Finite covers of graphs, their primitive homology, and representation theory

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Abstract

Consider a finite, regular cover $Y \to X$ of finite graphs, with associated deck group G. We relate the topology of the cover to the structure of $H_1(Y;\mathbb{C})$ as a G-representation. A central object in this study is the *primitive homology* group $H_1^{\text{prim}}(Y;\mathbb{C}) \subseteq H_1(Y;\mathbb{C})$, which is the span of homology classes represented by components of lifts of primitive elements of $\pi_1(X)$. This circle of ideas relates combinatorial group theory, surface topology, and representation theory.

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1 Introduction

Consider a regular covering $f : Y \to X$ of finite graphs, with associated deck group G. The goal of this article is to better understand $H_1(Y)$ as a G-representation.

The action of G on Y by homeomorphisms endows $H_1(Y;\mathbb{C})$ with the structure of a G-representation. Gaschütz (see [GLLM]) observed that the Chevalley-Weil formula [CW] for surfaces has the following analogue in this context: there is an isomorphism of G-representations

$$H_1(Y;\mathbb{C}) \cong \mathbb{C}[G]^{n-1} \oplus \mathbb{C} \tag{1}$$

where $\mathbb{C}[G]$ denotes the regular representation and $n = \mathrm{rank}(\pi_1(X))$ is the rank of the free group $\pi_1(X)$.

The isomorphism (1) hints at a broader dictionary between representation-theoretic information (on the right-hand side) and topological information (on the left-hand side). One of the goals of this paper is to begin an exploration of this dictionary.

Primitive homology of G-covers. Of course $H_1(Y;\mathbb{C})$ is spanned by a finite set of closed loops. But what if we demand that these loops project under f to be *primitive* in $\pi_1(X)$; that is, part of a free basis of the free group $\pi_1(X)$? Fixing a cover $f : Y \to X$ we define, following Boggi-Looijenga [BL] in the surface case (see § 8), the *primitive homology* of Y to be:

$$H_1^{\mathrm{prim}}(Y;\mathbb{C}) := \mathbb{C}\text{-span}\{[\gamma] \in H_1(Y;\mathbb{C}) : f(\gamma) \in \pi_1(X) \text{ is primitive}\}$$

By construction, $H_1^{\mathrm{prim}}(Y;\mathbb{C})$ is a G-subrepresentation of $H_1(Y;\mathbb{C})$. The main question we consider in this paper is the following, which we view as a fundamental question about finite covers of finite graphs.

Question 1.1 (Determining primitive homology). *What is the G-subrepresentation $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \subseteq H_1(Y;\mathbb{C})$? In particular, is $H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$ for every normal cover $f : Y \to X$?*

In order to state Question 1.1 more precisely, we let

$$\mathrm{Irr}(G) := \{V_0 := \mathbb{C}_{triv}, V_1, \ldots, V_r\}$$

be the set of (isomorphism classes of) complex, irreducible G-representations. In this notation, the Chevalley-Weil formula (1) can be restated as:

$$H_1(Y;\mathbb{C}) \cong \bigoplus_{V_i \in \mathrm{Irr}(G)} V_i^{(n-1)\dim(V_i)} \oplus \mathbb{C}_{triv}.$$

Since $H_1^{\mathrm{prim}}(Y;\mathbb{C})$ is a G-subrepresentation of $H^1(Y;\mathbb{C})$, Question 1.1 is equivalent to the following.

Question 1.2 (Which irreps occur?). *Which irreducible G-representations V_i occur in $H_1^{\mathrm{prim}}(Y;\mathbb{C})$? What are their multiplicities?*

In order to describe our progress on answering Question 1.2 we make the following definition. Note that the data of a normal G-cover $f : Y \to X$ is equivalent to a surjective homomorphism $\phi : \pi_1(X) = F_n \to G$. The following can therefore be viewed as a purely group-theoretic definition.

Definition 1.3 ($\mathbf{Irr^{pr}(\phi, G)}$)**.** Let G be a finite group and let $\phi : F_n \to G$ be a homomorphism. Let $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) \subset \mathrm{Irr}(G)$ denote the set of irreducible representations V of G with the property that $\phi(\gamma)(v) = v$ for some primitive element $\gamma \in F_n$ and some nonzero $v \in V$.

Our first main result gives a restriction on $H_1^{\mathrm{prim}}(Y; \mathbb{C})$ in terms of the purely algebraic data of $\mathrm{Irr}^{\mathrm{pr}}(\rho, G)$.

Theorem 1.4 (**Restricting primitive homology**)**.** *Let $f : Y \to X$ be a normal G-covering of finite graphs defined by $\phi : F_n \to G$, where G is a finite group and $\mathrm{rank}(\pi_1(X)) \geq 2$. Then*

$$H_1^{\mathrm{prim}}(Y; \mathbb{C}) \subseteq \bigoplus_{V_i \in \mathrm{Irr}^{\mathrm{pr}}(\phi, G)} V_i^{(n-1)\cdot \dim(V_i)} \oplus \mathbb{C}_{triv}$$

The proof of Theorem 1.4 is given in §2 below and it uses surface topology. After finishing this work, we learned that Malestein and Putman independently discovered this obstruction as well.

One can also obtain finer information about the structure of $H_1^{\mathrm{prim}}(Y; \mathbb{C})$ by studying the primitive homology of intermediate (nonregular) covers; see §2 for details. Theorem 1.4 suggests a strategy to search for examples with $H_1^{\mathrm{prim}}(Y; \mathbb{C}) \neq H^1(Y; \mathbb{C})$ via an algebraic question about G-representations.

Question 1.5. *Given a group G, does $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$ for every surjective homomorphism $\phi : F_n \twoheadrightarrow G$?*

We attack Question 1.5 from various angles, by restricting the class of groups we consider. In §3 we answer Question 1.5 in the affirmative for G abelian or 2–step nilpotent. One might thus expect that $H_1^{\mathrm{prim}}(Y; \mathbb{C}) = H^1(Y; \mathbb{C})$ in these cases. We can prove this in many instances.

Theorem 1.6 (**Abelian and 2-step nilpotent covers**)**.** *Let $Y \to X$ be a finite normal cover with deck group G defined by $\phi : F_n \to G$. Assume that $\mathrm{rank}(\pi_1(X)) \geq 3$. Suppose that G is either abelian or 2-step nilpotent. Then $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$.*

In the case of a nonabelian G, assume further that n is odd and every subgroup of the center of G has prime order. Then

$$H_1^{\mathrm{prim}}(Y; \mathbb{C}) = H^1(Y; \mathbb{C}).$$

The assumption $\mathrm{rank}(\pi_1(X)) \geq 3$ is in general necessary, even for reasonably simple G. In §7 we give a number of different examples that imply the following.

Theorem 1.7 (**Rank two counterexamples**)**.** *Let X be a wedge of two circles. There exist finite 2–step nilpotent groups G and G-covers $Y \to X$ given by surjections $\phi : \pi_1(X) = F_2 \to G$ so that $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) \subsetneq \mathrm{Irr}(G)$ and $H_1^{\mathrm{prim}}(Y; \mathbb{C}) \subsetneq H^1(Y; \mathbb{C})$.*

Primitives in the kernel. If $\phi : F_n \to G$ has any primitive element in the kernel, then trivially $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$ (and in fact, primitive homology is all of homology for the associated cover). Grunewald and Lubotzky [GL] study covers induced by such maps and call them *redundant*.

To find interesting examples in light of Question 1.5, we are thus lead to a search for ϕ without primitive elements in the kernel.

We discuss the "no primitives in the kernel" condition in §4, and relate it in §4.3 to the Product Replacement Algorithm. We connect this property to those discussed above, as follows : for a finite normal cover $Y \to X$ given by a surjection $\phi : \pi_1(X) = F_n \to G$:

$$\boxed{\ker(\phi) \text{ contains a primitive in } F_n} \Longrightarrow \boxed{H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H^1(Y;\mathbb{C})} \Longrightarrow \boxed{\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)}$$

The first implication is Lemma 2.4 below; the second is Theorem 1.4.

In general, the property that $\ker(\phi)$ does not contain any primitive element is far from sufficient to imply $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) \neq \mathrm{Irr}(G)$; for example, the standard mod-2 homology cover has the former property, but not the latter. In contrast, the following is easy to show from the definitions.

Observation 1.8. *Suppose that G is a finite group which acts freely and linearly on a sphere, and let $\phi : F_n \to G$ be any homomorphism. Then $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$ if and only if $\ker(\phi)$ contains a primitive element.*

At first glance, this makes the class of groups which act freely and linearly on spheres a likely candidate to answer Question 1.5 in the negative. However, in Section 5 we prove the following, which shows that in general this strategy does not work.

Theorem 1.9 (Groups acting freely on spheres). *There is a number $B \geq 3$ with the following property. Suppose that G is a finite group that acts freely and linearly on an odd-dimensional sphere. Then for all $n \geq B$ and all surjective homomorphisms $\phi : F_n \to G$, the kernel of ϕ contains a primitive element of F_n. Thus $\mathrm{Irr}(G) = \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$ for all ϕ and $H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H^1(Y;\mathbb{C})$ for every such G-cover.*

We remark that the finite groups G that have a free linear action on some sphere have been classified; see e.g. [W], [DM] or [N]. We use this classification in our proof of Theorem 1.9.

Algorithms. In §6 we consider the search for a surjection $\phi : F_n \to G$ with $\mathrm{Irr}(G) \neq \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$ from an algorithmic perspective. A priori, the question if $\mathrm{Irr}(G) = \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$ requires knowledge about an infinite set (of all primitives) in the free group. However, we will prove (Proposition 6.1) one can algorithmically compute the set of all elements of G that are images of primitive elements under ϕ. This allows computer-assisted searches for examples with $\mathrm{Irr}(G) \neq \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$.

Concerning Theorem 1.9 above, experiments suggest that the constant B is likely very small, maybe even 3. We have checked that $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$ for all such G and ϕ with $|G| \leq 1000$. On the other hand, in Proposition 7.1 we use a computer-assisted search to give an example that shows B must be at least 3.

The case of surfaces. There is an analogous theory to the above with the finite graph X replaced by a surface S_g, and primitive elements in $\pi_1(X)$ replaced by simple closed curves in $\pi_1(S_g)$. The "simple closed curve homology" is closely related to the problem of vanishing (or not) of the virtual first Betti number of the moduli space of genus $g \geq$ Riemann surfaces. We sketch this theory in §8.

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2 Representation Theory of H_1^{prim}

In this section we prove Theorem 1.4, and give a partial converse. We begin by setting up some notation that we will use throughout the article. X will denote the wedge of n copies of S^1.

Given a regular covering $Y \to X$, we fix a *preferred* lift $\hat{x}_0$ of the basepoint $x_0 \in X$. Given any loop γ in X, the *preferred elevation* $\hat{\gamma}$ of γ is the lift at $\hat{x}_0$ of the smallest power γ^k which does lift (with degree 1) to Y. Any image of $\hat{\gamma}$ under an element of the deck group will be called an *elevation* of γ.

We denote by G_γ the stabilizer in G of the preferred elevation of γ. Note that G_γ is cyclic. It is generated by the image of $[\gamma] \in \pi_1(X)$ under the surjection $\pi_1(X) \to G$. We denote this element by g_γ.

The key step in the proof of Theorem 1.4 is the following proposition, which can be considered as a new entry in the dictionary discussed in the introduction.

Proposition 2.1. *Let X be a wedge of n copies of S^1. Let $Y \to X$ be a regular cover with deck group G. Let l be a primitive loop on X and let $\widetilde{l}$ be the preferred elevation of l to Y. Then there is an isomorphism of G–representations:*

$$\mathrm{Span}_{H_1(X)}\left\{g \cdot [\widetilde{l}] : g \in G]\right\} \cong \mathrm{Ind}_{G_\ell}^{G} \mathbb{C}_{\mathrm{triv}}$$

The proof of Proposition 2.1 uses surface topology.

Proof of Proposition 2.1. We prove the proposition in three steps.

Step 1 (Reduction to the surface case): We choose an identification X with the core graph of a $(n+1)$–bordered sphere S so that l is freely homotopic to a simple closed curve α on S. This is possible since $\mathrm{Out}(F_n)$ acts transitively on the set of (conjugacy classes of) primitive elements of F_n. In addition, we can assume that each component of $S - \alpha$ contains at least one boundary component of S.

Let $F \to S$ denote the cover defined by $\pi_1(Y) < \pi_1(X) = \pi_1(S)$. Note that F is G–equivariantly homotopy equivalent to Y. Let $\hat{\alpha}_i$ be the elevations of α in F. The homology classes $[\hat{\alpha}_i]$ are exactly the homology classes defined by the elevations of l in Y.

To prove Proposition 2.1, it therefore suffices to consider the case of a cover $F \to S$ of surfaces with boundary, deck group G, and and l a simple closed curve α with elevations $[\hat{\alpha}_i]$ with the property that each component of $S - \alpha$ contains at least one boundary component of S. We assume this setup for the rest of the proof of the proposition.

Step 2 (Independence of Elevations): Let F be a surface with boundary, and let $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ be disjoint, pairwise nonisotopic simple closed curves on F. Suppose that each complementary component of $\hat{\alpha}_1 \cup \ldots \cup \hat{\alpha}_n$ in F contains at least one boundary component of F. We claim that $\{[\hat{\alpha}_1], \ldots, [\hat{\alpha}_n]\} \subset H_1(F; \mathbb{C})$ is linearly independent.

To see this, let $R_1, \ldots, R_k$ be the complementary components of $\hat{\alpha}_1 \cup \ldots \cup \hat{\alpha}_n$ in F. Let $\delta_i \subset R_i$ be the multicurve consisting of the union of all boundary components of F contained in R_i. Suppose that

$$a_1[\hat{\alpha}_1] + \ldots + a_n[\hat{\alpha}_n] = 0. \tag{2}$$

We can assume that all $a_i \neq 0$. Suppose that $\hat{\alpha}_1 \cup \ldots, \cup \hat{\alpha}_n$ separates the surface, since otherwise there is nothing to show. Thus, without loss of generality, we can assume that $\hat{\alpha}_n$ lies in $\partial R_1 \cap \partial R_2$. By adding a suitable multiple of ∂R_2 if necessary, we can rewrite (2) as

$$a_1'[\hat{\alpha}_1] + \ldots + a_{n-1}'[\hat{\alpha}_{n-1}] + b_2[\delta_2] = 0$$

for some a_i' and some b_2. Since $\hat{\alpha}_n$ does not in this equation, the support of this new relation contains $R_1 \cup R_2$ in its complement. Denote by R_1' the complementary component containing R_1. We can now repeat the argument: unless $\hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}$ has become nonseparating, there will be some curve which lies on the boundary of R_1' and a second boundary component R_j for some j. Repeating this modification a finite number of times we end up with

$$A_1[\hat{\alpha}_1] + \ldots + A_l[\hat{\alpha}_l] + b_2[\delta_2] + \ldots + b_k[\delta_k] = 0$$

where now $\hat{\alpha}_1, \ldots, \hat{\alpha}_l$ is nonseparating. Note that $[\delta_1]$ does not appear in this expression, and thus all involved classes are linearly independent. Thus, all coefficients are 0, which implies that the original linear combination was trivial as well. This proves the claim.

Step 3 (Identification of the representation): By construction, the element $g_\alpha \in G$ fixes the preferred elevation $\hat{\alpha}_1$ and permutes the other elevations $\hat{\alpha}_i$ in the same exact way that it permutes the the cosets of $G_\alpha = \langle g_\alpha \rangle$ in G. By the standard characterization of induced representations and the linear independence of the classes $[\hat{\alpha}_i]$, Proposition 2.1 follows. $\square$

Proof of Theorem 1.4. For any G-representations U, W, denote by $\langle U, W \rangle$ the inner product of the characters of the representations U and W. Let V be any irreducible G-representation. Proposition 2.1 followed by Frobenius Reciprocity gives:

$$\begin{aligned} \langle \mathrm{Span}_{H_1(Y)} \{[\hat{\alpha}_1], \ldots, [\hat{\alpha}_n]\}, V \rangle_G &= \langle \mathrm{Ind}_{G_\alpha}^G \mathbb{C}_{\mathrm{triv}}, V \rangle_G \\ &= \langle \mathrm{Res}_{G_\alpha}^G V, \mathbb{C}_{\mathrm{triv}} \rangle_{G_\alpha} \\ &= \dim(\mathrm{Fix}(G_\alpha)). \end{aligned}$$

Thus an irreducible representation V appears in $\mathrm{Span}_{H_1(Y)} \{[\hat{\alpha}_1], \ldots, [\hat{\alpha}_n]\}$ if and only if G_α has a nonzero fixed vector in V (equivalently, since G_α is cyclic, generated by g_α, this is equivalent to g_α having a non-fixed vector). Since every G-representation is a direct sum of irreducible representations, this implies that an irreducible representation V appears in $H_1^{\mathrm{prim}}(Y)$ only if $V \in \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$. $\square$

The rest of this section is devoted to a criterion which can in theory be used to determine the multiplicity of a given $V_i \in \mathrm{Irr}^{\mathrm{pr}}$ in the subrepresentation H_1^{prim}.

To begin, we first note the following simple consequence of transfer.

Lemma 2.2. *Let $Y \to X$ be a regular G–covering, where G is a finite group and X is a graph with* $\mathrm{rank}(\pi_1(X)) \geq 2$. *Let $g \in G$ be any element. Then*

$$H_1(Y;\mathbb{C})^{<g>} \cong H_1(Y/\langle g\rangle;\mathbb{C})$$

The same is true for primitive homology:

$$H_1^{\mathrm{prim}}(Y;\mathbb{C})^{<g>} \cong H_1^{\mathrm{prim}}(Y/\langle g\rangle;\mathbb{C})$$

Proof. To see the first claim, it suffices to note that the transfer map $H_1(Y/\langle g\rangle;\mathbb{C}) \to H_1(Y;\mathbb{C})$ has image in $H_1(Y;\mathbb{C})^{<g>}$ by construction. The second claim follows from the first since the transfer map respects primitive homology. □

Now suppose that $V_i \in \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$. Let $v \in V_i$ be a vector so that $g_x \cdot v = v$ for some primitive x. If we identify V_i as a subrepresentation of $H_1(Y;\mathbb{C})$ in any way, then $v \in H_1(Y;\mathbb{C})^{<g_x>}$. Also note that since $H_1^{\mathrm{prim}}(Y;\mathbb{C})$ is a G-subrepresentation, it contains any vector $v \in V_i$ if and only if it contains the complete representation V_i. Thus we conclude the following for the natural projection map $p : H_1(Y;\mathbb{C}) \to H_1(Y/\langle g\rangle;\mathbb{C})$.

Observation 2.3 (Transfer criterion). *Let $V_i \in \mathrm{Irr}(G)$ be a subrepresentation $V_i \subset H_1(Y;\mathbb{C})$. Then $V_i \subset H_1^{\mathrm{prim}}(Y;\mathbb{C})$ if and only if $0 \neq p(v) \in H_1^{\mathrm{prim}}(Y/\langle g_x\rangle;\mathbb{C})$ for any $v \in V_i$, and an element g_x which is the image of a primitive element in F_n.*

We expect that the primitive homology of the cover $Y/\langle g_x\rangle \to X$ should be easier to understand than that of $Y \to X$, being a cover of smaller degree. However, it is in general not a regular cover, and so the methods developed in this paper seem to be less adapted to studying it.

To give some evidence why $H_1^{\mathrm{prim}}(Y/\langle g_x\rangle;\mathbb{C})$ should be easier to understand, we have the following.

Lemma 2.4. *Suppose $Z \to X$ is a regular cover, and that some primitive loop in X lifts (with degree 1). Then*

$$H_1^{\mathrm{prim}}(Z;\mathbb{C}) = H_1(Z;\mathbb{C})$$

Proof. Up to applying an automorphism of F_n, assume that a_1 lifts. Let X' be the wedge of the loops $a_2, \ldots, a_n$. Then Z is the union of a connected cover of X' together with $|G|$ loops corresponding to the lifts of a_1. Since $a_1 w$ is primitive for any $w \in \pi_1(X') = F(a_2, \ldots, a_n)$, the lemma follows. □

In light of this we pose the following.

Question 2.5. *Suppose $Z \to X$ is a (not necessarily regular!) cover, and suppose that some primitive loop in X lifts to Z. Is it true that :*

$$H_1^{\mathrm{prim}}(Z;\mathbb{C}) = H_1(Z;\mathbb{C})?$$

By the discussion above, a positive answer to this question is equivalent to the statement that the inclusion in Theorem 1.4 is in fact an equality.

3 Abelian and nilpotent covers

In this section we use the point of view developed in Section 2 to study the primitive homology of covers whose deck groups are abelian or 2-step nilpotent groups.

3.1 Abelian covers

We begin with the following, which is an easy consequence of the standard fact that every representation of a finite abelian group factors through a cyclic group.

Proposition 3.1 (Abelian representations). *Suppose that $n \geq 2$, and G is any finite Abelian group. Then for any homomorphism $\phi : F_n \to G$, we have $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$.*

Proof. It is enough to prove this for the case $n = 2$ since we can otherwise simply restrict to a free factor of rank 2. Let $V \in \mathrm{Irr}(G)$ be given, and let $\rho : G \to \mathrm{GL}(V)$ be the corresponding representation. Since G is abelian and V is irreducible, V is 1-dimensional. Let $\{a, b\}$ be a free basis of F_2. There are numbers $n_a, n_b \geq 0, k > 0$ so that

$$\rho(\phi(a))(z) = e^{2\pi i n_a/k} z, \quad \rho(\phi(b))(z) = e^{2\pi i n_b/k} z$$

If either of n_a, n_b is zero, we are done. Otherwise, without loss of generality we can assume that $0 < n_a \leq n_b$. Note that $\{a, b' := ba^{-1}\}$ is also a free basis of F_2, and

$$\rho(\phi(ba^{-1}))(z) = e^{2\pi i (n_b - n_a)/k} z.$$

The proposition then follows by induction on $n_b + n_a$. $\square$

Proposition 3.1 can be improved to the following.

Proposition 3.2. *Let $Y \to X$ be a regular cover with finite abelian deck group G. Let X be a graph with $\mathrm{rank}(\pi_1(X)) \geq 2$. Then*

$$H_1^{\mathrm{prim}}(Y; \mathbb{C}) = H_1(Y; \mathbb{C}).$$

Proof. Proposition 3.1 gives that every $V \in \mathrm{Irr}(G)$ is also contained in $\mathrm{Irr}^{\mathrm{pr}}(G)$. Hence, we just need to show that the inclusion in Theorem 1.4 is an equality. Using Observation 2.3, it suffices to show that

$$H_1^{\mathrm{prim}}(Y/\langle g \rangle; \mathbb{C}) = H_1(Y/\langle g \rangle; \mathbb{C})$$

for all $g \in G$. However, since G is abelian, this is simply a consequence of Lemma 2.4. $\square$

3.2 2-step nilpotent covers

We next consider covers with finite nilpotent deck group.

Proposition 3.3 (2-step nilpotent representations). *Suppose that $n \geq 3$ and that G is finite 2-step nilpotent. Then $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) = \mathrm{Irr}(G)$ for any homomorphism $\phi : F_n \to G$.*

Proof. Since G is 2-step nilpotent, there is an exact sequence

$$0 \to A \to N \to Q \to 0$$

where A and Q are finite abelian. Assume that $n \geq 3$. By passing to a free factor if necessary, we can assume that $n = 3$.

We first claim that there is a free basis $\{a, b, c\}$ of F_3 so that $\rho(\phi([a, b]))v = v$ for some $v \in V$. To see this, consider a (1-dimensional) irreducible subrepresentation $W < V$ of A. Note that as G is 2-step nilpotent, we have $\phi([a, b]), \phi([b, c]) \in A$ and thus there are numbers $n_a, n_b \geq 0, k > 0$ so that for all $z \in W$

$$\rho(\phi([a, c]))(z) = e^{2\pi i n_a/k} z, \quad \rho(\phi([b, c]))(z) = e^{2\pi i n_b/k} z$$

Again using that G is 2-step nilpotent, note that

$$\phi([ba^{-1}, c]) = \phi([b, c])\phi([a, c])^{-1}$$

Thus, we have that

$$\rho(\phi([ba^{-1}, c]))(z) = e^{2\pi i (n_b - n_a)/k} z.$$

Applying the argument as in the case when G is abelian gives the claim.

Now define

$$V_0 := \{v \in V : \rho(\phi([a, b]))v = v\}.$$

Note that $V_0 \neq 0$ and V_0 invariant under $\rho(\phi(a)), \rho(\phi(b))$. Finally, note that the restrictions of $\rho(\phi(a))$ and $\rho(\phi(b))$ to the invariant subspace V_0 commute. Applying the case when G is abelian, we conclude that there is some primitive (which is a product of a, b) that has a nonzero fixed vector in V_0. □

To understand if $H_1^{\text{prim}}(Y; \mathbb{C}) = H_1(Y; \mathbb{C})$ for finite nilpotent covers $Y \to X$, we need a somewhat more precise understanding of the representations of finite nilpotent groups. We begin with the following, likely standard, lemma.

Lemma 3.4. *Let G be a finite group and let*

$$\rho : G \to \mathrm{GL}(V)$$

be any irreducible representation of G. Then ρ factors through a quotient of G which has cyclic center and acts faithfully via ρ.

Proof. First note that the lemma is true for G abelian: any irreducible representation of a finite Abelian group factors through a cyclic quotient. For general G, let Z be the center of G. Let $W := \mathrm{Res}_Z^G V$. Since Z is central, the Z–isotypic components of W are G–subrepresentations of V. Since V is irreducible as a G-representation, W therefore consists of a single Z-isotypic component.

By the abelian case applied to (the Z-irreducible summand of) W, there is a subgroup $K < Z$ so that Z/K is cyclic and W (as a Z-represenation) factors through a representation of Z/K. Since Z is the center of G we have that K is normal in G. Hence V factors through a representation of G/K, which has cyclic center. By taking a further quotient we can assume that ρ is faithful when restricted to the center. □

Proposition 3.5. *Let F_{2n+1} be a free group of odd rank at least 3. Let $\phi : F_n \to G$ be a surjection onto a 2-step nilpotent group G whose center has the property that each of its nontrivial cyclic subgroups has prime order. Then any irreducible representation of G factors through a quotient H of G so that the induced map $\phi : F_n \to H$ has a primitive element in the kernel.*

Proof of Proposition 3.5. Let V be an irreducible representation of G. By Lemma 3.4 we can assume that G has the form

$$1 \to Z \to G \to Q \to 1$$

where Z is central and cyclic. If Z is trivial, we are done by the Abelian case. Otherwise, by our assumption on the center of G, the order of Z is prime. Since G is 2-step nilpotent, $[G,G] \subset Z$, and therefore in fact $[G,G] = Z$.

We first aim to show that there is a free basis $a_1, \dots, a_{2n+1}$ of F_{2n+1} so that $\phi(a_{2n+1})$ is contained in Z. Namely, consider $\phi([a_1, a_i])$ for $i > 1$. If all of these elements are trivial, then $\phi(a_1) \in Z$ and we are done by relabeling. Otherwise, we can assume that $\phi([a_1, a_2])$ is nontrivial and hence a generator of Z. Next, we can arrange that $\phi([a_1, a_i]) = 1$ for all $i > 2$, by replacing a_i by $a_i a_2^{-k}$ for a suitable k. Similarly, we can arrange that $\phi([a_2, a_i]) = 1$ for all $i > 2$, by replacing a_i by $a_i a_1^{-l}$ for suitable l. Note that

$$\phi([a_1, a_i a_1^{-l}]) = \phi([a_1, a_i])$$

and hence after this modification we have arranged that

$$\phi([a_i, a_j]) = 1 \qquad \text{for any } 1 \leq i \leq 2 < j$$

Furthermore, note that performing Nielsen moves on $a_3, \dots, a_{2n+1}$ does not break this property. We can thus inductively continue, finding pairs $\phi([a_{2r+1}, a_{2r+2}])$ etc. which are nontrivial, but so that $\phi([a_i, a_k]) = 1$ for $k > 2r+2, i \leq 2r+2$. Since $2n+1$ is odd, after at most n steps we have thus found some a_l so that $\phi([a_i, a_l]) = 1$ for all i, and hence $\phi(a_l) \in K$.

If $\phi(a_l) = 1$ we are already done. Otherwise, since G is 2-step nilpotent, $\phi(a_l) \in Z = [G,G]$. Thus, there is some element $M \in [F_{2n+1}, F_{2n+1}]$ so that $\phi(M) = \phi(a_l)^{-1}$.

Next, consider the 2-step nilpotent quotient N of F_{2n+1}. By naturality, we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [F_{2n+1}, F_{2n+1}] & \longrightarrow & F_{2n+1} & \longrightarrow & H_1(F_{2n+1};\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^2 H_1(F_{2n+1};\mathbb{Z}) & \longrightarrow & N & \longrightarrow & H_1(F_{2n+1};\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

We denote by $[a_l]$ the image of a_l in N, and by $m \in \wedge^2 H_1(F_{2n+1})$ the image of M in N.

Note that $[a_l]m$ is the image of a primitive element $x \in F_n$: conjugating a_l by a_i sends $[a_l]$ to $[a_l]([a_l] \wedge [a_i])$; multiplying it by $[a_i, a_k]$ sends it to $[a_l]([a_i] \wedge [a_k])$. By construction (and naturality), we then have that $\phi(x) = 1$. $\square$

Remark 3.6. *The proof of Proposition 3.5 relies on an understanding of the image of primitive elements in the universal* 2*-step nilpotent quotient of a free group, or equivalently the action of the "Torelli group"* IA_n *on that quotient. As such, it is not entirely clear if a version of Proposition 3.5 remains true for groups of higher nilpotence degree (possibly increasing the rank of the free group). Whether or not this is the case is an interesting question for further research.*

Remark 3.7. *The assumption in Proposition 3.5 that the rank of the free group is odd is crucial: it is not true in general that there is a primitive element which maps into the center. However, there does not seem to be a reason to suspect that the conclusion of the proposition should be false for general free groups. Similarly, the condition on the center in Proposition 3.5 is used in the proof, but it is not clear if this assumption is really required.*

We are now able to deduce the following.

Corollary 3.8 **($H_1^{\mathrm{prim}} = H^1$ for certain nilpotent covers).** *Let G be a* 2*-step nilpotent group G whose center has the property that each nontrivial cyclic subgroup has prime order. Let $Y \to X$ be a finite normal G-cover with* $\mathrm{rank}(\pi_1(X)) = 2n+1, n \geq 1$*. Then*

$$H_1(Y;\mathbb{C}) = H_1^{\mathrm{prim}}(Y;\mathbb{C})$$

for any regular cover.

Proof. The assumption of the corollary together with Proposition 3.5 imply that some primitive element in $\pi_1(X)$ lifts to Y. Applying Lemma 2.4 gives the statement of the corollary. □

4 Primitives in the kernel

In this section we explore criteria to detect if homomorphisms $\phi : F_n \to G$ of a free group to a finite group have primitive elements in the kernel.

As indicated in the introduction, finding ϕ that do not have primitives in the kernel is a first step towards finding an example where $\mathrm{Irr}(G) \neq \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$. We will see that there are various group-theoretic obstructions that prevent a group G from having such a map.

4.1 Property KC_i

Here we introduce the following notion, which is a simple (yet sometimes effective!) tool to show that inductively constructed groups do not admit surjections without a primitive in the kernel.

Definition 4.1 **(Property KC_i).** Say that a finite group G has property KC_i (*kernel contains corank i*) if for any surjection

$$\phi : F_n \to G$$

there is a free factor $F < F_n$ of rank at least $n - i$ contained in $\ker \phi$.

We note the following easy consequence of this definition.

Lemma 4.2. *The following statements hold.*

1. *Finite cyclic groups have* KC_1*.*
2. *Every finite group* G *has* $\mathrm{KC}_{|G|+1}$*.*
3. *If* K *has* KC_i *and if* Q *has* KC_j*, and if*

$$1 \to K \to G \to Q \to 1$$

is exact, then G *has* KC_{i+j}*.*

Proof.

1. This is the Euclidean algorithm, as in the proof of the abelian case of Proposition 3.3.
2. This is the pidgeonhole principle: among any $|G|+1$ elements in a free basis at least 2 map to the same element in G; hence a Nielsen move can be applied to send one to the identity in G.
3. This follows from the fact that free factors in free factors are free factors.

□

4.2 p-groups

Recall that the *Frattini subgroup* $\Phi(G)$ of a group G is defined to be the intersection of all proper maximal subgroups of G. Elements of $\Phi(G)$ are often called *non-generators*, since any set generating G and containing such an element still generates G without it.

Theorem 4.3. *Let* G *be a* p*-group and let* F_n *a free group with free basis* $a_1, \dots, a_n$*. Denote by* $\pi : G \to G/\Phi(G)$ *the projection map. Then*

1. *The kernel of a surjection* $f : F_n \to G$ *contains no primitive element if and only if* $\{\pi(f(a_i)), i = 1, \dots, n\}$ *is a vector space basis of* $G/\Phi(G)$*.*
2. *There is a surjection* $f : F_n \to G$ *whose kernel contains no primitive element if and only if* $\dim_{\mathbb{F}_p}(G/\Phi(G)) = n$*.*

Proof. We will need the following classical result.

Lemma 4.4 (Burnside Basis Theorem)**.** *Let* p *be a prime and suppose that* P *is a* p*-group. Then* $V = P/\Phi(P)$ *is an* $\mathbb{F}_p$*–vector space, and :*

1. *A set* $S \subset P$ *generates* P *if and only if its image in* V *generates* V*.*
2. *A set* $S \subset P$ *is a minimal generating set if and only if its image in* V *is a vector space basis of* V*.*

Recall that any primitive element $x_1 \in F_n$ can be extended to a free basis $x_1, \dots, x_n$ (by definition), and any two free bases of F_n are related by a sequence of Nielsen moves. As such, the images $\pi(f(x_i))$ are related to $\pi(f(a_i))$ by a sequence of elementary transformations. Since these preserve the property of being a basis, we conclude that $\{\pi(f(a_i)), i = 1, \dots, n\}$

is a vector space basis of $G/\Phi(G)$ if and only if $\{\pi(f(x_i)), i = 1, \ldots, n\}$ is a vector space basis of $G/\Phi(G)$ for every free basis $x_1, \ldots, x_n$ of F_n.

We now prove the first statement of the theorem. If $\{\pi(f(a_i)), i = 1, \ldots, n\}$ is a vector space basis of $G/\Phi(G)$, then so is $\{\pi(f(x_i)), i = 1, \ldots, n\}$ for any other free basis x_i. In particular, no x_i lies in the kernel of f.

Conversely, suppose that no primitive element lies in the kernel of f, but assume that $\{\pi(f(a_i)), i = 1, \ldots, n\}$ is not a vector space basis of $G/\Phi(G)$. Then we can assume, without loss of generality, that $\pi(f(a_1))$ is a linear combination of $\pi(f(a_i)), i > 1$. In particular, there is a word $w \in F_n$ which only uses the letters $a_i, i > 1$ so that $\pi(f(w)) = -\pi(f(a_1))$. Then, as $a_1 w$ is primitive, there is a free basis $x_1, \ldots, x_n$ so that $\pi(f(x_n)) = 0$. In particular, by Claim 1 of the Burnside Basis Theorem, $f(x_1), \ldots, f(x_n)$ is not a minimal generating set of G (it is a generating set since f is surjective). However this again implies that (up to relabeling) $f(x_1) = f(v), v \in \langle x_2, \ldots, x_n \rangle$. Then $x_1 v^{-1}$ is a primitive element in the kernel of f.

The second statement of the theorem is a straightforward consequence of the first. $\square$

4.3 Connection to the Product Replacement Algorithm

The *Product Replacement Algorithm* is a common method used to generate random elements in finite groups. It is an active topic of research; see [P] or [Lu] for excellent surveys. In this section we sketch a connection between primitive elements in the kernel of a surjection $\pi : F_n \to G$ and the product replacement algorithm for G. It seems quite likely that there are many more avenues for investigation in this direction.

To begin, note that homomorphisms $\phi : F_n \to G$ are in 1–to–1–correspondence to n–tuples of elements in G.

A n–tuple $(g_1, \ldots, g_n)$ is called *redundant* if there is some i so that g_i is contained in the subgroup generated by $(g_1, \ldots, g_{-1}, g_{i+1}, \ldots, g_n)$.

Lemma 4.5. *If $(g_1, \ldots, g_n)$ is redundant, then the corresponding homomorphism $\phi : F_n \to G, \phi(a_i) = g_i$ has a primitive in the kernel. Here, a_i is a free basis of F_n.*

Proof. Assume without loss of generality that $\phi(a_1) \in \langle \phi(a_2), \ldots, \phi(a_n) \rangle$. Let w be a word in $a_2, \ldots, a_n$ so that $\phi(w) = \phi(a_1)^{-1}$. Then $a_1 w$ is a primitive element in the kernel of ϕ. $\square$

The converse to this lemma is false – for example, consider the map

$$\phi : F_2 \to \mathbb{Z}/6\mathbb{Z}, \quad \phi(a_1) = 2, \phi(a_2) = 3.$$

The set $(2, 3)$ is not redundant for $\mathbb{Z}/6\mathbb{Z}$, and yet ϕ contains a primitive element in the kernel $(\phi(a_1(a_2 a_1^{-1})^{-2}) = 0)$.

To clarify the relation between redundant elements and primitives in the kernel of homomorphisms, we need the following definition (see e.g. [P]).

The *Product Replacement Graph* $\Gamma_n(G)$ has a vertex for each n–element generating sets of G, and an edge connecting two generating sets that differ by a Nielsen move (multiply one of the elements of the tuple on the left or on the right by another element or its inverse). The *extended Product Replacement Graph* $\widetilde{\Gamma}_n(G)$ additionally has edges corresponding to inverting an element, or swapping two.

The question as to whether $\Gamma_n(G)$ or $\widetilde{\Gamma}_n(G)$ is connected for various values of n is an extremely challenging question; see Section 2 of [P] for a survey of known results. For us, the importance comes from the following, which is proved exactly like Theorem 4.3.

Lemma 4.6. *Let $\phi : F_n \to G$ be a surjective homomorphism, and $(\phi(a_1), \dots, \phi(a_n))$ the corresponding n–tuple. Then ϕ has a primitive element in the kernel if and only if the connected component of $\widetilde{\Gamma}_n(G)$ containing $(\phi(a_1), \dots, \phi(a_n))$ also contains a redundant tuple.*

Corollary 4.7. *Suppose that $\widetilde{\Gamma}_n(G)$ is connected. Then there is a surjection $f : F_n \to G$ with no primitive element in the kernel if and only if G cannot be generated by fewer than n elements.*

In particular, connectivity results are known for solvable groups and one obtains the following Corollary of a theorem of Dunwoody [P, Theorem 2.3.6]

Corollary 4.8. *Suppose that G is finite and solvable, and can be generated by k elements. Then, for any $n > k$ every surjection $f : F_n \to G$ has a primitive element in the kernel.*

Proof. By Dunwoody's theorem, $\Gamma_n(G)$ is connected as $n \geq d(G) + 1$. Since G can be generated by k elements, there is a redundant generating n–tuple. Hence, we are done by Lemma 4.6. □

We emphasize that the conclusion of this corollary is really nontrivial. The fact that G can be generated by fewer than n elements does not necessarily imply that *every* n–element generating set is redundant (compare again the example of $\mathbb{Z}/6\mathbb{Z}$ above). This discrepancy is exactly recorded by the Product Replacement Graph.

If the minimal size of a generating set for G is n, connectivity of $\Gamma_n(G)$ is particularly delicate. This seems to be the most interesting scenario from the point of view of trying to obtain surjections without primitives in the kernel.

5 Free Linear Actions

In this section we discuss groups which act freely and linearly on spheres. The importance of such groups to our goal stems from the following.

Lemma 5.1. *Suppose that G acts freely and linearly on some sphere. Then for the cover $Y \to X$ of a n-petal rose X defined by $\phi : F_n \to G$ we have*

$$H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C}) \quad \Leftrightarrow \quad \ker(\phi) \textit{ contains a primitive element}$$

Proof. Suppose that $\ker(\phi)$ does not contain a primitive element. Then, by assumption on G, there is a irreducible representation V of G where no element $1 \neq g$ fixes any vector. Hence, $V \neq \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$, and therefore $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$ by Theorem 1.4. The other direction is immediate from Lemma 2.4. □

Using the methods developed above we can show that (at least for large n) this does indeed always occur. Namely, we have the following.

Proposition 5.2. *There is a number $B \geq 3$ with the following property. Suppose that G is any group which admits a (complex) representation in which no non-identity element has a fixed vector. If $n \geq B$ and $\phi : F_n \to G$ is any surjection, then the kernel of ϕ contains a primitive element in F_n. In other words,* $\mathrm{Irr} = \mathrm{Irr}^{\mathrm{pr}}$ *for all such groups, and* $H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$ *for the associated covers.*

The proof of this ultimately relies on the classification of finite groups which act freely and linearly on spheres.

Proof of Proposition 5.2. Let G be as in the proposition. By the classification of finite groups acting freely linearly on spheres (compare page 233 of [DM] for the definition of types, and Theorem 1.29 to restrict the quotients in cases V, VI), there is a normal subgroup $N < G$ with the following properties:

1. N is metabelian, i.e. an split extension of a cyclic by a cyclic group.
2. G/N is either an extension of a cyclic by a cyclic group, or equal to one out of a list of four possible finite groups $Q_1, \ldots, Q_4$.

Thus the existence of the number B as desired follows from Lemma 4.2. $\square$

Remark 5.3. *While the proof of Proposition 5.2 given above is nonconstructive, we expect the minimal value for B to be fairly low (in fact, 3 or 4). Computer experiments (see below) are in agreement with this.*

6 Algorithms

In this section we explain that the question if $\mathrm{Irr}(G) = \mathrm{Irr}^{\mathrm{pr}}(\phi, G)$ for any given finite group G and surjection $\phi : F_n \to G$ is algorithmic, supposing that the irreducible representations are understood. The main ingredient here is the following:

Proposition 6.1. *Given a homomorphism $q : F_n \to G$ of a free group to a finite group G, there is an algorithm which computes the set of all elements in G which are the image of primitive elements in F_n in finite time.*

Proof. To begin, note that every primitive element in F_n is the image of the first standard free generator of F_n under some element of $\mathrm{Aut}(F_n)$. Additionally, $\mathrm{Aut}(F_n)$ is generated by Nielsen moves. Thus suggests the following algorithm:

- Start with a set L of elements in G^n, containing the images of the standard basis $q(a_1), \ldots, q(a_n)$ under q.
- For every element in L, apply all basic Nielsen moves to the tuple. Add all resulting tuples to L (unless they were already contained in L).
- If L grew in size during the last step, repeat the last step.
- The desired set of images is now the set of all entries of tuples in L.

To see that the resulting list actually contains every image of a primitive, note that if $g = q(x)$ for some primitive, then there is some sequence $M_1 \to M_2 \to \cdots \to M_k$ of Nielsen moves of the standard free basis $a_1, \ldots, a_k$ ending in a basis $x_1, \ldots, x_k$ containing x. Now observe that by construction every image of $q(a_1), \ldots, q(a_k)$ under any sequence of moves M_i is contained in L; thus $q(x)$ is one of the entries of a tuple in L. $\square$

We stress that we do not claim that the algorithm is particularly fast. Also note that [CG] contains a different algorithm that works for non-regular covers, but seems harder to implement in practice.

For groups as discussed in Section 5 experiments with the above algorithm show that already for a free group of rank 3, any surjection contains a primitive element for all groups of order at most 1000 that can act freely on spheres.

With Proposition 6.1 in hand, the claim made at the beginning of this section is immediate: given $\phi : F_n \to G$ one can first compute the images of all primitive elements of F_n under ϕ in G. For each irreducible representation one can then check if any of the associated matrices have eigenvalue 1.

7 Rank 2 examples

In this section we give various examples with $\mathrm{rank}(\pi_1(X)) = 2$ and where $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) \subsetneq \mathrm{Irr}(G)$ and $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \subsetneq H^1(Y;\mathbb{C})$. These show in particular that our results assuming $\mathrm{rank}(\pi_1(X)) \geq 3$ are sharp.

7.1 A group acting freely on a sphere

Proposition 7.1. *There are surjections $\phi : F_2 \to G$ with the following two properties:*

i) The group G acts freely and linearly on a sphere S^N.

ii) No primitive element of F_2 is contained in the kernel of ϕ.

In particular, for the associated cover we have $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \neq H^1(Y;\mathbb{C})$.

Proof. Consider the group G defined by the relations

$$G = \langle a, b | a^3 = 1, b^8 = 1, bab^{-1} = a^2 \rangle$$

This group appears as Type i), with parameters $n = 8, m = 3, r = 2$ in the list in [N], and hence acts linearly and freely on some sphere.

Using the algorithm described in Section 6, one can check that the map $q : F(a, b) \to G$ has no primitive element in the kernel. Hence $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \neq H^1(Y;\mathbb{C})$ for the associated cover. $\square$

7.2 A 2-step nilpotent group

We now give examples which show that the obstruction given by $\mathrm{Irr}^{\mathrm{pr}}(\phi, G)$ is indeed stronger than requiring that G act linearly and freely on spheres.

Proposition 7.2. *There is a surjection $\phi : F_2 \to G$, and a representation ρ of G with the following properties:*

i) No primitive element of F_2 is contained in the kernel of ϕ.

ii) For every primitive element $x \in F_2$, the matrix $\rho(\phi(x))$ does not have eigenvalue 1.

iii) There is some some $y \in F_n$, so that $\rho(\phi(y))$ is the identity.

In particular, we have $\mathrm{Irr}^{\mathrm{pr}}(\phi, G) \neq \mathrm{Irr}(G)$ and for the associated cover we have $H_1^{\mathrm{prim}}(Y;\mathbb{C}) \neq H^1(Y;\mathbb{C})$, yet G does not act freely and linearly on a sphere via ρ.

Proof. Consider the 2-step nilpotent group Γ which fits into the sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma \to \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \to 1$$

and is obtained from the canonical mod-4, 2-step nilpotent quotient of F_2 by taking a further rank 2 quotient of the center.

Thus, there is a surjection $\phi : F_2 \to \Gamma$ which induces the mod-4-homology map $F_2 \to H_1(F_2;\mathbb{Z}/4\mathbb{Z})$ in abelianizations. Γ is generated by two elements α, β which are images of a free basis a, b of F_2.

Consider the representation

$$\rho : \Gamma \to GL(\mathbb{C}^2)$$

given by

$$\alpha \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Applying the algorithm described in Section 6, one checks that if $x \in F_2$ is primitive, then $\rho(\phi(x))$ does not have eigenvalue 1. Hence, the representation ρ is not an element of $\mathrm{Irr}^{\mathrm{pr}}(\phi, \Gamma)$. However, $\phi(a^2[a,b]) \neq 1$, yet $\rho(\phi(a^2[a,b])) = 1$, so ρ is not faithful. In particular, Γ does not act on a sphere linearly and freely via ρ. This proves the proposition. □

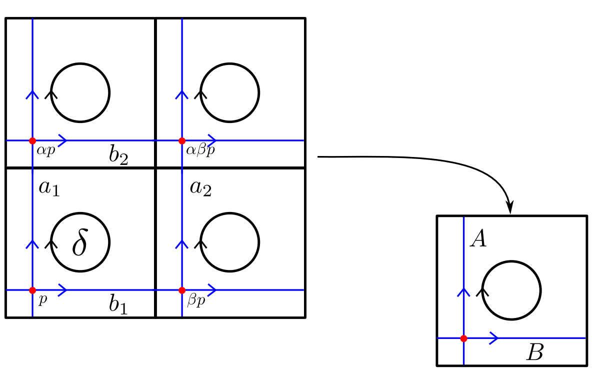
7.3 Torus homology cover

The goal of this section is to give a more geometric construction of a torus cover with nontrivial primitive homology. Namely, we will show

Proposition 7.3. *There is a cover $Y \to T$ of the torus T with one boundary component, which is obtained as an iterated homology cover, and where*

$$H_1^{\mathrm{prim}}(Y;\mathbb{C}) \neq H_1(Y;\mathbb{C})$$

Choose a basepoint on the torus T^2 and identify the fundamental group based at that point with the free group on A, B. We first consider the mod-2 homology cover $X \to T^2$. Denote by α, β the deck group elements corresponding to A, B respectively.



The homology $H_1(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^5$; we choose an explicit basis

$$a_1, a_2, b_1, b_2, \delta$$

where a_1, a_2 are the two components of the preimage of A, and b_1, b_2 are the two components of the preimage of B, and δ is a lift of the boundary component (to the preferred basepoint).

Lemma 7.4. *This is indeed a basis.*

Proof. We show that is a generating set. Namely, we have $a_1 - a_2 = \delta + \alpha\delta$, and hence $\alpha\delta$ is in the span V of the set in question. Similarly, $\beta\delta$ is in V, and since $\delta + \alpha\delta + \beta\delta + \alpha\beta\delta = 0$ all four boundary components are in V. Furthermore, a_1, b_1 correspond two curves intersecting once, and hence V is everything. $\square$

Lemma 7.5. *The deck transformations α, β act as vertical (resp. horizontal) translations on X. Their matrices with respect to our basis are*

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Proof. As α acts as a vertical translation, it fixes both a_1, a_2 and interchanges b_1, b_2. We have

$$\delta + \alpha\delta + a_2 - a_1 = 0$$

as those four curves are the boundary of the "left" two-holed annulus. This explains the first matrix. Similarly, β acts as a horizontal translation. Thus, b_1, b_2 are fixed and a_1, a_2 are exchanged. We have

$$\delta + \beta\delta + b_1 - b_2 = 0$$

as those four are the boundary of the "lower" two-holed annulus. This explains the second matrix. $\square$

To analyse which elements of homology are components of lifts of simple closed curves, we use the following lemma.

Lemma 7.6. *If $x \in H_1(X;\mathbb{Z}/p^n\mathbb{Z})$ is equal to a multiple of a component of the preimage of a nonseparating simple closed curve, then one of $\alpha x, \beta x, \alpha\beta x$ is equal to x and*

$$\dim_{\mathbb{F}_{p^n}} \mathrm{Span}\{x, \alpha x, \beta x, \alpha\beta x\} = 2.$$

Proof. Suppose that $x = [\widetilde{\gamma}]$ is defined by a component $\widetilde{\gamma}$ of the preimage of a simple closed curve γ. First note that there is a mapping class φ of T^2 and a lift $\widetilde{\varphi}$ so that $\widetilde{\varphi} = a_1$. Namely, we can choose φ so that $\varphi(\gamma) = A$. Then, for any lift $\widetilde{\varphi}$ we have that $\widetilde{\varphi}(\widetilde{\gamma})$ is a component of the preimage of A. We can choose a lift so that this component is a_1.

Since $\widetilde{\varphi}$ is equivariant with respect to the deck group action, it therefore maps the G-span of x to the G-span of a_1. In particular, the G-span of x has rank 2. Since a_1 is fixed by α, we have that x is fixed by $\widetilde{\varphi}_*^{-1}(\alpha)$ as claimed. $\square$

Consider an element

$$x = r_1a_1 + r_2a_2 + s_1b_1 + s_2b_2 + d\delta$$

and its orbit under G:

$$\begin{aligned}
\alpha x &= (r_1 + d)a_1 + (r_2 - d)a_2 + s_2b_1 + s_1b_2 - d\delta \\
\beta x &= r_2a_1 + r_1a_2 + (s_1 - d)b_1 + (s_2 + d)b_2 - d\delta \\
\alpha\beta x &= (r_2 - d)a_1 + (r_1 + d)a_2 + (s_2 + d)b_1 + (s_1 - d)b_2 + d\delta
\end{aligned}$$

We collect the coefficients in the matrix:

$$\begin{pmatrix}
r_1 & r_2 & s_1 & s_2 & d \\
r_1 + d & r_2 - d & s_2 & s_1 & -d \\
r_2 & r_1 & s_1 - d & s_2 + d & -d \\
r_2 - d & r_1 + d & s_2 + d & s_1 - d & d
\end{pmatrix}$$

If x is defined by a component of the lift of a simple closed curve, then by Lemma 7.6 the first row has to be equal to one of the other rows. We distinguish cases, depending on which rows are equal.

Row 1 equal to Row 2 This immediately implies $d = 0$ and then $s_1 = s_2$. This leaves us with the matrix

$$\begin{pmatrix}
r_1 & r_2 & s_1 & s_1 & 0 \\
r_1 & r_2 & s_1 & s_1 & 0 \\
r_2 & r_1 & s_1 & s_1 & 0 \\
r_2 & r_1 & s_1 & s_1 & 0
\end{pmatrix}$$

which needs to have rank 2. There are two subcases: if $s_1 \neq 0$ then the matrix has rank 2 if and only if $r_1 \neq r_2$. If $s_1 = 0$, then the matrix has rank 2 if and only if $r_1 \neq \pm r_2$.

$d = 0$, $s_1 = s_2$, $r_1 \neq r_2$. If $s_1 = s_2 = 0$ then also $r_1 \neq -r_2$.

Row 1 equal to Row 3 This immediately implies $d = 0$ and then $r_1 = r_2$. This leaves us with the matrix

$$\begin{pmatrix}
r_1 & r_1 & s_1 & s_2 & 0 \\
r_1 & r_1 & s_2 & s_1 & 0 \\
r_1 & r_1 & s_1 & s_2 & 0 \\
r_1 & r_1 & s_2 & s_1 & 0
\end{pmatrix}$$

which needs to have rank 2. There are two subcases: if $r_1 \neq 0$ then the matrix has rank 2 if and only if $s_1 \neq s_2$. If $r_1 = 0$, then the matrix has rank 2 if and only if $s_1 \neq \pm s_2$.

$d = 0$, $r_1 = r_2$, $s_1 \neq s_2$. If $r_1 = r_2 = 0$ then also $s_1 \neq -s_2$.

Row 1 equal to Row 4 This leads to $r_1 = r_2 - d$ and $s_1 = s_2 + d$. Simplifying the matrix yields

$$\begin{pmatrix} r_1 & r_1+d & s_1 & s_1-d & d \\ r_1+d & r_1 & s_1-d & s_1 & -d \\ r_1+d & r_1 & s_1-d & s_1 & -d \\ r_1 & r_1+d & s_1 & s_1-d & d \end{pmatrix}$$

If $d = 0$ this has rank 1, so this case never happens. The matrix has rank ≤ 1 if and only if the first two rows are dependent. As $d \neq 0$ the only way this can happen is if the second row is the negative of the first row. This leads to

$$-r_1 = r_1 + d, -s_1 = s_1 - d$$

Hence the matrix has rank 2 if one of the above is false, leading to

$$\boxed{d \neq 0,\ r_1 = r_2 - d,\ s_1 = s_2 + d \text{ and } d \neq -2r_1 \text{ or } d \neq 2s_1.}$$

Consequence

Lemma 7.7. *The representation of* $(\mathbb{Z}/p)^5$ *defined by*

$$\rho(r_1, r_2, s_1, s_2, d) = \left(z \mapsto \zeta_p^{r_1 - r_2 + s_1 - s_2 + d} z\right)$$

has the property that no $x \in (\mathbb{Z}/p)^5$ *which is equal to the component of a preimage of a simple closed curve acts as the identity.*

Proof. We consider in turn the three possibilities for such x. All exponents are seen as elements of $\mathbb{Z}/p$. The equalities and inequalities also are in this field.

1. $d = 0$, $s_1 = s_2$, $r_1 \neq r_2$. If $s_1 = s_2 = 0$ then also $r_1 \neq -r_2$.

   In that case the element acts as

   $$z \mapsto \zeta_p^{r_1 - r_2 + s_1 - s_2 + d} = \zeta_p^{r_1 - r_2} z \neq z$$

2. $d = 0$, $r_1 = r_2$, $s_1 \neq s_2$. If $r_1 = r_2 = 0$ then also $s_1 \neq -s_2$.

   In that case the element acts as

   $$z \mapsto \zeta_p^{r_1 - r_2 + s_1 - s_2 + d} = \zeta_p^{s_1 - s_2} z \neq z$$

3. $d \neq 0$, $r_1 = r_2 - d$, $s_1 = s_2 + d$ and $d \neq -2r_1$ or $d \neq 2s_1$.

   In that case the element acts as

   $$z \mapsto \zeta_p^{r_1 - r_2 + s_1 - s_2 + d} = \zeta_p^{d} z \neq z$$

□

The following corollary finishes the proof of the main proposition of this section.

Corollary 7.8. *Consider the group G defined by the extension*

$$1 \to (\mathbb{Z}/p)^5 \to G \to (\mathbb{Z}/2)^2 \to 1$$

via iterated homology covers of G, and the corresponding $\phi : F_2 \to G$. Then there is a irreducible representation V of G so that no image of a simple closed curve in G fixes any vector.

Proof. We consider the representation induced from the representation ρ of $(\mathbb{Z}/p)^5$ to G. While this may not be irreducible, it does have the property that no image of a simple closed curve has any fixed vectors. Namely, suppose that v would be fixed by the image of γ. Then, for γ^n the smallest power so that γ^n maps into $(\mathbb{Z}/p)^5$, a component of v would also have a fixed vector. By the previous lemma, this is impossible unless $v = 0$. □

8 The case of surfaces and simple closed curves

There is a close analogy of our discussion for finite regular covers $f : Y \to X$ where X is a genus $g \geq 2$ surface. In this section we sketch this analogy.

8.1 Simple homology

The analog for surfaces of primitive homology for graphs is the following.

Definition 8.1 (simple homology). Let $f : Y \to X$ be a finite cover. The *simple closed curve homology* (or *simple homology*) $H_1^{\mathrm{scc}}(Y;\mathbb{C})$ corresponding to this finite cover is defined to be:

$$H_1^{\mathrm{scc}}(Y;\mathbb{C}) := \mathbb{C}\text{-span}\{W_\gamma : \gamma \in \pi_1(X) \text{ is a simple closed curve}\}$$

Using work of Putman-Wieland [PW], Boggi-Looijenga [BL] conjecture that vanishing of the virtual first Betti numbers of the moduli spaces of Riemann surfaces (which is a well-known open question of Ivanov; see [PW] for precise statements) is closely related to the question if $H_1(Y;\mathbb{C}) = H_1^{\mathrm{scc}}(Y;\mathbb{C})$.

Again, the very basic question if simple homology is ever a proper subrepresentation is at the current point wide open. For homology with integral coefficients, examples have been constructed where simple homology is a proper subspace; see [I, KS]. For the covers Y constructed in [KS], it is not known if $H_1^{\mathrm{scc}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$.

Here, we want to again study a representation-theoretic version of simple homology.

Definition 8.2 ($\mathbf{Irr}^{\mathbf{scc}}(\boldsymbol{\phi}, \boldsymbol{G})$). Let G be a finite group. Fix a homomorphism $\phi : \pi_1(X) \to G$. Let $\mathrm{Irr}^{\mathrm{scc}}(\phi, G)$ denote the set of irreducible representations V of G with the property that $\phi(\gamma)(v) = v$ for some element γ which can be represented by a simple closed curve and some nonzero $v \in V$.

We will prove the analog of Theorem 1.4 for surfaces.

Theorem 8.3 (Restricting simple closed curve homology). *Let $f : Y \to X$ be a regular G-covering of surfaces defined by $\phi : \pi_1(X) \to G$, where G is a finite group and X has genus at least 2. Then*

$$H_1^{\mathrm{scc}}(Y;\mathbb{C}) \subseteq \bigoplus_{V_i \in \mathrm{Irr}^{\mathrm{scc}}(\phi,G)} V_i^{(2g-2)\cdot\dim(V_i)} \oplus \mathbb{C}^2_{triv}$$

As with primitive homology for graphs, one can understand covers with particularly easy deck groups. For example, one has

Proposition 8.4. *Let $f : Y \to X$ be a regular G-covering of surfaces for a finite Abelian group G. Then*

$$H_1^{\mathrm{scc}}(Y;\mathbb{C}) = H_1(Y;\mathbb{C})$$

This can be shown fairly immediately using the point of view used in [L]: one can first reduce to cyclic covers, as every representation of an Abelian group factors through a cyclic quotient. For such surfaces, one then explicitly shows which curves one needs to lift (compare also [I]).

As in the case of primitive homology one can perform computer experiments to test if covers satisfy the representation–theoretic obstruction for groups acting freely and linearly on spheres. The examples from Propostion 7.1 still apply; but the case of simple closed curves is more restrictive than the primitive case, as the following shows (compare the end of the section for a sketch of the proof):

Proposition 8.5. *Let $\Sigma_{1,2}$ be a torus with two marked points. There are surjections $\phi : \pi_1(\Sigma_{1,2}) \to G$ with the following two properties:*

i) The group G acts freely and linearly on a sphere S^N.

ii) No element of $\pi_1(\Sigma_{1,2})$ representing a simple closed curve is contained in the kernel of q.

iii) There are primitive elements contained in the kernel of ϕ.

In particular, we have $\mathrm{Irr}^{\mathrm{scc}}(\phi, G) \neq \mathrm{Irr}(G)$ and for the associated cover we have $H_1^{\mathrm{scc}}(Y;\mathbb{C}) \neq H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H^1(Y;\mathbb{C})$.

On the contrary, already for closed genus 2 surfaces we were unable to find with computer experiments an example with $\mathrm{Irr}^{\mathrm{scc}}(\phi, G) \neq \mathrm{Irr}(G)$. Note that in contrast to the primitive case it is not clear if this condition suffices to ensure that $H_1^{\mathrm{scc}}(Y;\mathbb{C}) = H^1(Y;\mathbb{C})$.

Proof of Theorem 8.3. We follow with the same argument as in Section 2, with the following lemma replacing Step 1b.

Lemma 8.6. *Let F be a closed surface with no boundary components. Let $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ be disjoint pairwise nonisotopic simple closed curves on F. Let $R_1, \ldots, R_m$ be the set of subsurfaces bounded by $\hat{\alpha}_1 \cup \ldots \cup \hat{\alpha}_n$. We have an exact sequence*

$$0 \to \mathbb{C}\left[\sum \partial R_j\right] \to \bigoplus_{j=1}^{m} \mathbb{C}\left[\partial R_j\right] \to \bigoplus_{i=1}^{n} \mathbb{C}\left[\hat{\alpha}_i\right] \to \mathrm{Span}_{H_1(X)}\left\{\hat{\alpha}_1, \ldots, \hat{\alpha}_n\right\} \to 0$$

where the maps are induced by the natural identifications of multicurves in the various direct sums.

Proof. We check exactness at the various places individually. The surjectivity of $\phi : \bigoplus_{i=1}^{n} \mathbb{C}\left[\hat{\alpha}_i\right] \to \mathrm{Span}_{H_1(X;\mathbb{C})}\left\{\hat{\alpha}_1, \ldots, \hat{\alpha}_n\right\}$ is clear by definition.

Next, consider the image of $\psi : \bigoplus_{j=1}^{m} \mathbb{C}\left[\partial R_j\right] \to \bigoplus_{i=1}^{n} \mathbb{C}\left[\hat{\alpha}_i\right]$. The fact that the image of ψ lies in the kernel of ϕ follows since each generator ∂R_j of $\mathbb{C}\left[\partial R_j\right]$ is mapped to a boundary by definition.

To see that the image of ψ is the complete kernel of ϕ, suppose that some linear combination of the $\hat{\alpha}_i$ is zero in homology:

$$a_1[\hat{\alpha}_1] + \ldots + a_n[\hat{\alpha}_n] = 0$$

First note that if the multicurve $\hat{\alpha}_1 \cup \ldots \cup \hat{\alpha}_n$ is nonseparating, then all a_i are zero, and there is nothing to show.

If not, then our goal is to use the relations $\psi(\partial R_j)$ to modify the relation so that the involved curves become nonseparating. To this end, consider the subsurface R_1. We can assume that there is a curve $\hat{\alpha}_i$ which appears only once in the boundary of R_1; if there is no such curve, then R_1 itself is the complement of $\hat{\alpha}_1 \cup \ldots \cup \hat{\alpha}_n$ and the latter is nonseparating.

Let now R_2 be the subsurface on the other side of $\hat{\alpha}_i$. Using the element $\psi(\partial R_2)$ to modify the linear combination, we can change the coefficient a_i of $\hat{\alpha}_i$ to be 0. We interpret the resulting linear combination as one involving only the curves $\hat{\alpha}_i, j \neq i$, where now $R_1 \cup R_2$ in place of R_1 is one of the complementary subsurfaces of the relation $a_2[\hat{\alpha}_2] + \ldots + a_n[\hat{\alpha}_n] = 0$. Induction on the complexity of R_1 now yields the claim.

Injectivity of $\rho : \mathbb{C}\left[\sum \partial R_j\right] \to \bigoplus_{j=1}^m \mathbb{C}\left[\partial R_j\right]$ is clear. The fact that the image of ρ is exactly the kernel of ψ follows since each $\hat{\alpha}_i$ appears exactly twice amongst the curves appearing in ∂R_j (with opposite signs): on the one hand this immediately implies that the sum of the ∂R_j maps to 0 under ψ. On the other hand, consider any linear combination of a proper subset of the ∂R_j. In that case, there is at least one $\hat{\alpha}_i$ which appears in exactly one of the ∂R_j of the subset. For the combination to be in the kernel of ψ the coefficient of that ∂R_j needs to be 0. Inductively it now follows that the combination is trivial. $\square$

This completes the proof of the theorem. $\square$

8.2 Algorithms

It is a basic fact in surface topology that every nonseparating simple closed curve is the image of the first standard generator under a suitable element of the mapping class group. Since the mapping class group is generated by finitely many (explicit) elements, one can adapt the algorithm given in Section 6 to yield the following.

Proposition 8.7. *Given a homomorphism $q : \pi_1(\Sigma) \to G$ of the fundamental group of a surface to a finite group G, there is an algorithm which computes the set of all elements in G which are the image of elements representable by nonseparating simple closed curves.*

Let $\Sigma_{1,2}$ be a torus with two marked points. Define a group G via the relations

$$G = \langle a, b, c | a^3 = 1, b^4 = 1, bab^{-1} = a^2, c^2 = b^2, cac^{-1} = a, cbc^{-1} = b^3 \rangle$$

This group appears as Type ii), with parameters $n = 4, m = 3, r = 2, l = 1, k = 3$ in the list in [N], and hence acts linearly and freely on some sphere.

We choose a basis a, b, c of $\pi_1 \Sigma_{1,2}$ of simple closed curves, where a surrounds one of the punctures, and b, c generate $H_1(\Sigma_{1,2})$. Let $q : \pi_1(\Sigma_{1,2}) \to G$ be the obvious map induced by the labels.

Using the algorithms described, one can verify:

i) The group G acts freely and linearly on a sphere S^N.

ii) No element of $\pi_1(\Sigma_{1,2})$ representing a simple closed curve is contained in the kernel of q.

iii) There are primitive elements contained in the kernel of q.

In particular, for the associated cover we have

$$H_1^{\mathrm{scc}}(Y;\mathbb{C}) \neq H_1^{\mathrm{prim}}(Y;\mathbb{C}) = H^1(Y;\mathbb{C})$$

This shows Proposition 8.4.

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