

Math 312, Autumn 2007
Problem Set 4

Rudin, Chapter 4: 2, 3, 5, 6, 7, 9, 13, 14, 17, 18
Probability Notes: Exercise 4.5

Exercise 1 Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $H_{\mathcal{F}}$ denote the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Suppose $\mathcal{G} \subset \mathcal{F}$ is a sub σ -algebra, and let $H_{\mathcal{G}}$ denote the corresponding Hilbert space of \mathcal{G} -measurable, square-integrable random variables.

1. Show that $H_{\mathcal{G}}$ is a closed subspace of $H_{\mathcal{F}}$.
2. If $X \in H_{\mathcal{F}}$ denote by $\mathcal{E}(X | \mathcal{G})$ the projection of X onto $H_{\mathcal{G}}$. Show that for all events $A \in \mathcal{G}$,

$$\mathbf{E}[X 1_A] = \mathbf{E}[\mathcal{E}(X | \mathcal{G}) 1_A] \quad (1)$$

3. Suppose X is a random variable in $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}[|X|] < \infty$ (note that we do not assume that $\mathbf{E}[X^2] < \infty$). Show that there is an integrable random variable $\mathcal{E}(X | \mathcal{G})$ such that (1) holds for all $A \in \mathcal{G}$. (You may wish to consider $X \geq 0$ first. You may not use the Radon-Nikodym theorem.)
4. Suppose that X, Y are square-integrable random variables and Y is \mathcal{G} -measurable. Show that

$$\mathcal{E}(XY | \mathcal{G}) = Y \mathcal{E}(X | \mathcal{G}).$$

5. Suppose $\tilde{\mathcal{G}}$ is independent of \mathcal{G} and X is $\tilde{\mathcal{G}}$ -measurable. What is $\mathcal{E}(X | \mathcal{G})$?
6. Show that

$$\mathbf{E}[\mathcal{E}(X | \mathcal{G})^2] \leq \mathbf{E}[X^2],$$

where we allow infinity as a possible value for the expectation. Give an example to show that the left-hand side can be finite and the right-hand side infinite.