpretations for various aspects of this uniformization, for example for the degenerations of C_t into 10 lines (resp. 5 conics) whose intersection graph is the Petersen graph (resp. K_5).

In the second half of this paper we give an explicit arithmetic uniformization of the Wiman curve C_0 itself as the quotient $\Lambda\backslash\mathfrak{H}$, where Λ is a principal level 5 subgroup of a certain "unit spinor norm" group of Möbius transformations. We then prove that C_0 is a certain moduli space of Hodge structures, endowing it with the structure of a Shimura curve of indefinite quaternionic type.

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1 Introduction

In 1895 Wiman [11] discovered a smooth, projective curve C_0 of genus 6 with automorphism group the symmetric group $\mathfrak{S}_5$; indeed this is the unique such curve (see Theorem 3.3 of [5] or Theorem 2.1 below). In 1981 Edge [6] placed C_0 in a family $C_t, t \in \mathcal{B}$ (now called the *Wiman-Edge pencil*) of smooth, projective curves C_t of genus 6 with a faithful action of the icosahedral group $\mathfrak{A}_5$; it is universal among all such families (see Theorem 3.3 of [5] or Theorem 1.1 below). As explained by Edge [6] (see [5] for a more modern treatment), the family C_t of curves appears naturally on a quintic del Pezzo surface S, with the Wiman curve "a uniquely special canonical curve of genus 6" on S: the standard action of $\mathfrak{S}_5$ on S leaves C_t invariant and leaves invariant exactly the curve C_0. The base $\mathcal{B}$ of the Wiman pencil appears also as the moduli space of K3 surfaces with (a certain) faithful $\mu_2 \times \mathfrak{A}_5$ action; see §5.3 of [7]. For a number of recent papers on the Wiman-Edge pencil, see [2, 3, 5, 7, 12].

The problem of finding uniformizations of moduli spaces is a classical one, but it is typically a difficult task. The first main result of this paper is to give an explicit uniformization of the smooth locus of the base of the Wiman pencil. We also give modular interpretations for various aspects of this uniformization, for example for the degenerations of the family.

Theorem 1.1 (Uniformization of the universal icosahedral family). *There exists a torsion-free subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ of index 18 such that $\mathcal{B}^\circ := \Gamma\backslash\mathfrak{H}$ underlies the base of a universal family $\mathscr{C}_{\mathcal{B}^\circ} \to \mathcal{B}^\circ$ of compact Riemann surfaces of genus 6 endowed with a faithful $\mathfrak{A}_5$-action such that:*

(i) The map which assigns to a member of $\mathscr{C}_{\mathcal{B}^\circ} \to \mathcal{B}^\circ$ its $\mathfrak{A}_5$-quotient is represented by the natural map $\Gamma\backslash\mathfrak{H} \to \mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$.

(ii) The natural completion $\mathcal{B}$ of $\mathcal{B}^\circ$ is of genus zero and has five cusps, one of width 2, two of width 3 and two of width 5.

(iii) The family $\mathscr{C}_{\mathcal{B}^\circ} \to \mathcal{B}^\circ$ extends naturally to a Deligne-Mumford stable family $\mathscr{C}_{\mathcal{B}} \to \mathcal{B}$ of $\mathfrak{A}_5$-curves such that the fiber over each cusp has (after normalization) only rational irreducible components. It is one of the following types, labeled by its dual intersection graph (see Figure 1):

Petersen curve: *The Petersen graph (cusp width 2).*

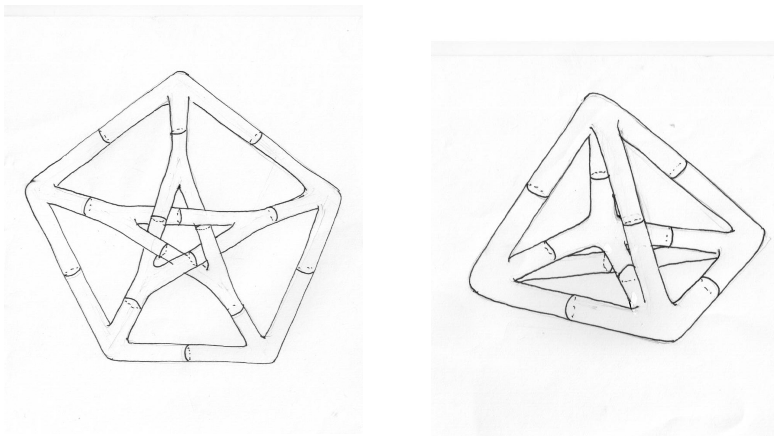

Figure 1: Two genus 6 surfaces with $\mathfrak{A}_5$-symmetry. The embedded circles are the vanishing cycles for the Petersen degeneration (cusp width 2) resp. the K_5 degeneration (cusp width 3).

K_5 **curve:** *The complete graph on 5 vertices (cusp width 3).*

Dodecahedral curve: *The graph with one vertex and six loops attached (cusp width 5).*

(iv) The involution ι of $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ coming from the unique nontrivial outer automorphism of $\mathfrak{A}_5$ is defined by conjugation with an element of order two in $\mathrm{PSL}_2(\mathbb{Z})$ which normalizes Γ; it exchanges the two cusps of the same width (see Figure 3).

As we explain in Remark 2.4 below, the group Γ in Theorem 1.1 is not a congruence subgroup. We prove Theorem 1.1 in §2.

Our next main result is the content of §3, where we find explicit affine, piecewise-Euclidean and hyperbolic metrics on each member of the family C_t. These metrics allow us to view the 5 degenerations into singular members of the family. One example of a piecewise-affine metric on a member of C_t is given in Figure 2.

The second half of this paper concerns the Wiman curve C_0 itself; again, this is the unique smooth, genus 6 curve with faithful $\mathfrak{S}_5$ action. In §4 we determine C_0 as a hyperbolic triangle group, and we prove that the Wiman curve is actually an *arithmetic curve*, that is, the quotient of the hyperbolic disk by an arithmetic lattice. Since the details are somewhat involved, we give here only a rough statement; see Theorem 4.2 for an exact statement.

Theorem 1.2 (**Arithmeticity of the Wiman curve).** *The Wiman curve is biholomorphic to the quotient of the hyperbolic disk by the principal level* 5 *subgroup of a certain "unit spinor norm" group of Möbius transformations.*

In §5 we give a modular interpretation to the Wiman curve C_0 itself as a certain moduli space of Hodge structures. We use this as well as the arithmetic description of C_0 to prove the following (see Theorem 5.4 for a precise statement).

Theorem 1.3 (**Modular description of the Wiman curve).** *The Wiman curve naturally supports a family of abelian surfaces A for which* $\mathrm{End}(A)$ *is an indefinite quaternion algebra,*

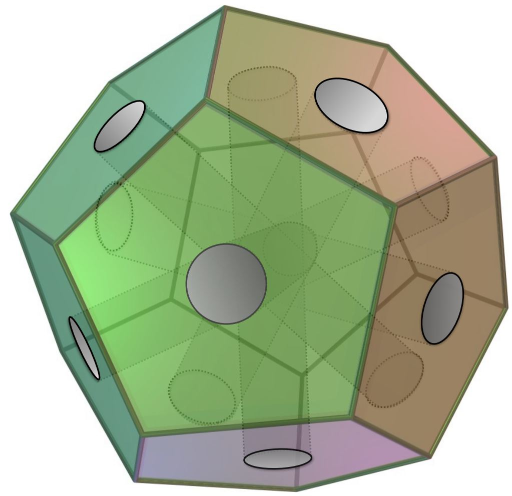

Figure 2: A genus 6 surface with $\mathfrak{A}_5$-action obtained as a dodecahedron with interior disks removed and opposite boundary components identified (as suggested by the cylinders). The piecewise-linear metric determines a conformal structure without singularities. The six antipodal pairs of boundary components define as many vanishing cycles of a degeneration into an irreducible stable $\mathfrak{A}_5$-surface with an irreducible singular fiber (cusp width 5).

which is endowed with an isomorphism $\mathbb{F}_5 \otimes \mathrm{End}(A)$ *with a fixed* $\mathbb{F}_5$*-algebra and for which* $H_1(A;\mathbb{Z})$ *is a principal* $\mathrm{End}(A)$*-module. This gives the Wiman curve the structure of a Shimura curve of indefinite quaternionic type.*

Acknowledgements. The present paper grew out of our joint work [5] with Igor Dolgachev. It is a pleasure to thank Igor for sharing with us his knowledge and insights on this topic. We also thank Amie Wilkinson for making Figure 2.

2 Riemann surfaces with $\mathfrak{A}_5$-action and their orbifolds

2.1 Reconstructing a surface with symmetry from its orbifold

This section is closely related to the material in §3.2 of [5], but we take here an orbifold point of view. This is in a sense a bottom up approach. We will use the language of branched covers and 2-dimensional orbifolds; see, e.g., §2 of [10]. Denote by $(g; m_1, \dots, m_r)$ a complex-analytic (or hyperbolic when specified) orbifold of genus g with r orbifold points of orders $m_1, \dots, m_r$. We use the classical terminology and call this the *genus* of the orbifold.

Our point of departure is Proposition 3.2 of [5], where we observe that this proposition is essentially topological in nature. It says among other things that if C is compact Riemann surface of genus 6 endowed with a faithful action of the alternating group $\mathfrak{A}_5$ (resp. the symmetric group $\mathfrak{S}_5$), then it gives rise to an orbifold of type $(0; 3, 2, 2, 2)$ (resp. $(0; 6, 4, 2)$). We also noted that in the case of an $\mathfrak{S}_5$-action, the passage from the $\mathfrak{A}_5$-orbit space to the $\mathfrak{S}_5$-orbit space defines a degree 2 cover from an orbifold of type $(0; 3, 2, 2, 2)$ to one of type $(0; 6, 4, 2)$ which ramifies over the orbifold points of order 6 and 4.

We investigate here to what extent the orbifold determines the Riemann surface with its group action. Our findings in this section are summed up in Theorem 1.1 in the introduction. We will deduce this theorem from two results. The first reproves in an elementary fashion the uniqueness assertion of Corollary 3.6 of [5].

Theorem 2.1 (Genus 6 surfaces with $\mathfrak{S}_5$ action). *There is, up to covering transformations, exactly one compact connected Riemann surface of genus 6 endowed with a faithful $\mathfrak{S}_5$-action.*

The surface whose uniqueness and existence it asserts is called the *Wiman curve*, and we shall denote it by C_0.

In order to state the other two results, let $(P; q_1, q_2, q_3, q_4)$ be an orbifold of type $(0; 3, 2, 2, 2)$. We first address the question 'How many Riemann surfaces of genus 6 with faithful $\mathfrak{A}_5$-action give rise to this orbifold?'. In order to state our answer, we fix an embedded segment γ in $P \smallsetminus \{q_1, q_2\}$ connecting the order 2 orbifold points q_3 and q_4. The embedded segments in $P \smallsetminus \gamma$ connecting q_1 with q_2 then belong to a single isotopy class; let α be such a segment. Clearly, α is a deformation retract of $P \smallsetminus \gamma$.

Let $f : C \to P$ be any connected *smooth* $\mathfrak{A}_5$-covering; by smooth we mean here that f 'resolves' the orbifold points in the sense that the order of the orbifold point is the order of ramification of f over this point. Then $f^{-1}\gamma$ is a disjoint union of embedded circles and a deformation retraction of $P \smallsetminus \gamma$ onto α lifts to a deformation retraction of $C \smallsetminus f^{-1}\gamma$ onto $f^{-1}\alpha$. So each connected component of $f^{-1}(P \smallsetminus \gamma)$ has a connected component of $f^{-1}\alpha$ as deformation retract, and since $f^{-1}\alpha$ is a trivalent graph (with vertex set the preimage of q_1), this component will have first Betti number ≥ 2, and therefore negative Euler characteristic.

Denote by t_γ the simple braid generator in the mapping class group of the orbifold P supported by a regular neighborhood of γ in $P \smallsetminus \{q_1, q_2\}$. So a homeomorphism representing t_γ interchanges q_3 and q_4. Note that t_γ takes an $f : C \to P$ as above to an $\mathfrak{A}_5$-covering $f' : C' \to P$ of the same *type*, by which we mean that this comes with orientation-preserving $\mathfrak{A}_5$-equivariant diffeomorphism $(C, f^{-1}\gamma) \cong (C', f'^{-1}\gamma)$.

The following is the second main result of this section. It answers the above question.

Theorem 2.2 (Genus 6 surfaces with $\mathfrak{A}_5$ action). *Up to P-isomorphism, there are 18 $\mathfrak{A}_5$-coverings of P that resolve the orbifold singularities; these coverings are necessarily of genus 6. They come in three types and make up five orbits under the group $\langle t_\gamma \rangle$ generated by t_γ. More precisely, they are:*

Petersen configuration *The intersection graph of $(C, f^{-1}\gamma)$ is the Petersen graph; this is a single $\langle t_\gamma \rangle$-orbit which has two elements. The $\mathfrak{A}_5$-stabilizer of a connected component of $C \smallsetminus f^{-1}\gamma$ is conjugate to $\mathfrak{S}_3^{\mathrm{ev}}$ (or equivalently, is the $\mathfrak{A}_5$-centralizer of a transposition of $\mathfrak{S}_5$).*

K_5 configuration *The intersection graph of $(C, f^{-1}\gamma)$ is the complete graph on 5 vertices; it has two $\langle t_\gamma \rangle$-orbits, each having three elements. The $\mathfrak{A}_5$-stabilizer of a connected component of $C \smallsetminus f^{-1}\gamma$ is conjugate to the subgroup $\mathfrak{A}_4 \subset \mathfrak{A}_5$.*

Dodecahedral configuration *The intersection graph of* $(C, f^{-1}\gamma)$ *has a single vertex with five loops; it has two* $\langle t_\gamma \rangle$*-orbits, each having five elements* ([1])*.*

An outer automorphism of $\mathfrak{A}_5$ *leaves the orbit type invariant; it exchanges the two elements of the Petersen case and exchanges the two orbits in the last two cases.*

Remark 2.3. The topological interpretation of the t_γ-action is as follows. If U_γ is a thin regular neighborhood U_γ of γ, then each connected component of $f^{-1}U_\gamma$ has the same degree d over U_γ and for the three cases listed d equals 2, 3 and 5 respectively. The mapping class t_γ^d takes C to itself, but induces a Dehn twist along each connected component of $f^{-1}\gamma$. In particular, its restriction to $C \smallsetminus f^{-1}\gamma$ is isotopic to the identity.

We prove at the same time a universal property. This will yield a different (and perhaps more elementary) proof of Theorem 3.4 and Corollary 3.5 of [5].

For P as above, the double cover of P branched at the four orbifold points is a genus one Riemann surface, which we make an elliptic curve by taking the order 3 point as origin. Then the three other points of ramification are the points of order 2 of this elliptic curve. Thus the isomorphism type of P determines and is determined by an element of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$, the moduli space of elliptic curves.

Before proceeding to the proofs of the theorems above, we make a few remarks.

Remark 2.4. Recall that a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ is called a *congruence subgroup* if for some positive integer m it is the preimage of a subgroup of $\mathrm{PSL}_2(\mathbb{Z}/m)$ under the mod m reduction $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/m)$. It was shown by H. Larcher (Theorem C in [8]) that the set of cusp widths of such a group is closed under taking *lcm* and *gcd*. As this is evidently not the case for the group Γ in Theorem 1.1, we conclude that Γ is *not* a congruence subgroup.

Remark 2.5 (Universal family versus Deligne-Mumford stack). We established in Theorem 3.4 of [5] that $\mathcal{B}^\circ$ parametrizes the smooth, projective, genus 6 curves endowed with a faithful $\mathfrak{A}_5$-action. It is given by the part of the base of the Wiman-Edge pencil over which we have smooth fibers; there are five singular fibers, each of which is Deligne-Mumford stable, and which have dual intersection graphs as described here. In other words, our notation is compatible with the notation employed in [5]. In particular, $\mathcal{B}$ can be identified with the base of the Wiman-Edge pencil.

A comment regarding Theorem 3.4 of [5]—if not a correction—is in order. The $\mathfrak{S}_5$-action on the Wiman curve implies that as a $\mathfrak{A}_5$-curve it admits an automorphism that is not inner. So if we want to attribute to the family over $\mathcal{B}^\circ$ a universal property, as does Theorem 3.4, then we must work in the setting of Deligne-Mumford stacks. To be precise, while it is true that every family of smooth geometrically connected $\mathfrak{A}_5$-curves of genus 6 fits in a cartesian square with the Wiman-Edge pencil on the right, there are a few (rare) cases for which there exist more than one choice for the top arrow. A similar phenomenon occurs at the Petersen curve.

Remark 2.6. Since Γ is torsion free and contained in $\mathrm{PSL}_2(\mathbb{Z})$, it is in fact free, and it can be identified with the fundamental group of $\mathcal{B}^\circ$, with the customary ambiguity: we need to choose a base point. Here a natural choice is the point $c_o \in \mathcal{B}$ defining the Wiman curve–and

[1]The name *dodecahedral configuration* will become clear later.

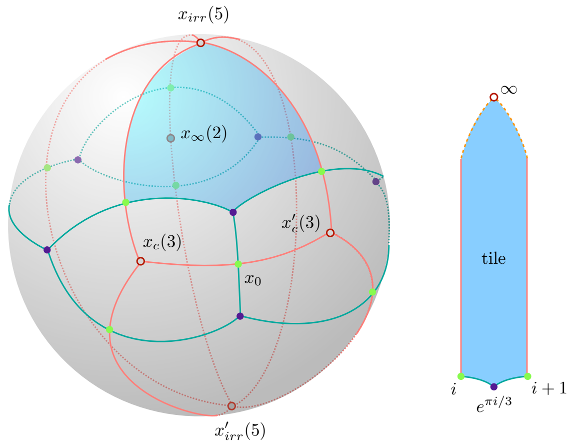

Figure 3: The base $\mathcal{B}$ of the Wiman-Edge pencil $\mathscr{C}$. It is a 2-sphere with 5 marked points (colored red in the figure) corresponding to the 5 singular members of $\mathscr{C}$. As explained in Theorem 1.1, the complement $\mathcal{B}^\circ$ of these 5 points in $\mathcal{B}$, which represents the moduli space of smooth genus 6 curves with an $\mathfrak{A}_5$ action, has a finite area hyperbolic metric with 5 cusps, one for each deleted point. The cusp x_∞ has width 2, the cusps x_c and x'_c have width 3 and the cusps x_{ir} and x'_{ir} have width 5. The point x_o defines the Wiman curve and ι is the reflection in the axis through x_o and x_∞.

an isomorphism is then given up to inner automorphism. The base $\mathcal{B}^\circ$ is a 5-punctured sphere and so Γ is free on 4 generators.

In order to prove the theorems stated above, we need some preparation. We do part of this in greater generality than is needed here, as we believe that this makes the discussion more transparent, while it involves no additional effort.

2.2 Coverings of orbifolds of genus zero

We denote by $\pi_{0,n}$ the group with generators $a_1, \dots, a_n$ subject to the relation $a_1 \dots a_n = 1$. This is of course the fundamental group of an n-punctured genus zero surface, where a_i represents a simple loop around the ith puncture. We begin by introducing notation for data that are useful for encoding a finite group action on a Riemann surface.

Definition 2.7 (The set $\boldsymbol{G(p)}$)**.** Given a group G and a descending sequence of positive integers $\mathbf{p} = (p_1 \geq \cdots \geq p_n)$ of length $n \geq 3$ with $p_n > 1$, we denote by $\tilde{G}(\mathbf{p})$ the set of surjective group homomorphisms $\mathbf{g} : \pi_{0,n} \to G$ such that $\mathbf{g}(a_i)$ has order p_i $(i = 1, \dots, n)$. We can of course think of $\mathbf{g}$ as an n-tuple $(g_1, \dots, g_n) \in G^n$ with $g_1 g_2 \dots g_n = 1$ and whose members generate G. Note that $\mathrm{Aut}(G)$ acts on $\tilde{G}(\mathbf{p})$ by postcomposition on the left. In particular G acts via $\mathrm{Inn}(G)$ on $\tilde{G}(\mathbf{p})$ by simultaneous conjugation. Define $G(\mathbf{p})$ to be the quotient

$$G(\mathbf{p}) := G\backslash\tilde{G}(\mathbf{p})$$

so that the $\mathrm{Aut}(G)$-action on $\tilde{G}(\mathbf{p})$ induces an $\mathrm{Out}(G)$-action on $G(\mathbf{p})$.

Denote by $\widetilde{\mathrm{Mod}}_{0,\mathbf{p}}$ the group of automorphisms of $\pi_{0,n}$ that permute the conjugacy classes of the a_i in such a manner that if a_i is mapped to the conjugacy class of a_j, then $p_i = p_j$. It acts in an obvious manner (by precomposition, so on the right) on $\tilde{G}(\mathbf{p})$, and this action evidently commutes with the $\mathrm{Aut}(G)$ action. Note that $\widetilde{\mathrm{Mod}}_{0,\mathbf{p}}$ contains the group of inner automorphisms of $\pi_{0,n}$ as a normal subgroup. We write $\mathrm{Mod}_{0,\mathbf{p}}$ for the quotient group. So $\mathrm{Mod}_{0,\mathbf{p}}$ is a group of outer automorphisms of $\pi_{0,n}$; we may regard it as the mapping class group of an orbifold of type $(0; p_1, \dots, p_n)$.

The outer automorphisms of $\pi_{0,n}$ that preserve the conjugacy class of *each* a_i form a subgroup of $\mathrm{Mod}_{0,\mathbf{p}}$ that can be identified with the mapping class group $\mathrm{Mod}_{0,n}$ of an n-punctured sphere that preserves the punctures. It is a normal subgroup of $\mathrm{Mod}_{0,\mathbf{p}}$ with factor group the subgroup $\mathfrak{S}_{\mathbf{p}} \subset \mathfrak{S}_n$ of $\sigma \in \mathfrak{S}_n$ which fix the map $i \mapsto p_i$, so that we have a short exact sequence

$$1 \to \mathrm{Mod}_{0,n} \to \mathrm{Mod}_{0,\mathbf{p}} \to \mathfrak{S}_{\mathbf{p}} \to 1.$$

These groups also have an interpretation as (orbifold) fundamental group: $\mathrm{Mod}_{0,n}$ is the fundamental group of the fine moduli space $\mathcal{M}_{0,n}$ of n-pointed genus zero curves (the space of injective maps $\{1, \dots, n\} \to \mathbb{P}^2$ modulo projective equivalence) and $\mathrm{Mod}_{0,\mathbf{p}}$ is the orbifold fundamental group of the space $\mathcal{M}_{0,\mathbf{p}}$ parametrizing orbifolds of type $(0; p_1, \dots, p_n)$. The latter is in general not a fine moduli space, but underlies a Deligne-Mumford stack.

For any $\mathbf{g} \in \tilde{G}(\mathbf{p})$, precomposition of $\mathbf{g}$ with the inner automorphism defined by $a \in \pi_{0,n}$ is the same as postcomposition with the inner automorphism defined by $\mathbf{g}(a) \in \pi_{0,n}$, and so the right action of $\widetilde{\mathrm{Mod}}_{0,\mathbf{p}}$ on $\tilde{G}(\mathbf{p})$ decends to a right action of $\mathrm{Mod}_{0,\mathbf{p}}$ on $G(\mathbf{p})$ that commutes with the action of $\mathrm{Out}(G)$.

We use this setup to encode the data of a G-cover of an orbifold of type $(0; p_1, \dots, p_n)$. We shall see that the set $G(\mathbf{p})/\mathrm{Mod}(\mathbf{p})$ classifies the closed connected surfaces with a faithful G-action for which the associated orbifold is of type $(0; p_1, \dots, p_n)$. Points of $G(\mathbf{p})/\mathrm{Mod}_{0,n}$ enumerate in addition the irregular orbits (the ith orbit has size $|G|/p_i$).

Constructing the moduli space

We now explain how one constructs the entire moduli space of Riemann surfaces endowed with a faithful action of a finite group G whose associated orbifold is of type $(0; p_1, \dots, p_n)$. We shall here assume that $n \geq 3$ and that each p_i is ≥ 2. Let P be a copy of the Riemann sphere. Given an injection $\mathbf{q} : \{1, \dots, n\} \to P$, $i \mapsto q_i$, put $U_{\mathbf{q}} := P \smallsetminus \{q_1, \dots, q_n\}$. Choose a basepoint $q_0 \in U_{\mathbf{q}}$ and a set of standard generators of $\pi_1(U_{\mathbf{q}}, q_0)$ that identifies $\pi_1(U_{\mathbf{q}}, q_0)$ with $\pi_{0,n}$. So a_i is represented by a simple (positively oriented) loop around q_i $(i = 1, \dots, n)$ such that $a_1 \dots a_n = 1$ is the only relation (we traverse composite loops in the given order, so from left to right).

Given $\mathbf{g} \in \tilde{G}(\mathbf{p})$, then a homomorphism $\pi_1(U_{\mathbf{q}}, q_0) \to G$ is defined by $a_i \mapsto g_i$. This homomorphism is surjective and hence defines a connected G-covering $U_{\mathbf{q}}(\mathbf{g}) \to U_{\mathbf{q}}$. By the theory of coverings, $\mathbf{g}' \in \tilde{G}(\mathbf{p})$ is in the same G-orbit as $\mathbf{g}$ precisely if there exists a G-isomorphism $U_{\mathbf{q}}(\mathbf{g}') \cong U_{\mathbf{q}}(\mathbf{g})$ over $U_{\mathbf{q}}$. This isomorphism need not be unique, because a G-covering $U_{\mathbf{q}}(\mathbf{g}) \to U_{\mathbf{q}}$ may have automorphisms over $U_{\mathbf{q}}$. But such an automorphism must be a deck transformation which commutes with all other deck transformations, in other words, must belong to the center of G. Therefore, *we assume from now on that G has trivial center.* Note that this holds both for $\mathfrak{A}_5$ and for $\mathfrak{S}_5$.

We apply the Riemann extension theorem to extend $U_{\mathbf{q}}(\mathbf{g}) \to U_{\mathbf{q}}$ uniquely to a ramified G-covering $P_{\mathbf{q}}(\mathbf{g}) \to P$, so that $P_{\mathbf{q}}(\mathbf{g})$ is a nonsingular projective curve and with ramification order p_i over q_i. Because of the above assumption, a G-isomorphism $P_{\mathbf{q}}(\mathbf{g}') \cong P_{\mathbf{q}}(\mathbf{g})$ exists over P if and only if $\mathbf{g}'$ and $\mathbf{g}$ lie in the same G-orbit. So if we regard $(P, \mathbf{q})$ as an orbifold of type $(0; p_1, \dots, p_n)$, then the smooth connected G-covers of $(P, \mathbf{q})$ (so those which resolve the orbifold singularities) form a $G(\mathbf{p})$-torsor.

If we now allow $\mathbf{q}$ to vary, then the P-isomorphism types of these covers define a covering of the configuration space $P^{(n)}$ of injective maps $\mathbf{q} : \{1, \dots, n\} \hookrightarrow P$ with fiber isomorphic to $G(\mathbf{p})$. The action of $\mathrm{Aut}(P)$ on P lifts to this covering, despite the fact that $\mathrm{Aut}(P) \cong \mathrm{PSL}_2(\mathbb{C})$ is not simply-connected. The reason is that traversing a nontrivial loop in $\mathrm{Aut}(P)$ based at the unit acts on $\pi_1(U_{\mathbf{q}}, q_0)$ as an inner automorphism and so will act trivially on the cover. When we divide out by this $\mathrm{Aut}(P)$-action (which is easily accomplished by fixing the first three components of $P^{(n)}$) we obtain a diagram

$$\mathscr{C}_{\widehat{B}G(\mathbf{p})} \to \widehat{B}G(\mathbf{p}) \to \mathcal{M}_{0,n}, \tag{2.1}$$

in which the first morphism has the interpretation of a fine moduli space of systems $(C, \phi; \mathcal{O}_1, \dots, \mathcal{O}_n)$, where C is a connected compact Riemann surface, $\phi : G \hookrightarrow \mathrm{Aut}(C)$ an injective homomorphism such that $G\backslash C$ has genus zero and $\mathcal{O}_1, \dots, \mathcal{O}_n$ is a faithful enumeration of the irregular orbits such that $\mathcal{O}_i$ has order $|G|/p_i$. Here we declare $(C, \phi; \mathcal{O}_1, \dots, \mathcal{O}_n)$ and $(C', \phi'; \mathcal{O}'_1, \dots, \mathcal{O}'_n)$ to be equivalent if there exist a G-isomorphism of C onto C' over P that respects the indexing of the irregular orbits. Since G is centerless, this isomorphism is unique, so that the fibers of $\mathscr{C}_{\widehat{B}G(\mathbf{p})} \to \widehat{B}G(\mathbf{p})$ are smooth curves endowed with a G-action. The second morphism is given by formation of the G-orbifold, where we use the numbering of the irregular orbits.

The finite group $\mathfrak{S}_{\mathbf{p}}$ acts on $P^{(n)}$ by permuting the factors. This commutes with the $\mathrm{Aut}(P)$-action, but the product action need no longer be free, for some nontrivial element in $\mathrm{Aut}(P)$ could permute nontrivially the ramification points $(q_1, \dots, q_n)$, while preserving their weights. In other words, the residual action of the finite group $\mathfrak{S}_{\mathbf{p}}$ on the diagram (2.1) may have fixed points in $\mathcal{M}_{0,n}$ and can act nontrivially on a fiber over such a point. So the quotient by this action is a priori a Deligne-Mumford stack

$$\mathscr{C}_{\mathbb{B}G(\mathbf{p})} \to \mathbb{B}G(\mathbf{p}) \to \mathbb{M}_0(\mathbf{p}). \tag{2.2}$$

Its modular interpretation is that of (2.1), except that the irregular orbits are no longer numbered. Observe that a fiber of $\mathbb{B}G(\mathbf{p}) \to \mathbb{M}_0(\mathbf{p})$ over a non-orbifold point is as a $\mathrm{Out}(G) \times \mathrm{Mod}(\mathbf{p})$-set identified with $G(\mathbf{p})$, where the action $\mathrm{Mod}(\mathbf{p})$ is as a group of covering transformations of $\mathbb{B}G(\mathbf{p}) \to \mathbb{M}_0(\mathbf{p})$, so that the connected components of $\mathbb{B}G(\mathbf{p})$ are indexed by the set $G(\mathbf{p})/\,\mathrm{Mod}(\mathbf{p})$.

Remark 2.8. We have to resort to Deligne-Mumford stacks, because there might exist a G-curve admitting an automorphism which nontrivially permutes its irregular orbits. We shall see that this happens for the Wiman curve.

2.3 Proofs of the three theorems

In view of the previous discussion, Theorem 2.1 is an immediate consequence to the following lemma.

Lemma 2.9. *The set $\mathfrak{S}_5(6, 4, 2)$ (in the notation of §2.2) is a principal $\mathfrak{S}_5$-orbit. An orbit representative is $((123)(45), (1245), (14)(23))$.*

Proof. Since the elements of $\mathfrak{S}_5$ of order 6 make up a single conjugacy class, we can assume that $g_1 = (123)(45)$. Suppose that $g_2, g_3 \in \mathfrak{S}_5$ is such that (g_1, g_2, g_3) satisfy the definition of $\mathfrak{S}_5(6,4,2)$. Note that g_2 must be a 4-cycle. We must show that any two choices of g_2 are conjugate via an element of the centralizer $Z(g_1)$ of g_1 in $\mathfrak{S}_5$. Note that $Z(g_1)$ is generated by g_1. The $Z(g_1)$-conjugacy classes of 4-cycles are represented by (1234), (1324), (1245) and (2145). If g_2 is one of these elements, then we find that g_1g_2 is of order 5, 3, 2 and 3 respectively. So (1245) is the unique representative with the required property. We then find that $g_3 = (14)(23)$. $\square$

We now turn to the proofs of Theorems 2.2 and 1.1. So here $n = 4$ and $(p_1, p_2, p_3, p_4) = (3, 2, 2, 2)$. We begin with three combinatorial lemmas.

Lemma 2.10. *Let $\mathfrak{A}_5(r)$ denote the set of elements of $\mathfrak{A}_5$ of order r. Consider the action of $\mathfrak{A}_5$ on $\mathfrak{A}_5(3) \times \mathfrak{A}_5(2)$ by simultaneous conjugation. Then this action is free, and every orbit is represented by precisely one of the following pairs (we name each case after the conjugacy class of the subgroup of $\mathfrak{A}_5$ generated by the pair):*

$\mathfrak{S}_3^{\mathrm{ev}}$ $((123), (23)(45))$, *so that any pair (g_1, g_2) in this orbit generates a subgroup conjugate to $\mathfrak{S}_3^{\mathrm{ev}}$ and g_1g_2 has order 2,*

$\mathfrak{A}_4$ *either* $((123), (12)(34))$ *or* $((123), (12)(35))$, *so that any pair (g_1, g_2) in one of these orbits generates a subgroup conjugate to $\mathfrak{A}_4$ and g_1g_2 has order 3,*

$\mathfrak{A}_5$ *either* $((123), (14)(25))$ *or* $((123), (15)(24))$, *so that any pair (g_1, g_2) in one of these orbits generates a subgroup conjugate to $\mathfrak{A}_5$ and g_1g_2 has order 5.*

Proof. It is helpful to picture a k-cycle of $\mathfrak{S}_r$ as an oriented k-polygon with vertex set a k-element subset of $\{1, \dots, r\}$ (for $k = 2$ this amounts to an unoriented edge). So an element $(g_1, g_2) \in \mathfrak{A}_5(3) \times \mathfrak{A}_5(2)$ is represented by oriented triangle and an unordered pair of disjoint edges. We then see that only three types are possible:

$\mathfrak{S}_3^{\mathrm{ev}}$ the pair of disjoint edges meets the triangle in an edge only (the other edge is then the unique edge disjoint with the triangle),

$\mathfrak{A}_4$ the pair of disjoint edges meets the triangle in the union of an edge and the opposite vertex,

$\mathfrak{A}_5$ the pair of disjoint edges meets the triangle in two distinct vertices.

This graph-like classification (where we ignore the labeling of the vertices) is a geometric way of describing the $\mathfrak{S}_5$-orbits in $\mathfrak{A}_5(3) \times \mathfrak{A}_5(2)$. So there are three of these. It is straightforward to check that in the first case the $\mathfrak{S}_5$-orbit is also a $\mathfrak{A}_5$-orbit, and in the two other cases splits into two such orbits. The other assertions are also straightforward. $\square$

Lemma 2.11. *Fix $h \in \mathfrak{A}_5(r)$ and consider the set of pairs $(h_1, h_2) \in \mathfrak{A}_5(2) \times \mathfrak{A}_5(2)$ such that $h = h_1h_2$. For $r = 2, 3, 5$, this set consists of exactly r items: for $r = 2$, h_1 and h_2 must commute and the two pairs only differ by their order (they generate a Kleinian Vierergruppe), whereas for $r = 3$ and $r = 5$ this is a free $\langle h \rangle$-orbit (acting by simultaneous conjugation).*

The proof of this lemma is left as an exercise.

Proposition 2.12. *The* $(g_1,g_2,g_3,g_4)\in\mathfrak{A}_5(3,2,2,2)$ *come in three types, according to the order* r *of* g_1g_2*: two elements with* $r=2$*, six elements with* $r=3$ *and ten elements with* $r=5$*. The stabilizer of* a_1a_2 *in* $\mathrm{Mod}(3,2,2,2)$ *preserves each type, acts transitively on the set of elements with* $r=2$*, and has two orbits for* $r=3,5$*.*

The nontrivial element of $\mathrm{Out}(\mathfrak{A}_5)\cong\mathfrak{S}_5/\mathfrak{A}_5$ *acts on* $\mathfrak{A}_5(3,2,2,2)$ *by exchanging the last two items for* $r=2$ *and exchanging the two* $\mathrm{Mod}(3,2,2,2)$*-orbits for* $r=3,5$*.*

The action of $\mathrm{Mod}(3,2,2,2)$ *on* $\mathfrak{A}_5(3,2,2,2)$ *is transitive.*

Proof. For the first assertion we must count the $\mathfrak{A}_5$-orbits in $\tilde{\mathfrak{A}}_5(3,2,2,2)$. Lemma 2.10 allows us represent each orbit by a $\mathbf{g}=(g_1,g_2,g_3,g_4)$ with (g_1,g_2) as in that lemma. This leads us to the three types, according to the order r of g_1g_2: for $r=2$ we have one case and for $r=3,5$ we have two cases to consider. Lemma 2.10 also asserts that simultaneous $\mathfrak{A}_5$-conjugation acts freely on $\mathfrak{A}_5(3)\times\mathfrak{A}_5(2)$ and so for a given $(g_1,g_2)\in\mathfrak{A}_5(3)\times\mathfrak{A}_5(2)$, it remains to count the number of $(g_3,g_4)\in\mathfrak{A}_5(2)\times\mathfrak{A}_5(2)$ for which $g_3g_4=(g_1g_2)^{-1}$. This is what Lemma 2.11 does for us, and thus the first assertion follows. Note that a transposition in $\mathfrak{S}_5$ (which represents the nontrivial element of $\mathrm{Out}(\mathfrak{A}_5)$ exchanges the two pairs (g_3,g_4) that we get for $r=2$ and exchanges the two orbits that we get for $r=3,5$.

The stabilizer of a_1a_2 in $\mathrm{Mod}(3,2,2,2)$ will of course preserve the type. One such element is given by

$$(a_1,a_2,a_3,a_4)\mapsto(a_1,a_2,(a_1a_2)a_3(a_1a_2)^{-1},(a_1a_2)a_4(a_1a_2)^{-1}).$$

This has on $\mathfrak{A}_5(3,2,2,2)$ of course a similar effect with a_i replaced by g_i. Lemma 2.11 shows that for $r=3$ and $r=5$ we get all the elements in $\tilde{\mathfrak{A}}_5(3,2,2,2)$ with (g_1,g_2) prescribed and of order 3 or 5. When $r=2$ we use the element of $\mathrm{Mod}(3,2,2,2)$ given by $(a_1,a_2,a_3,a_4)\mapsto(a_1,a_2,a_3a_4a_3^{-1},a_3)$, which has the effect on (g_1,g_2,g_3,g_4) of exchanging g_3 and g_4. This proves the second assertion.

The last assertion will follow if we prove that $\mathrm{Mod}(3,2,2,2)$ acts transitively on the types. We start with $\mathbf{g}=((123),(14)(25),(12)(34),(15)(24))$, for which the product of the first two items has order 5. The element of $\mathrm{Mod}(3,2,2,2)$ defined by

$$\begin{aligned}&(a_1,a_2,a_3,a_4)\mapsto(a_1,a_3,a_3^{-1}a_2a_3,a_4)\text{ and its inverse}\\&(a_1,a_2,a_3,a_4)\mapsto(a_1,a_2a_3a_2^{-1},a_2,a_4)\end{aligned}$$

takes $\mathbf{g}$ to $((123),(12)(34),\dots)$ resp. $((123),(13)(45),\dots)$ for which the product of the first two items has order 3 resp. 2. □

Proof of Theorem 2.2. The number and the labeling of 18 cases follow from Proposition 2.12. It remains to show that the intersection graphs are as asserted. We do this with the help of Lemma 2.10: it tells us that the vertices are in bijection the $\mathfrak{A}_5$-left cosets of $\langle g_1,g_2\rangle\subset\mathfrak{A}_5$ and the edges are in bijection with the $\mathfrak{A}_5$-left cosets of the cyclic subgroup $\langle g_1g_2\rangle$, the incidence relation being given by inclusion. This is then straightforward. □

Proof of Theorem 1.1. Let us first observe that local charts of the moduli space $\mathcal{B}^\circ$ of compact, connected Riemann surfaces of genus 6 with faithful $\mathfrak{A}_5$-action are obtained by a total order on the four orbifold points and taking their cross ratio. This implies that the map

$\mathcal{B}^\circ \to \mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$ resolves the orbifold singularities of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$. In other words, this identifies $\mathcal{B}^\circ$ with a quotient $\Gamma\backslash\mathfrak{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ is a torsion-free subgroup. The index of Γ in $\mathrm{PSL}_2(\mathbb{Z})$ will of course be the degree of $\mathcal{B}^\circ \to \mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$, i.e., 18. It follows from Proposition 2.12 that Γ has five cusps with width 2, 3, 3, 5 and 5. The Riemann-Hurwitz formula shows that if we fill in each of the five cusps, the resulting Riemann surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\widehat{\mathfrak{H}}$ has genus zero.

The degenerations in question are obtained by shrinking γ (for this is how we tend to the cusp of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$). The extension of $\mathscr{C}_{\mathcal{B}^\circ} \to \mathcal{B}^\circ$ to $\mathscr{C}_{\mathcal{B}} \to \mathcal{B}$ is formally taken care of by the general theory of Hurwitz schemes [1], but as the present case is a relatively simple instance of this, we briefly indicate how this is done. By shrinking γ we make of course also the connected components of $f^{-1}\gamma$ shrink and each such a 'vanishing component' creates a node. To be precise, a local model at such a point is the double cover $w^2 = z^2 - t$ (with ramification points the two roots of $z^2 - t$) and when t moves in the unit disk, then over $t = 0$ the cover acquires an ordinary node. In global terms, an $\mathfrak{A}_5$-cover of $(P; q_1, q_2, q_3, q_4)$ becomes a stable curve when q_3 and q_4 coalesce and its dual intersection graph is then as described above. That these graphs are as asserted then follows from Theorem 2.2.

Applying an outer automorphism of $\mathfrak{A}_5$ must define an involution ι in $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$. It will have the point of $\mathcal{B}^\circ$ defining the Wiman curve as fixed point for which the orbifold is representable $(\mathbb{P}^1; \infty, -1, 0, 1)$. This involution is therefore representable by a Möbius transformation in the $\mathrm{PSL}_2(\mathbb{Z})$-conjugacy class of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It must normalize Γ and so it generates with Γ a subgroup $\tilde{\Gamma} \subset \mathrm{PSL}_2(\mathbb{Z})$ which contains Γ as a subgroup of index 2. The involution ι fixes the cusp of width 2, interchanges the two cusps of width 3, and interchanges the two cusps of width 5. $\square$

Remark 2.13. The theorems in this section do not include the assertion that Figure 3 is the correct representation of $\Gamma\backslash\widehat{\mathfrak{H}}$. We know however that the latter has the structure of a polygonal complex with the solid n-gons in bijective correspondence with the cusps of width n in such a manner that in each vertex exactly three of these meet. In our case we have two pentagons, two triangles and one 'bigon' and then the reader will easily find that this figure is essentially the only way these can fit together to yield a closed surface (necessarily of genus zero).

3 Geometric models of the Wiman-Edge pencil

In this section we first give a concrete geometric picture of each type of degeneration of the Wiman-Edge pencil. This is followed by three geometric descriptions of the Wiman-Edge pencil: the first one is directly based on the main results of Section 2 and the other two come from putting a natural piecewise flat resp. hyperbolic structure on the fibers.

3.1 Geometric models for the three degeneration types

Recall that we have essentially three types of nodal degenerations for genus 6 curves with $\mathfrak{A}_5$-symmetry, distinguished by their dual intersection graphs, or what amounts to the same, their configurations of vanishing cycles: the Petersen curve, the K_5-curve and the dodecahedral curve. For each of these cases we give a geometric model and use polar coordinates (of Fenchel-Nielsen type) to describe the degeneration.

For the Petersen and the K_5-curve this is accomplished by the boundary of a regular neighborhood of an embedding the graph in real 3-space. Over each midpoint of an edge we have a (vanishing) circle and these circles decompose the surface into ten 3-holed spheres resp. five 4-holed spheres, see Figure (1). The $\mathfrak{A}_5$-stabilizer of such a holed sphere S acts transitively on its boundary components and is a copy of $\mathfrak{S}_3$ resp. $\mathfrak{A}_4$.

Thus a $\mathfrak{A}_5$-invariant hyperbolic structure on the surface such that each vanishing circle becomes a geodesic, gives on S a hyperbolic metric invariant for which ∂S is geodesic and invariant under its stabilizer group. All boundary components of S thus have the same length r, and it is not hard to check that r is a complete invariant of the isometry type of S. Conversely, if we are given $r > 0$, then a number of copies of a hyperbolic holed sphere with symmetry and geodesic boundary length r can be assembled to produce a closed hyperbolic surface with $\mathfrak{A}_5$-symmetry, although this involves an additional (angular) parameter θ which prescribes how boundary components of spheres are identified. The degeneration in question is obtained by letting r tend to zero.

For the irreducible degeneration we proceed as follows. Let D be a regular dodecahedron and choose an isomorphism of its isometry group with $\mathfrak{A}_5$. Then the group $\mathfrak{A}_5$ acts transitively on the set of 12 faces of D, each of which is a regular pentagon. From each pentagonal face of D we remove an open disk of a small radius (the same for every face) centered at its barycenter so that the resulting surface D' is a $\mathfrak{A}_5$-invariant 12-holed sphere. Next we identify opposite boundary components of D' in a $\mathfrak{A}_5$-equivariant manner, for example by means of the antipodal map. Since there are 6 such pairs, the resulting surface Σ has genus 6 and comes with a faithful action of $\mathfrak{A}_5$ by orientation preserving homeomorphisms. Notice that the image of each face of D' makes up a cylinder in Σ. These 12 cylinders pave Σ in particular way: one boundary component of a cylinder meets 5 others (let us call this the *pentagonal component*) and the other boundary component (the *smooth component*) meets it opposite copy.

The surface Σ also inherits from D a flat metric which has the 20 vertices of D as its cone points, each with angular defect $\pi/5$. The underlying conformal structure extends across such points and makes it a Riemann surface with $\mathfrak{A}_5$-action. The interior of each cylinder will be even a cylinder in the conformal sense, but its closure has only symmetry under a cyclic group of order 5: along the pentagonal component it has now at each vertex of the pentagon an interior angle of size $2\pi/3$. A fundamental domain for this symmetry group is conformally equivalent to a hyperbolic 4-gon with angles $\pi/2, \pi/2, \pi/3, \pi/3$. Its isometry type is determined by the length ℓ of its first edge (between the straight angles). Note that it has an obvious axis of symmetry. Five copies of this 4-gon make up a cylinder $Z(r)$ of the desired type; the geodesic boundary component has length $r = 5\ell$. We have no freedom in the way these 12 copies of $Z(r)$ are glued onto one another along their pentagonal components, so that r is a complete invariant of the 12-holed sphere. But we have an additional angular parameter θ when identifying opposite boundary components. The degeneration is defined by letting r tend to zero.

In all three cases, the pair (r, θ) is a pair of Fenchel-Nielsen coordinates.

3.2 Geometric interpretation of the tesselation of $\mathcal{B}$

Let P be a complex orbifold of type $(0; 3, 2, 2, 2)$. We observed that if we choose an embedded segment γ that connects two orbifold points of order 2, then for every smooth $\mathfrak{A}_5$-cover

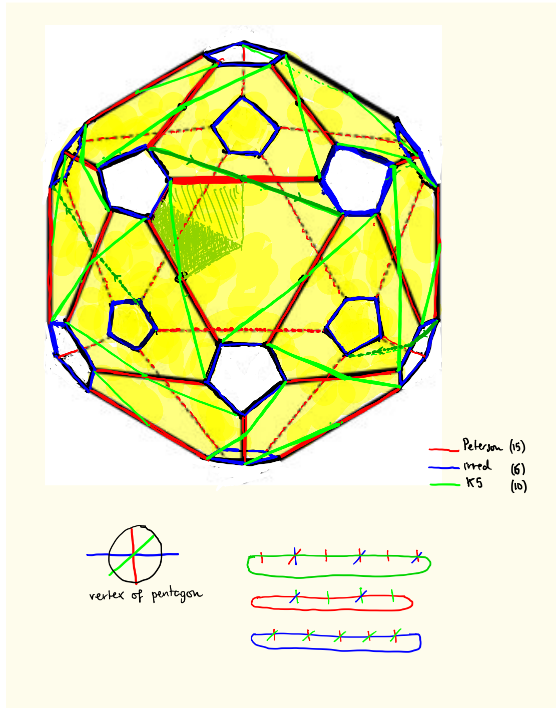

Figure 4: The three types of vanishing cycles depicted on an icosahedral surface of genus 6 (opposite pentagons must be identified).

$f : C \to P$, $f^{-1}\gamma$ is a closed submanifold of dimension one which decomposes C into genus zero surfaces with boundary. If γ' is another such embedded segment that meets γ in just one orbifold point, but has a different tangent direction half-line at that point, then the two one-dimensional submanifolds $f^{-1}\gamma$ and $f^{-1}\gamma'$ will meet transversally. We show how to do this in a prescribed and consistent manner so that we end up with a combinatorial model of sorts of the Wiman-Edge pencil.

Denote by $q_\infty \in P$ the order 3 orbifold point. We regard $P \smallsetminus \{q_\infty\}$ is an affine complex line. The set Q of three orbifold points of P of order 2 then make up a (possibly degenerate) triangle in this affine line.

Observe that the group $\mathrm{PSL}_2(\mathbb{Z})$ has in $\mathfrak{H}$ the fundamental domain the hyperbolic quadrangle (with one improper point at infinity) $F \subset \mathfrak{H}$ defined by

$$F = \{z : 0 \le \Re(z) \le 1, |z| \ge 1 \text{ and } |z-1| \ge 1\}.$$

This gives rise to a tesselation of $\mathfrak{H}$ with two $\mathrm{PSL}_2(\mathbb{Z})$-orbits of vertices, namely of $\rho := \exp(\pi\sqrt{-1}/3)$ and of $\sqrt{-1}$; and two $\mathrm{PSL}_2(\mathbb{Z})$-orbits of edges, namely of the (bounded) arc E_b centered at 0 connecting ρ and $\sqrt{-1}$, and the unbounded half line E_u on the imaginary axis with imaginary part > 1. Observe that F is bounded by E_u, E_b and the translates $E_b' := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) E_b$ and $E_u' := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) E_u$. We also note that the $\mathrm{PSL}_2(\mathbb{Z})$-orbit of $\overline{E}_b$ is the familiar trivalent graph that is a $\mathrm{PSL}_2(\mathbb{Z})$-equivariant deformation retract of $\mathfrak{H}$. So its image in $\Gamma\backslash\mathfrak{H}$ (which we shall denote by K) will be a deformation retract of $\Gamma\backslash\mathfrak{H}$.

When we interpret $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$ as the coarse moduli space of elliptic curves, the points of E_b and E_u represent elliptic curves that admit a *real structure*; that is, elliptic curves that have an antiholomorphic involution. Such an antiholomorphic involution is here given as an antiholomorphic involution of P. It is then determined by its fixed-point set in P. This is a real form of P (a real projective line). In our case this fixed-point set must pass through q_∞, and so is given in $P \smallsetminus \{q_\infty\}$ as a straight line. To see this, let us enumerate the members $\{q_1, q_2, q_3\}$ of Q in such a manner that $[q_2, q_3]$ is a side of minimal length and (q_1, q_2, q_3) defines the complex orientation of $P \smallsetminus \{q_\infty\}$ when the triangle is nondegenerate. This labeling is almost always unique; the situations for which this is not the case correspond to edges or vertices of F, namely: E_u parameterizes the situations for which q_3 lies on the segment $[q_1, q_2]$; E'_u for when q_2 lies on the segment $[q_1, q_3]$; E_b for when q_3 is equidistant to q_1 and q_2; E'_b for when q_2 is equidistant to q_1 and q_3, and (hence) ρ corresponds to q_1, q_2, q_3 making up an equilateral triangle, i to the case when $q_3 = \frac{1}{2}q_1 + \frac{1}{2}q_2$, and $i+1$ the situation for which $q_2 = \frac{1}{2}q_1 + \frac{1}{2}q_3$. In these cases the line defining the antiholomorphic symmetry is obvious: it is either a line spanned by two points of Q or a line of points equidistant to two points of Q.

3.3 A model based on affine geometry

The three sides of Q have a well-defined length ratio. Denote by γ_P the union of the sides of this triangle of minimal length. So either γ_P consists of a single edge, or consists of two edges of equal length making an angle $> \pi/3$, or consists of the three sides of an equilateral triangle. So over E_b, $\gamma(P)$ adds to $[q_2, q_3]$ the side $[q_1, q_3]$, over E'_b the side $[q_1, q_3]$ and hence over ρ we have all three sides. The map that assigns to P the isomorphism type of γ_P defines on the moduli space of such orbifolds a 'spine' which, as we will now explain, is familiar via the identification of this moduli space with $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$.

If $f : C \to P$ is an $\mathfrak{A}_5$-cover and γ_P consists of one edge, then the preimage $f^{-1}\gamma_P$ is as described by Theorem 2.2: this is a codimension one submanifold that decomposes C into genus zero surfaces-with-boundary. But if γ_P consists of two edges, then a deformation retraction $P \smallsetminus \gamma_P$ onto q_∞ lifts to $C \smallsetminus f^{-1}\gamma_P$, and this shows that every connected component of $C \smallsetminus f^{-1}\gamma_P$ is a topological disk. If γ_P consists of three sides, then a similar argument shows that the same is true for a connected components of $C \smallsetminus f^{-1}\gamma_P$.

3.4 A model based on piecewise flat structures

The theory of Jenkins-Strebel differentials provides another combinatorial model of the Wiman-Edge pencil. This amounts to putting on the complement of the size 20 orbit of every fiber a complete flat structure with a finite number of singularities.

Let P be an orbifold P of type $(0; 3, 2, 2, 2)$ in the holomorphic category. Denote the orbifold point of order 3 by q_∞ and regard $P^\circ := P \smallsetminus \{q_\infty\}$ as an affine complex line. We denote the 3-element set of order 2 orbifold points by Q. For any $a \in P^\circ$ there exists a unique meromorphic quadratic differential $\eta_{P,a}$ on P characterized by the property that it has divisor $-(2q_\infty) - Q + a_P$ and with double residue at q_∞ equal to -1. So if we choose an affine coordinate z for P°, then

$$\eta_{P,a} = -\frac{z - z(a)}{\prod_{q \in Q}(z - z(q))} dz^2.$$

At a point where $\eta_{P,a}$ has neither a zero nor a pole, there exists a local coordinate w such that $\eta_{P,a} = dw^2$. So $|\eta_{P,a}|$ defines a flat metric at such a point. If $\eta_{P,a}$ has order $k \geq -1$ at

z, then this metric extends there, and makes the space there locally a Euclidean cone with total angle $(k+2)\pi$. For $k=-2$, the metric does not extend and a punctured neighborhood will be metrically an infinite cylinder. But $\eta_{P,a}$ gives more than a metric. If we regard it as a meromorphic function on the tangent bundle of P, then the tangent vectors on which $\eta_{P,a}$ takes a real nonnegative value define a foliation on P with singularities.

In order that $\eta_{P,a}$ be a Jenkins-Strebel differential, all the leaves of this foliation save for a finite number must be compact. The theory asserts that this happens for exactly one choice of a. We then write a_P for a and η_P for $\eta_{P,a}$. The compact leaves then encircle the point at infinity and have length 2π (when the length is measured via η_P), and the noncompact leaves make up a tree D_P connecting the three orbifold points of order 2 with total length π. So the metric $|\eta_P|$ makes $P^\circ \smallsetminus D_P$ isometric to the product of the unit circle and an open Euclidean half line.

When $a_P \notin Q$, the tree D_P has three edges connecting a_P with a point of Q, otherwise D_P degenerates in an obvious way into a graph with two edges. The form η_P endows P° with a flat Euclidean structure which has singularities in Q (with angular excess $-\pi$ for any $q \in Q$ and $+\pi$ at a, albeit that the singularities at $q \in Q$ and a_P cancel each other out when $a_P = q$. This gives P a *ribbon structure* with D_P the associated "ribbon tree"; see for example [9]. The ribbon structure manifests itself only when $a_P \notin Q$ and then amounts to a cyclic ordering of Q.

Since the metrized ribbon tree D_P is a complete invariant of P (see for example [9]), the moduli space of complex orbifolds of type $(0;3,2,2,2)$ is thus identified with the space of triples of nonnegative real numbers with sum π of which at most one is zero and which are cyclically ordered when all are nonzero. By dividing these lengths by π, we can interpret such triples as the barycentric coordinates of a 2-simplex; we thus have this moduli space identified with the standard 2-simplex modulo its cyclic group of automorphisms (of order 3) minus the image of the point that represents the vertex set of the simplex. This space is homeomorphic to a sphere minus a point, the missing point corresponding to the case when (exactly) two points of Q coalesce to a single point q_0; this is evidently a degenerate case. The associated Jenkins-Strebel differential is then

$$\eta_P = -dz^2/(z - z(q_0))(z - z(q_1)), \tag{degen}$$

where q_1 denotes the remaining point (we carry P along in the notation, but P should now be thought of as a degenerate orbifold) and the associated graph D_P will be an arc connecting q_0 with q_1.

If $f : C \to P$ is a smooth $\mathfrak{A}_5$-cover (so that $f^{-1}q_\infty$ is a size 20-orbit), then $f^*\eta_P$ defines on $f^{-1}P^\circ = C \smallsetminus f^{-1}q_\infty$ a complete $\mathfrak{A}_5$-invariant flat Euclidean structure with singularities. The singular locus of this metric is $f^{-1}a_P$, for the angular excess of the preimage of an order 2 orbifold point $\neq a_P$ will vanish because of the branching. When $a_P \notin Q$, the angular excess at any point of $f^{-1}a_P$ is $+\pi$, otherwise it will be 2π, because of the ramification. In fact, $f^*\eta_P$ is a Jenkins-Strebel differential associated to the pair $(C, f^{-1}q_\infty)$. It is characterized by the property that its double residue at each point of $f^{-1}q_\infty$ is -9. Its associated metrized ribbon graph is $f^{-1}D_P$ and has $f^{-1}a_P$ as its set of genuine vertices (i.e., vertices that are not of order 2). A connected component of $C \smallsetminus f^{-1}D_P$ triply covers the open disk $P \smallsetminus D_P$ with ramification over q_∞ and hence is itself an open disk. When $a_P \notin Q$, the closure of such a component has the structure of a one point compactification of a metrical product of a circle of circumference 6π and a Euclidean half line $\mathbb{R}_{\geq 0}$. Its boundary is a 9-gon contained in $f^{-1}D_P$ which has as vertex set its intersection with $f^{-1}a_P$. Each of its nine edges contains a ramification point of f. So this gives C a canonical $\mathfrak{A}_5$-invariant cell structure.

Let us see what happens when two points of Q coalesce to a point q_0. We denote by q_1 the inert point so that now $Q_0 = \{q_0, q_1\}$ and (in anticipation of the orbifold structure we have yet to define) we denote the projective line with the three points q_∞, q_0, q_1 by P_0. Then η_{P_0} and D_{P_0} have the simple form (degen) displayed above. The $\mathfrak{A}_5$-covers over such a degenerate orbifold P_0 are nodal curves such that the preimage of q_0 is the set of nodes. Let C_0 be the normalization of an irreducible component of the associated cover. Theorem 2.2 tells us that C_0 is a copy of $\mathbb{P}^1$ and has stabilizer conjugate to $\mathfrak{S}_3^{\mathrm{ev}}$ (Petersen graph), $\mathfrak{A}_4$ (K_5 graph) or $\mathfrak{A}_5$ (irreducible case). The map $f_0 : C_0 \to P_0$ has ramification of order 3 over q_∞ and of order 2 over q_1 as before, but the ramification over q_0 depends on the case.

When C is smooth, any point of C lying over Q has stabilizer of order 2. When two such points are forced to merge in a single branch point of C_0 then this point on C_0 has as its stabilizer the subgroup of the generated by two elements of order 2 in the stabilizer of C_0. This will be of order 2 in the Petersen case, 3 in the K_5 case and 5 in the dodecahedral (irreducible) case. In other words, we may regard q_0 as an orbifold point for the cover $C_0 \to P_0$ of order 2, 3 or 5 respectively.

The pull-back $f_0^*\eta_{P_0}$ will still be a Jenkins-Strebel differential, and its order of vanishing in a point of $f_0^{-1}q_0$ will be respectively 0, 1 and 3. In the last two cases, the associate graph is just $f_0^{-1}D_{P_0}$, but in the first case this graph is in fact empty: each connected component is an infinite cylinder foliated by circles, one of these being $f_0^{-1}D_{P_0}$. We therefore put $D_{C_0} := f_0^{-1}D_{P_0}$. So in the Petersen case, D_{C_0} is a circle containing the three points of $f_0^{-1}q_0$ and the three points of $f_0^{-1}q_1$. In the K_5 case resp. dodecahedral case, each connected component of D_{C_0} is the 1-skeleton of a regular tetrahedron resp. icosahedron having $f_0^{-1}q_0$ as vertex set and $f_0^{-1}q_1$ as the set of midpoints of the edges. These regular polyhedra are however punctured at the barycenters of their faces and the piecewise flat metric makes every closed face deprived from its barycenter a half cylinder. In other words, in each of the three cases every face—a solid equilateral triangle—is replaced by the product of its boundary and a Euclidean half line. (So from this perspective we end up with an icosahedral model rather than a dodecahedral model as depicted in Figure 2.)

We thus obtain another combinatorial model of the Wiman-Edge pencil. It differs from the one we found in §3.3. For example, if Q has cyclic symmetry of order 3 (so that we can take our affine coordinate z such that $z(Q)$ are the third roots of unity), then the graph considered in §3.3 is the triangle spanned by Q, whereas D_P consists of the three rays emenating from the barycenter of this triangle to its vertices. But the underlying tesselation of $\mathcal{B}$ is the same as for the model based on affine geometry, because the loci of exceptional orbifolds has the same characterization. For example, when $\ell_1 = \ell_2$ means that q_3 is equidistant to q_1 and q_2. We then have a real structure on P for which $\{q_1, q_2\}$ is a conjugate pair and η_P will be defined over $\mathbb{R}$ relative to this structure. The situation is similar when the three points of Q are collinear.

3.5 A model based on hyperbolic geometry

We here endow each smooth member C_t of the Wiman-Edge pencil with the unique hyperbolic metric compatible with its complex structure. The $\mathfrak{A}_5$-action on C_t by biholomorphic automorphisms then preserves this hyperbolic metric, giving its orbit space the structure of a hyperbolic orbifold. We therefore begin our discussion with considering such orbifolds.

So let P be an orbifold of type $(0; 3, 2, 2, 2)$ in the holomorphic category, but endowed with the associated complete hyperbolic structure: so the total angle at the order 3 point q_∞ is $2\pi/3$ and the total angle at an order 2 point $q \in Q$ is π. Let δ be a geodesic arc in P connecting two orbifold points of order 2 which does not pass through the other orbifold points. If $f : C \to P$ is an $\mathfrak{A}_5$-cover, then $f^{-1}\delta$ consists of a collection of pairwise disjoint closed geodesics. These geodesics decompose C into hyperbolic genus zero surfaces-with-boundary. Note that if we let the two points of Q connected by δ coalesce to a single point q_0, then we obtain we have a complete orbifold hyperbolic structure on $P_0 \smallsetminus \{q_0, q_\infty\}$ (which induces a complete hyperbolic structure on its orbifold covering).

The distance between δ and the remaining point q_1 of Q will be realized be a geodesic δ'. If δ' hits δ in its interior, then it will do so orthogonally. If δ' meets δ in an endpoint, then since point is an orbifold point of order 2, the passage to a local orbifold cover shows that δ' meets δ in the opposite direction (so that both lie on a complete geodesic). Since q_∞ is on orbifold point of order 3, δ' does not pass through q_∞. So $U := P \smallsetminus \{\delta \cup \delta'\}$ is a topological disk having q_∞ as an interior point. This hyperbolic decomposition of P gives rise to a paving of C into hyperbolic right-angled hexagons whose alternating sides have equal length, namely $2\ell(\delta)$ and $2\ell(\delta')$. To see this, note that each connected component $\tilde{U}$ of $f^{-1}U$ triply covers U with the unique point $\tilde{q}_\infty \in \tilde{U}$ that lies over q_∞ as the only point of ramification. In particular, the boundary $\partial\tilde{U}$ triply covers $\delta \cup \delta'$ and we then easily find that $\tilde{U}$ is indeed the interior of a hyperbolic right-angled hexagon of the asserted type.

Noteworthy is the case when P is invariant under an involution, as this yields the Wiman curve. Then there exists an affine coordinate z for $P \smallsetminus \{q_\infty\}$ such that $z(Q) = \{-1, 0, 1\}$, the involution taking z to $-z$. We take here $\delta \cup \delta' = [-1, 1]$ (which is indeed the union of two geodesics). So then our right-angled hexagons have all sides of equal length. The right angles are at points of the orbit $f^{-1}(0)$ and the two orbits $f^{-1}(1)$ and $f^{-1}(-1)$ consists of midpoints of edges. When the involution lifts to C, then this lift and the covering transformations endow C with an $\mathfrak{S}_5$-action, so that C will be a copy of the Wiman curve.

If we succeed in shrinking δ to a point q_0, while taking along a δ' as above, then α' becomes a geodesic ray α'_0 from q_1 to the cusp q_0. Now $f_0^{-1}\alpha'_0$ will be a tesselation of $C_0 \smallsetminus f_0^{-1}q_0$ into ideal triangles (which may be regarded as degenerate right angled hexagons). But we do not know whether we can do this in a uniform manner as we did for piecewise flat structures on P using Jenkins-Strebel differentials.

4 The Wiman curve via hyperbolic triangle groups

The goal of this section is to investigate the natural hyperbolic structure on the Wiman curve. We shall later establish that it is in fact a Shimura curve of indefinite quaternionic type.

4.1 Spinor norm

This subsection merely serves to recall the definition and a few simple properties related to the notion of spinor norm.

Let V be a finite-dimensional vector space over a field F endowed with a quadratic form $q : V \to F$. This means that $q(\lambda v) = \lambda^2 q(v)$ for $\lambda \in F$ and $v \in V$ and that the map $b : V \times V \to F$ given by
$$b(v, v') := q(v + v') - q(v) - q(v')$$
is bilinear. Note that $b(v, v) = 2q(v)$, so that we can recover q from b if $\mathrm{char}(F) \neq 2$. Given $u \in V$ with $q(u) \neq 0$, then the orthogonal reflection $s_u : V \to V$ with respect to u is the element of $\mathrm{O}(V)$ defined by

$$s_u(v) := v - q(u)^{-1} b(v, u)u.$$

It is clear that $s_u = s_{u'}$ if and only if $u' = \lambda u$ for some $\lambda \in F^\times$ and so the image of $q(u)$ in $F^\times/(F^\times)^2$ is an invariant of s_u. The *spinor norm* is a group homomorphism $\epsilon : \mathrm{O}(V) \to F^\times/(F^\times)^2$ characterized by the property that it assigns to s_u the image of $q(u)$ in $F^\times/(F^\times)^2$. We denote its kernel by $\mathrm{O}^\#(V)$. When every element of F is a square (which is for instance the case when $F = \mathbb{F}_2$ or $F = \mathbb{C}$), then $\mathrm{O}(V) = \mathrm{O}^\#(V)$ and this notion is devoid of interest.

This is not so when $F = \mathbb{R}$, for we may then identify $F^\times/(F^\times)^2$ with $\{\pm 1\}$. In that case we have the following alternate definition of the spinor norm. If q has signature (p, n), then the negative definite linear subspaces $W \subset V$ of dimension n make up a contractible open subset $\mathbb{D}(q)$ in the Grassmannian (it is the symmetric space of $\mathrm{O}(V)$) and hence the tautological n-plane bundle over $\mathbb{D}(q)$ is trivial. So this bundle can be oriented. Any reflection $s_u : V \to V$ will leave invariant a negative definite n-plane N. It will act on N as the identity resp. as a reflection (and hence be orientation-reversing in N) if $q(u) > 0$ resp. $q(u) < 0$. Thus the spinor norm of $g \in \mathrm{O}(V)$ is 1 if and only if the action of g on this bundle is orientation preserving. The spinor norms attached to q and $-q$ define a homomorphism $\mathrm{O}(V) \to \{\pm 1\} \times \{\pm 1\}$ whose kernel is the identity component of $\mathrm{O}(V)$; if q is indefinite, then this map is also onto. Note that the product of these two spinor norms is the determinant $\det : \mathrm{O}(V) \to \{\pm 1\}$. In particular, the two spinor norms coincide on $\mathrm{SO}(V)$, and the common value on $g \in \mathrm{SO}(V)$ is 1 if and only if it induces an orientation preserving diffeomorphism of $\mathbb{D}(q)$.

4.2 Triangle groups of compact hyperbolic type

For a triple (p_1, p_2, p_3) with $p_i \in \{2, 3, 4, \dots, \infty\}$ (and here ordered as $p_1 \geq p_2 \geq p_3$), the *triangle group* $\Delta(p_1, p_2, p_3)$ is the Coxeter group defined by generators s_1, s_2, s_3 subject to the relations $s_i^2 = (s_{i-1}s_{i+1})^{p_i} = 1$, where we let i run over $\mathbb{Z}/(3)$. The Tits reflection representation (as described for example in Bourbaki) makes $\Delta(p_1, p_2, p_3)$ a subgroup of $\mathrm{GL}_3(\mathbb{R})$, as follows: we define a symmetric bilinear form $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $b(e_i, e_i) = 1$ and $b(e_i, e_j) := -\cos(\pi/p_{i+j})$ for $i \neq j$; we then assign to s_i the b-orthogonal reflection with respect to e_i:
$$s_i(x) = x - 2b(x, e_i)e_i.$$

The reflection representation is actually defined to be the dual of this representation: a fundamental chamber is then given by the $\xi \in (\mathbb{R}^3)^\vee$ that are positive on each e_i. When b is nondegenerate, the passage to the dual is not necessary, as this representation is then self-dual and we can take as the fundamental chamber the set defined by $b(x, e_i) > 0$. The index two subgroup $\Delta^+(p_1, p_2, p_3) \subset \Delta(p_1, p_2, p_3)$ of orientation-preserving elements is generated by $g_i := s_{i-1}s_{i+1}$ $(i \in \mathbb{Z}/3)$ and has as a complete set of relations $g_i^{p_i} = 1$ (for $p_i = \infty$ read this as the empty relation) and $g_1g_2g_3 = 1$.

Now assume that we are in the compact hyperbolic case: this means that each p_i is finite and that $1/p_1 + 1/p_2 + 1/p_3 < 1$. Then b is indeed nondegenerate with signature $(2, 1)$ and there is a unique connected component C^+ of the of set $b(x, x) < 0$ for which the closure of the fundamental chamber is contained in $C^+ \cup \{0\}$. So the projection of C^+ in the real projective plane $\mathbb{P}^2(\mathbb{R})$ is a hyperbolic disk $\mathbb{D}$ (which is in fact the symmetric space of $\mathrm{O}(b)$) and the closed fundamental chamber defines a solid geodesic triangle Π in $\mathbb{D}$. The classical Schwarz theory (which is here subsumed by the theory of reflection groups) tells us that $\Delta(p_1, p_2, p_3)$ acts faithfully and properly discretely on $\mathbb{D}$ and has the compact Π as a strict fundamental domain. So for any $i \in \mathbb{Z}/(3)$, the hyperbolic quadrangle $\Pi \cup s_i\Pi$ serves as a fundamental domain for $\Delta^+(p_1, p_2, p_3)$.

4.3 The triangle group attached to the Wiman curve

Let us now focus on the case of interest here, where $(p_1, p_2, p_3) = (6, 4, 2)$. So then

$$\begin{aligned} b(e_2, e_3) &= -\cos(\pi/6) = -\tfrac{1}{2}\sqrt{3}, \\ b(e_3, e_1) &= -\cos(\pi/4) = -\tfrac{1}{2}\sqrt{2}, \\ b(e_1, e_2) &= -\cos(\pi/2) = 0. \end{aligned}$$

This can be made part of system of root data in several ways. We take $\alpha_1 := 2e_1$ and $\alpha_2 := \sqrt{6}e_2$, $\alpha_3 := \sqrt{2}e_3$ so that

$$b(\alpha_i, \alpha_j) = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 2 \end{pmatrix}.$$

This allows us to define a generalized root system for which the α_i are the simple roots and $\alpha_i{}^\vee := 2b(\alpha_i, -)/b(\alpha_i, \alpha_i)$ is the coroot corresponding to α_i. Then $\alpha_i{}^\vee(\alpha_j)$ (an entry of the Cartan matrix) is integral and $s_i(x) = x - 2\alpha_i{}^\vee(x)\alpha_i$. So the *root lattice*

$$L := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$$

is invariant under $\Delta(6, 4, 2)$ and b is an even, integral, symmetric bilinear form on L. In particular, we have an associated quadratic form $q : L \to \mathbb{Z}$, $q(v) = \frac{1}{2}b(v, v)$. (We could have taken α_1 to be e_1 instead of $2e_1$, but then b would not be even and so q would not be $\mathbb{Z}$-valued.) The discriminant of b is easily computed to be -12.

Let us observe that if we write $x = \sum_i x_i\alpha_i$, then

$$2q(x) = b(x, x) = 4x_1^2 + 6x_2^2 + 2x_3^2 - 4x_1x_3 - 6x_2x_3 = (2x_1 - x_3)^2 - 3x_2^2 + (x_3 - 3x_2)^2,$$

If we reduce this modulo 3, we get $(2x_1 - x_3)^2 + x_3^2$ and this quadric has no solution in $\mathbb{P}^2(\mathbb{F}_3)$. So the conic $K_q \subset \mathbb{P}^2$ defined by q has no rational point.

We thus get $\Delta(6, 4, 2) \hookrightarrow \mathrm{O}(q)$ and hence $\Delta^+(6, 4, 2) \hookrightarrow \mathrm{SO}(q)$. Vinberg's theory of hyperbolic reflection groups shows that $\Delta(6, 4, 2)$ the $\mathrm{O}(q)$-stabilizer of the cone C^+, that is, to $\mathrm{O}^\#(q)$. So $\Delta^+(6, 4, 2) = \mathrm{SO}^\#(q)$.

Let us consider the reduction mod 5 of q,

$$q_{\mathbb{F}_5} : L_{\mathbb{F}_5} \to \mathbb{F}_5.$$

Since b has discriminant -12, we know that $b_{\mathbb{F}_5}$ is nondegenerate, but as -12 is not a square mod 5, it is not equivalent to the standard diagonal form modulo 5. The identity

$$q(-\alpha_1 + 2\alpha_2 + 2\alpha_3) = 10$$

shows that $K_q(\mathbb{F}_5)$ is nonempty, and so we can identify $K_q(\mathbb{F}_5)$ with $\mathbb{P}^1(\mathbb{F}_5)$. We thus obtain a homomorphism

$$\mathrm{O}(q) \to \mathrm{Aut}(K_b(\mathbb{F}_5)) \cong \mathrm{PGL}_2(\mathbb{F}_5).$$

It is known that $\mathrm{PGL}_2(\mathbb{F}_5) \cong \mathfrak{S}_5$.

Lemma 4.1. *The natural homomorphism* $\mathrm{SO}^{\#}(q) \to \mathrm{Aut}(K_b(\mathbb{F}_5)) \cong \mathfrak{S}_5$ *is surjective with kernel the principal level* 5 *subgroup of* $\mathrm{SO}^{\#}(q)$ *(denoted here by* Γ_5*). The group* Γ_5 *is torsion free so that the image of each* g_i *in* $\mathrm{PGL}_2(\mathbb{F}_5)$ *has the same order as* g_i*.*

Proof. Let us first show that the kernel is as asserted. Suppose $g \in \mathrm{SO}^{\#}(q)$ acts as the identity on $K_b(\mathbb{F}_5)$. Let $\overline{\mathbb{F}}_5$ be an algebraic closure of $\mathbb{F}_5$. An automorphism of $\mathbb{P}^1(\overline{\mathbb{F}}_5)$ which is the identity on $\mathbb{P}^1(\mathbb{F}_5)$ must be the identity (as it fixes three distinct points). So g acts as the identity on $K_b(\overline{\mathbb{F}}_5)$. Since we have a natural identification of $\mathrm{Sym}^2(K_b(\overline{\mathbb{F}}_5))$ with $\check{\mathbb{P}}^2(\overline{\mathbb{F}}_5)$ (a line in $\mathbb{P}^2(\overline{\mathbb{F}}_5)$ meets $K_b(\overline{\mathbb{F}}_5)$ in a degree 2 divisor), it follows that g acts as the identity on $\check{\mathbb{P}}^2(\overline{\mathbb{F}}_5)$ and hence also as the identity on $\mathbb{P}(L_{\mathbb{F}_5})$. So $g \in \mathrm{SO}^{\#}(q)$ will act on $L_{\mathbb{F}_5}$ as scalar multiplication, say by $\lambda \in \mathbb{F}_5^\times$. Since $\det(g) = 1$, we must have $\lambda^3 = 1$. Since, $\mathbb{F}_5^\times$ is cyclic of order 4, it follows that $\lambda = 1$.

For the remaining part we use Serre's observation that if an element of $\mathrm{GL}(n, \mathbb{Z})$ has finite order, then for any $\ell \geq 3$, its mod ℓ reduction has the same order. We thus find that g_1, g_2 define in $\mathfrak{S}_5$ elements of order 6 and 4 respectively such that $g_1 g_2$ has order 2. This does not yet imply that g_1 and g_2 generate $\mathfrak{S}_5$: we must exclude the possibility that they generate a subgroup conjugate to $\mathfrak{S}_3 \times \mathfrak{S}_2$). So it suffices to find an element in $\Delta^+(6, 4, 2)$ whose image in $\mathrm{Aut}(K_b(\mathbb{F}_5))$ has order 5, or in view of the preceding, whose image in $\mathrm{SO}^{\#}(q_{\mathbb{F}_5})$ has order 5. One checks that $(s_1 s_2 s_3)^2 \in \Delta^+(6, 4, 2)$ has this property. $\square$

It follows that Γ_5 acts freely on $\mathbb{D}$, so that $\Gamma_5\backslash\mathbb{D}$ is a compact orientable hyperbolic surface. It comes with a faithful action of $\mathrm{SO}^{\#}(L_{\mathbb{F}_5}) \cong \mathfrak{S}_5$ and a Riemann-Hurwitz count shows that it has genus 6. We conclude:

Theorem 4.2. *If we endow* $\mathbb{D}$ *with an orientation such that* $\Gamma_5\backslash\mathbb{D}$ *becomes a compact connected Riemann surface of genus* 6 *with a faithful action of* $\mathrm{SO}^{\#}(L_{\mathbb{F}_5}) \cong \mathfrak{S}_5$*, then* $\Gamma_5\backslash\mathbb{D}$ *is isomorphic to the Wiman curve.*

Remark 4.3 (The antiholomorphic involution). The orientation reversing elements of $\Delta(6, 4, 2)$ are reflections. There are three conjugacy classes of these, represented by s_1, s_2, s_3. That these are indeed distinct is a consequence of the evenness of the arguments of $\Delta(6, 4, 2)$, for this implies that we have a surjection $\Delta(6, 4, 2) \to (\mathbb{Z}/2)^3$ which sends s_i to the ith generator. Each s_i determines a $\mathfrak{S}_5$-conjugacy class of antiholomorphic involutions of C. We can interpret this by saying that there are exactly 3 isomorphism types of nonsingular projective curves of genus 6 with $\mathfrak{S}_5$-action that are defined over $\mathbb{R}$.

5 Modular interpretation of the Wiman curve

The goal of this section is to prove Theorem 1.3. As mentioned in the introduction we prove in fact somewhat more precise result (Theorem 5.4). We continue with the terminology used in §4.3.

5.1 Hodge structures parametrized by the Wiman curve

We fix an orientation on $L_{\mathbb{R}}$. As we will explain, this turns $\mathbb{D}$ into symmetric domain for $\mathrm{SO}^{\#}(L_{\mathbb{R}})$ which parametrizes weight zero Hodge structures on L.

We first observe that an element of $\mathbb{D}$ is uniquely represented by a vector f in the cone C^+ (defined in Subsection 4.3 with $q(f)=-1$. We shall use this to identify $\mathbb{D}$ with a connected component of the hyperboloid in $L_{\mathbb{R}}$ defined by this identity. Note that for $f\in\mathbb{D}$ the orthogonal complement of f is positive definite. So if (e',e'') is an orthogonal *oriented basis* of this orthogonal complement then the complex structure on the tangent space of $\mathbb{D}$ at f is given by the transformation which takes e' to e'' and e'' to $-e'$. Note that $e:=e'+\sqrt{-1}e''$ is the $-\sqrt{-1}$ eigenspace for this complex structure and that the map

$$T_f\mathbb{D}\to\mathbb{C}\otimes_{\mathbb{R}}T_f\mathbb{D}\to\mathbb{C}\otimes_{\mathbb{R}}T_f\mathbb{D}/\mathbb{C}e$$

is then an isomorphism of complex lines. This shows that the map $\mathbb{D}\to\mathbb{P}(L_{\mathbb{C}})$ which assigns to f the complex-linear span of e is holomorphic. Note that e has the property that $b_{\mathbb{C}}(e,e)=0$ and $b_{\mathbb{C}}(e,\bar{e})>0$. This defines a weight zero Hodge structure on L which only depends on f: $L_f^{1,-1}$, $L_f^{-1,1}$, $L_f^{0,0}$ is spanned by e, $\bar{e}$, f respectively. This Hodge structure is polarized by $-b$. The locus in $\mathbb{P}(L_{\mathbb{C}})$ defined $b_{\mathbb{C}}(e,e)=0$ and $b_{\mathbb{C}}(e,\bar{e})>0$ has two connected components and the map just defined $f\in\mathbb{D}\mapsto[L_f^{1,-1}]\in\mathbb{P}(L_{\mathbb{C}})$ identifies $\mathbb{D}$ with one of these in the holomorphic category. In particular, $\mathbb{D}$ becomes a symmetric domain for $\mathrm{SO}^{\#}(L_{\mathbb{R}})$.

Thus the Wiman curve parametrizes Hodge structures of this type. There exists an isometric embedding of L in a $K3$-lattice (this is an even unimodular lattice of signature $(3,19)$) and any two such differ by an orthogonal transformation of the $K3$-lattice. This enables us to let the Wiman curve parametrize $K3$-surfaces (in fact, Kummer surfaces), but we prefer to set up things in a more canonical fashion which for instance takes explicitly into account the $\mathfrak{A}_5$-symmetry. This involves a Kuga-Satake construction in the spirit of Deligne [4].

5.2 The Clifford algebra associated to our triangle group

The Clifford algebra associated to the quadratic lattice (L,q) is by definition the quotient $C(q)$ of the tensor algebra of L by the 2-sided ideal generated by the even tensors of the form $v\otimes v-q(v)$, $v\in L$. This algebra clearly comes with an action of $\mathrm{O}(q)$. The anti-involution which reverses the order of a pure tensor, $a=v_1\otimes\cdots\otimes v_r\mapsto a^*:=v_r\otimes\cdots\otimes v_1$, descends to an anti-involution in $C(q)$. In view of the fact that $aa^*=q(v_1)\cdots q(v_r)$, the map $a\in C(q)\to aa^*\in\mathbb{Z}$ is a quadratic function which extends q. We therefore continue to denote it by q; it is known as the *norm* on $C(q)$. So if $q(a)\neq 0$, then a is invertible in $C(q)_{\mathbb{Q}}$ with 2-sided inverse $q(a)^{-1}a^*$.

For $r=0,1,2,\ldots$, the images of the tensors of order $\leq r$ define a filtration of $C(q)$ as an algebra whose associated graded algebra is the exterior algebra of L. In particular,

$C(q)$ is a free $\mathbb{Z}$-module of rank $2^3 = 8$. The splitting into even and odd tensors defines a corresponding splitting $C(q) = C_+(q) \oplus C_-(q)$, with $C_+(q)$ being a subalgebra $C(q)$ of rank 4.

Lemma 5.1. *The algebra $C_+(q)$ is a division algebra of indefinite quaternion type. In other words, for every $a \in C_+(q) \smallsetminus \{0\}$, $q(a) \neq 0$ and $C_+(q)_\mathbb{R}$ is isomorphic to* $\mathrm{End}_\mathbb{R}(\mathbb{R}^2)$

Proof. The fact that q does not represent zero implies that $C_+(q)$ is a division algebra. For this we must show that $q(a) \neq 0$ when $a \in C_+(q) \smallsetminus \{0\}$. If $\beta \in L_\mathbb{Q}$ is nonzero, then extend β to an orthogonal basis $\beta = \beta_1, \beta_2, \beta_3$ of $L_\mathbb{Q}$. Then it is clear that right multiplication with β takes the basis $1, \beta_1\beta_2, \beta_2\beta_3, \beta_3\beta_1$ of $C_-(V)_\mathbb{Q}$ to a basis of $C_-(V)_\mathbb{Q}$. In particular, $a\beta \neq 0$. It follows that $aL_\mathbb{Q}$ is a 3-dimensional subspace of the 4-dimensional subspace $C_-(V)_\mathbb{Q}$. So $\dim(L_\mathbb{Q} \cap aL_\mathbb{Q}) \geq 2$. Now choose $\beta \in L_\mathbb{Q}$ such that $a\beta$ is a nonzero element of $L_\mathbb{Q}$. Then $0 \neq q(a\beta) = q(a)q(\beta)$ and so $q(a) \neq 0$.

The assertion that $C_+(q)_\mathbb{R} \cong \mathrm{End}_\mathbb{R}(\mathbb{R}^2)$ follows from the fact that $q_\mathbb{R}$ has hyperbolic signature. □

Note that if $v, x \in L$, then in $C(q)$ we have

$$vxv = v(-vx + b(x,v)) = -q(v)x + b(x,v)v,$$

which in case $q(v) \neq 0$, is just $-q(v)$ times the orthogonal reflection s_v in v.

We define the group $\mathrm{Spin}(q_\mathbb{R})$ as the group of units u of $C_+(q)_\mathbb{R}$ with $q(u) = 1$ and with the property that conjugation with u preserves $L_\mathbb{R}$. This conjugation acts in $C(q)_\mathbb{R}$ as an orthogonal transformation and the resulting group homomorphism $\mathrm{Spin}(q_\mathbb{R}) \to \mathrm{O}(q_\mathbb{R})$ has image $\mathrm{SO}^\#(q_\mathbb{R})$ and kernel $\{\pm 1\}$. The group $\mathrm{Spin}(q_\mathbb{R})$ acts (faithfully) on $C_+(q)_\mathbb{R}$ by left multiplication (this is the spinor representation) and also by right multiplication after inversion. The two actions clearly commute and as we noted, the diagonal action (so given by conjugation) factors through $\mathrm{SO}^\#(q_\mathbb{R})$.

We write $\mathrm{Spin}(q)$ for the preimage of $\mathrm{SO}^\#(q)$ in $\mathrm{Spin}(q_\mathbb{R})$ so that we have an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(q) \to \mathrm{SO}^\#(q) \to 1.$$

We can use the root basis $\alpha_1, \alpha_2, \alpha_3$ of L to obtain presentations of $C(q)$ and $C_+(q)$. A presentation for $C(q)$ has these basis elements as generators and is such that $\alpha_i\alpha_j + \alpha_j\alpha_i = b(\alpha_i, \alpha_j)$. So

$$\alpha_1\alpha_1 = 2,\ \alpha_2\alpha_2 = 3,\ \alpha_3\alpha_3 = 1,$$
$$\alpha_1\alpha_2 + \alpha_2\alpha_1 = 0,\ \alpha_2\alpha_3 + \alpha_3\alpha_2 = -3,\ \alpha_3\alpha_1 + \alpha_1\alpha_3 = -2.$$

In particular, $\alpha_1^{-1} = \frac{1}{2}\alpha_1$, $\alpha_2^{-1} = \frac{1}{3}\alpha_2$ and $\alpha_3^{-1} = \alpha_3$. For $x \in L$,

$$\alpha_1 x \alpha_1 = -2s_1(x),\ \alpha_2 x \alpha_2 = -3s_2(x),\ \alpha_3 x \alpha_3 = -s_3(x).$$

Note that $a_1 := \alpha_2\alpha_3$, $a_2 := \alpha_3\alpha_1$ and $a_3 := \alpha_1\alpha_2$ generate $C_+(q)$. In fact, a_1 and a_2 will do, since $a_1a_2 = -a_3$. We also have:

$$a_1^2 + 3a_1 + 3 = 0,\ a_2^2 + 2a_2 + 2 = 0,\ a_3^2 + 6 = 0$$

Note that for $x \in L$, $a_i x a_i^{-1} = g_i(x)$, so that $\tilde g_i := a_i/\sqrt{q(a_i)} \in \mathrm{Spin}(q_{\mathbb{R}})$ is a lift of g_i and hence lies in $\mathrm{Spin}(q)$. This implies that $\tilde g_1^6$, $\tilde g_2^4$ and $\tilde g_3^2$ are all equal to -1. One checks that in addition, $\tilde g_1 \tilde g_2 \tilde g_3 = -1$. Since the g_i generate $\Delta^+(6,4,2) = \mathrm{SO}^{\#}(q)$, the $\tilde g_i$ will generate $\mathrm{Spin}(q)$.

We identified $\Gamma_5 \subset \mathrm{SO}^{\#}(q)$ as the principal level 5 subgroup of $\mathrm{SO}^{\#}(q)$. So its preimage $\hat\Gamma_5$ in $\mathrm{Spin}(q)$, which is central a extension of Γ_5 by $\{\pm 1\}$, is the subgroup of $\mathrm{Spin}(q)$ that will act as the identity on the mod 5 reduction of L. Hence it will act trivially on the mod 5 reduction of $C_+(q)$. The converse holds up to sign, in the sense that if $g \in \mathrm{SO}^{\#}(q)$ acts as the identity on $C_+(q) \otimes \mathbb{F}_5$, then $g \in \{\pm 1\}.\Gamma_5$. To see this, note that we may identify $C_+(q)/\mathbb{Z}$ with $\wedge^2 L$. The assertion is then a consequence of the fact that if g acts trivially on $(\wedge^2 L)_{\mathbb{F}_5} \cong \wedge^2_{\mathbb{F}_5} L_{\mathbb{F}_5}$, then g acts as ± 1 on $L_{\mathbb{F}_5}$.

5.3 Abelian surfaces parametrized by the Wiman curve

We noted that every $f \in \mathbb{D}$ defines a weight zero Hodge structure L_f on L polarized by $-b$. We extend this to a Hodge structure on the tensor algebra of L. The 2-sided ideal we divide out by to get $C(q)$ is a subHodge structure, and thus $C(q)$ receives a Hodge structure. The decomposition $C(q) = C_+(q) \oplus C_-(q)$ is then one into Hodge structures. This Hodge structure has bidegrees $(1,-1)$, $(0,0)$ and $(-1,0)$ only. To be concrete, let $e = e' + \sqrt{-1}e'' \in L_{\mathbb{C}}$ be as in Subsection 5.1, so that e spans $L_f^{1,-1}$ and $b(e,\bar e) > 0$. Then f spans $L_f^{0,0}$, $\bar e$ spans $L_f^{-1,1}$ and $(ef, e\bar e, \bar e e, \bar e f)$ is a basis of $C_+(q)_{\mathbb{C}}$ (note that $e\bar e + \bar e e = 1$) with bidegrees $(1,-1)$, $(0,0)$, $(0,0)$ and $(-1,1)$ respectively. Now $j_f := e'e'' \in C_+(q)_{\mathbb{R}}$ satisfies the identity $\frac{1}{2}e\bar e = 1 - \sqrt{-1}j_f$ and hence only depends on $L_f^{1,-1}$, is of bidegree $(0,0)$ and has the property that $j_f j_f = e'e''. - e''e' = -1$. So *right* multiplication by j_f defines a complex structure J_f on $C_+(q)_{\mathbb{R}}$. It takes e' to e'' and e'' to $-e'$ and so J_b lifts the complex structure in $T_f\mathbb{D}$ under the covering projection $\mathrm{Spin}(q_{\mathbb{R}}) \to \mathrm{SO}^{\#}(q_{\mathbb{R}})$.

It is clear that this complex structure is preserved by the action of $C_+(q)_{\mathbb{R}}$ on the left of $C_+(q)_{\mathbb{R}}$, so that this turns $(C_+(q)_{\mathbb{R}}, J_f)$ into a complex representation of $C_+(q_{\mathbb{C}})$ of dimension 2.

Lemma 5.2. *Every complex structure on $C_+(q)_{\mathbb{R}}$ which leaves the left $C_+(q)_{\mathbb{R}}$-module structure invariant is equal to J_f or $-J_f$ for some $f \in \mathbb{D}$.*

Proof. This is perhaps best seen by choosing an $\mathbb{R}$-algebra isomorphism $C_+(q)_{\mathbb{R}}$ with $\mathrm{End}_{\mathbb{R}}(W)$, where W is a real vector space of dimension 2. Via the natural identification $\mathrm{End}_{\mathbb{R}}(W) \cong W \otimes_{\mathbb{R}} W^\vee$, this identifies $\mathrm{Spin}(q_{\mathbb{R}})$ with $\mathrm{SL}(W) \times \mathrm{SL}(W^\vee)$. A complex structure on $\mathrm{End}_{\mathbb{R}}(W)$ which preserves the left $\mathrm{End}_{\mathbb{R}}(W)$-module structure amounts to a complex structure on $W^\vee$. The group $\mathrm{SL}(W^\vee)$ permutes these complex structures and two are in the same orbit if and only if they define the same orientation. $\square$

We note that the generator e of $L_f^{1,-1}$ satisfies

$$J_f(e) = (e' + \sqrt{-1}e'')e'e'' = e'' - \sqrt{-1}e' = -\sqrt{-1}e,$$

so that e lies in the $-\sqrt{-1}$-eigenspace of J_f. It is in fact straightforward to verify that the $-\sqrt{-1}$-eigenspace of J_f in $C_+(q)_{\mathbb{C}}$ equals $C_-(q)_{\mathbb{C}} L_f^{1,-1} \subset C_+(q)_{\mathbb{C}}$. This implies that

$C_-(q)_{\mathbb{C}} L_f^{1,-1}$ depends holomorphically on f. Hence so does the complex torus

$$A_f := C_+(q) \backslash C_+(q)_{\mathbb{C}} / C_-(q)_{\mathbb{C}} L_f^{1,-1}.$$

Observe that its first homology group $H_1(A_f)$ is naturally identified with the lattice underlying $C_+(q)$ (with its weight -1 Hodge structure); we may of course also think of A_f as $C_+(q) \backslash C_+(q)_{\mathbb{R}}$, where $C_+(q)_{\mathbb{R}}$ has the complex structure defined by J_f.

Consider the map

$$h : H_1(A_f) \times H_1(A_f) = C_+(q)^2 \to C_+(q), \quad (a, b) \mapsto ab^*.$$

This can be regarded as a $C_+(q)$-valued hermitian form on $H_1(A_f)$ with respect to the anti-involution $a \mapsto a^*$. The elements $h(a,b) - h(b,a)$ clearly lie in the (-1)-eigenspace of this involution in $C_+(q)_{\mathbb{Q}}$ and in fact span that space over $\mathbb{Q}$ (for $h(a,1) - h(1,a) = a - a^*$). This is a subspace of $C_+(q)_{\mathbb{Q}}$ of dimension 3 which supplements $\mathbb{Q}$. If we identify $C_+(q)$ with $H_1(A_f)$ as above, then after dualization we get the map

$$E : C_+(q)^\vee \to H^1(A_f) \otimes H^1(A_f) \xrightarrow{\cup} H^2(A_f)$$

The preceding implies that E has rank 3.

Lemma 5.3. *The left action of $C_+(q)$ on itself yields an embedding of algebras $C_+(q) \hookrightarrow \mathrm{End}(A_f)$ and the image E lies in the Neron-Severi group of A_f.*

Proof. The first part of this assertion is clear. For the second statement, it suffices to show that the image of E lies in $H^{1,1}(A_f)$, or equivalently, that h is zero on $C_-(q)_{\mathbb{C}} L_f^{1,-1} \times C_-(q)_{\mathbb{C}} L_f^{1,-1}$. But the vector e which spans $L_f^{1,-1}$ is isotropic and so $ee = 0$ in $C(q)_{\mathbb{C}}$. It follows that $h(xe, ye) = xeey^* = 0$ for all $x, y \in C(q_{\mathbb{C}})$. $\square$

We thus obtain a family $\mathscr{A}_{\mathbb{D}}$ over $\mathbb{D}$ with an action of $\mathrm{Spin}(q)$. Note that the action of $\mathrm{Spin}(q)$ on $\mathbb{D}$ factors through $\mathrm{SO}^\#(q) = \mathrm{Spin}(q)/\{\pm 1\}$ and that $-1 \in \mathrm{Spin}(q)$ acts in each A_f as the natural involution. With the help of Lemma 5.2 we then find:

Theorem 5.4. *The orbit space of this action by $\hat{\Gamma}_5$ yields a moduli stack of Kummer surfaces $\{\pm 1\} \backslash \mathscr{A}_{C_0} \to C_0$. To be precise, it classifies abelian surfaces A endowed with an isomorphism $\mathrm{End}(A)_{\mathbb{F}_5} \cong C_+(q_{\mathbb{F}_5})$ which is the mod 5 reduction of an algebra isomorphism $\mathrm{End}(A) \cong C_+(q)$ and for which $H_1(A; \mathbb{Z})$ is a principal $\mathrm{End}(A)$-module. This affords C_0 with the structure of a Shimura curve of indefinite quaternionic type.*

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