Math 312, Autumn 2007
Problem Set 1

Rudin: pp 31–32, 1, 5, 6, 9, 12

Exercise 1 *Consider the symmetric group S_n, i.e., the set of permutations on $\{1, 2, \ldots, n\}$. Consider the probability measure on S_n that assigns measure $1/n!$ to each permutation σ. For each permutation, let X denote the number of fixed points, i.e., the number of j such that $\sigma(j) = j$. Compute $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$. (Hint: $X = X_1 + \cdots + X_n$ where X_j is the indicator function of the event $\{\sigma(j) = j\}$.)*

Exercise 2 *Call I a* basic interval *in $[0, 1)$ if $I = \emptyset$, or $I = [a, b)$ for some $0 \leq a < b \leq 1$.*

1. *Let $\mathcal{F}_0$ be the set of all sets A of the form*

$$A = A_1 \cup \cdots \cup A_n$$

where n is a positive integer and $A_1, \ldots, A_n$ are disjoint basic intervals. Show that $\mathcal{F}_0$ is an algebra but not a σ-algebra.

2. *Show that $\sigma(\mathcal{F}_0)$ is the Borel subsets of $[0, 1)$.*

3. *Let $F : \mathbb{R} \to [0, \infty)$ be an increasing (i.e., nondecreasing) function. Define μ on $\mathcal{F}_0$ by $\mu(\emptyset) = 0$, $\mu([a, b) = F(b-) - F(a-)$ and if $A = A_1 \cup \cdots \cup A_n$, with $A_1, \ldots, A_n$ disjoint then*

$$\mu(A) = \mu(A_1) + \cdots + \mu(A_n).$$

Show that μ is well defined and countably additive on $\mathcal{F}_0$.

4. *We will show that this implies that μ can be extended uniquely to the Borel subsets. Assume this. What is $\mu(\{a\})$?*

Exercise 3 *Suppose $\mathcal{F}$ is a collection of subsets of a set X containing the empty set, closed under complementation, and satisfying the following: if $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

1. *Show that if $\mathcal{F}$ is also an algebra, then $\mathcal{F}$ is a σ-algebra.*

2. *Does $\mathcal{F}$ have to be an algebra?*

Exercise 4 *Suppose $f : \mathbb{R} \to [0, \infty)$ is a continuous function satisfying*

$$\int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty.$$

Let

$$F(s) = \int_{-\infty}^{\infty} cos(sx)\, f(x)\, dx.$$

Show that F is differentiable and compute the derivative. (Hint: use the dominated convergence theorem to justify the interchange of derivative and integral.)

Exercise 5 *If X is a set, then an outer measure on X is a function μ^* from the set of subsets of X to $[0, \infty]$ satisfying: $\mu^*(\emptyset) = 0$; if $A \subset B$. then $\mu^*(A) \leq \mu^*(B)$; and countable subadditivity, i.e., for all $A_1, A_2, \ldots,$*

$$\mu^* \left[\bigcup_{n=1}^{\infty} A_n \right] \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Given an outer measure, let $\mathcal{F}$ denote the collection of all subsets A such that for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Show that $\mathcal{F}$ is a σ-algebra. (Hint: it is probably easiest to first show it is an algebra and then to show it is closed under disjoint countable unions.)