The Casson invariant and the word metric on the Torelli group

Nathan Broaddus, Benson Farb, Andrew Putman *

July 13, 2007

Abstract

We bound the value of the Casson invariant of any integral homology 3-sphere M by a constant times the distance-squared to the identity, measured in any word metric on the Torelli group $\mathcal{I}$, of the element of $\mathcal{I}$ associated to any Heegaard splitting of M. We construct examples which show this bound is asymptotically sharp.

1 Introduction

The *Casson invariant* $\lambda(M) \in \mathbb{Z}$ is a fundamental and well-studied invariant of integral homology 3-spheres M. Roughly speaking, $\lambda(M)$ is half the algebraic number of conjugacy classes of irreducible representations of $\pi_1(M)$ into SU(2). See [1] for a thorough exposition of the Casson invariant.

The *mapping class group* Mod_g of a closed, orientable, genus g surface Σ_g is the group of homotopy classes of orientation-preserving homeomorphisms of Σ_g. The subgroup of Mod_g consisting of elements acting trivially on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is called the *Torelli group*, and is denoted by $\mathcal{I}_g$.

Let M be an integral homology 3-sphere, and let $f : \Sigma_g \to M$ be a Heegaard embedding. For any $\phi \in \mathcal{I}_g$, denote by M_ϕ the homology 3-sphere obtained by cutting M along $f(\Sigma_g)$ and gluing back the resulting two handlebodies M^+ and M^- along their boundaries via the homeomorphism ϕ. Note that any integral homology 3-sphere can be obtained from $M = S^3$ in this way.

In this note we give a sharp asymptotic bound on $|\lambda(M_\phi)|$ in terms of the word metric on $\mathcal{I}_g$. To explain our result, we fix $g > 2$ and pick once and for all a finite set S of generators for $\mathcal{I}_g$; the fact that $\mathcal{I}_g$ is finitely generated when $g > 2$ is a deep result of D. Johnson (see [3]). Denote by $\|\cdot\|$ the induced word norm on $\mathcal{I}_g$; i.e. $\|\phi\|$ is the length of the shortest word in $S^{\pm 1}$ which equals ϕ. Different choices of finite generating sets for $\mathcal{I}_g$ give word norms whose ratios are bounded by a constant. For a fixed Heegaard embedding $f : \Sigma_g \to M$, Morita [4] has defined a kind of *normalized Casson invariant* $\lambda_f : \mathcal{I}_g \to \mathbb{Z}$ via

$$\lambda_f(\phi) := \lambda(M_\phi) - \lambda(M).$$

*The first two authors are supported in part by the NSF

In particular, if $M = S^3$ and $h : \Sigma_g \to S^3$ is the unique genus g Heegaard embedding then $\lambda(S^3) = 0$, so the normalized Casson invariant λ_h satisfies $\lambda_h(\phi) = \lambda(S^3_\phi)$.

Theorem 1.1 *Let M be an oriented integral homology 3-sphere, let $g > 2$, and let $f : \Sigma_g \to M$ be a Heegaard embedding. Then there exists a constant $C > 0$ so that $|\lambda_f(\phi)| \leq C\|\phi\|^2$ for every $\phi \in \mathcal{I}_g$. This bound is sharp in the sense that there exists an infinite sequence $\{\phi_n\}$ of elements of $\mathcal{I}_g$ so that for some constant $K > 0$ we have $|\lambda_f(\phi_n)| \geq K\|\phi_n\|^2$ for all n.*

2 Morita's formula

Our proof of Theorem 1.1 relies in an essential way on a beautiful formula due to Morita [4] for $\lambda_f(\phi)$, which we now explain (following §4 of [4]). This formula measures the extent to which λ_f fails to be a homomorphism. This failure is encoded as a function $\delta_f : \mathcal{I}_g \times \mathcal{I}_g \to \mathbb{Z}$ defined as follows. Let $\mathcal{I}_{g,1}$ denote the Torelli group of an oriented, genus g surface with one boundary component $\Sigma_{g,1}$. In other words, $\mathcal{I}_{g,1}$ is the group of homotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ which fix the boundary pointwise, modulo homotopies which do the same and where the homeomorphisms act trivially on $H := \mathrm{H}_1(\Sigma_g; \mathbb{Z})$. Gluing a disc to $\partial\Sigma_{g,1}$ induces a natural surjective homomorphism $\pi : \mathcal{I}_{g,1} \to \mathcal{I}_g$, and there is a corresponding commutative diagram of *Johnson homomorphisms* (c.f. [2] for a discussion of these homomorphisms τ and their remarkable properties):

$$\begin{array}{ccc} \mathcal{I}_{g,1} & \xrightarrow{\tau} & \wedge^3 H \\ \pi \downarrow & & \downarrow \\ \mathcal{I}_g & \xrightarrow{\tau} & \wedge^3 H / H \end{array}$$

The map $f : \Sigma_g \to M$ induces homomorphisms $H \to \mathrm{H}_1(M^{\pm}; \mathbb{Z})$ whose kernels we denote by H^+ and H^-, respectively. It is then easy to see that $H^+ \otimes \mathbb{R}$ and $H^- \otimes \mathbb{R}$ are maximal isotropic subspaces of the symplectic vector space $H \otimes \mathbb{R}$, and that

$$H = H^+ \oplus H^-.$$

Moreover, since M is an integral homology 3-sphere, there is a symplectic basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ for H with $x_i \in H^+$ and $y_i \in H^-$. Now, given any two $\phi, \psi \in \mathcal{I}_g$, choose any lifts $\tilde{\phi}, \tilde{\psi}$ to $\mathcal{I}_{g,1}$. Using the obvious basis for $\wedge^3 H$ coming from our choice of basis for H, we can write

$$\tau(\tilde{\phi}) = \Big[\sum_{i<j<k} a_{ijk} \; y_i \wedge y_j \wedge y_k \Big] + \text{other terms},$$

$$\tau(\tilde{\psi}) = \Big[\sum_{i<j<k} b_{ijk} \; x_i \wedge x_j \wedge x_k \Big] + \text{other terms}$$

for some $a_{ijk}, b_{ijk} \in \mathbb{Z}$. Morita defines

$$\delta_f(\phi, \psi) = \sum_{i<j<k} a_{ijk} b_{ijk}$$

and proves that $\delta_f(\phi,\psi)$ does not depend on either the choice of lifts $\tilde{\phi},\tilde{\psi}$ or the choice of symplectic basis for H. Morita then proves, as Theorem 4.3 of [4], that the following formula holds for all $\phi,\psi \in \mathcal{I}_g$:

$$\lambda_f(\phi\psi) = \lambda_f(\phi) + \lambda_f(\psi) + 2\delta_f(\phi,\psi). \tag{1}$$

3 Proof of Theorem 1.1

Let $\{x_1,\ldots,x_g,y_1,\ldots,y_g\}$ be the standard basis for $H := \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ discussed in the previous section. For any vector $v \in \wedge^3 H$, we denote by $\ell(v)$ the maximum of the absolute values of the coefficients of v with respect to the induced basis for $\wedge^3 H$.

We want to relate $\lambda_f(\phi)$ to the word length of ϕ in $\mathcal{I}_g$, but Morita's formula (1) is computed using elements of $\mathcal{I}_{g,1}$, not of $\mathcal{I}_g$. To address this point, we first recall that gluing a disk to $\partial\Sigma_{g,1}$ induces an exact sequence

$$1 \to \pi_1(T^1\Sigma_g) \to \mathcal{I}_{g,1} \xrightarrow{\pi} \mathcal{I}_g \to 1$$

where $T^1\Sigma_g$ is the unit tangent bundle of Σ_g. For each generator $s \in S$ of $\mathcal{I}_g$, choose a single lift $\tilde{s} \in \mathcal{I}_{g,1}$, and denote by $\widetilde{S}$ the union of these elements. We can then choose as a generating set for $\mathcal{I}_{g,1}$ the set $\widetilde{S}$ together with a finite generating set for $\pi_1(T^1\Sigma_g)$. With these choices of generating sets, we note that each $\phi \in \mathcal{I}_g$ has some lift $\tilde{\phi}$ so that

$$\|\tilde{\phi}\|_{\mathcal{I}_{g,1}} = \|\phi\|_{\mathcal{I}_g}. \tag{2}$$

This equality follows by writing out ϕ as a product of elements of S, then lifting generator by generator. Henceforth whenever we choose a lift of an element $\phi \in \mathcal{I}_g$, we will always choose a lift $\tilde{\phi}$ satisfying (2). The main point is that in computing with (1), we are allowed to choose any lifts, since Morita proves that $\delta_f(\phi,\psi)$ does not depend on the choice of lifts. Thus we can choose lifts which do not alter word length.

Now since $\widetilde{S}$ is finite, there exists C_1 so that

$$\ell(\tau(\tilde{s})) \leq C_1 \quad \text{for all } s \in \widetilde{S}^{\pm 1}. \tag{3}$$

Since τ is a homomorphism to the abelian group $\wedge^3 H$, it follows from (3) that

$$\ell(\tau(\tilde{\phi})) \leq C_1\|\tilde{\phi}\| \quad \text{for all } \tilde{\phi} \in \mathcal{I}_{g,1}. \tag{4}$$

Finally, consider $\phi,\psi \in \mathcal{I}_g$ together with lifts $\tilde{\phi},\tilde{\psi}$ satisfying (2). If a_{ijk} (resp. b_{ijk}) are the coordinates of $\tau(\tilde{\phi})$ (resp. $\tau(\tilde{\psi})$) as in the previous section, then

$$
\begin{aligned}
|\delta_f(\phi,\psi)| &= \Big| \sum_{i<j<k} a_{ijk} b_{ijk} \Big| \\
&\le \Big| \sum_{i<j<k} \ell(\tau(\tilde{\phi}))\ell(\tau(\tilde{\psi})) \Big| \\
&\le \sum_{i<j<k} C_1^2 \, \|\phi\| \, \|\psi\| \\
&\le C_2 \|\phi\| \, \|\psi\|
\end{aligned}
\tag{5}
$$

where $C_2 = \binom{2g}{3} C_1^2$.

Now given any $\phi \in \mathcal{I}_g$, write $\phi = s_1 \cdots s_n$, where each s_i is an element of $S^{\pm 1}$ and where $n = \|\phi\|$. An iterated use of Morita's formula (1) gives

$$
\begin{aligned}
\lambda_f(\phi) &= \lambda_f(s_1) + \lambda_f(s_2 \cdots s_n) + 2\delta_f(s_1, s_2 \cdots s_n) \\
&= \lambda_f(s_1) + \lambda_f(s_2) + \lambda_f(s_3 \cdots s_n) + 2\delta_f(s_1, s_2 \cdots s_n) + 2\delta_f(s_2, s_3 \cdots s_n) \\
&\;\;\vdots \\
&= \sum_{i=1}^{n} \lambda_f(s_n) + 2\sum_{i=1}^{n-1} \delta_f(s_i, s_{i+1} \cdots s_n).
\end{aligned}
\tag{6}
$$

Since S is finite, there exists $C_3 > 0$ so that $|\lambda_f(s)| \le C_3$ for every $s \in S$. For some $C > 0$, we thus have

$$
\begin{aligned}
|\lambda_f(\phi)| &\le \sum_{i=1}^{n} |\lambda_f(s_n)| + 2\sum_{i=1}^{n-1} |\delta_f(s_i, s_{i+1} \cdots s_n)| \\
&\le C_3 n + 2\sum_{i=1}^{n-1} C_2 \cdot 1 \cdot (n-i) \\
&\le C n^2 = C\|\phi\|^2.
\end{aligned}
$$

The first claim of the theorem follows.

We now consider the second claim. Johnson proved (see, e.g. [2]) that the homomorphisms τ are surjective. Hence there exists some $\nu \in \mathcal{I}_g$ so that for some lift $\tilde{\nu} \in \mathcal{I}_{g,1}$ we have

$$
\tau(\tilde{\nu}) = x_1 \wedge x_2 \wedge x_3 + y_1 \wedge y_2 \wedge y_3,
$$

and hence

$$
\tau(\tilde{\nu}^n) = n(x_1 \wedge x_2 \wedge x_3) + n(y_1 \wedge y_2 \wedge y_3). \tag{7}
$$

Note that the choice of ν depends in a nontrivial way on the Heegaard embedding $f : \Sigma_g \to M$, so ν is not given explicitly. By equation (6), we have

$$\lambda_f(\nu^n) = \sum_{i=1}^{n} \lambda_f(\nu) + 2\sum_{i=1}^{n-1} \delta_f(\nu, \nu^{n-i}). \tag{8}$$

Now let $K_1 = |\lambda_f(\nu)|$, which is a constant since ν is fixed. By (7) and the definition of δ_f, we have for any $m > 0$ that $\delta_f(\nu, \nu^m) = m$. Thus by equation (8) there is some N such that for all $n \geq N$

$$\begin{aligned} |\lambda_f(\nu^n)| &= \Big| \sum_{i=1}^{n} \lambda_f(\nu) + 2\sum_{i=1}^{n-1}(n-i) \Big| \\ &\geq 2\sum_{i=1}^{n-1}(n-i) - \sum_{i=1}^{n} K_1 \\ &\geq K_2 n^2 \end{aligned}$$

for some $K_2 > 0$. If $\|\nu\| = K_3$, then clearly $\|\nu^n\| \leq K_3 n$. Thus

$$|\lambda_f(\nu^n)| \geq K_2 n^2 \geq \frac{K_2}{K_3^2}\|\nu^n\|^2 \qquad \text{for all } n \geq N.$$

Setting $K = \frac{K_2}{K_3^2}$ we get the desired sequence $\{\nu^n\}_{n=N}^{\infty}$ establishing the asymptotic tightness of the upper bound.

References

[1] S. Akbulut and J. D. McCarthy, *Casson's invariant for oriented homology* 3*-spheres*, Princeton Univ. Press, Princeton, NJ, 1990.

[2] D. Johnson, A survey of the Torelli group, in *Low-dimensional topology (San Francisco, Calif., 1981)*, 165–179, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.

[3] D. Johnson, The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$, Ann. of Math. (2) **118** (1983), no. 3, 423–442.

[4] S. Morita, On the structure of the Torelli group and the Casson invariant, Topology **30** (1991), no. 4, 603–621.

Department of Mathematics, University of Chicago
5734 S. University Ave.
Chicago, IL 60637 USA

Emails: broaddus@math.uchicago.edu *(Nathan Broaddus)*,
farb@math.uchicago.edu *(Benson Farb)*,
andyp@math.uchicago.edu *(Andrew Putman)*