Math 312, Autumn 2007
Problem Set 2

Rudin: Chapter 2: 3,7,8,9,13,15,24,25

Exercise 1 *Given an example of an algebra $\mathcal{F}$ on a set X and a countably additive measure μ on $\mathcal{F}$ for which there is more than one extension of μ to $\sigma(\mathcal{F})$ that is a measure. (Recall that this implies that μ is not σ-finite.) In your example, state which extension arises from the outer measure as in the proof of the Carathéodory Extension Theorem.*

Exercise 2 *Does there exist a countably additive probability measure on the Borel subsets of $[0, 1)$ such that every open interval gets positive measure and the collection of measurable sets is all subsets of $[0, 1)$?*

Exercise 3 *Consider $[0, 1]$ with Lebesgue measure as a probability space. Write each $\omega \in [0, 1]$ in its dyadic expansion $\omega = \omega_1\omega_2 \cdots$, i.e.,*

$$\omega = \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}.$$

This expansion is unique for almost every x (which suffices for this exercise). Define the random variable:

$$Y(\omega) = \sum_{n=1}^{\infty} \frac{2\omega_n}{3^n}.$$

1. *What is the distribution function for Y? Show it is continuous.*

2. *Find a Borel subset B of $[0, 1]$ which has zero Lebesgue measure for which $\mathbf{P}\{Y \in B\} = 1$.*

Exercise 4 *Suppose $f : (X, \mathcal{F}) \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable. Let*

$$\sigma(f) = \{f^{-1}(B) : B \text{ Borel } \}$$

Show that $\sigma(f)$ is a σ-algebra and that $\sigma(f)$ is the smallest σ-algebra $\mathcal{G}$ for which $f : (X, \mathcal{G}) \to \mathbb{R}$ is measurable. Show that $\sigma(g \circ f) \subset \sigma(f)$ and give an example to show that the inclusion can be strict.

Exercise 5 *Suppose X is a nonnegative random variable and $\alpha > 0$.*

1. *Show that*

$$\mathbf{E}[X^\alpha] = \alpha \int_0^\infty x^{\alpha-1}\, \mathbf{P}\{X \geq x\}\, dx.$$

2. *Show that* $\mathbf{E}[X^\alpha] < \infty$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\alpha-1}\mathbf{P}\{X \geq n\} < \infty.$$

Exercise 6 *Consider a pertmutation* σ *chosen at random from the uniform distribution on the symmetric group of* n *elements. Let*

$$X = \#\{j : \sigma(j) = j\}.$$

1. *Show that*

$$\left|\mathbf{P}\{X = 0\} - \frac{1}{e}\right| \leq \frac{1}{(n+1)!}. \tag{1}$$

2. *Show that if* k *is a positive integer,*

$$\lim_{n\to\infty} \mathbf{P}\{X = k\} = \frac{1}{k!\,e}.$$

3. *A popular science magazine a number of years ago gave a problem to determine*

$$\lim_{n\to\infty} \mathbf{P}\{X = 0\}.$$

The official solution was that the probability was exactly

$$\left(1 - \frac{1}{n}\right)^n$$

and hence had limit $1/e$*. Show that the answer above is NOT exactly correct for large* n *by estimating how close it is to* $1/e$*.*