# A stability conjecture for the unstable cohomology of mapping class groups, $\mathrm{SL}_n\,\mathbb{Z}$, and $\mathrm{Aut}(F_n)$

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**Abstract**

In this paper we give a conjectural picture of a large piece of the unstable rational cohomology of mapping class groups, of $\mathrm{SL}_n\,\mathbb{Z}$, and of $\mathrm{Aut}(F_n)$.

For each of the sequences of groups in the title, the $i$-th rational cohomology is known to be independent of $n$ in a linear range $n \geq Ci$. Furthermore, this "stable cohomology" has been explicitly computed in each case. In contrast, very little is known about the unstable cohomology. In this paper we conjecture a new kind of stability in the cohomology of these groups. These conjectures concern the unstable cohomology, in a range near the "top dimension".

## 1 Stability in the unstable cohomology of mapping class groups

Let $\mathrm{Mod}_g$ be the mapping class group of a closed, oriented, genus $g \geq 2$ surface, and let $\mathcal{M}_g$ be the moduli space of genus $g$ Riemann surfaces. It is well-known (see, e.g., [FM, Theorem 12.13]) that for each $i \geq 0$,

$$H^i(\mathrm{Mod}_g;\mathbb{Q}) \approx H^i(\mathcal{M}_g;\mathbb{Q}). \tag{1}$$

Harer [Ha1] proved that for each $i \geq 0$ the group $H^i(\mathrm{Mod}_g;\mathbb{Z})$ does not depend on $g$ for $g \gg i$. Harer also proved [Ha2] that the *virtual cohomological dimension* of $\mathrm{Mod}_g$ is

$$\mathrm{vcd}(\mathrm{Mod}_g) = 4g - 5.$$

This implies that $H^k(\mathrm{Mod}_g;\mathbb{Q}) = 0$ for all $k > 4g - 5$.

**Conjecture 1.** *For each $i \geq 0$ the group $H^{4g-5-i}(\mathrm{Mod}_g;\mathbb{Q})$ does not depend on $g$ for $g \gg i$.*

The form of Conjecture 1 may appear surprising to readers familiar with the known examples of homological stability, so we now explain some of the intuition behind the conjecture. The moduli space $\mathcal{M}_g$ is an orbifold of dimension $6g - 6$. Thus if $\mathcal{M}_g$ were compact, Poincaré duality combined with (1) would imply that $H^{6g-6-i}(\mathrm{Mod}_g;\mathbb{Q})$ was independent of $g$ for $g \gg i$. Of course $\mathcal{M}_g$ is

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| $g$ | vcd | $H^1(\mathrm{Mod}_g;\mathbb{Q})$ | | | $H^2(\mathrm{Mod}_g;\mathbb{Q})$ | | | $\cdots$ | | $H^{\mathrm{vcd}}(\mathrm{Mod}_g;\mathbb{Q})$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | | | | | | | | | | |
| 2 | 3 | 0 | 0 | 0 | | | | | | | | |
| 3 | 7 | 0 | $\mathbb{Q}$ | 0 | 0 | 0 | $\mathbb{Q}$ | 0 | | | | |
| 4 | 11 | 0 | $\mathbb{Q}$ | 0 | $\mathbb{Q}$ | $\mathbb{Q}$ | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1: The cohomology groups of $\mathrm{Mod}_g$ for $1 \leq g \leq 4$. For $g = 1$ these are classical; for $g = 2$ they were calculated by Igusa [I]; for $g = 3$ they were calculated by Looijenga [Lo]; and for $g = 4$ they were calculated by Tommasi [T]. The classes in $H^6(\mathrm{Mod}_3;\mathbb{Q})$ and $H^5(\mathrm{Mod}_4;\mathbb{Q})$ are unstable.

not compact and does not satisfy Poincaré duality. The more general notion of Bieri–Eckmann duality allows us to repair this gap, and also sheds light on how Conjecture 1 might be proved. This philosophy was recently applied in [CFP] to prove Conjecture 1 for $i = 0$ (this was also proved independently by Morita–Sakasai–Suzuki [MSS] using different methods, and had been announced some years ago by Harer).

**Computational evidence.** Complete calculations of $H^*(\mathrm{Mod}_g;\mathbb{Q})$ are only known for $1 \leq g \leq 4$. These calculations are summarized in Table 1. They are consistent with Conjecture 1.

**Possible approaches.** An important feature of Conjecture 1 is that there are natural candidates for "stabilization maps" between $H^{4g-5-i}(\mathrm{Mod}_g;\mathbb{Q})$ and $H^{4(g+1)-5-i}(\mathrm{Mod}_{g+1};\mathbb{Q})$ which could realize the isomorphisms conjectured in Conjecture 1.

**Mess stabilization.** We first give a topological construction of a stabilization map

$$H^{4(g+1)-5-i}(\mathrm{Mod}_{g+1};\mathbb{Q}) \to H^{4g-5-i}(\mathrm{Mod}_g;\mathbb{Q}).$$

Let $S_g^1$ be a compact oriented genus $g$ surface with one boundary component and let $\mathrm{Mod}_g^1$ be its mapping class group. Johnson proved that there is a short exact sequence

$$1 \to \pi_1(T^1 S_g) \to \mathrm{Mod}_g^1 \to \mathrm{Mod}_g \to 1 \tag{2}$$

where $T^1 S_g$ is the unit tangent bundle of the closed surface $S_g$. Since $T^1 S_g$ is a 3–manifold and $\mathrm{Mod}_g$ acts trivially on $H_3(T^1 S_g;\mathbb{Q})$, we obtain a Gysin map $H^k(\mathrm{Mod}_g^1;\mathbb{Q}) \to H^{k-3}(\mathrm{Mod}_g;\mathbb{Q})$.

Similarly, there is a Gysin map $H^k(\mathrm{Mod}_g^1 \times \mathbb{Z};\mathbb{Q}) \to H^{k-1}(\mathrm{Mod}_g^1;\mathbb{Q})$ coming from the trivial extension

$$1 \to \mathbb{Z} \to \mathrm{Mod}_g^1 \times \mathbb{Z} \to \mathrm{Mod}_g^1 \to 1.$$

Finally, the injection $\mathrm{Mod}_g^1 \times \mathbb{Z} \hookrightarrow \mathrm{Mod}_{g+1}$ given by sending the generator of $\mathbb{Z}$ to a nonseparating Dehn twist supported in $S_{g+1} \setminus S_g^1$ induces the restriction $H^k(\mathrm{Mod}_{g+1};\mathbb{Q}) \to H^k(\mathrm{Mod}_g^1 \times \mathbb{Z};\mathbb{Q})$. Consider the composition:

$$H^k(\mathrm{Mod}_{g+1};\mathbb{Q}) \to H^k(\mathrm{Mod}_g^1 \times \mathbb{Z};\mathbb{Q}) \to H^{k-1}(\mathrm{Mod}_g^1;\mathbb{Q}) \to H^{k-4}(\mathrm{Mod}_g;\mathbb{Q})$$

Taking $k = 4(g+1) - 5 - i$, we obtain a map

$$H^{4(g+1)-5-i}(\mathrm{Mod}_{g+1};\mathbb{Q}) \to H^{4g-5-i}(\mathrm{Mod}_g;\mathbb{Q}) \tag{3}$$

which we conjecture is an isomorphism for $g \gg i$.

**Remark.** We refer to the map (3) as *Mess stabilization* because this construction was first used by Mess in [Me] to construct a subgroup $K < \mathrm{Mod}_g$ isomorphic to the fundamental group of a closed aspherical $(4g-5)$-manifold. This gives an explicit witness for the lower bound $\mathrm{vcd}(\mathrm{Mod}_g) \geq 4g-5$, as follows. An elementary argument (sometimes called Shapiro's Lemma) shows that

$$H^*(\mathrm{Mod}_g; M) \approx H^*(K;\mathbb{Q})$$

where $M \coloneqq \mathrm{Hom}_{\mathbb{Q}K}(\mathbb{Q}\,\mathrm{Mod}_g, \mathbb{Q})$. We thus have $H^{4g-5}(\mathrm{Mod}_g; M) \approx H^{4g-5}(K;\mathbb{Q}) \approx \mathbb{Q}$, demonstrating that $\mathrm{vcd}(\mathrm{Mod}_g) \geq 4g-5$. Despite this, it follows from [CFP] that the fundamental class $[K] \in H_{4g-5}(\mathrm{Mod}_g;\mathbb{Q})$ itself vanishes rationally.

**Duality groups.** Our second approach to Conjecture 1 would give a map in the other direction, namely a map

$$H^{4g-5-i}(\mathrm{Mod}_g;\mathbb{Q}) \to H^{4(g+1)-5-i}(\mathrm{Mod}_{g+1};\mathbb{Q}). \tag{4}$$

Recall that a group $\Gamma$ is a *duality group* if there is an integer $\nu$ and a $\mathbb{Z}\Gamma$-module $D$, called the *dualizing module* for $\Gamma$, with the property that there are isomorphisms

$$H^{\nu-i}(\Gamma; M) \approx H_i(\Gamma; M \otimes_{\mathbb{Z}} D)$$

for any $\mathbb{Z}\Gamma$-module $M$. No group which contains torsion can be a duality group. To remedy this, we say that a group $\Gamma$ is a *virtual duality group* if it has some finite index subgroup which is a duality group. This implies that there exists a *rational dualizing* $\mathbb{Q}\Gamma$*-module* $D$ so that

$$H^{\nu-i}(\Gamma; M) \approx H_i(\Gamma; M \otimes_{\mathbb{Q}} D)$$

for any $\mathbb{Q}\Gamma$-module $M$. The integer $\nu$ equals the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$. See [BE] or [Bro, VIII.10] for details.

**Duality for $\mathrm{Mod}_g$.** Let $\mathcal{C}_g$ be the *curve complex*, which is the simplicial complex whose $k$-simplices consist of $(k+1)$-tuples of isotopy classes of mutually disjoint simple closed curves on $S_g$. Harer [Ha2, Theorem 3.5] proved that $\mathcal{C}_g$ has the homotopy type of an infinite wedge of $(2g-2)$-dimensional spheres. The *Steinberg module* of $\mathrm{Mod}_g$ is defined to be

$$\mathrm{St}(\mathrm{Mod}_g) \coloneqq H_{2g-2}(\mathcal{C}_g;\mathbb{Q}).$$

Since $\mathrm{Mod}_g$ acts on $\mathcal{C}_g$ by simplicial automorphisms, $\mathrm{St}(\mathrm{Mod}_g)$ is a $\mathbb{Q}\,\mathrm{Mod}_g$-module. Harer [Ha2, Theorem 4.1] proved that $\mathrm{Mod}_g$ is a virtual duality group with dualizing module $\mathrm{St}(\mathrm{Mod}_g)$ and $\nu = \mathrm{vcd}(\mathrm{Mod}_g) = 4g-5$. Given this, Conjecture 1 has the following equivalent restatement.

**Conjecture 1, restated.** *For each $i \geq 0$ the group $H_i(\mathrm{Mod}_g; \mathrm{St}(\mathrm{Mod}_g))$ does not depend on $g$ for $g \gg i$.*

In this form the conjecture looks like a standard formulation of homological stability. However, the devil is in the details of the coefficient module $\mathrm{St}(\mathrm{Mod}_g)$, which itself is changing with $g$.

**Proving the restated conjecture.** In order to prove the restated version of Conjecture 1, one wants to construct directly a homomorphism

$$H_i(\mathrm{Mod}_g; \mathrm{St}(\mathrm{Mod}_g)) \to H_i(\mathrm{Mod}_{g+1}; \mathrm{St}(\mathrm{Mod}_{g+1}))$$

that one can prove is an isomorphism for large enough $g$ (depending on $i$). Here we encounter our first technical issue: there is no natural map $\mathrm{Mod}_g \to \mathrm{Mod}_{g+1}$ (or vice versa). This issue already arises in proving ordinary homological stability for $\mathrm{Mod}_g$. The solution there is to consider surfaces with boundary, since there is a map $\mathrm{Mod}_g^1 \to \mathrm{Mod}_{g+1}^1$ induced by embedding $S_g^1$ into $S_{g+1}^1$. There is also a natural surjection $\mathrm{Mod}_g^1 \to \mathrm{Mod}_g$ induced by gluing a disc to the boundary component of $S_g^1$. Harer proved homological stability for $\mathrm{Mod}_g$ by showing that both the induced maps $H_i(\mathrm{Mod}_g^1) \to H_i(\mathrm{Mod}_g)$ and $H_i(\mathrm{Mod}_g^1) \to H_i(\mathrm{Mod}_{g+1}^1)$ are isomorphisms for $g \gg i$.

The same tactic could be applied to our conjecture. Applying [BE, Theorem 3.5] to (2) shows that $\mathrm{Mod}_g^1$ is a duality group with $\nu = 4g - 2$ and the same dualizing module $\mathrm{St}(\mathrm{Mod}_g)$, on which $\mathrm{Mod}_g^1$ acts via the projection $\mathrm{Mod}_g^1 \twoheadrightarrow \mathrm{Mod}_g$. Thus a first step towards the reformulation of Conjecture 1 would be to prove the following.

**Conjecture 2.** *For each $i \geq 0$, the natural map $H_i(\mathrm{Mod}_g^1; \mathrm{St}(\mathrm{Mod}_g)) \longrightarrow H_i(\mathrm{Mod}_g; \mathrm{St}(\mathrm{Mod}_g))$ is an isomorphism for $g \gg i$.*

Since the coefficient modules are the same in this case, this seems fairly tractable. For example, this property of the coefficient modules automatically implies Conjecture 2 for $i = 0$ (even without the vanishing result proved in [CFP] that implies that both sides are zero). The next step would be to construct some natural map

$$H_i(\mathrm{Mod}_g^1; \mathrm{St}(\mathrm{Mod}_g)) \to H_i(\mathrm{Mod}_{g+1}^1; \mathrm{St}(\mathrm{Mod}_{g+1}))$$

and prove that it is an isomorphism for $g \gg i$. Since we have the inclusion $\mathrm{Mod}_g^1 \hookrightarrow \mathrm{Mod}_{g+1}^1$, to construct the desired map it is therefore enough to construct a $\mathrm{Mod}_g^1$–equivariant map $\mathrm{St}(\mathrm{Mod}_g) \to \mathrm{St}(\mathrm{Mod}_{g+1})$. Such a map could be described concretely using Broaddus' resolution of $\mathrm{St}(\mathrm{Mod}_g)$ in terms of certain pictorial *chord diagrams* [Br, Prop. 3.3] (compare with the map (7) for $\mathrm{SL}_n \mathbb{Z}$ defined below). However, the natural first guess for the stabilization map for $\mathrm{St}(\mathrm{Mod}_g)$ turns out to be the zero map (see [Br, Proposition 4.5]), so a new idea is necessary. We thus pose the following open problem, which would give a stabilization map as in (4).

**Problem 3.** Construct a natural nonzero $\mathrm{Mod}_g^1$-equivariant map $\mathrm{St}(\mathrm{Mod}_g) \to \mathrm{St}(\mathrm{Mod}_{g+1})$.

| $n$ | vcd | $H^1(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$ | $H^2(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$ | $\cdots$ | $H^{\mathrm{vcd}}(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$ | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | | | | | | | | | | | | | | | | | | | | |
| 3 | 3 | 0 | 0 | 0 | | | | | | | | | | | | | | | | | | |
| 4 | 6 | 0 | 0 | $\mathbb{Q}$ | 0 | 0 | 0 | | | | | | | | | | | | | | | |
| 5 | 10 | 0 | 0 | 0 | 0 | $\mathbb{Q}$ | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | |
| 6 | 15 | 0 | 0 | 0 | 0 | $\mathbb{Q}^2$ | 0 | 0 | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | 0 | 0 | 0 | 0 | 0 | | | | | | |
| 7 | 21 | 0 | 0 | 0 | 0 | $\mathbb{Q}$ | 0 | 0 | 0 | $\mathbb{Q}$ | 0 | 0 | 0 | 0 | $\mathbb{Q}$ | $\mathbb{Q}$ | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2: The rational cohomology of $\mathrm{SL}_n \mathbb{Z}$ for $2 \leq n \leq 7$. For $n = 2$ these are classical; for $n = 3$ they were calculated by Soulé [So]; for $n = 4$ they were calculated by Lee–Szczarba [LS2]; and for $5 \leq n \leq 7$ they were calculated by Elbaz-Vincent–Gangl–Soulé [EVGS]. The classes in $H^3(\mathrm{SL}_4 \mathbb{Z};\mathbb{Q})$, $H^8(\mathrm{SL}_6 \mathbb{Z};\mathbb{Q})$, $H^{10}(\mathrm{SL}_6 \mathbb{Z};\mathbb{Q})$, and $H^{15}(\mathrm{SL}_7 \mathbb{Z};\mathbb{Q})$, as well as one dimension in $H^5(\mathrm{SL}_6 \mathbb{Z};\mathbb{Q})$, are unstable.

## 2 Stability in the unstable cohomology of $\mathrm{SL}_n \mathbb{Z}$

There has been a long-standing and fruitful analogy between $\mathrm{Mod}_g$ and arithmetic groups such as $\mathrm{SL}_n \mathbb{Z}$. This analogy is particularly strong with respect to cohomological properties, and many of the results we have described for $\mathrm{Mod}_g$ were first proved for the arithmetic group $\mathrm{SL}_n \mathbb{Z}$. Borel [Bo] proved that $H^i(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$ does not depend on $n$ for $n \gg i$. Moreover Borel–Serre [BS] proved that $\mathrm{SL}_n \mathbb{Z}$ is a virtual duality group with $\nu = \mathrm{vcd}(\mathrm{SL}_n \mathbb{Z}) = \binom{n}{2}$. Motivated by Conjecture 1, we make the following conjecture on the unstable cohomology of $\mathrm{SL}_n \mathbb{Z}$.

**Conjecture 4.** *For each $i \geq 0$ the group $H^{\binom{n}{2}-i}(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$ does not depend on $n$ for $n \gg i$.*

For $i = 0$, Conjecture 4 is a theorem of Lee–Szczarba [LS1], who proved that $H^{\binom{n}{2}}(\mathrm{SL}_n \mathbb{Z};\mathbb{Q}) = 0$ for all $n \geq 3$.

**Computational evidence.** The rational cohomology groups of $\mathrm{SL}_n \mathbb{Z}$ are all known only for $2 \leq n \leq 7$. These calculations are summarized in Table 2. The data in this table supports Conjecture 4; in fact, it seems to suggest that possibly $H^{\binom{n}{2}-i}(\mathrm{SL}_n \mathbb{Z};\mathbb{Q}) = 0$ for $n > i + 1$.

**Stabilization maps.** There are clear analogues for $\mathrm{SL}_n \mathbb{Z}$ of both of our approaches to stabilization maps for $\mathrm{Mod}_g$. These yield two possible candidates for stabilization maps realizing the conjectured isomorphisms between $H^{\binom{n}{2}-i}(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$ and $H^{\binom{n+1}{2}-i}(\mathrm{SL}_{n+1} \mathbb{Z};\mathbb{Q})$.

We first construct a map

$$H^{\binom{n+1}{2}-i}(\mathrm{SL}_{n+1} \mathbb{Z};\mathbb{Q}) \to H^{\binom{n}{2}-i}(\mathrm{SL}_n \mathbb{Z};\mathbb{Q})$$

as follows. The stabilizer in $\mathrm{SL}_{n+1} \mathbb{Z}$ of the subspace $\mathbb{Q}^n < \mathbb{Q}^{n+1}$ is isomorphic to the semi-direct product $\mathbb{Z}^n \rtimes \mathrm{SL}_n \mathbb{Z}$, where the normal subgroup $\mathbb{Z}^n$ consists of those automorphisms that restrict to the identity on $\mathbb{Q}^n$. Note that the action of $\mathrm{SL}_n \mathbb{Z}$ on $\mathbb{Z}^n$ in this semi-direct product is the standard one; in particular $\mathrm{SL}_n\,\mathbb{Z}$ acts trivially on $H_n(\mathbb{Z}^n;\mathbb{Z})$. The extension

$$1 \to \mathbb{Z}^n \to \mathbb{Z}^n \rtimes \mathrm{SL}_n\,\mathbb{Z} \to \mathrm{SL}_n\,\mathbb{Z} \to 1$$

therefore yields a Gysin map $H^k(\mathbb{Z}^n \rtimes \mathrm{SL}_n\,\mathbb{Z}) \to H^{k-n}(\mathrm{SL}_n\,\mathbb{Z})$. Taking $k = \binom{n+1}{2} - i$ yields the composition

$$H^{\binom{n+1}{2}-i}(\mathrm{SL}_{n+1}\,\mathbb{Z};\mathbb{Q}) \to H^{\binom{n+1}{2}-i}(\mathbb{Z}^n \rtimes \mathrm{SL}_{n+1}\,\mathbb{Z};\mathbb{Q}) \to H^{\binom{n}{2}-i}(\mathrm{SL}_n\,\mathbb{Z};\mathbb{Q}) \tag{5}$$

where the first map is restriction. We conjecture that (5) is an isomorphism for $n \gg i$.

**Remark.** Iterating this process starting with $\mathrm{SL}_1\,\mathbb{Z}$ yields the group of strictly upper-triangular matrices $N_n$. This is the fundamental group of an $\binom{n}{2}$-dimensional nil-manifold, and thus provides an explicit witness for the lower bound $\mathrm{vcd}(\mathrm{SL}_n\,\mathbb{Z}) \geq \binom{n}{2}$.

**Stabilization and the Steinberg module.** For $\mathrm{Mod}_g$ we sketched a construction of a stabilization map that increases dimension in cohomology, contingent upon resolving some technical difficulties. For $\mathrm{SL}_n\,\mathbb{Z}$ these difficulties can be avoided, and we construct a stabilization map

$$H^{\binom{n}{2}-i}(\mathrm{SL}_n\,\mathbb{Z};\mathbb{Q}) \to H^{\binom{n+1}{2}-i}(\mathrm{SL}_{n+1}\,\mathbb{Z};\mathbb{Q})$$

as follows. The *(spherical) Tits building* $\mathcal{B}(\mathbb{Q}^n)$ is the complex of flags of nontrivial proper subspaces of $\mathbb{Q}^n$. By the Solomon–Tits Theorem, $\mathcal{B}(\mathbb{Q}^n)$ is homotopy equivalent to an infinite wedge of $(n-2)$-dimensional spheres, and the rational dualizing module for $\mathrm{SL}_n\,\mathbb{Z}$ is the Steinberg module $\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z}) \coloneqq H_{n-2}(\mathcal{B}(\mathbb{Q}^n);\mathbb{Q})$ [BS, Theorem 11.4.2]. By definition, $\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z})$ satisfies

$$H^{\binom{n}{2}-i}(\mathrm{SL}_n\,\mathbb{Z};\mathbb{Q}) \cong H_i(\mathrm{SL}_n\,\mathbb{Z};\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z}))$$

for all $i \geq 0$. This gives the following equivalent formulation of Conjecture 4.

**Conjecture 4, restated.** *For each $i \geq 0$ the group $H_i(\mathrm{SL}_n\,\mathbb{Z};\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z}))$ does not depend on $n$ for $n \gg i$.*

**Remark.** Dwyer [D] (see also van der Kallen [vdK]) proved that the homology of $\mathrm{SL}_n\,\mathbb{Z}$ stabilizes with respect to families of twisted coefficient systems satisfying certain growth conditions. However, the coefficient systems $\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z})$ do not satisfy Dwyer's condition.

We now construct an explicit candidate for a stabilization map

$$H_i(\mathrm{SL}_n\,\mathbb{Z};\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z})) \to H_{i+1}(\mathrm{SL}_n\,\mathbb{Z};\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z})). \tag{6}$$

The map $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ defines an inclusion $\mathrm{SL}_n\,\mathbb{Z} \hookrightarrow \mathrm{SL}_{n+1}\,\mathbb{Z}$. To define (6) it therefore suffices to construct an $\mathrm{SL}_n\,\mathbb{Z}$-equivariant map $\varphi\colon \mathrm{St}(\mathrm{SL}_n\,\mathbb{Z}) \to \mathrm{St}(\mathrm{SL}_{n+1}\,\mathbb{Z})$. This construction uses the following description of $\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z})$ in terms of "modular symbols", which was given by Ash–Rudolph [AR] following work of Lee–Szczarba [LS1].

**Theorem 5** (Ash–Rudolph [AR]). *The $\mathbb{Q}$-vector space $\mathrm{St}(\mathrm{SL}_n \mathbb{Z})$ has the following presentation.*

- *The generators are $n$-tuples $[v_1, \ldots, v_n]$ of vectors in $\mathbb{Q}^n$.*
- *The relations are:*
  1. *If $v_1, \ldots, v_n \in \mathbb{Q}^n$ are not linearly independent, then $[v_1, \ldots, v_n] = 0$.*
  2. *For nonzero $c \in \mathbb{Q}$, we have $[cv_1, v_2, \ldots, v_n] = [v_1, \ldots, v_n]$.*
  3. *For $\sigma \in S_n$, we have $[v_{\sigma\cdot 1}, \ldots, v_{\sigma\cdot n}] = (-1)^{|\sigma|}[v_1, \ldots, v_n]$.*
  4. *For $v_1, \ldots, v_{n+1} \in \mathbb{Q}^n$, we have $\sum_{i=1}^{n+1}(-1)^{i-1}[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}] = 0$.*

Let $e_1, \ldots, e_{n+1}$ be the standard basis for $\mathbb{Q}^{n+1}$. Regarding $\mathbb{Q}^n$ as a subspace of $\mathbb{Q}^{n+1}$ in the natural way, define $\varphi$ on the generators of $\mathrm{St}(\mathrm{SL}_n \mathbb{Z})$ from Theorem 5 by the formula

$$\varphi([v_1, \ldots, v_n]) = [v_1, \ldots, v_n, e_{n+1}] \qquad (v_1, \ldots, v_n \in \mathbb{Q}^n). \tag{7}$$

To see that this is well-defined, we must check that it takes each of the four families of relations in Theorem 5 to relations in $\mathrm{St}(\mathrm{SL}_{n+1} \mathbb{Z})$. The first three of these are obvious, so we concentrate on the fourth. Given $v_1, \ldots, v_{n+1} \in \mathbb{Q}^n$, let us abbreviate the fourth relation above by $\partial[v_1, \ldots, v_{n+1}] = 0$. We have

$$\begin{aligned}\varphi\big(\partial[v_1, \ldots, v_{n+1}]\big) &= \varphi\Big(\sum_{i=1}^{n+1}(-1)^{i-1}[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}]\Big) \\ &= \sum_{i=1}^{n+1}(-1)^{i-1}[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}, e_{n+1}].\end{aligned}$$

Since the vectors $v_1, \ldots, v_{n+1}$ all lie in $\mathbb{Q}^n$, they are not linearly independent. This implies that $[v_1, \ldots, v_{n+1}] = 0$ as an element of $\mathrm{St}(\mathrm{SL}_{n+1} \mathbb{Z})$. Adding this term to the right side, we obtain

$$\begin{aligned}\varphi(\partial[v_1, \ldots, v_{n+1}]) &= \sum_{i=1}^{n+1}(-1)^{i-1}[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}, e_{n+1}] + (-1)^n[v_1, \ldots, v_{n+1}, \widehat{e_{n+1}}] \\ &= \partial[v_1, \ldots, v_{n+1}, e_{n+1}] = 0\end{aligned}$$

as desired.

**Cocompact lattices.** The lattice $\mathrm{SL}_n \mathbb{Z}$ is not cocompact in $\mathrm{SL}_n \mathbb{R}$. However, there are natural families of cocompact lattices in $\mathrm{SL}_n \mathbb{R}$. We will prove below that the analogue of Conjecture 4 holds for these families, providing another piece of evidence for Conjecture 4.

Since examples of such families are not so well known, we begin by giving an explicit construction of a family of cocompact lattices $\Gamma_n$ in $\mathrm{SL}_n \mathbb{R}$. Let $\sqrt[4]{2}$ denote the positive real fourth root of 2. Given $x \in \mathbb{Z}[\sqrt[4]{2}]$, define $||x||^2 \in \mathbb{Z}[\sqrt{2}]$ by writing $x = a + b\sqrt[4]{2}$ for some $a, b \in \mathbb{Z}[\sqrt{2}]$ and defining

$$||x||^2 = (a + b\sqrt[4]{2})(a - b\sqrt[4]{2}) = a^2 - \sqrt{2}b^2.$$

Define $\Gamma_n$ to be the group of matrices with entries in $\mathbb{Z}[\sqrt[4]{2}]$ that preserve the corresponding Hermitian form; that is, let

$$\Gamma_n := \mathrm{SU}_n\big(||x_1||^2 + \cdots + ||x_n||^2; \mathbb{Z}[\sqrt[4]{2}]\big).$$

Then $\Gamma_n$ is a cocompact lattice in $\mathrm{SL}_n(\mathbb{R})$, as we now explain. The group $\Gamma_n$ is the $\mathbb{Z}[\sqrt{2}]$-integer points of the simple algebraic group $G$ defined over $\mathbb{Q}(\sqrt{2})$ given by

$$G := \mathrm{SU}_n\big(||x_1||^2 + \cdots + ||x_n||^2; \mathbb{Q}(\sqrt[4]{2})\big).$$

The group $G$ is only algebraic over $\mathbb{Q}(\sqrt{2})$, not over $\mathbb{Q}(\sqrt[4]{2})$, for the same reason that $\mathrm{SU}(n)$ is only a real Lie group, not a complex Lie group. A well-known theorem of Borel and Harish-Chandra (see [PR, Theorem 4.14]) states that the $\mathbb{Z}$-points of a semisimple algebraic group $G$ over $\mathbb{Q}$ form a lattice in the real points $G(\mathbb{R})$. In our situation, the corresponding theorem states that $\Gamma_n = G(\mathbb{Z}[\sqrt{2}])$ is a lattice in the product $G(\mathbb{R}) \times G^\sigma(\mathbb{R})$, where $G(\mathbb{R})$ and $G^\sigma(\mathbb{R})$ are obtained from $G$ by the two embeddings of $\mathbb{Q}(\sqrt{2})$ into $\mathbb{R}$ (see [PR, §2.1.2]). For a basic example of this phenomenon, note that although $\mathbb{Z}[\sqrt{2}]$ is not a discrete subset of $\mathbb{R}$, when it is embedded in $\mathbb{R} \times \mathbb{R}$ by $a + b\sqrt{2} \mapsto (a + b\sqrt{2}, a - b\sqrt{2})$ its image is discrete and indeed a lattice.

By [PR, Proposition 2.15(3)], $G(\mathbb{R}) \approx \mathrm{SL}_n \mathbb{R}$. Since the other embedding $\sigma$ sends $\sqrt{2} \mapsto -\sqrt{2}$, we have

$$G^\sigma = \mathrm{SU}_n(||x_1||^2_\sigma + \cdots + ||x_n||^2_\sigma; \mathbb{Q}(\sqrt{-\sqrt{2}})),$$

where $||x||^2_\sigma$ is defined by writing $x \in \mathbb{Q}(\sqrt{-\sqrt{2}})$ as $x = a + b\sqrt{-\sqrt{2}}$ for $a, b \in \mathbb{Q}(\sqrt{2})$ and defining

$$||x||^2_\sigma = (a + b\sqrt{-\sqrt{2}})(a - b\sqrt{-\sqrt{2}}) = a^2 + \sqrt{2}b^2.$$

It is clear from this description that when we pass from $\mathbb{Q}(\sqrt{2})$ to $\mathbb{R}$, we obtain

$$G^\sigma(\mathbb{R}) = \mathrm{SU}_n(||x_1||^2_\sigma + \cdots + ||x_n||^2_\sigma; \mathbb{C}) = \mathrm{SU}(n).$$

We conclude that $\Gamma_n$ embeds as a lattice in $\mathrm{SL}_n \mathbb{R} \times \mathrm{SU}(n)$. Since $\sigma(\Gamma_n)$ is a subgroup of the compact group $\mathrm{SU}(n)$, it contains no unipotent elements, and so neither does $\Gamma_n$. This implies that $\Gamma_n$ acts cocompactly on $\mathrm{SL}_n \mathbb{R} \times \mathrm{SU}(n)$ (see [PR, §2.1.4 and Theorem 4.17(3)]). But since $\mathrm{SU}(n)$ is compact, the projection of $\Gamma_n$ to the first factor $\mathrm{SL}_n \mathbb{R}$ remains discrete and cocompact. We conclude that $\Gamma_n = G(\mathbb{Z}[\sqrt{2}])$ is a cocompact lattice in $G(\mathbb{R}) = \mathrm{SL}_n \mathbb{R}$. Note that there are natural inclusions $\Gamma_n \to \Gamma_{n+1}$ for each $n \geq 1$.

Since $\Gamma_n$ is a cocompact lattice in $\mathrm{SL}_n \mathbb{R}$, it acts properly discontinuously and cocompactly on the contractible symmetric space $\mathrm{SL}_n \mathbb{R}/\mathrm{SO}(n)$. By Selberg's Lemma, $\Gamma_n$ has a finite index torsion-free subgroup, which acts freely on $\mathrm{SL}_n \mathbb{R}/\mathrm{SO}(n)$. Thus

$$\mathrm{vcd}(\Gamma_n) = \dim \mathrm{SL}_n \mathbb{R} - \dim \mathrm{SO}(n) = (n^2 - 1) - \binom{n}{2} = \binom{n+1}{2} - 1.$$

The analogue of Conjecture 4 for the family of cocompact lattices $\Gamma_n$ is thus the following theorem.

**Theorem 6.** *For each $i \geq 0$ the group $H^{\binom{n+1}{2}-1-i}(\Gamma_n; \mathbb{Q})$ does not depend on $n$ for $n \gg i$.*

*Proof.* For any lattice $\Gamma_n$ in $\mathrm{SL}_n \mathbb{R}$, let $X_n$ be the locally symmetric space

$$X_n \coloneqq \Gamma_n \backslash \mathrm{SL}_n \mathbb{R} / \mathrm{SO}(n).$$

Since $\Gamma_n$ acts on the contractible space $\mathrm{SL}_n \mathbb{R}/\mathrm{SO}(n)$ with finite stabilizers we have $H^*(X_n;\mathbb{Q}) \approx H^*(\Gamma_n;\mathbb{Q})$. Since $\Gamma_n$ is cocompact, the above remarks imply that $X_n$ is a finite quotient of a closed aspherical manifold. Thus its rational cohomology satisfies Poincaré duality, which gives:

$$H^{\binom{n+1}{2}-1-i}(\Gamma_n;\mathbb{Q}) \approx H^i(\Gamma_n;\mathbb{Q}) \quad \text{for each } i \geq 0 \tag{8}$$

The real cohomology of the compact symmetric space $\mathrm{SU}(n)/\mathrm{SO}(n)$ is isomorphic to the space of $\mathrm{SL}_n \mathbb{R}$-invariant forms on $\mathrm{SL}_n \mathbb{R}/\mathrm{SO}(n)$. These forms are closed and indeed harmonic. Being $\mathrm{SL}_n \mathbb{R}$-invariant, these forms are *a fortiori* $\Gamma_n$-invariant, and so they descend to harmonic forms on $X_n$. Thus for any lattice $\Gamma_n$ we obtain a map

$$\iota\colon H^*(\mathrm{SU}(n)/\mathrm{SO}(n);\mathbb{R}) \to H^*(X_n;\mathbb{R}) \approx H^*(\Gamma_n;\mathbb{R})$$

If $\Gamma_n$ is cocompact, applying Hodge theory to $X_n$ implies that $\iota$ is injective in all dimensions. Moreover a theorem of Matsushima [Ma] implies in this case that $\iota$ is in fact surjective in a linear range of dimensions. Thus for $n \gg i$ we have for any cocompact $\Gamma_n$ (see, e.g., [Bo, §11.4]):

$$H^i(\Gamma_n;\mathbb{R}) \approx H^i(\mathrm{SU}(n)/\mathrm{SO}(n);\mathbb{R}) \approx H^i(\mathrm{SU}/\mathrm{SO};\mathbb{R}) \approx \mathrm{gr}^i {\textstyle\bigwedge}^* \langle e_5, e_9, e_{13}, e_{17}, \ldots \rangle$$

In particular $H^i(\Gamma_n;\mathbb{R})$ is independent of $n$ for $n \gg i$. Applying (8) completes the proof. □

We remark that Borel's proof of homological stability for $H^i(\mathrm{SL}_n \mathbb{Z};\mathbb{R})$ mentioned earlier was accomplished by showing that $\iota$ is an isomorphism for non-cocompact lattices as well, albeit in a smaller range of dimensions.

**Automorphic forms.** We close this section by briefly mentioning a connection to automorphic forms. We recommend [Bo2], [Sch2], and [St, Appendix A] for general surveys of the connection between automorphic forms and the cohomology of arithmetic groups. Generalizing a classical result of Eichler–Shimura, Franke [Fr] proved that the groups $H^*(\mathrm{SL}_n \mathbb{Z};\mathbb{C})$ are isomorphic to spaces of certain automorphic forms on $\mathrm{SL}_n \mathbb{R}$ (those of "cohomological type"). This had previously been a conjecture of Borel. This space of automorphic forms is the direct sum of two pieces, the cuspidal cohomology and the Eisenstein cohomology. However, it was observed by Borel, Wallach, and Zuckermann that the cuspidal cohomology is all concentrated around the middle range of the cohomology (see [Sch, Proposition 3.5] for a precise statement). This implies that in the range described by Conjecture 4, the cohomology consists entirely of Eisenstein cohomology. From this perspective, our conjecture is related to assertions regarding which Eisenstein series contribute to cohomology and how Eisenstein series for different $n$ are related by induction.

| $n$ | vcd | $H^i(\mathrm{Aut}(F_n);\mathbb{Q})$ | vcd | $H^i(\mathrm{Out}(F_n);\mathbb{Q})$ |
|---|---|---|---|---|
| 2 | 2 | 0 0 | 1 | 0 |
| 3 | 4 | 0 0 0 0 | 3 | 0 0 0 |
| 4 | 6 | 0 0 0 $\mathbb{Q}$ 0 0 | 5 | 0 0 0 $\mathbb{Q}$ 0 |
| 5 | 8 | 0 0 0 0 0 0 $\mathbb{Q}$ 0 | 7 | 0 0 0 0 0 0 0 |
| 6 | | | 9 | 0 0 0 0 0 0 0 $\mathbb{Q}$ 0 |

Table 3: The rational cohomology of $\mathrm{Aut}(F_n)$ for $2 \leq n \leq 5$ and of $\mathrm{Out}(F_n)$ for $2 \leq n \leq 6$. For $1 \leq i \leq 6$ these were computed in both cases by Hatcher–Vogtmann [HV]; $H^7(\mathrm{Aut}(F_5);\mathbb{Q})$ and $H^8(\mathrm{Aut}(F_5);\mathbb{Q})$ were computed by Gerlits (see [CKV]); and $H^7(\mathrm{Out}(F_5);\mathbb{Q})$ and $H^*(\mathrm{Out}(F_6);\mathbb{Q})$ were computed by Ohashi [O]. All rational cohomology classes are unstable.

# 3 Stability in the unstable cohomology of $\mathrm{Aut}(F_n)$

The analogy between $\mathrm{Mod}_g$ and $\mathrm{SL}_n \mathbb{Z}$ is well-known to extend to the automorphism group $\mathrm{Aut}(F_n)$ of the free group $F_n$ of rank $n \geq 2$. Hatcher–Vogtmann (and later with Wahl, see [HW]) proved that $H^i(\mathrm{Aut}(F_n);\mathbb{Z})$ is independent of $n$ for $n \gg i$. Culler–Vogtmann [CuV] proved that $\mathrm{vcd}(\mathrm{Aut}(F_n)) = 2n-2$.

**Conjecture 7.** *For each $i \geq 0$ the group $H^{2n-2-i}(\mathrm{Aut}(F_n);\mathbb{Q})$ only depends on the parity of $n$ for $n \gg i$.*

This conjecture is perhaps more speculative than Conjectures 1 and 4, and it remains an open question even for $i = 0$. However, known conjectures on sources of unstable cohomology are consistent with Conjecture 7 for $i = 1$ and $i = 2$, as we explain below. The closely related group $\mathrm{Out}(F_n)$ has virtual cohomological dimension $2n-3$, and we conjecture that $H^{2n-3-i}(\mathrm{Out}(F_n);\mathbb{Q})$ only depends on the parity of $n$ for $n \gg i$.

**Computational evidence.** The rational cohomology groups of $\mathrm{Aut}(F_n)$ have been computed for $2 \leq n \leq 5$, and the rational cohomology groups of $\mathrm{Out}(F_n)$ have been computed for $2 \leq n \leq 6$. These calculations are summarized in Table 3. They are consistent with Conjecture 7.

**Unstable classes and graph homology.** When $n$ is even, Morita [Mo, §6.5] constructed cycles in $H_{2n-4}(\mathrm{Out}(F_n);\mathbb{Q}) = H_{\nu-1}(\mathrm{Out}(F_n);\mathbb{Q})$, and Conant–Vogtmann [CoV] showed that these cycles can be lifted to $H_{2n-4}(\mathrm{Aut}(F_n);\mathbb{Q}) = H_{\nu-2}(\mathrm{Aut}(F_n);\mathbb{Q})$. These classes are known to be nonzero in $H_4(\mathrm{Out}(F_4);\mathbb{Q})$ and $H_4(\mathrm{Aut}(F_4);\mathbb{Q})$ [Mo], in $H_8(\mathrm{Out}(F_6);\mathbb{Q})$ and $H_8(\mathrm{Aut}(F_6);\mathbb{Q})$ [CoV], and in $H_{12}(\mathrm{Out}(F_8);\mathbb{Q})$ and $H_{12}(\mathrm{Aut}(F_8);\mathbb{Q})$ [Gr]. They are conjectured to be nonzero for all even $n$.

Galatius [G] proved that for $n \gg i$ we have $H^i(\mathrm{Aut}(F_n);\mathbb{Q}) = 0$ and $H^i(\mathrm{Out}(F_n);\mathbb{Q}) = 0$, so *all* the rational homology of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ is unstable. The Morita cycles are known to be immediately unstable: Conant–Vogtmann [CoV3] proved that the Morita cycles vanish after stabilizing once from $H_{2n-4}(\mathrm{Aut}(F_n);\mathbb{Q})$ to $H_{2n-4}(\mathrm{Aut}(F_{n+1});\mathbb{Q})$. However, Conjecture 7 provides a sense in which these classes might be stable after all.

Similarly, when $n$ is odd, Conant–Kassabov–Vogtmann [CKV] have recently constructed classes in $H_{2n-3}(\mathrm{Aut}(F_n);\mathbb{Q}) = H_{\nu-1}(\mathrm{Aut}(F_n);\mathbb{Q})$, which are known to be nonzero in $H_7(\mathrm{Aut}(F_5);\mathbb{Q})$ and $H_{11}(\mathrm{Aut}(F_7);\mathbb{Q})$ and conjectured to be nonzero for all odd $n$. All known nonzero rational homology classes for $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ fit into one of these families. Finally, the Morita cycles were generalized by Morita and by Conant–Vogtmann [CoV, §6.1] to produce, for every graph of rank $r$ with $k$ vertices all of odd valence, a cycle in $H_{\nu-(k-1)}(\mathrm{Out}(F_{r+k});\mathbb{Q})$. The Morita cycles in $H_{\nu-1}(\mathrm{Out}(F_n);\mathbb{Q})$ correspond to the graph with 2 vertices connected by $n-1$ parallel edges. Can all odd-valence graphs be naturally grouped into families which contribute to $H_{\nu-i}(\mathrm{Out}(F_n);\mathbb{Q})$ for some fixed $i$?

**Stabilization and duality.** Bestvina–Feign [BF] proved that $\mathrm{Out}(F_n)$ is a virtual duality group, so by [BE, Theorem 3.5] $\mathrm{Aut}(F_n)$ is a virtual duality group as well. The rational dualizing module $\mathrm{St}(\mathrm{Aut}(F_n))$ can be understood in terms of the topology at infinity of Culler–Vogtmann's *Outer space* (see [BF, §5] for details), but it has not been described explicitly.

**Problem 8.** Construct a resolution for $\mathrm{St}(\mathrm{Aut}(F_n))$ analogous to Ash–Rudolph's resolution of $\mathrm{St}(\mathrm{SL}_n\,\mathbb{Z})$ in terms of modular symbols, and analogous to Broaddus's resolution of $\mathrm{St}(\mathrm{Mod}_g)$) in terms of chord diagrams.

We do not know an analogue for $\mathrm{Aut}(F_n)$ of the stabilization map (6) that we constructed for $\mathrm{SL}_n\,\mathbb{Z}$.

**Problem 9.** Define a nontrivial, natural $\mathrm{Aut}(F_n)$-equivariant map $\mathrm{St}(\mathrm{Aut}(F_n)) \to \mathrm{St}(\mathrm{Aut}(F_{n+1}))$.

Conant–Vogtmann used Bestvina–Feign's bordification of Outer space to construct a complex of *filtered graphs* that computes the homology of $\mathrm{Aut}(F_n)$ [CoV2, §7.3]. This should yield a resolution of $\mathrm{St}(\mathrm{Aut}(F_n))$. However, from this perspective it is not clear to us how to define a stabilization map $\mathrm{St}(\mathrm{Aut}(F_n)) \to \mathrm{St}(\mathrm{Aut}(F_{n+1}))$.

**Abelian cycles.** Consider the subgroup $K < \mathrm{Aut}(F_n)$ generated by $x_i \mapsto x_i x_1$ and by $x_i \mapsto x_1 x_i$ for $1 < i \leq n$. This subgroup is isomorphic to $\mathbb{Z}^{2n-2}$ and thus provides an explicit witness for the lower bound $\mathrm{vcd}(\mathrm{Aut}(F_n)) \geq 2n-2$.

**Question 10.** Under the inclusion $i\colon \mathbb{Z}^{2n-2} \approx K \hookrightarrow \mathrm{Aut}(F_n)$ of the subgroup $K$, is the image of the fundamental class nonzero for some $n \geq 5$? That is, is it ever true that

$$i_*[\mathbb{Z}^{2n-2}] \neq 0 \in H_{2n-2}(\mathrm{Aut}(F_n);\mathbb{Q})?$$

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## References

[AR] A. Ash and L. Rudolph, The modular symbol and continued fractions in higher dimensions, *Invent. Math.* 55 (1979), no. 3, 241–250.

[BE] R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, *Invent. Math.* 20 (1973), 103–124.

[BF] M. Bestvina and M. Feign, The topology at infinity of $\mathrm{Out}(F_n)$, *Invent. Math.* 140 (2000), 651–692. Available at: `http://www.math.utah.edu/~bestvina/eprints/duality.ps`

[Bo] A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup.* (4) 7 (1974), 235–272 (1975).

[Bo2] A. Borel, Introduction to the cohomology of arithmetic groups, in *Lie groups and automorphic forms*, 51–86, AMS/IP Stud. Adv. Math., 37 Amer. Math. Soc., Providence, RI.

[BS] A. Borel and J.-P. Serre, Corners and arithmetic groups, *Comment. Math. Helv.* 48 (1973), 436–491.

[Br] N. Broaddus, Homology of the curve complex and the Steinberg module of the mapping class group, *Duke Math. Journal*, to appear. arXiv:0711.0011.

[Bro] K. Brown, *Cohomology of groups*, Springer GTM Series, Vol. 87, 1982.

[CFP] T. Church, B. Farb and A. Putman, The rational cohomology of the mapping class group vanishes in its cohomological dimension, *Inter. Math. Res. Notices*, to appear. arXiv:1108.0622.

[CKV] J. Conant, M. Kassabov, and K. Vogtmann, Hairy graphs and the unstable homology of $\mathrm{Mod}(g, s)$, $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$, preprint (2011), arXiv:1107.4839.

[CoV] J. Conant and K. Vogtmann, Morita classes in the homology of automorphism groups of free groups, *Geom. Topol.* 8 (2004), 1471–1499. arXiv:math/0406389.

[CoV2] J. Conant and K. Vogtmann, On a theorem of Kontsevich, *Algebr. Geom. Topol.* 3 (2003), 1167–1224. arXiv:math/0208169.

[CoV3] J. Conant and K. Vogtmann, Morita classes in the homology of $\mathrm{Aut}(F_n)$ vanish after one stabilization, *Groups Geom. Dyn.* 2 (2008) 1, 121–138. arXiv:math/0606510.

[CuV] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups, Invent. Math. **84** (1986), no. 1, 91–119.

[D] W. G. Dwyer, Twisted homological stability for general linear groups, Ann. of Math. (2) **111** (1980), no. 2, 239–251.

[EVGS] P. Elbaz-Vincent, H. Gangl, and C. Soulé, Perfect forms and the cohomology of modular groups, preprint (2010). Most recent version (v3) available at:
`http://hal.archives-ouvertes.fr/hal-00443899/fr/`

[FM] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, Vol. 49, Princeton Univ. Press, 2012. Pre-publication version available at:
`http://www.math.utah.edu/~margalit/primer/`

[Fr] J. Franke, Harmonic analysis in weighted $L_2$-spaces, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279.

[G] S. Galatius, Stable homology of automorphism groups of free groups, *Annals of Math.*, to appear. arXiv:math/0610216.

[Gr] J. Gray, On the homology of automorphism groups of free groups, Ph.D. thesis, University of Tennessee, 2011. Available at: `http://trace.tennessee.edu/utk_graddiss/974/`

[Ha1] J. Harer, Stability of the homology of the mapping class groups of orientable surfaces, *Annals of Math.* (2) 121 (1985), no. 2, 215–249.

[Ha2] J. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, *Invent. Math.* 84 (1986), no. 1, 157–176.

[HV] A. Hatcher and K. Vogtmann, Rational homology of $\mathrm{Aut}(F_n)$, *Math. Res. Lett.* 5 (1998), 759–780. Available at: `http://www.math.cornell.edu/~vogtmann/papers/Rational/`

[HW] A. Hatcher and N. Wahl, Erratum to: "Stabilization for the automorphisms of free groups with boundaries", *Geom. Topol.* 12 (2008), no. 2, 639–641. arXiv:math/0608333.

[I] J. Igusa, Arithmetic variety of moduli for genus two, Ann. of Math. (2) **72** (1960), 612–649.

[J] D. Johnson, The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$, Ann. of Math. (2) **118** (1983), no. 3, 423–442.

[LS1] R. Lee and R.H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* 33 (1976), no. 1, 15–53.

[LS2] R. Lee and R.H. Szczarba, On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$, Duke Math. J. 45 (1978), 101129.

[Lo] E. Looijenga, Cohomology of $\mathbf{M}_3$ and $\mathbf{M}_3^1$, in *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, 205–228, Contemp. Math., 150 Amer. Math. Soc., Providence, RI. Available at: `http://www.staff.science.uu.nl/~looij101/mthreenew.ps`

[Mo] S. Morita, Structure of the mapping class groups of surfaces: a survey and a prospect, *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, 349–406, Geom. Topol. Monogr. 2, Geom Topol Publ., Coventry, 1999. math.GT/9911258.

[PR] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Pure and Applied Mathematics 139, Academic Press, Inc., Boston, MA, 1994.

[Ma] Y. Matsushima, On Betti numbers of compact, locally symmetric Riemannian manifolds, *Osaka Math. J.* 14 (1962), 1–20.

[Me] G. Mess, Unit tangent bundle subgroups of mapping class groups, IHES preprint, 1990.

[MSS] S. Morita, T. Sakasai, and M. Suzuki, Abelianizations of derivation Lie algebras of free associative algebra and free Lie algebra, preprint (2011), arXiv:1107.3686.

[O] R. Ohashi, The rational homology group of $\mathrm{Out}(F_n)$ for $n \leq 6$, *Experiment. Math.* 17 (2008) 2, 167–179.

[Sch] J. Schwermer, Holomorphy of Eisenstein series at special points and cohomology of arithmetic subgroups of $\mathrm{SL}_n(\mathbb{Q})$, J. Reine Angew. Math. **364** (1986), 193–220.

[Sch2] J. Schwermer, Geometric cycles, arithmetic groups and their cohomology, *Bull. Amer. Math. Soc.* 47 (2010) 2, 187–279.

[St] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, 79, Amer. Math. Soc., Providence, RI, 2007.

[So] C. Soulé, The cohomology of $\mathrm{SL}_3(\mathbb{Z})$, Topology 17, no. 1 (1978), 1–22.

[T] O. Tommasi, Rational cohomology of the moduli space of genus 4 curves, Compos. Math. **141** (2005), no. 2, 359–384. arXiv:math/0312055.

[vdK] W. van der Kallen, Homology stability for linear groups, *Invent. Math.* 60 (1980), no. 3, 269–295.

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