On being Thurstonized

Being a Thurston student was inspiring and frustrating – often both at once. At our second meeting I told Bill that I had decided to work on understanding fundamental groups of negatively curved manifolds with cusps. In response I was introduced to the famous "Thurston squint", whereby he looked at you, squint his eyes, give you a puzzled look, then gaze into the distance (still with the squint). After two minutes of this he turned to me and said: "Oh, I see, it's like a froth of bubbles, and the bubbles have a bounded amount of interaction." Being a diligent graduate student, I dutifully wrote down in my notes: "Froth of bubbles. Bounded interaction." After our meeting I ran to the library to begin work on the problem. I looked at the notes. Froth? Bubbles? Is that what he said? What does that mean? I was stuck.

Three agonizing years of work later I solved the problem. It's a lot to explain in detail, but if I were forced to summarize my thesis in five words or less, I'd go with: "Froth of bubbles. Bounded interaction."

A Thurston lecture would typically begin by Bill drawing a genus 4 surface, slowly erasing a hole, adding it back in, futzing with the lines, and generally delaying things while he quickly thought up the lecture he hadn't prepared. Why did we all still attend? The answer is that once in a while we would receive a beautiful insight that was absolutely unavailable via any other source.

Here's an example. Consider a Tinker Toy set of rigid unit-length rods, bolts and hinges. Rods can have one end bolted to a table and can hinged to each other. For any given Tinker Toy T, bolted down on a table at one point, we have the space $C(T)$ of all possible configurations of T. If T is a single rod then $C(T)$ is a circle. If one adds a hinged rod on the end of T, the resulting configuration space is the torus. What other smooth, compact manifolds can you get with this method? I still remember the communal thrill when Bill explained to us how to obtain all compact, smooth manifolds as a component of some $C(T)$. Further, every smooth map between manifolds can actually be represented via some rods connecting the two associated tinker toys.

Thurston completely transformed several areas of mathematics, including 3-manifold theory, foliation theory, geometric group theory, and the theory of rational maps. His papers contain a dizzying array of deep, original, influential ideas. All of this is well known. However, in my opinion Thurston's influence is underrated: it goes far beyond the (enormous) content of his mathematics. As Bill wrote in his paper "On proof and progress in mathematics":

"What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds."

We did learn his ways of thinking, or at least some approximation of them. Bill changed our idea of what it means to "encounter" and "interact with" a mathematical object. The phrase "I understand X" has taken a whole new meaning. Mathematical symbols and even pictures are not sufficient for true understanding, especially in geometry and topology. We must strive to live somehow inside the objects we study, to experience them as 3-dimensional beings. I think that this change is now almost invisible; it has become a structural feature of the way many of us do mathematics.[1] This kind of pervasive influence can be likened to the way that Grothendieck changed the way many people think about mathematics, even on topics Grothendieck himself never touched.

The change in viewpoint described above was taken beyond topology by many of Thurston's students, who went out and "Thurstonized" a number of other areas of mathematics, changing those areas in a notable way. Oded Schramm's work is a case in point. Early in his career Schramm solved many of the major open problems about circle packings. This theory gives a way to really understand (in the Thurstonian sense) the Riemann Mapping Theorem as the limit of an iterative process. Schramm then moved on to apply his geometric insight to understand the scaling limit for many two-dimensional lattice models in statistical physics. The Schramm-Loewner evolution gives a geometric, "what it looks like" understanding of these limits.

Bill was probably the best geometric thinker in the history of mathematics. Thus it came as a surprise when I found out that he had no stereoscopic vision, that is, no depth perception. Perhaps the latter was responsible somehow for the former? I once mentioned this theory to Bill. He disagreed with it, claiming that all of his skill arose from his decision, apparently as a first grader, to "practice visualizing things" every day.

I can't end this note without addressing a fundamental misunderstanding that many people seem to have about Thurston's work. In particular the completeness of proofs in his later work have been questioned. Such complaints are not justified. One can point to Thurston's sometimes lack of proper attributions, and to some brevity in his mathematical arguments. But for the most part he gave complete, albeit concise, proofs. Whenever I asked Bill to explain a theorem to me, he was always willing and able to provide as much detail as I needed, right down to the epsilons. Thurston is one of the few mathematicians I know who never (as far as I can tell) wrote any incorrect statement, or made a conjecture that turned out to be wrong. What people without personal knowledge have picked up on is the frustration one can feel in not understanding what Bill was trying to communicate, and the desire for more detail, only to realize after understanding things that the details were there all along.

I had a rocky relationship with Bill. However, like so many other people, my mathematical viewpoint was shaped by his way of thinking. In interacting with other mathematical greats, one gets the feeling that these people are like us but just 100 (ok, 500) times better. In contrast, Thurston was a singular mind. He was an alien. There is no multiplicative factor here; Thurston was simply orthogonal to everyone. Mathematics loses a dimension with his death.

[1] This reminds me of the story of the old fish who passes by two young fish and says: "Morning, boys. How's the water?" The two young fish look at each other, and one asks the other : "What the heck is water?"